Distribution of Noncentral Indefinite Quadratic Forms in Complex Normal Variables

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***Abstract*— A new series expansion is developed for the probability distribution function and the cumulative distribution function for indefinite noncentral Hermitian quadratic forms in complex normal random variables. The moment generating function is inverted by contour integration using the residue theorem. The function is separated into two parts, one part, containing an essential singularity, is expanded by Laurent series and the other part is expanded by Taylor series. The series are combined for evaluating the residue of the complete function. Several different series can be obtained by modifications of the basic approach. The series are computationally efficient and normally fast converging. The convergence rate depends on the separation of the eigenvalues. Multiple eigenvalues are allowed, and can be used to approximately replace a close pair of eigenvalues.**

***Index Terms*—Quadratic forms, probability distributions, series expansions, noncoherent detection, diversity combining.**

I. Introduction

The quadratic form (QF) in normal random variables [1], [2, ch. 29] appears in many applications in statistics and in communication theory. The applications in communication theory include noncoherent detection [3]–[5], diversity combining [6, pp. 416–489, 590–595], [7, pp. 777–792, 887–896], multiple access, array processing, estimation of power spectra [29], and many others. Applications in statistics include χ^2 tests and analysis of variance [2, ch. 29]. Applications in optimal control are described in [33]. Much attention was given over the years to the problem of evaluating the distribution of a QF, and various series expansions and approximations were developed [8]–[35]. However, most of the authors considered positive-definite or semidefinite cases. Relatively little attention has been devoted to the problem of obtaining the distribution of an indefinite QF. Several series expansions or approximations for the indefinite case were developed in [1], [2, ch. 29], [8]–[17]. Because most of these are very complex, their practical usefulness is limited. Some references provide a solution for restricted cases [7, pp. 882–886], [16] or the central case [1, sec. 4.3b]. Numerical methods like numerical integration [4, Appendix B], [18] are commonly used for evaluation of the distribution of an indefinite QF. Here we present a new technique for developing series expansions which results in some computationally very efficient expansions.

Let A be an $N \times N$ Hermitian matrix, and $\boldsymbol{r}$ be a complex normal random vector with covariance matrix V and expected value η. A complex QF in $\boldsymbol{r}$ is defined by

$$Y = \tfrac{1}{2}\boldsymbol{r}^{\dagger} A \boldsymbol{r} \tag{1}$$

where "†" denotes transpose conjugate. A QF is said to be definite or indefinite depending on whether the matrix A is definite or indefinite. If $\eta = 0$, the QF is said to be central, otherwise it is noncentral. The QF in complex random vectors is equivalent to a special case of a real QF ($\boldsymbol{r}$ is real).

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II. Series Expansion

Let the matrix L be any nonsingular factorization of V, such that $V = 2L^{\dagger}L$. According to [1] and [2, ch. 29], the QF in (1) can be expressed as

$$Y = \tfrac{1}{2} Z^{\dagger}\Lambda Z = \tfrac{1}{2}\sum_{i=0}^{N-1}\Lambda_i |Z_i|^2 \tag{2}$$

where Z is a Gaussian vector with independent components whose real and imaginary parts both have a unit variance and whose expected value is $E[Z] = Q^{\dagger}L^{-1\dagger}\eta$. The matrix Λ is the diagonal eigenvalue matrix of $LAL^{\dagger}$ with diagonal elements Λ_i, and Q is the corresponding unitary eigenvector matrix. In the same manner, a real QF can be expressed as

$$Y = \sum_{j=0}^{N^{(R)}-1} \lambda_j^{(R)} W_j^2 \tag{3}$$

where W_i are real independent normal random variables with unit variance and mean ω_i. We can immediately see that a complex QF is equivalent to a real QF in which the eigenvalues are given in equal pairs. Here we develop new series expansions for the pdf (probability density function) and for the cdf (cumulative distribution function) of the complex QF.

Let us reorder and group together the eigenvalues so that there are N' different eigenvalues $\lambda_k, k = 0, \cdots, N'-1$, each with multiplicity m_k. Evidently $N = \Sigma m_k$. In addition, define

$$\mu_k^2 = \sum_{m_k} |E[Z_i]|^2 \tag{4}$$

where the sum is performed on the set of m_k indices corresponding to λ_k. The QF is then a weighted sum of independent χ^2 random variables of order $2m_k$ and noncentrality parameters μ_k^2.

