On Disjoint Chains of Subsets

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Abstract

We prove the following theorem concerning the poset of all subsets of $[n]$ ordered by inclusion. Consider any two equal-size families of subsets of $[n]$, $\mathcal{S}$ and $\mathcal{R}$, where within each family all subsets have the same number of elements. Suppose there exists a bijection $\phi : \mathcal{S} \mapsto \mathcal{R}$ such that $A \supset f(A)$ for all $A \in \mathcal{S}$. Then there exist $|\mathcal{S}|$ disjoint saturated chains containing all the subsets in $\mathcal{S}$ and $\mathcal{R}$.

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1 Introduction

We consider the poset of all subsets of $X = [n]$ ordered by inclusion and denoted $\mathcal{P}(X)$. The family of all subsets of X with cardinality i is denoted $X^{(i)}$. In what follows we shall be interested only in *saturated* chains[1] in $\mathcal{P}(X)$, and for sake of succinctness we refer to them simply as chains. We prove the following theorem.

Theorem 1 *Let $\mathcal{S} \subseteq X^{(s)}$ and $\mathcal{R} \subseteq X^{(r)}$, where $|\mathcal{S}| = |\mathcal{R}| = m$ and $r < s$. Assume there exists a bijection $\phi : \mathcal{S} \mapsto \mathcal{R}$, such that $A \supset f(A)$ for all $A \in \mathcal{S}$. (In other words, for each pair of subsets $A \in \mathcal{S}$ and $\phi(A) \in \mathcal{R}$, there is a chain that contains both.) Then there exist m* disjoint *chains that contain all subsets in $\mathcal{S}$ and $\mathcal{R}$.*

The above theorem was initially proved and applied in the context of analyzing an algorithm for "testing monotonicity" [3].[2] While subsequently a simpler technique was found for proving the correctness of the testing algorithm [4], we believe that Theorem 1 is of independent interest and related results could be useful in the context of *packet routing* on the hypercube network.

Chains that correspond to the matching. Note that Theorem 1 *does not* hold if one requires that the disjoint chains exactly correspond to the given bijection ϕ. A counter-example (suggested by Dan Kleitmann) to this stronger claim is depicted in Figure 1. Here $m = 8$, but there are no 8 disjoint (saturated) chains that correspond to the given bijection. More generally, it can be shown [2] that if the chains are required to correspond to a particular bijection then the number of disjoint chains can be as small as $O(m/n)$.

[1]A chain $\mathcal{C}$ is said to be saturated if for every two subsets $A, B \in \mathcal{C}$ such that $A \subset B$, *every* subset D, such that $A \subset D \subset B$ also belongs to the chain $\mathcal{C}$.

[2]An algorithm for testing monotonicity is given query-access to a function $f : \{0,1\}^n \to \{0,1\}$, and a parameter ϵ. It is required to determine whether f is a monotone function or whether f is *ϵ-far* from being monotone (that is, more than an ϵ-fraction of its values must be altered so that it become monotone).

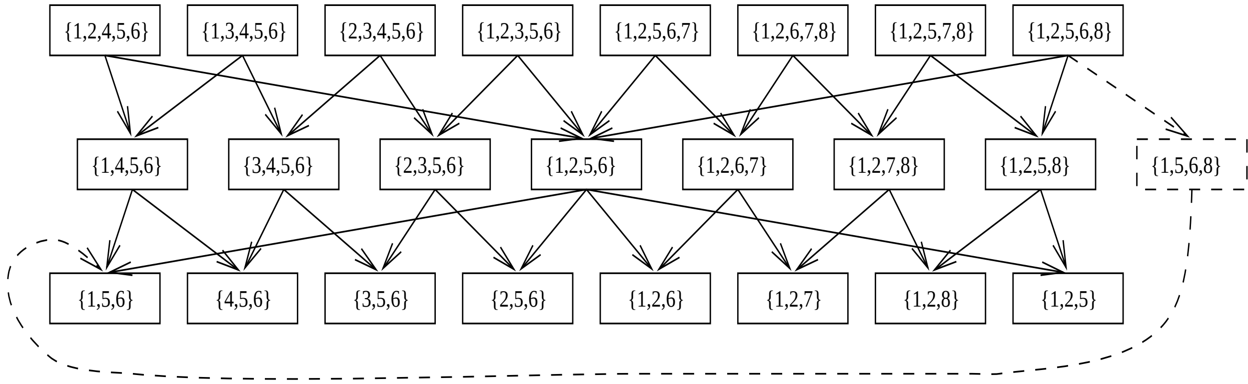

Figure 1: An example in which there *aren't* enough disjoint chains that correspond to a *particular* bijection. Each subset at the top level is mapped to the subset directly below in the bottom level. Since the union of all chains that correspond to the bijection contains only 7 subsets from the intermediate level, there exist no 8 disjoint chains corresponding to the bijection. However, there are other bijections between the two families for which disjoint chains do exist. For example, consider the "circular shift-to-right" association and use the auxiliary subset on the right.

