A new high performance turbo encoder scheme for bandwidth efficient modulation

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1 Introduction

Constellation shaping can provide an energy saving called shaping gain. The idea behind constellation shaping is that signals with large norm are used less frequently than signals with small norm, thus improving the overall gain by adding shaping gain to the original coding gain [1]. Theoretically, when constellation points are selected according to a continuous Gaussian distribution at every dimension, the maximum shaping gain of $\frac{\pi e}{6} = 1.53\ \mathrm{dB}$ can be achieved in the limit for infinite transmission rates. Practically, a smaller gain can be achieved in finite constellations.

Many shaping techniques for the Gaussian channel are known in the literature. It was first shown by Gallager [2] that binary codes can be used for mappings of nonequiprobable letters that achieve capacity on an arbitrary discrete memoryless channel. Calderbank and Ozarow [3] introduced subconstellation partitioning. In this approach a signal constellation is partitioned into several subconstellations, points in the same subconstellation are used equiprobably, and a shaping code selects the subconstellations with various frequencies. Forney [4] introduced trellis shaping. This method uses a search through the trellis diagram of a shaping convolutional code for selecting the constellation points. Kschischang and Pasupathy [5] took a different approach by mapping simple variable-length prefix codes onto the constellation. A similar method that uses subdivision of the signal constellation into variable-size regions was proposed by Livingston [6].

For Gaussian channels, turbo coded modulation techniques can be broadly classified into binary schemes and Turbo Trellis Coded Modulation (TTCM) [7]. The first group can be further divided into "pragmatic" schemes with a single component binary turbo code, and multilevel binary turbo codes [8]. The pragmatic approach [9] is simple and versatile and is much less complex to design and to implement than TTCM.

In this paper we combine nonuniform signaling with the pragmatic binary turbo coded modulation environment. Our shaping method follows the technique proposed by Gallager in his 1968 book on information theory [2]. This technique can be easily applied to any binary turbo or turbo-like code, including parallel, serial and Low Density Parity Check (LDPC) codes. The nonequiprobable letters are obtained by using a table that maps b–bits equiprobable input words onto nonequiprobable m–bits M–ary PAM (or equivalently M^2 QAM) symbols, at rates 2 and 3 bits/dim.

2 Signal shaping in a finite constellation

We would like to find the maximum achievable shaping gain, when using practical signal sets with finite constellation points and a discrete input distribution. First we turn to the calculation of the *capacity gain*, which is the optimization of the mutual information of the additive white Gaussian noise channel (AWGN) with discrete inputs. Consider an AWGN channel having discrete inputs $\mathbf{c}$ taking on the values $\{c_j\}$ for $j = 0 \ldots J-1$, with probabilities $\mathrm{P_r}(\mathbf{c}) = \{\mathrm{P_r}(c_0), \mathrm{P_r}(c_1), \ldots, \mathrm{P_r}(c_{J-1})\}$. Denoting the noise variance by σ^2, we can express the output probability density functions as

$$Q(Y|c_j) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(Y-c_j)^2} \tag{1}$$

The capacity of the discrete input channel is given by the maximum of the mutual information

$$\begin{aligned} C &= \max_{\mathrm{P_r}(\mathbf{c})} I(\mathbf{c};Y) \\ &= \max_{\mathrm{P_r}(\mathbf{c})} \sum_{j=0}^{J-1} \mathrm{P_r}(c_j) \int_{-\infty}^{\infty} Q(Y|c_j)\, \log \frac{Q(Y|c_j)}{\sum\limits_{i=0}^{J-1} \mathrm{P_r}(c_i) Q(Y|c_i)} dY. \end{aligned} \tag{2}$$

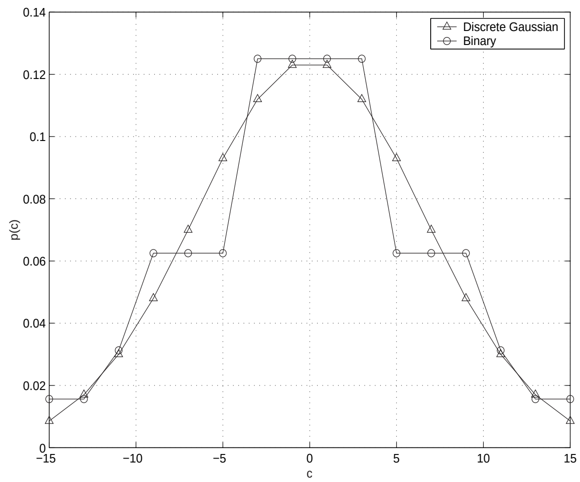

Figure 1: Optimal Binary discrete distribution for a 16-PAM constellation along with the Maxwell-Boltzmann distribution for R=3.0 Bits/symbol and $\lambda = 1/21$.