The moment-generating function of Y in (2) from [19] rearranged using (4) is[1]

$$\Phi(s) = \exp\left\{-\frac{1}{2}\sum_j \mu_j^2 + \frac{1}{2}\sum_j \frac{\mu_j^2}{1-\lambda_j s}\right\} \cdot \prod_j \frac{1}{(1-\lambda_j s)^{m_j}}. \tag{5}$$

Mathai [1, sec. 4.3b] has used the residue theorem for inverting the moment generating function in the central indefinite case. The finite-order poles in that case allow a finite sum expression. In the noncentral case there are essential singularities, and we are going to express the residues as infinite sums. By the Laplace inversion theorem [36, pp. 223–224], we get the pdf

$$p(y) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\sigma-iR}^{\sigma+iR} e^{-sy}\Phi(s)\,ds. \tag{6}$$

We choose $\sigma = 0$, where the function is always analytic. Since $\Phi(s)$ vanishes at infinity, we can replace the integral by an integral along

[1] In all the following summations and products, when the range is not specified, it is assumed to be from 0 to $N'-1$.

contour C circling the entire right half plane for $y \geq 0$ or the left half plane for $y < 0$. Furthermore, we can use the residue theorem for evaluating this contour integral. For $y \geq 0$

$$\begin{aligned} p(y) &= -\frac{1}{2\pi i} \oint_C e^{-sy} \Phi(s)\, ds \\ &= -\sum_{\lambda_k > 0} \operatorname{Res} \exp(-\lambda_k^{-1} y) \Phi(\lambda_k^{-1}) \end{aligned} \tag{7}$$

and for $y < 0$

$$p(y) = \sum_{\lambda_k < 0} \operatorname{Res} \exp(-\lambda_k^{-1} y) \Phi(\lambda_k^{-1}). \tag{8}$$

Let us proceed with the case of $y > 0$. We can alternatively write

$$p(y) = -\sum_{\lambda_k > 0} \operatorname{Res} F_k(0) \tag{9}$$

where

$$F_k(s) = \exp\{-(\lambda_k^{-1} + s)y\} \Phi(\lambda_k^{-1} + s). \tag{10}$$

For each function F_k, the pole at λ_k^{-1} was translated to $s = 0$ for convenience. The following derivation should be repeated for each eigenvalue λ_k. Recall that the eigenvalues are distinct.

$$\begin{aligned} F_k(s) &= \exp\left\{ -\frac{1}{2}\sum_j \mu_j^2 + \frac{1}{2}\sum_j \frac{\mu_j^2}{1 - \lambda_j(\lambda_k^{-1} + s)} - \lambda_k^{-1} y - sy \right\} \\ &\quad \cdot \prod_j \frac{1}{[1 - \lambda_j(\lambda_k^{-1} + s)]^{m_j}} \\ &= \exp\left\{ -\frac{1}{2}\sum_j \mu_j^2 + \frac{1}{2}\sum_{j\neq k} \frac{\mu_j^2}{\alpha_{kj} - \lambda_j s} - \frac{\mu_k^2}{2\lambda_k s} - \lambda_k^{-1} y - sy \right\} \\ &\quad \cdot \frac{1}{(-\lambda_k s)^{m_k}} \cdot \prod_{j\neq k} \frac{1}{(\alpha_{kj} - \lambda_j s)^{m_j}} \end{aligned} \tag{11}$$

where $\alpha_{kj} = 1 - \lambda_j/\lambda_k$. Let us define

$$\begin{aligned} g_k(s,y) &= \exp\left\{ \frac{1}{2}\sum_{j\neq k} \frac{\mu_j^2}{\alpha_{kj} - \lambda_j s} - sy \right\} \\ &\quad \cdot \prod_{j\neq k} \frac{1}{(\alpha_{kj} - \lambda_j s)^{m_j}} \end{aligned} \tag{12}$$

and

$$f_k(s) = \frac{1}{s^{m_k}} \exp\left\{ -\frac{\mu_k^2}{2\lambda_k s} \right\}. \tag{13}$$

Then

$$F_k(s) = \frac{1}{(-\lambda_k)^{m_k}} \exp\left\{ -\frac{1}{2}\sum_j \mu_j^2 - \lambda_k^{-1} y \right\} \cdot g_k(s,y) f_k(s). \tag{14}$$

We write $f_k(s)$ as a Laurent series and $g_k(s,y)$ as a Taylor series in s

$$f_k(s) = \sum_{n=-\infty}^{\infty} b_{kn} s^{-n-1} \tag{15}$$

$$g_k(s,y) = \sum_{n=0}^{\infty} a_{kn}(y) s^n, \qquad a_{kn}(y) = \frac{1}{n!} g_k^{(n)}(0,y). \tag{16}$$