2 Proof of Theorem 1

It will be useful for our purposes to consider the *Hasse diagram* of $\mathcal{P}(X)$, which we denote by H_n. This is the directed graph with vertex set consisting of all subsets of X and with a directed edge from vertex (set) A to vertex B if and only if $B = A \setminus \{i\}$ for some $i \in A$. By a "path" in H_n, we shall always mean a directed path. Thus, there is a path from each $A \in \mathcal{S}$ to $\phi(A) \in \mathcal{R}$. We would like to show that there are $m = |\mathcal{S}| = |\mathcal{R}|$ *vertex-disjoint* paths in H_n between $\mathcal{S}$ and $\mathcal{R}$.

We shall prove Theorem 1 by induction on the cardinality m of the families $\mathcal{R}$ and $\mathcal{S}$, and on the distance between them, $d = s - r$. The base cases, i.e., the case where $m = 1$ and $d \geq 1$, and the case where $d = 1$ and $m \geq 1$, clearly hold. Consider general $m > 1$ and $d > 1$, and assume by induction that the claim holds for every pair m' and d' such that either $m' < m$ and $d' \leq d$ or $m' \leq m$ and $d' < d$. Recall that $\mathcal{S} \subseteq X^{(s)}$, and $\mathcal{R} \subseteq X^{(r)}$. Let $\mathcal{Q}$ be the set of vertices in $X^{(s-1)}$ that are on a path going from some vertex in $\mathcal{S}$ to some vertex in $\mathcal{R}$, and let $\mathcal{P}$ be the set of vertices in $X^{(r+1)}$ that are on such directed paths from $\mathcal{S}$ to $\mathcal{R}$ (see Figure 2). We shall prove the induction claim in two steps. In the first step, we use the induction hypothesis (for $m' < m$ and $d' = d$) to show that either $|\mathcal{Q}| \geq m$ or $|\mathcal{P}| \geq m$ (or both). In the second step, we use this fact together with the induction hypothesis (for $m' < m$ and $d' = d$ *and* for $m' = m$ and $d' < d$) to prove the induction claim.

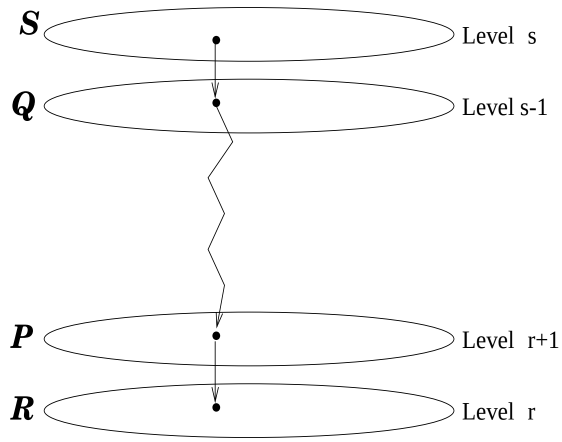

Figure 2: The families $\mathcal{S}$, $\mathcal{R}$, $\mathcal{Q}$, and $\mathcal{P}$.

Step 1: Either $|\mathcal{Q}| \geq m$ or $|\mathcal{P}| \geq m$.

Proof: Consider the subgraph H'_n of H_n consisting of $\mathcal{S}$, $\mathcal{R}$ and all vertices and edges on paths between $\mathcal{S}$ and $\mathcal{R}$. For any vertex A in H'_n, let $\mathrm{out}(A)$ denote the outdegree of A in H'_n, and let $\mathrm{in}(A)$ denote its indegree in H'_n.

Claim 1.1: *Let A and B be vertices in H'_n such that there is a path from A to B. Then* $\mathrm{out}(A) \geq \mathrm{out}(B)$ *and* $\mathrm{in}(A) \leq \mathrm{in}(B)$.