Optimizing the mutual information with respect to the input probabilities will give the lowest average input power when transmitting at various rates $R = C$ bits/dim. We consider the power reduction, compared to equiprobable transmission, as the desired capacity gain. The optimization procedure for all possible input probabilities is difficult. Therefore we use a distribution that can be practically implemented and also produces reasonable gains. Consider the binary probabilities 2^{-k} where $k = 3, 4, 5$ and 6. We apply these probabilities to the 16-ary PAM constellation, where $\mathbf{c}$ is the set of the one-dimensional signals $\{-15, -13, -11, \ \dots \ , -1, 1, 3, 5, \dots, 13, 15\}$, in the following way:

$$\begin{aligned} \mathrm{P_r}(c = \pm 1, \pm 3) &= 2^{-3}, \mathrm{P_r}(c = \pm 5, \pm 7, \pm 9) = 2^{-4}, \\ \mathrm{P_r}(c = \pm 11) &= 2^{-5}, \mathrm{P_r}(c = \pm 13, \pm 15) = 2^{-6}. \end{aligned} \tag{3}$$

These probabilities are derived by rounding the values of the optimal discrete Gaussian distribution

$$\mathrm{P_r}(c_j) = K(\lambda) \cdot e^{-\lambda |c_j|^2}, \qquad \lambda \geq 0 \quad , \tag{4}$$

to the nearest 2^{-k} value while maintaining $\sum \mathrm{P_r}(c_j) = 1$. The discrete Gaussian distribution is also known as the *Maxwell-Botlzmann* distribution. It is optimal because it maximizes the bit rate for a fixed energy or alternatively, minimizes the average energy for a fixed bit rate [5]. The parameter $\lambda \geq 0$ governs the trade-off between bit rate and average energy and

$$K(\lambda) = \Big(\sum_{c_j} e^{-\lambda |c_j|^2}\Big)^{-1} \tag{5}$$

is the distribution normalization factor. We demonstrate the rounding of the discrete Gaussian distribution into the binary one in Fig. 1, which depicts the distribution of (3) along with the Maxwell-Boltzmann distribution, optimized ($\lambda = 1/21$) for a transmission rate of $R = 3.0$ Bits/dim. Using the distribution of (3) in (2), we calculated the maximum theoretical shaping gains for the transmission rates $R = 2.0$ and 3.0 bits/dim, which resulted in 0.682 dB and 0.948 dB respectively.

The binary probabilities are realized by a table that maps equiprobable 6-bits input words onto nonequiprobable 4-bits output words such as Table 1. The columns $b_0, b_1 \dots b_5$ represent the input bits. The notation $[\times 2\ ]$ in the table means that a certain input bit in that location can take on the values 0 or 1. In this way four words are mapped, for example, to the signal point 9.0, and eight are mapped to 1.0 and so on. Clearly, the probability of each input word is $1/64$, whereas probabilities of the output signals become $8/64$, $4/64$, $2/64$ and $1/64$. As desired, the output probabilities are equal to the binary probabilities in (3).

3 System model

We have applied the nonequiprobable distribution derived above to turbo coded modulation. In pragmatic binary turbo coded modulation [9] a single binary turbo code of rate $1/3$ is used as the component code. Its encoder outputs are suitably multiplexed, punctured and mapped onto an M-PSK or M-QAM signal set using Gray code or the above

Table 1: Signal mapper with binary probabilities of the type 2^{-k}, $k = 3, 4, 5$ and 6, for a 16-PAM constellation.

Signal Point	b_0	b_1	b_2	b_3	b_4	b_5
15	0	0	0	0	0	0
13	0	0	0	0	0	1
11	0	0	0	0	1	$\times 2$
9	0	0	0	1	$\times 2$	$\times 2$
7	0	0	1	1	$\times 2$	$\times 2$
5	0	0	1	0	$\times 2$	$\times 2$
3	0	1	1	$\times 2$	$\times 2$	$\times 2$
1	0	1	0	$\times 2$	$\times 2$	$\times 2$
-1	1	1	0	$\times 2$	$\times 2$	$\times 2$
-3	1	1	1	$\times 2$	$\times 2$	$\times 2$
-5	1	0	1	0	$\times 2$	$\times 2$
-7	1	0	1	1	$\times 2$	$\times 2$
-9	1	0	0	1	$\times 2$	$\times 2$
-11	1	0	0	0	1	$\times 2$
-13	1	0	0	0	0	1
-15	1	0	0	0	0	0

mapping table. The receiver calculates the log-likelihood function for each encoded binary digit based on the received noisy symbols. The stream of the bit likelihood values is then bit deinterleaved, demultiplexed and depunctured before passing to the binary turbo decoder which is based on MAP algorithm e.g., [12]. As suggested in [10], we also included the bit LLR calculation block in the turbo decoder iterations to achieve further improvement in performance.