The notation $g_k^{(n)}(s,y)$ denotes the nth derivative with respect to s. Since $f_k(s)$ is analytic for all $s \neq 0$, for any ϵ, (15) converges absolutely in an annulus $|s| > \epsilon > 0$. The expansion (16) converges absolutely in a circle of radius R around $s = 0$ for some R. By substituting (15) and (16) in (14) we can obtain a power series in s which converges absolutely in the annulus $R > |s| > \epsilon$. Hence, the coefficient of the s^{-1} term in the Laurent series expansion of $F_k(s)$, which is the residue of $F_k(s)$ at $s = 0$, can be expressed as

$$\operatorname{Res} F_k(0) = \frac{1}{(-\lambda_k)^{m_k}} \exp\left\{ -\frac{1}{2}\sum_j \mu_j^2 - \lambda_k^{-1} y \right\} \sum_{n=0}^{\infty} b_{kn} a_{kn}. \tag{17}$$

Next, we find a recursive formula for $g_k^{(n)}(s,y)$ with a technique similar to that in [1, sec. 4.3b].

$$\ln g_k(s,y) = \frac{1}{2}\sum_{j\neq k} \frac{\mu_j^2}{\alpha_{kj} - \lambda_j s} + \sum_{j\neq k} m_j \ln\left(\frac{1}{\alpha_{kj} - \lambda_j s} \right) - sy. \tag{18}$$

$$[\ln g_k(s,y)]^{(1)} = \frac{1}{2}\sum_{j\neq k} \frac{\lambda_j \mu_j^2}{(\alpha_{kj} - \lambda_j s)^2} + \sum_{j\neq k} \frac{m_j \lambda_j}{\alpha_{kj} - \lambda_j s} - y. \tag{19}$$

By induction we get

$$[\ln g_k(s,y)]^{(n)} = \frac{1}{2}\sum_{j\neq k} \frac{n! \lambda_j^n \mu_j^2}{(\alpha_{kj} - \lambda_j s)^{n+1}} + \sum_{j\neq k} \frac{(n-1)! \lambda_j^n m_j}{(\alpha_{kj} - \lambda_j s)^n}. \tag{20}$$

Taking the $(n-1)$st derivative of both sides of $g_k^{(1)} = g_k(\ln g_k)^{(1)}$ yields

$$g_k^{(n)} = \sum_{l=0}^{n-1} \binom{n-1}{l} g_k^{(l)} (\ln g_k)^{(n-l)}, \qquad n \geq 1. \tag{21}$$

Now we can recursively compute $g_k^{(n)}(0,y)$ from

$$g_k(0,y), g_k^{(1)}(0,y), \cdots, g_k^{(n-1)}(0,y).$$

Let $\beta_k = \mu_k^2/(2\lambda_k)$, then

$$f_k(s) = \frac{1}{s^{m_k}} \exp(-\beta_k/s) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-\beta_k)^n}{s^{n+m_k}}. \tag{22}$$

Hence

$$b_{kn} = \begin{cases} \dfrac{1}{(n - m_k + 1)!} (-\beta_k)^{(n-m_k+1)}, & \text{if } n \geq m_k - 1 \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

Substituting (23) into (17) we get

$$\begin{aligned} \operatorname{Res} F_k(0) &= \frac{1}{(-\lambda_k)^{m_k}} \exp\left\{ -\frac{1}{2}\sum_j \mu_j^2 - \lambda_k^{-1} y \right\} \\ &\quad \cdot \sum_{n=m_k-1}^{\infty} \frac{g_k^{(n)}(0,y)}{n!(n - m_k + 1)!} (-\beta_k)^{(n-m_k+1)}. \end{aligned} \tag{24}$$

Using (9), we get

$$\begin{aligned} p(y) &= -\exp\left\{ -\frac{1}{2}\sum_j \mu_j^2 \right\} \sum_{\lambda_k > 0} \frac{1}{(-\lambda_k)^{m_k}} \exp(-\lambda_k^{-1} y) \\ &\quad \cdot \sum_{n=m_k-1}^{\infty} g_k^{(n)}(0,y) \frac{(-\beta_k)^{(n-m_k+1)}}{n!(n - m_k + 1)!}. \end{aligned} \tag{25}$$

