New conditions for delay-derivative-dependent stability\textsuperscript{a,\textsuperscript{b}}

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\textbf{A B S T R A C T}

Two recent Lyapunov-based methods have significantly improved the stability analysis of time-delay systems: the delay-fractioning approach of Gouaisbaut and Peaucelle (2006) for systems with constant delays and the convex analysis of systems with time-varying delays of Park and Ko (2007). In this paper we develop a convex optimization approach to stability analysis of linear systems with interval time-varying delay by using the delay partitioning-based Lyapunov–Krasovskii Functionals (LKFs). Novel LKFs are introduced with matrices that depend on the time delays. These functionals allow the derivation of stability conditions that depend on both the upper and lower bounds on delay derivatives.

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1. Introduction

Over the past decades, much effort has been invested in the analysis and design of systems with time delays (see e.g. Fridman & Shaked, 2002; Hale & Lunel, 1993; He, Wang, Lin, & Wu, 2007; Kolmanovskii & Myshkis, 1999; Niculescu, 2001; Richard, 2003). Among the recent advances in this area, two Lyapunov-based methods should be mentioned that significantly improved the stability analysis: the convex analysis of systems with time-varying delays of Park and Ko (2007) and the delay-fractioning approach of Gouaisbaut and Peaucelle (2006) for systems with constant delays.

These recent methods inspired the present work, where we extend the delay partitioning approach to systems with interval time-varying delay in a convex way. We introduce novel LKFs with matrices that depend on the time delays. This enables us to derive LMI conditions that depend not only on the upper, but also on the lower bound of the delay derivative. The efficiency of the new stability criteria is demonstrated via numerical examples.

2. Stability of systems with time-varying delay

Consider the system

\[
\dot{x}(t) = Ax(t) + A_1 x(t - \tau(t)),
\]

where \(\tau(t) \in [h_a, h_b] \), \(h_a \geq 0\) and where \(A\) and \(A_1\) are constant matrices. The delay is assumed to be either differentiable with

\[
d_1 \leq \dot{\tau}(t) \leq d_2,
\]

where \(d_1\) and \(d_2\) are given bounds, or fast-varying (with no restrictions on the delay derivative). The initial condition is given by \(x(t_0 + \theta) = \phi(\theta), \theta \in [-h_a, 0], \phi \in W\), where \(W\) is the space of absolutely continuous functions \(\phi : [-h_a, 0] \to \mathbb{R}^n\) with the square integrable derivative and with the norm

\[
\|\phi\|_{W_2} = |\phi(0)|^2 + \int_{-h_0}^0 (|\phi(s)|^2 + |\dot{\phi}(s)|^2)\,ds.
\]

2.1. A delay partitioning approach to stability

We divide the delay interval \([h_a, h_b]\) into two segments: \([h_1, h_2]\) and \([h_2, h_b]\), where we denote \(h_1 = h_a\), \(h_3 = h_b\) and \(h_2 = (h_a + h_b)/2\). Then, (1) can be represented as

\[
\dot{x}(t) = Ax(t) + \chi(h_1, h_2)(\tau(t))A_1 x(t - \tau(t)) + [1 - \chi(h_1, h_2)(\tau(t))]A_1 x(t - \tau(t)),
\]

where \(\chi(h_1, h_2) : \mathbb{R} \to [0, 1]\) is the characteristic function of \([h_1, h_2]\)

\[
\chi(h_1, h_2)(s) = \begin{cases} 1, & \text{if } s \in [h_1, h_2] \\ 0, & \text{otherwise.} \end{cases}
\]