Proof: We prove the claim concerning outdegrees; the other inequality is proved similarly. Suppose that there is an edge from B to $B \setminus \{i\}$ in H'_n. We will show that there is a corresponding edge from A to $A \setminus \{i\}$ in H'_n by exhibiting a path from a vertex in $\mathcal{R}$ to a vertex in $\mathcal{S}$ that contains this edge. The claim follows. We construct such a path by concatenating a path from some vertex in

$\mathcal{R}$ to A (which exists since A is in H'_n), the edge from A to $A \setminus \{i\}$, a path from $A \setminus \{i\}$ to $B \setminus \{i\}$ (obtained by deleting i from each vertex on the path from A to B), and a path from $B \setminus \{i\}$ to an element of $\mathcal{R}$ (which exists since $B \setminus \{i\}$ is in H'_n by supposition). □

The above claim can be strengthened to show that $\mathrm{out}(A)$ exceeds $\mathrm{out}(B)$ by at least the length of the path joining A and B. (And similarly for indegrees.) This is because there are also edges from A to $A \setminus \{i\}$ for each $i \in A \setminus B$.

Claim 1.2: *If both $|\mathcal{Q}| < m$ and $|\mathcal{P}| < m$ then (by the induction hypothesis), $\sum_{A\in\mathcal{S}} \mathrm{out}(A) > \sum_{B\in\mathcal{P}} \mathrm{out}(B)$, and $\sum_{C\in\mathcal{R}} \mathrm{in}(C) > \sum_{D\in\mathcal{Q}} \mathrm{in}(D)$,*

Proof: Again, we prove the claim concerning outdegrees, and the claim about indegrees is proved analogously. First, suppose all vertices in $\mathcal{S}$ have the same outdegree. By Claim 1.1 this outdegree is an upper bound on the outdegree of vertices in $\mathcal{P}$. Since $|\mathcal{P}| < m$, the claim follows.

Otherwise, let $\mathcal{S}_1 \subset \mathcal{S}$ be the set of all vertices with minimum outdegree, denoted k_1. Since $m_1 = |\mathcal{S}_1| < m$, we may apply the induction hypothesis and obtain that there are m_1 vertex disjoint paths between $\mathcal{S}_1$ and $\phi(\mathcal{S}_1)$. The number of vertices from $\mathcal{P}$ that reside on these paths is m_1, and by Claim 1.1, they all have outdegree at most k_1. In other words, the number of vertices in $\mathcal{P}$ that have outdegree at most k_1 is greater or equal to the number of vertices in $\mathcal{S}$ that have outdegree (at most) k_1.

By the same reasoning we get that for *every* degree q, the number of vertices in $\mathcal{P}$ that have outdegree at most q is greater or equal to the number of vertices in $\mathcal{S}$ that have outdegree at most q. But since $|\mathcal{P}| < m$ while $|\mathcal{S}| = m$, and $\max_{B\in\mathcal{P}} \mathrm{out}(B) \leq \max_{A\in\mathcal{S}} \mathrm{out}(A)$, Claim 1.2 follows.
□

By combining the two parts of Claim 1.2, we have that if both $|\mathcal{P}| < m$ and $|\mathcal{Q}| < m$, then

$$\sum_{A\in\mathcal{S}} \mathrm{out}(A) + \sum_{C\in\mathcal{R}} \mathrm{in}(C) > \sum_{B\in\mathcal{P}} \mathrm{out}(B) + \sum_{D\in\mathcal{Q}} \mathrm{in}(D) \tag{1}$$

However, the number of edges going out of vertices in $\mathcal{S}$ must equal the number of edges entering vertices in $\mathcal{Q}$, and similarly the number of edges entering vertices in $\mathcal{R}$ must equal the number of edges going out of $\mathcal{P}$. In other words, $\sum_{A\in\mathcal{S}} \text{out}(A) = \sum_{D\in\mathcal{Q}} \text{in}(D)$ and $\sum_{C\in\mathcal{R}} \text{in}(C) = \sum_{B\in\mathcal{P}} \text{out}(B)$, contradiction Equation 1. Thus, we must have that either $|\mathcal{P}| \geq m$ or $|\mathcal{Q}| \geq m$, and the proof of Step 1 is completed. ■

Step 2: There exist vertex disjoint paths from $\boldsymbol{\mathcal{S}}$ to $\boldsymbol{\mathcal{R}}$.

Proof: From Step 1 we have that either $|\mathcal{Q}| \geq m$ or $|\mathcal{P}| \geq m$. Assume the former is true — we shall see that this can be done without loss of generality. We next show that (1) there exists a bijection between $\mathcal{S}$ and some $\mathcal{Q}' \subseteq \mathcal{Q}$ (where here each corresponding pair simply has an edge between them); and (2) there exists a bijection ϕ' from $\mathcal{Q}'$ to $\mathcal{R}$ so that there is a path from each $A \in \mathcal{Q}'$ to $\phi'(A)$. Given (2) we can apply the induction hypothesis for $d' = d - 1$ (and $m' = m$) on $\mathcal{Q}'$ and $\mathcal{R}$, and by combining with (1) we get the desired paths from $\mathcal{S}$ to $\mathcal{R}$.