It is difficult to integrate this expression for $p(y)$ in order to obtain the cdf, since $g_k^{(n)}(0,y)$ depends on y. However, we can easily get the cdf series directly. For a contour C circling the right half plane, but excluding the imaginary axis

$$\begin{aligned}\Pr\{Y \geq y\} &= \int_y^\infty p(\tau)\, d\tau \\ &= -\frac{1}{2\pi i}\oint_C \frac{1}{s}e^{-sy}\Phi(s)\, ds \end{aligned} \tag{26}$$

where $y > 0$ is assumed (for $y < 0$ the integration range should be from $-\infty$ to y). Let us define

$$\hat{F}_k(s) = \frac{1}{\lambda_k^{-1}+s}\exp\{-(\lambda_k^{-1}+s)y\}\Phi(\lambda_k^{-1}+s) \tag{27}$$

so

$$\int_y^\infty p(\tau)\, d\tau = -\sum_{\lambda_k>0} \operatorname{Res}\hat{F}_k(0) \tag{28}$$

and define

$$\begin{aligned}\hat{g}_k(s,y) = \exp&\left\{\frac{1}{2}\sum_{j\neq k}\frac{\mu_j^2}{\alpha_{kj}-\lambda_j s} - sy\right\}\frac{1}{\lambda_k^{-1}+s} \\ &\cdot \prod_{j\neq k}\frac{1}{(\alpha_{kj}-\lambda_j s)^{m_j}}.\end{aligned} \tag{29}$$

Leaving the function $f_k(s)$ unchanged, we obtain, similarly to (14)

$$\hat{F}_k(s) = \frac{1}{(-\lambda_k)^{m_k}}\exp\left\{-\frac{1}{2}\sum_j \mu_j^2 - \lambda_k^{-1}y\right\}\cdot \hat{g}_k(s,y)\cdot f_k(s). \tag{30}$$

We then follow the same procedure as before to reach the final expression. By defining $\alpha_{kk} \triangleq -1$ (α_{kk} has not been used previously), we obtain an expression very close to (12), and such that the expressions (18)–(21) can be used to evaluate $\hat{g}_k^{(n)}(0,y)$ with only a minor change in the range variable:.

$$\hat{g}_k(s,y) = -\lambda_k\exp\left\{\frac{1}{2}\sum_{j\neq k}\frac{\mu_j^2}{\alpha_{kj}-\lambda_j s} - sy\right\}\prod_j \frac{1}{(\alpha_{kj}-\lambda_j s)^{m_{kj}}} \tag{31}$$

where $m_{kj} = m_j$ for $j \neq k$, and $m_{kk} = 1$. Finally

$$\begin{aligned}\Pr\{Y\geq y\} = -\exp&\left\{-\frac{1}{2}\sum_j \mu_j^2\right\}\sum_{\lambda_k>0}\frac{1}{(-\lambda_k)^{m_k-1}}\exp(-\lambda_k^{-1}y) \\ &\cdot \sum_{n=m_k-1}^{\infty}\hat{g}_k^{(n)}(0,y)\frac{(-\beta_k)^{(n-m_k+1)}}{n!(n-m_k+1)!}.\end{aligned} \tag{32}$$

III. An Alternative Series Expansion

The expression for $p(y)$ that we derived has one disadvantage, namely, that the dependence on y is complicated, or not explicit. This may cause difficulties in using the expression in further analysis, e.g., integration to find the mean of a function of Y.

Let us assume $m_k = 1$ for simplicity. We use the same technique as in the previous section, but partition $F_k(s)$ differently. Let us define

$$\tilde{g}_k(s) = \exp\left\{\frac{1}{2}\sum_{j\neq k}\frac{\mu_j^2}{\alpha_{kj}-\lambda_j s}\right\}\cdot\prod_{j\neq k}\frac{1}{\alpha_{kj}-\lambda_j s} \tag{33}$$

and

$$\tilde{f}_k(s,y) = \frac{1}{s}\exp\left\{-\frac{\mu_k^2}{2\lambda_k s} - sy\right\}. \tag{34}$$

Note that we moved the e^{-sy} term from g to f. Equation (14) still holds with the new variables. Since $\tilde{g}_k(s) = g_k(s,0)$, the same method that has been described for computing $g_k^{(n)}(0,y)$ can be applied to $\tilde{g}_k(s)$ with $y = 0$.

Let us move on to evaluate b_{kn} by contour integration. Since the function $\tilde{f}_k(s.y)$ is analytic everywhere except at $s = 0$, we can arbitrarily choose any contour that circles the point $s = 0$. We choose a circular contour around the pole at $s = 0$

$$s = \varepsilon e^{i\phi} \qquad ds = \varepsilon e^{i\phi} i\, d\phi. \tag{35}$$

Then

$$\begin{aligned}b_{kn} &= \frac{1}{2\pi i}\oint s^n \tilde{f}_k(s,y)\, ds \\ &= \frac{1}{2\pi}\int_0^{2\pi}\varepsilon^n \exp\left\{in\phi - \varepsilon y e^{i\phi} - \frac{\mu_k^2}{2\lambda_k\varepsilon}e^{-i\phi}\right\}d\phi.\end{aligned} \tag{36}$$