Consider the following Lyapunov functional:

\[
V(t, x, \dot{x}) = x^T(t)P(\tau(t))x(t) + \int_{t-h_1}^{t} x^T(s)Qx(s)\,ds + \int_{t-h_2}^{t} x^T(s)S_0 x(s)\,ds + \int_{t-h_2}^{t} \xi^T(s)\begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{13} \end{bmatrix} \xi(s)\,ds
\]

\[
+ \int_{t-h_2}^{t} \xi^T(s)\begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{13} \end{bmatrix} \xi(s)\,ds
\]
Moreover, we seek $P(t)$ of the form

$$P(t) = \chi(hi, \bar{h}) (t) \left[ \frac{\tau(t) - h_1}{h_2 - h_1} p_1 + \frac{h_2 - \tau(t)}{h_2 - h_1} p_2 \right]$$

i.e.,

$$P(t) = \begin{cases} \frac{\tau(t) - h_1}{h_2 - h_1} p_1 + \frac{h_2 - \tau(t)}{h_2 - h_1} p_2, & 1 \leq \tau(t) \leq h_1, \\ \frac{\tau(t) - h_2}{h_2 - h_1} p_1 + \frac{h_2 - \tau(t)}{h_2 - h_1} p_2, & h_1 < \tau(t) \leq h_2, \\ \frac{\tau(t) - h_2}{h_2 - h_1} p_1 + \frac{h_2 - \tau(t)}{h_2 - h_1} p_2, & \tau(t) > h_2, \end{cases}$$

where $p_k > 0, \ k = 1, 2, 3.$ Note that the function $\bar{P}(t)$ is continuous in $t,$ since $lim_{t \to h_i} \bar{P}(t) = P_i.$

Following [Hale & Lunel, 1995], we define

$$\bar{V}(t, x, \dot{x}_\tau) = \lim sup_{\tau \to \tau_+} \frac{1}{\tau} \left[ V(t + \tau, x, \dot{x}, \dot{x}_\tau) \right]$$

for some scalar $\alpha > 0.$ We first consider $\tau \neq h_2.$ We have

$$\dot{\bar{P}}(t) = x^T(t) \bar{P}(t)x(t),$$

and find

$$\dot{\bar{V}}_{\tau \neq h_2} = x^T(t) \dot{\bar{P}}(t)x(t) + 2x^T(t) \left[ \chi \left( \frac{\tau(t) - h_1}{h_2 - h_1} p_1 \right) + (1 - \chi) \left( \frac{\tau(t) - h_2}{h_2 - h_1} p_1 \right) \right] \dot{x}(t).$$

Moreover,

$$\frac{d}{dt} \left[ \sum_{i=0}^{2} (hi+1 - hi) \int_{t-h_i}^{t} \left[ \dot{x}^T(s)R_i \dot{x}(s) \right] ds \right] = \dot{x}^T(t) \left[ \sum_{i=0}^{2} (hi+1 - hi) R_i \right] \dot{x}(t)$$