4 Simulation Results

The results are reported in Fig.1 and Fig.2. We used the standard turbo-encoder, made up of two elementary encoders with the same constraint length $K = 5$ and the same generator polynomials (23,35). Turbo decoding was performed in 18 iterations on information blocks of 32,768 bits using random interleaving. We applied two schemes for spectral efficiencies of 2 and 3 bits/dim and compared performances between our nonequiprobable signaling scheme and the conventional equiprobable pragmatic turbo coded modulation [9]. We can notice that for rate of 2.0 bits/dim, the nonequiprobable scheme produces a gain of 0.6 dB compared to the equiprobable one at BER $P_b(e) = 10^{-5}$. The continuous input channel capacity limit of the AWGN channel is $E_b/N_0 = 5.74$ dB. The performance of our decoder at $P_b(e) = 10^{-5}$ is about 1.1 dB from this limit. At 3.0 bits/dim, the nonequiprobable scheme produces a gain of 0.93 dB compared to the equiprobable one at BER $P_b(e) = 10^{-5}$, the channel capacity limit is $E_b/N_0 = 10.2$ dB, and we achieve $P_b(e) = 10^{-5}$ within 1.2 dB of this limit.

5 Conclusions

We presented a new scheme for improving the performance of pragmatic binary turbo coded modulation by using nonequiprobable signaling. We described a nonequiprobable signaling technique that makes it possible to approach the maximum theoretical shaping gain of a finite constellation AWGN channel. Our nonuniform signaling scheme is very easy to implement and adds negligible load on the turbo decoder. We showed for an example of 6 bits/QAM symbol, a gain of 0.93 dB out of the available shaping gain of 0.948 dB, and transmission within 1.2 dB of the Shannon limit.

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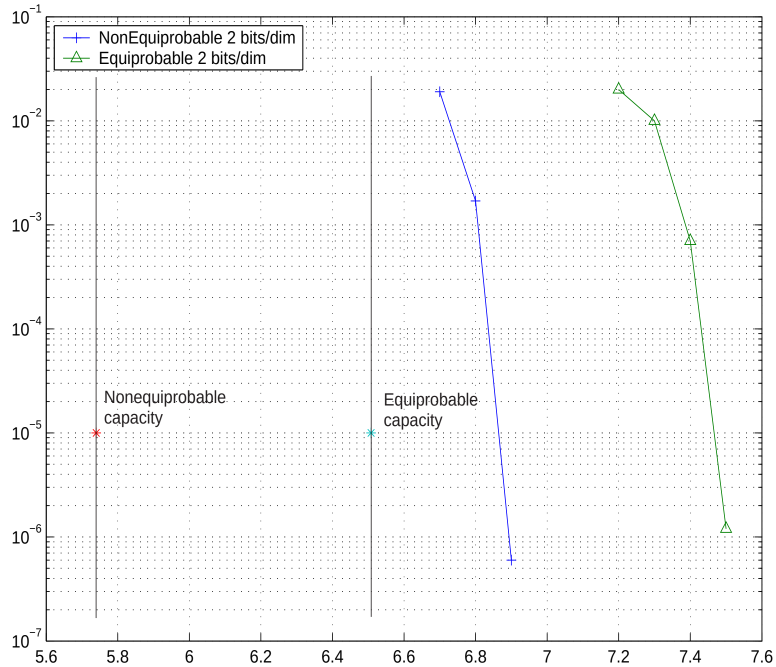

Figure 2: Performance comparison between two schemes of nonequiprobable and equiprobable signaling at rate 2 bits/dim using pragmatic binary turbo coded modulation with 18 iterations and information block length $N = 32,768$ bits. Channel capacity limit is 5.74 dB.

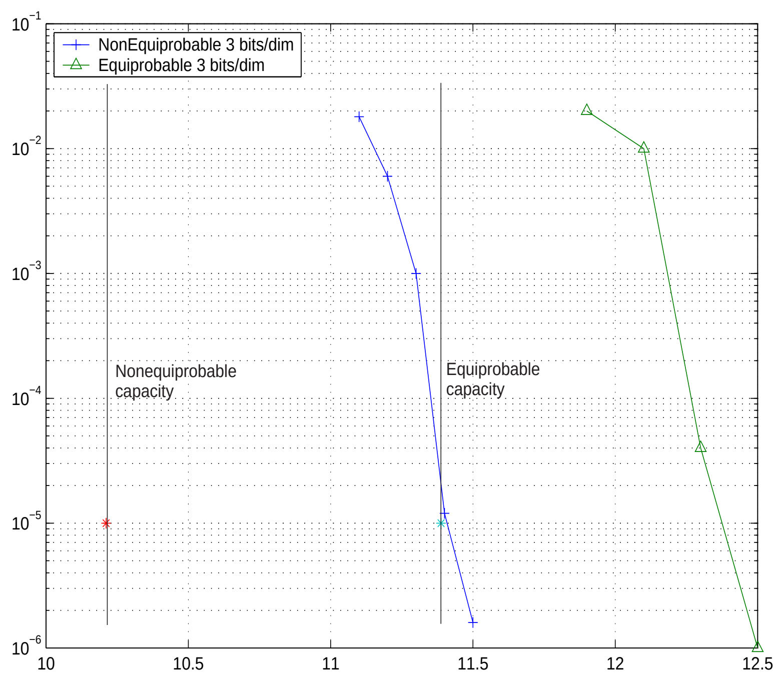

Figure 3: Performance comparison between two schemes of nonequiprobable and equiprobable signaling at rate 3 bits/dim using pragmatic binary turbo coded modulation with 18 iterations and information block length $N = 32,768$ bits. Channel capacity limit is 10.2 dB.