Choosing $\varepsilon = \mu_k/\sqrt{2y\lambda_k}$, we get

$$\begin{aligned}b_{kn} &= \frac{\varepsilon^n}{2\pi}\int_0^{2\pi}\exp\left\{in\phi - \sqrt{\frac{2\mu_k^2 y}{\lambda_k}}\cos\phi\right\}d\phi \\ &= \varepsilon^n I_n\left(-\sqrt{\frac{2\mu_k^2 y}{\lambda_k}}\right).\end{aligned} \tag{37}$$

Here $I_n(x)$ is the modified Bessel function of the first kind of order n. Finally

$$\begin{aligned}p(y) &= -\sum_{\lambda_k>0}\operatorname{Res}F_k(0) \\ &= \exp\left\{-\frac{1}{2}\sum_k |\mu_k|^2\right\}\sum_{\lambda_k>0}\lambda_k^{-1} \\ &\quad\cdot \exp(-\lambda_k^{-1}y)\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{g}_k^{(n)}(0)\left(\frac{\mu_k^2}{2y\lambda_k}\right)^{n/2} I_n\left(-\sqrt{\frac{2\mu_k^2 y}{\lambda_k}}\right).\end{aligned} \tag{38}$$

It is interesting to show the result of the cdf obtained by integration of (38) from zero to infinity. Let

$$\begin{aligned}B_{kn} &= \lambda_k^{-1}\int_0^\infty \exp(-\lambda_k^{-1}y)\left(\frac{\mu_k^2}{2y\lambda_k}\right)^{n/2} I_n\left(-\sqrt{\frac{2\mu_k^2 y}{\lambda_k}}\right) \\ &= \begin{cases}\exp\left\{\frac{1}{2}\mu_k^2\right\}, & \text{if } n = 0\\ (-\lambda_k)^{-n}\cdot\frac{1}{(n-1)!}\exp\left\{\frac{1}{2}\mu_k^2\right\}\gamma\left(n, \frac{1}{2}\mu_k^2\right), & \\ \text{otherwise} & \end{cases}\end{aligned} \tag{39}$$

where $\gamma(n.x)$ is the incomplete gamma function

$$\gamma(n,x) = \int_0^x t^{n-1}e^{-t}\, dt. \tag{40}$$

Then

$$\int_0^\infty p(y)\, dy = \exp\left\{-\frac{1}{2}\sum_k \mu_k^2\right\}\cdot\sum_{\lambda_k>0}\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{g}_k^{(n)}(0)B_{kn}. \tag{41}$$

Now substitute $y + \gamma$ for y in (11), resulting in

$$F_k(s) = -\frac{1}{\lambda_k}\exp\left\{-\frac{1}{2}\sum_j \mu_j^2 - \frac{y+\gamma}{\lambda_k}\right\}\cdot g_k(s,y)\tilde{f}_k(s,\gamma). \tag{42}$$

By using this $F_k(s)$, (41) is changed to

$$\begin{aligned}\Pr\{Y \geq y\} &= \int_0^\infty p(y+\gamma)\,d\gamma \\ &= \exp\left\{-\frac{1}{2}\sum_k \mu_k^2\right\} \sum_{\lambda_k>0} \exp(-\lambda_k^{-1} y) \\ &\quad \cdot \sum_{n=0}^\infty \frac{1}{n!} g_k^{(n)}(0,y) B_{kn}. \end{aligned} \tag{43}$$

Note that the resulting series is computationally less efficient than (32) owing to the additional computation of the gamma functions.

By differentiating (43) with respect to y, one obtains an additional, but computationally less efficient, alternative series expansion for $p(y)$.

IV. Convergence Rate

The right-hand side of (20) is a multivariate polynomial in $\delta_{kj} = \lambda_k \alpha_{kj}^{-1}$. Therefore, for large n, $|[\ln g_k(0,y)]^{(n)}|$ can be approximated by $Kn!\delta^n$, where $\delta = |\lambda_{\min}\alpha_{\min}^{-1}|$, $K = |\mu_{\min}^2 \alpha_{\min}^{-1}/2|$, and $\alpha_{\min}$ is the smallest α_{jk} with the corresponding μ_j^2 and λ_j. Note that for $y > 0$ only the positive eigenvalues should be used for computing $\alpha_{\min}$. It was assumed that $\mu_{\min}^2$ is not close to zero, else the second term in (20) would dominate, leading to a very similar approximation. We are going to show that asymptotically for large n

$$|g_k^{(n)}(0,y)| < cn!\gamma^n \tag{44}$$

where $\gamma = \delta(K+1)$ and c is a constant. Then, the absolute value of the summand in (25) or (32) (in the sum over n) is approximately bounded by $c(\gamma\beta_{\min})^n/n!$. Since $\gamma \cong \mu_{\min}^2 \lambda_{\min} \alpha_{\min}^{-2}/2$, there is a strong dependence of the convergence rate on $\alpha_{\min}$.