We actually prove both (1) and (2) together. Consider the following auxiliary directed graph, K. It has a single source vertex s, a single target vertex t, and the rest of the vertices are partitioned into three layers corresponding to $\mathcal{S}$, $\mathcal{Q}$ and $\mathcal{R}$, respectively. There is an edge from s to each of the vertices in $\mathcal{S}$, and from each of the vertices in $\mathcal{R}$ to t. The edges between $\mathcal{S}$ and $\mathcal{Q}$ are as in H'_n and edges between $\mathcal{Q}$ and $\mathcal{R}$ correspond to all (not necessarily disjoint) directed paths in H'_n. We show that the minimum s–t vertex-separator in K has size m. Items (1) and (2) follow by one of the variations of Menger's Theorem (see [1, Thm. 11.6]), which guarantees the existence of m disjoint paths from s to t in the graph K.

Assume in contradiction that there exists a vertex-separator $\mathcal{C}$ of size smaller than m in K. Let $m_1 \stackrel{\text{def}}{=} |\mathcal{C} \cap \mathcal{S}|$, $m_2 \stackrel{\text{def}}{=} |\mathcal{C} \cap \mathcal{Q}|$, and $m_3 \stackrel{\text{def}}{=} |\mathcal{C} \cap \mathcal{R}|$. Consider the subset of vertices $\mathcal{S}' \subseteq \mathcal{S}$ that do not belong to $\mathcal{C}$ and are not mapped by ϕ to vertices in $\mathcal{R} \cap \mathcal{C}$. The size of $\mathcal{S}'$ is at least

$$m' = m - (m_1 + m_3) > |\mathcal{C}| - (m_1 + m_3) = m_2$$

Let $\mathcal{R}' \stackrel{\text{def}}{=} \phi(\mathcal{S}')$, and let $\mathcal{Q}'$ be the subset of vertices in $\mathcal{Q}$ that are on directed paths in H_n' going from vertices in $\mathcal{S}'$ to vertices in $\mathcal{R}'$.

We consider two cases. If $\mathcal{S}' = \mathcal{S}$ (i.e., $\mathcal{C} \subseteq \mathcal{Q}$) then $\mathcal{Q}' = \mathcal{Q}$, and since $|\mathcal{C}| < m \leq |\mathcal{Q}|$, there exists at least one vertex in $\mathcal{Q} \setminus \mathcal{C}$ on a path from a vertex in $\mathcal{S}$ to a vertex in $\mathcal{R}$, contradicting the assumption that $\mathcal{C}$ is a vertex separator. If $\mathcal{S}' \subset \mathcal{S}$, then by the induction hypothesis (for $m' = |\mathcal{S}'| < m$ and $d' = d$), there exist vertex disjoint paths in H_n' from $\mathcal{S}'$ to

$calR'$ and hence necessarily $|\mathcal{Q}'| \geq |\mathcal{S}'| > m_2$. Since $|\mathcal{C} \cap \mathcal{Q}| = m_2$, we again reach contradiction to the assumption that $\mathcal{C}$ is a vertex separator. ■ (Step 2 and Theorem 1)

3 Open Problems

An elegant problem related to the one addressed here remains open. As before, let $\mathcal{S} \subseteq X^{(s)}$ and $\mathcal{R} \subseteq X^{(r)}$, where $|\mathcal{S}| = |\mathcal{R}|$ and $r < s$. Let $\phi : \mathcal{S} \mapsto \mathcal{R}$ be a bijection such that $A \supset \phi(A)$ for all $A \in \mathcal{S}$. Must there exist a set of *edge-disjoint* paths in the Hasse diagram H_n containing a path from A to $\phi(A)$ for every $A \in \mathcal{S}$?

This question differs from the previous one in two respects. First, we require a set of paths consistent with the given bijection ϕ, as opposed to an arbitrary one. Second, the earlier condition that paths be vertex-disjoint is relaxed; now, only edge-disjointness is required. (Without this relaxation, the answer would be negative, as shown in Figure 1.) It appears that a counter-example must be sizable, if one exists at all.

In fact, a stronger assertion may hold; must there still exist edge-disjoint paths if $\mathcal{S}$ and $\mathcal{R}$ are arbitrary, equal-sized subsets of X? A positive answer would imply more efficient monotonicity testing and progress on some long-standing questions regarding routing in the hypercube network.

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