Some Useful Probabilistic Facts

For a random variable χ we denote the expected value of χ by $\mathrm{Exp}[\chi]$.

Linearity of Expectation

For any two random variables χ_1 and χ_2, $\mathrm{Exp}[\chi_1 + \chi_2] = \mathrm{Exp}[\chi_1] + \mathrm{Exp}[\chi_2]$.

Union Bound

Let E_1 and E_2 be two events over the same probability space. Then $\Pr[E_1 \cup E_2] \leq \Pr[E_1] + \Pr[E_2]$.

Markov's Inequality

This is the most basic inequality on the deviation of a random variable from its expectation: it assumes very little (but is also weaker than the others that follow below). Let χ be a non-negative random variable. Then for any positive k,

$$\Pr[\chi \geq k \cdot \mathrm{Exp}[\chi]] \leq \frac{1}{k}$$

Chebishev's Inequality

Recall the definition of the variance of a random variable χ:

$$\mathrm{Var}[\chi] \stackrel{\mathrm{def}}{=} \mathrm{Exp}[(\chi - \mathrm{Exp}[\chi])^2] = \mathrm{Exp}[\chi^2] - (\mathrm{Exp}[\chi])^2$$

(where the equality follows from the linearity of the expectation). Then for any $t > 0$,

$$\Pr[|\chi - \mathrm{Exp}[\chi]| \geq t \cdot \mathrm{Var}^{1/2}[\chi]] \leq \frac{1}{t^2}$$

In particular, consider the case that $\chi = \sum_{i=1}^{m} \chi_i$ where the χ_i's are *pairwise* independent random variables. Let $p \stackrel{\mathrm{def}}{=} \frac{1}{m} \sum_{i=1}^{m} \mathrm{Exp}[\chi_i]$. Then for any $\gamma > 0$ we can get that:

$$\Pr\left[\left|\frac{1}{m} \sum_{i} \chi_i - p\right| \geq \gamma\right] \leq \frac{\sum_i \mathrm{Var}[\chi_i]}{m^2 \gamma^2}$$

Chernoff/Heoffding Inequalities

Let $\chi_1, \ldots, \chi_m$ be m independent random variables where $\chi_i \in [0,1]$ for every $1 \le i \le m$. Let $p \stackrel{\text{def}}{=} \frac{1}{m} \sum_i \mathrm{Exp}[\chi_i]$. (A special case, which occurs quite often, is when $\chi_i \in \{0,1\}$ (*Bernoully* random variables), and $\Pr[\chi_i = 1] = p$ for every i (i.e., the random variables are equally distributed).) Then, for every $\gamma \in (0,1]$, the following bounds hold:

- (Additive Form)

  $$\Pr\left[\frac{1}{m} \cdot \sum_{i=1}^{m} \chi_i \;>\; p + \gamma\right] < \exp\left(-2\gamma^2 m\right)$$

  and

  $$\Pr\left[\frac{1}{m} \cdot \sum_{i=1}^{m} \chi_i \;<\; p - \gamma\right] < \exp\left(-2\gamma^2 m\right)$$

- (Multiplicative Form)

  $$\Pr\left[\frac{1}{m} \cdot \sum_{i=1}^{m} \chi_i \;>\; (1+\gamma)p\right] < \exp\left(-\gamma^2 pm/3\right)$$

  and

  $$\Pr\left[\frac{1}{m} \cdot \sum_{i=1}^{m} \chi_i \;<\; (1-\gamma)p\right] < \exp\left(-\gamma^2 pm/2\right)$$

Also, for every $k > 1$ we have

$$\Pr\left[\frac{1}{m} \cdot \sum_{i=1}^{m} \chi_i > k \cdot p\right] < \left(\frac{e^{k-1}}{k^k}\right)^{pm}$$