We prove (44) by induction. Assuming that (44) is true for $n-1$, we shall prove that it is true for n. Note that the proof is based on approximations for large n. From (21)

$$\begin{aligned}|g_k^{(n)}(0,y)| &< \sum_{l=0}^{n-1} \binom{n-1}{l} |g_k^{(l)}||(\ln g_k)^{(n-l)}| \\ &< Kc \sum_{l=0}^{n-1} \frac{(n-1)!}{l!(n-1-l)!} l!\gamma^l (n-l)!\delta^{(n-l)} \\ &= Kc(n-1)!\delta^n \sum_{l=0}^{n-1} (n-l)(\gamma\delta^{-1})^l \\ &< Kcn!\delta^n \frac{(\gamma\delta^{-1})^n - 1}{\gamma\delta^{-1}-1} \cong Kcn!\gamma^n \frac{1}{\gamma\delta^{-1}-1} = cn!\gamma^n \end{aligned} \tag{45}$$

□

If there is a pair of eigenvalues close together (only among the positive ones for $y > 0$ or negative ones for $y < 0$), $\alpha_{\min} \ll 1$, and then the convergence will be slow. However, close pairs of eigenvalues can be well approximated by a multiple eigenvalue. A suggested approximation method is given in the Appendix, and some numerical examples are given in the following section.

If the eigenvalues are separated by at least a ratio of 1.5 we observe a very fast convergence, and only a few terms suffice for practical applications. For ratios of less than 1.2, coalescing to a multiple eigenvalue is recommended.

V. Practical Example

In [4] a multiple-symbol noncoherent decoder for coded PSK symbols and additive white Gaussian channel (AWGN) operates by maximizing the following metric over all possible transmitter sequences $\{x^{(n)}\}$:

$$\Psi(x^{(n)}) = \sum_{k=-\infty}^{\infty} \left| \sum_{i=0}^{S-1} r_{k+i}^* x_{k+i}^{(n)} \right|^2 \tag{46}$$

where r is the received signal (in complex baseband) and S is the observation length in symbols. In this example, we compute the error probability of this decoder for a system with two possible transmitted sequences carrying QPSK symbols for $S = 2$

$$x^{(0)} = \{1,1,1,1\}, \quad x^{(1)} = \{1,+j,-j,1\}, \qquad j = \sqrt{-1}.$$

Define

$$Y = \sum_{k=0}^{2} \left| \sum_{i=0}^{1} r_{k+i}^* x_{k+i}^{(1)} \right|^2 - \sum_{k=0}^{2} \left| \sum_{i=0}^{1} r_{k+i}^* x_{k+i}^{(0)} \right|^2. \tag{47}$$

Then the decoder error probability, given that $x^{(0)}$ is transmitted, is $\Pr\{Y \geq 0\}$. After some algebra, (47) is rewritten as $Y = \frac{1}{2}\boldsymbol{r}^\dagger A \boldsymbol{r}$, where $\boldsymbol{r} = \{r_0, \cdots, r_3\}$, and

$$A = \left\{ \begin{array}{cccc} 0 & -2-2j & 0 & 0 \\ -2+2j & 0 & -4 & 0 \\ 0 & -4 & 0 & -2-2j \\ 0 & 0 & -2+2j & 0 \end{array} \right\}.$$

In addition, the r_i's are independent normal random variables, normalized so that both their real and imaginary parts have a unit variance and their expected values are $\eta_i = E[r_i] = \sqrt{2E_s/N_0}, \forall i$, where E_s/N_0 is the signal-to-noise ratio per symbol. Let us arbitrarily choose $E_s/N_0 = 4$ dB. Since the r_i's are independent, L is the identity matrix. Next, Q and Λ_i are computed. The results are

$$\lambda_i = \Lambda_i = \{-5.464, -1.464, 1.464, 5.464\}$$

(no multiple eigenvalues). Using $E[Z] = Q^\dagger \eta$ and $\mu_k^2 = |E[Z_i]|^2$ we get

$$\mu_k^2 = \{14.787, 3.962, 0.284, 1.062\}.$$

After we obtain the required parameters, we can evaluate the error probability via (32) with $y = 0$. We start with the first positive eigenvalue $\lambda_2 = 1.464$. The recursion for $g_k^{(n)}(s,y)$ is initialized by computing $g_k(0,0) = 0.409$. Next $[\ln g_k(0,0)]^{(1)}$ is computed via (19), and the result is -7.491. Then $g_k^{(1)}(0,0)$ is computed by (21) using the previously computed results, and the result is -3.064. Next $[\ln g_k(0,0)]^{(2)} = 11.68$ is computed, and so on. Table I provides the result of (32) as a function of the number of terms used in the infinite sum over the index n.