$$- \sum_{i=0}^{2} (hi+1 - hi) \int_{t-h_i}^{t} \dot{x}^T(s)R_i \dot{x}(s) ds.$$
\( (h_2 - \tau(t))Y_{21} \), \( (h_2 - \tau(t))Y_{31} \), \( 0 \), \( R_2 + W_3 - S_{12} \), \( \phi_{0}^{(1)} \) \( (h_2 - \tau(t))T_1 \), \( 0 \), \( R_2 + W_3 - S_{12} \), \( \phi_{0}^{(1)} \) \( (h_2 - \tau(t))T_1 \), \( 0 \), \( R_2 + W_3 - S_{12} \), \( \phi_{0}^{(1)} \) where
\[
\begin{align*}
\Omega_{11} &= A^T P_2 + P_2^T A + S_0 - R_0, \\
\Omega_{12} &= \begin{bmatrix} \tau(t) - h_1 \cdot t_1 + h_2 - \tau(t) \cdot t_2 \end{bmatrix} - P_{21}^T A^T P_3, \\
\Omega_{22} &= -P_{33} - P_{33}^T + \sum_{i=0}^2 (h_{i+1} - h_i)^2 R_i, \\
\Omega_{17} &= Y_{12}^2 + P_{2}^T A_1, \\
\Omega_{27} &= Y_{22}^2 + P_{2}^T A_1, \\
\phi_3^1 &= -(S_0 + R_0 - S_{11} - Q), \\
\phi_4^1 &= -(S_{11} + R_2 - S_{13}), \\
\phi_5^1 &= -(h_2 - h_1) \tau(t) \cdot R_1, \\
\phi_6^1 &= -(h_2 - h_1) (h_2 - \tau(t)) R_1, i, j = 1, 2, \end{align*}
\]
The latter LMI leads for \( \tau(t) \rightarrow h_1 \) and for \( \tau(t) \rightarrow h_2 \) to the following LMIs:
\[
\begin{align*}
\Psi_1 &= \begin{bmatrix} \Omega_{11} + \varepsilon(t) (P_1 - P_2) \\
\Omega_{12} \left( \tau(t) \equiv h_1 \right) \cdot R_0 - Y_{11}^2 \\
\Omega_{12}^2 - Y_{12}^2 - Z_{11}^2 \\
\phi_3^{(1)} S_{12} \\
\phi_4^{(1)} S_{12} \\
\phi_5^{(1)} S_{12} \\
\phi_6^{(1)} (h_2 - h_1) W_3 \\
* \\
* \\
* \\
* \\
* \\
* \\
* \\
* \\
* \end{bmatrix} < 0, \\
\Psi_2 &= \begin{bmatrix} \Omega_{11} + \varepsilon(t) (P_1 - P_2) \\
\Omega_{12} \left( \tau(t) \equiv h_2 \right) \\
\Omega_{12}^2 - Y_{12}^2 - Z_{11}^2 \\
\phi_3^{(1)} S_{12} \\
\phi_4^{(1)} S_{12} \\
\phi_5^{(1)} S_{12} \\
\phi_6^{(1)} (h_2 - h_1) W_3 \\
* \\
* \\
* \\
* \\
* \\
* \\
* \\
* \end{bmatrix} < 0.
\end{align*}
\]
It is easy to see that \( \Psi_1 \) results from \( \Psi_{\tau \geq h_2}, \tau \equiv h_1 \), where we have deleted the zero row and the zero column. Denoting:
\[
\eta_3(t) = \text{col}(x(t), \dot{x}(t), x(t-h_2), x(t-h_2), y_j), \\
\eta(t - \tau(t), x(t-h_1)), i = 1, 2,
\]
the latter two LMIs imply (16) because
\[
\begin{align*}
\eta_2(t) &= \theta_{12}(t) \Psi_{\tau < h_1} \Psi_{\tau = h_1} + \tau(t) - h_1 \cdot Y_{11} \nonumber \\
&= \eta_2^1(t) \Psi_{\tau < h_2} \eta_2^1(t) \\
&= \eta_2^1(t) \Psi_{\tau < h_2} \eta_2^1(t) \\
&\leq -\eta_3(t) \eta_3(t)^T
\end{align*}
\]
and \( \Psi_{\tau < h_2} \) is thus convex in \( \tau(t) \in [h_1, h_2) \).

LMI (19) leads for \( \tau(t) = d_i, i = 1, 2 \) to the following:
\[
\begin{align*}
\text{LMI (19) leads for } \tau(t) = d_i, i = 1, 2 \text{ to the following:}
\Psi_{\tau = d_i} &= \begin{bmatrix} \Psi_{\tau = d_1} \\
\Psi_{\tau = d_2} \\
\end{bmatrix}
< 0,
\end{align*}
\]
and \( \Psi_{\tau = d_i} \) is thus convex for \( \tau(t) \in [d_i, d_{i+1}) \).

LMI (19) implies (19) because
\[
\begin{align*}
d_2 - \tau(t) \Psi_{\tau = d_1} + \tau(t) - d_1 \Psi_{\tau = d_1} &= \Psi_{\tau = d_1} < 0,
\end{align*}
\]
and \( \Psi_{\tau = d_i} \) is thus convex in \( \tau(t) \in [d_i, d_{i+1}) \).