TABLE I
Series Result for a Numerical Example

No. Terms	Result
1	0.0543
2	0.015399
3	0.0213005
4	0.02315728
5	0.02353806
6	0.02359475
7	0.02360129
8	0.02360191
9	0.02360196
10	0.02360196

We should like to demonstrate the accuracy lost when a close pair of eigenvalues is coalesced to a multiple eigenvalue. Let us first change the value of λ_2 from 1.464 to 4.6 to be within about 20% of $\lambda_3 = 5.464$. Next, λ_2 and λ_3 are combined by the method given

TABLE II
COALESCING EIGENVALUES: FIRST EXAMPLE

y	$\Pr\{Y \geq y\}$, separated	$\Pr\{Y \geq y\}$, coalesced	ϵ
0	0.04371318	0.04371684	-0.0008
30	$9.7801 \cdot 10^{-4}$	$9.7482 \cdot 10^{-4}$	$+0.02$
100	$3.1899 \cdot 10^{-8}$	$3.0473 \cdot 10^{-8}$	$+0.3$

TABLE III
COALESCING EIGENVALUES: SECOND EXAMPLE

y	$\Pr\{Y \geq y\}$, separated	$\Pr\{Y \geq y\}$, coalesced	ϵ
0	0.05893852	0.05893834	$+0.00002$
30	$1.086822 \cdot 10^{-3}$	$1.086642 \cdot 10^{-3}$	$+0.001$
100	$2.8685 \cdot 10^{-8}$	$2.8678 \cdot 10^{-8}$	$+0.0015$

in the Appendix. The results before and after the approximation are shown in Table II, where ϵ is the change in y which would cause the same deviation from the true value as caused by the approximation. This example is a worst case one, since the modified eigenvalues are the largest; the largest eigenvalues dominate the distribution of a QF.

Next, to check the case where the coalesced eigenvalues are not the largest, we change λ_1 from -1.464 to 1.1 and set λ_2 to its original value 1.464. We coalesce λ_1 with λ_2, and the results are shown in Table III. A very small error is observed even though λ_2 is as far as 33% from λ_1.

VI. CONCLUSION

Convenient series expansions were developed for the pdf and for the cdf of a quadratic form in complex normal variables. The rate of convergence of the series depends upon the separation between the eigenvalues. For eigenvalues well separated from each other, all series are fast converging and computationally efficient. For other cases, a good approximation is proposed. The coefficients in the expansions are computed recursively.

APPENDIX

In this Appendix, a method for approximating a close pair of eigenvalues by a multiple eigenvalue is described. Let λ_1 and λ_2 be a pair of eigenvalues with corresponding μ_1^2, m_1, μ_2^2, and m_2. It is desired to replace this pair with an eigenvalue λ_e with multiplicity $m_e = m_1 + m_2$ and a corresponding noncentrality parameter μ_e^2. A good criterion for choosing λ_e and μ_e^2 is to keep both the mean and the variance of the QF unchanged. The mean of a QF is given by

$$E[Y] = \sum_k \lambda_k (m_k + \mu_k^2/2) \tag{48}$$

and the variance by

$$\mathrm{Var}\,[Y] = \sum_k \lambda_k^2 (m_k + \mu_k^2). \tag{49}$$

Two equations are formed, one for the mean and one for the variance.

$$\lambda_e(m_e + \mu_e^2/2) = \lambda_1(m_1 + \mu_1^2/2) + \lambda_2(m_2 + \mu_2^2/2) \triangleq P \tag{50}$$

$$\lambda_e^2(m_e + \mu_e^2) = \lambda_1^2(m_1 + \mu_1^2) + \lambda_2^2(m_2 + \mu_2^2) \triangleq Q \tag{51}$$

The solution of this set of equations is

$$\lambda_e = \frac{1}{2m_e}(P - \sqrt{P^2 - m_e Q}) \tag{52}$$

$$\mu_e^2 = \sqrt{P^2 - m_e Q}/\lambda_e. \tag{53}$$

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The Shape of Low Minima of the Envelope of Narrowband Gaussian Noise

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***Abstract*— Low minima of the envelope of narrowband noise are shown to be hyperbolic in shape. They are accompanied by large, short impulses in the instantaneous frequency of the noise. The statistics of these phenomena are described, and they are related to Rice's formula for the rate of zero-crossings of Gaussian noise.**

***Index Terms*—Narrowband Gaussian noise; low minima of envelope: shape, depth, and frequency; impulses in instantaneous frequency: shape, height, and frequency; and zero-crossing frequency.**

I. Introduction

Large excursions of Gaussian noise are known [1], [2] to have the shape of the correlation function of the noise. It is also interesting and useful to know the behavior of Gaussian noise in the neighborhood of unusually small values. Minima of the absolute value of the noise itself do not have a very interesting shape, as the noise passes through zero approximately linearly. Low minima of the amplitude $A(t)$ of narrowband Gaussian noise

$$A(t)\cos[2\pi Ft+\phi(t)] = x(t)\cos 2\pi Ft - y(t)\sin 2\pi Ft \quad (1)$$

are more complicated; they result from nearly simultaneous zeros of the $x(t)$ and $y(t)$ components of the noise phasor

$$A(t)e^{j\phi(t)} = x(t)+jy(t). \quad (2)$$

It is most convenient to take as the reference frequency the centroid

$$F=\int_0^\infty fS(f)\,df/\sigma^2 \quad (3)$$

of the noise spectrum, where $S(f)$ is the one-sided power spectral density of the noise, and

$$\sigma^2=\int_0^\infty S(f)\,df$$

is the mean-squared value (average power) of the noise. Two other important noise parameters are its mean-squared frequency

$$\varrho^2=\int_0^\infty f^2S(f)\,df/\sigma^2 \quad (4)$$

and its mean-squared bandwidth

$$\rho^2=\varrho^2-F^2, \quad (5)$$

ρ being the radius of gyration of $S(f)$ around its centroid F. Since $\frac{1}{2}[S(f-F)+S(-f-F)]$ is the power spectral density of $x(t)$ and of $y(t)$ [3, ch. 4], ρ^2 is also the mean-squared frequency in the spectra of $x(t)$ and $y(t)$. Rice [4] showed that the average number of zero crossings per second of Gaussian noise is 2ϱ.

The noise phasor (2) has a Rayleigh-distributed length A with mean-squared value $2\sigma^2$ (half coming from x and half from y) and an independently uniformly distributed direction ϕ. Its speed of motion has a Rayleigh distribution also, with mean-squared value $8\pi^2\rho^2\sigma^2$ (half coming from $\dot{x}=dx/dt$ and half from $\dot{y}=dy/dt$) and an independently uniformly distributed direction, both of which are statistically independent of A and ϕ [3] at any instant. If F in (1) were not the centroid (3) of the spectrum of the noise, $\dot{x}$ and $\dot{y}$ would not be statistically independent of x and y. There are, in any case, correlations between these quantities at different times that depend on the shape of $S(f)$ or, equivalently, on its Fourier transform, the autocorrelation function of the noise.

The durations of the phenomena studied here, however, are so short that those correlations play no role; $\dot{x}$ and $\dot{y}$ can be treated as constant during each event. As this phasor meanders over the complex plane, rarely straying more than a few σ from the origin, it occasionally (at times forming a Poisson process) passes the origin at close range, say $|m|\ll\sigma$. Its speed then is likely to be of the order of $\sqrt{8}\pi\rho\sigma$. Hence $\phi(t)$ is likely to change by almost $\pm\pi$ during a time of the order of $|m|/(2\pi\rho\sigma)$; i.e., the instantaneous frequency deviation $\dot{\phi}/2\pi=(d\phi/dt)/2\pi$ of the noise will exhibit a pulse with this sort of duration whose peak value is of the order of $\rho\sigma/|m|$.

II. Minima of the Noise Envelope

During a brief interval when the amplitude $A(t)$ is small in comparison with the rms value σ of the noise, $x(t)$ and $y(t)$ can be linearly approximated as

$$x(t)=x_0+\dot{x}_0(t-t_0) \quad\text{and}\quad y(t)=y_0+\dot{y}_0(t-t_0). \quad (6)$$

Hence

$$\begin{aligned}A(t)&\approx\sqrt{[x_0+\dot{x}_0(t-t_0)]^2+[y_0+\dot{y}_0(t-t_0)]^2}\\&=\sqrt{s^2(t-t_1)^2+m^2}\end{aligned} \quad (7)$$

where

$$s=\sqrt{\dot{x}_0^2+\dot{y}_0^2} \quad (8)$$

is the speed of motion of the phasor (2),

$$m=\frac{x_0\dot{y}_0-\dot{x}_0y_0}{s}, \quad (9)$$

and

$$t_1=t_0-\frac{x_0\dot{x}_0+y_0\dot{y}_0}{s^2} \quad (10)$$

is the time of occurrence of the minimum value $|m|$ of (7). The 3-dB halfwidth of this minimum is $|m|/s$. Equation (7) describes the upper branch of a hyperbola with a vertical transverse axis. Its asymptotes $A=\pm s(t-t_1)$ intersect at the point $(t_1, 0)$. Such a

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