Similarly to (15), the expression with \( j = 2 \) is added to \( \dot{V} \). We then arrive at the following:
\[
\dot{V}_{\tau < h_2} \leq \eta_2^1(t) \Psi_{\tau \geq h_2} \eta_2^1(t),
\]
where
\[
\begin{align*}
\Psi_{\tau \geq h_2} &= \begin{bmatrix} \Omega_{11} + \varepsilon(t) (P_1 - P_2) \\
\Omega_{12} \left( \tau(t) = h_1 \right) \cdot R_0 - Y_{11}^2 \\
\Omega_{12}^2 - Y_{12}^2 - Z_{11}^2 \\
\phi_3^{(2)} R_1 - W_{21} + S_{12} \\
\phi_4^{(2)} R_1 - W_{21} + S_{12} \\
\phi_5^{(2)} R_1 - W_{21} + S_{12} \\
\phi_6^{(2)} (h_2 - h_1) W_3 \\
* \\
* \\
* \\
* \\
* \\
* \\
* \end{bmatrix} < 0,
\end{align*}
\]
and
\[
\begin{align*}
\Omega_{12}^2 &= \begin{bmatrix} \tau(t) - h_2 \cdot t_1 + h_3 - \tau(t) \cdot t_2 \\
\Omega_{12} \left( \tau(t) = h_3 \right) \\
\Omega_{12}^2 - Y_{12}^2 - Z_{11}^2 \\
\phi_3^{(2)} S_{13} \\
\phi_4^{(2)} S_{13} \\
\phi_5^{(2)} S_{13} \\
\phi_6^{(2)} (h_2 - h_1) T_1 \\
* \\
* \\
* \\
* \\
* \\
* \\
* \end{bmatrix} < 0.
\end{align*}
\]
and where
\[
\begin{align*}
\phi_3^{(2)} &= -(S_0 + R_0 + R_3 - S_{11} - Q), \\
\phi_4^{(2)} &= -(S_{11} + R_1 - S_{13}), \\
\phi_5^{(2)} &= -(h_2 - h_3) (h_2 - \tau(t)) R_2, \\
\phi_6^{(2)} &= -(h_3 - h_2) (h_3 - \tau(t)) R_2.
\end{align*}
\]
We note that \( \Psi_{\tau \geq h_2} \) is convex in \( \tau(t) \in (h_2, h_3] \) and, thus, for the feasibility of LMI(22), it is sufficient to verify this LMI for \( \tau(t) \rightarrow h_2 \) and for \( \tau(t) \rightarrow h_3 \). Denote the resulting LMIs by
Let there exist $n$ and $\psi_{03} < 0$, respectively, where the zero columns and rows are deleted from $\psi_{02}$ and $\psi_{03}$. Clearly $\psi_{02}$ and $\psi_{03}$ are also convex in $\tau(t) \in [d_1, d_2]$. Along the system (3), we therefore have

$$V_{\tau(h_2)} \leq \chi(h_{1,2})(\tau(t))\psi_1(t)\psi_{1,h_2,\eta_1}(t) + [1 - \chi(h_{1,2})(\tau(t))]\psi_2(t)\psi_{1,h_2,\eta_2}(t) \leq -\alpha|\tau(t)|^2 \quad (24)$$

for some positive scalar $\alpha > 0$.

For $\tau = h_2$, taking into account the definition (7) of $V$, we find

$$\dot{V}_{\tau(h_2)} \leq \max\{\psi_1(t)\psi_{1,h_2,\eta_1}(t), \psi_2(t)\psi_{1,h_2,\eta_2}(t)\} - \alpha|\tau(t)|^2. \quad (25)$$

By using Theorem 8.1.6 of (Kolmanovskii & Myshkis, 1999), we finally obtain the following:

**Theorem 1.** Let there exist $n \times n$-matrices $Q, R_i(i = 0, 1, 2)$, $S_0, S_{11, 12, 13}$ satisfying (5), and $n \times n$-matrices $P^k > 0$, $k = 1, 2, 3, W_1, W_2, P_{3, 3}, P_{3, 2}, P_{3, 1}, W_1, W_2, T$ and $Z_1, Y_1, Z_1$ such that the eight LMIs (17), for $\tau(t) \rightarrow h_1$ and $\tau(t) \rightarrow h_2$, and (22), for $\tau(t) \rightarrow h_2$ and $\tau(t) \rightarrow h_3$, where $\tau(t) = d_1, d_2$, with notations given in (18) and (23), are feasible. Then (1) is asymptotically stable for all differentiable delays $\tau(t) \in [h_1, h_3]$ with $d_1 \leq \tilde{\tau}(t) \leq d_2$.

For unknown $d_1$, by substituting $P = P^1 = P^2 = P^3$ and $\dot{\tau}(t) = d_2$ into (17) and (22), we arrive at the following:

**Corollary 1.** Let there exist $n \times n$-matrices $Q, R_i(i = 0, 1, 2)$, $S_0, S_{11, 12, 13}$ satisfying (5), and $n \times n$-matrices $P > 0, W_1, W_2, P_{3, 3}, P_{3, 2}, P_{3, 1}, Y_1, Y_1, T_1, T_2, Z_1$ and $Z_2, j \in 1, 2$ such that the four LMIs (17), for $\tau(t) \rightarrow h_1$ and $\tau(t) \rightarrow h_2$, and (22), for $\tau(t) \rightarrow h_2$ and $\tau(t) \rightarrow h_3$, where $\dot{\tau}(t) = d_2$, with notations given in (18) and (23), are feasible. Then (1) is asymptotically stable for all differentiable delays $\tau(t) \in [h_1, h_3]$ with $\dot{\tau}(t) \leq d_2$. Moreover, if the above LMIs are feasible with $Q = 0$, then (1) is asymptotically stable for all fast-varying delays in $[h_1, h_3]$.

2.2. On other possibilities for delay partitioning

If $h_3$ is big enough, delay partitioning of $[0, h_3]$ can improve the result. For simplicity we combine delay partitioning of $[0, h_3]$ with a non-partitioned $[h_2, h_3]$, where $h_3 = h_1$ and $h_2 = h_2$. We apply

$$V(t, x, \dot{x}) = x^T(t)P(t)\dot{x}(t),$$

$$+ \int_{h_2}^{\tau(t)} x^T(s)Q_1x(s)ds + \int_{h_2}^{\tau(t)} x^T(s)Q_2x(s)ds + \int_{h_2}^{\tau(t)} x^T(s)Q_3x(s)ds,$$

where $\dot{\tau}(t) \in [h_1, h_2]$ and where

$$P(\tau(t)) = \frac{\tau(t) - h_1}{h_2 - h_1}P^1 + \frac{h_2 - (\tau(t))}{h_2 - h_1}P^2, \quad P^1 > 0, P^2 > 0. \quad (27)$$

$$Q \geq 0, R_0 > 0, R_1 > 0, S_1 > 0, \left[ \begin{array}{cc} S_0 & S_{02} \\ * & S_{03} \end{array} \right] > 0. \quad (28)$$

We have:

$$\frac{d}{dt}x^T(t)P(\tau(t))x(t) = x^T(t)(P^1 - P^2)x(t) + 2x^T(t)\frac{\tau(t) - h_1}{h_2 - h_1}P^1 + \frac{h_2 - (\tau(t))}{h_2 - h_1}P^2x(t).$$

By using arguments of Theorem 1 (without partitioning of $[h_2, h_3]$, we obtain that for some $\alpha > 0$

$$\dot{V} \leq -\alpha|\tau(t)|^2, \quad (29)$$

where $\tau(t) = \mathrm{col}(x(t), \dot{x}(t), x(t - h_1), x(t - h_2), v_1, v_2, x(t - \tau(t)), x(t - h_3))$, if the LMI

$$\Phi = \begin{bmatrix} \Phi_{11} + \frac{\tau(t)}{h_2 - h_1}P^1 - \Phi_{13} & \Phi_{12} & \Phi_{14} \\ \Phi_{21} & -Y_1^T & \Phi_{24} \\ \Phi_{31} & \Phi_{32} & -Y_2^T \\ \Phi_{41} & \Phi_{42} & \Phi_{43} \end{bmatrix} < 0 \quad (30)$$

holds, where

$$\Phi_{11} = A^T\Phi_2 + \Phi_{11}^T + \Phi_{01} - R_0, \quad \Phi_{12} = \frac{\tau(t) - h_1}{h_2 - h_1}P^1 + \frac{h_2 - \tau(t)}{h_2 - h_1}P^2, \quad \Phi_{13} = -\frac{P_1}{4}A_0^T, \quad \Phi_{14} = -\frac{P_1}{4}A_0^T,$$

$$\Phi_{22} = -P_3 - \frac{1}{4}h_2^2R_0 + (h_2 - h_1)^2R_1, \quad \Phi_{33} = -h_3 - h_2 - (h_3 - h_2)R_1,$$

The following result is then obtained.

**Theorem 2.** Let there exist $n \times n$-matrices $Q, R_0, S_0, S_{01, 02, 03}, R_1, S_1$, satisfying (28), and $n \times n$-matrices $P^i > 0, P^i > 0, P_2, P_3, T, Y_1, Y_2, Z_1, Z_2$ such that the four LMIs (30), for $\tau(t) \rightarrow h_1$ and $\tau(t) \rightarrow h_2$, where $\dot{\tau}(t) = d_1, d_2$, with notations given in (31), are feasible. Then (1) is asymptotically stable for all differentiable delays $h_1 \leq \tau(t) \leq h_2$ satisfying $d_1 \leq (\tilde{\tau}(t) \leq d_2$.

When $d_1$ is unknown, by setting $P = P^1 = P^2$, $\dot{\tau}(t) = d_2$ in (3), we obtain the following

**Corollary 2.** Let there exist $n \times n$-matrices $Q, R_0, S_0, S_{01, 02, 03}, R_1, S_1$, satisfying (28), and $n \times n$-matrices $P^1 > 0, P^2 > 0, P_2, P_3, T, Y_1, Y_2, Z_1, Z_2$ such that the two LMIs (30), for $\tau(t) \rightarrow h_1$ and $\tau(t) \rightarrow h_2$, where $\dot{\tau}(t) = d_1, d_2$, with notations given in (31), are feasible. Then (1) is asymptotically stable for all fast-varying delays in $[h_1, h_2]$.

**Remark 1.** The examples below illustrate that for big enough $h_0$ (for big enough $h_0 - h_3$) the delay partitioning of $[0, h_0]$ (of $[h_2, h_3]$) improve the result. Thus, in Example 2, the results by Corollary 2 are worse for $h_0 \leq 1$ and better for $h_0 \geq 2$ than the ones by Corollary 1. Partitioning of the above intervals into $n \geq 2$ subintervals may lead to further improvements. In (Jiang & Han, 2008) another delay partitioning was introduced, which corresponded to the partitioning into two subintervals of $[0, h_0]$ and of $[0, h_0]$. Example 2 below shows that our approach yields less conservative results.
For $h_a = 1, 2, 3, 4, 5$ and fast-varying delays we obtain, by applying Corollaries 1 and 2, the maximum values of $h_b$ given in Table 4. These results are favorably compared with the existing ones.

For $d_2 = 1$, choosing $d_1$ and $h_a$ as in Table 5 we apply Theorems 1 and 2. The results by Theorem 1 are $d_1$-dependent (see Table 5), where the value of $h_b$ grows for $d_1 \to 0$. For unknown $d_1$ the results coincide with those of Corollary 1. The results by Theorem 2 are $d_1$-independent and coincide with those by Corollary 2.

3. Conclusions

In this paper new LKFs with matrices depending on the time delay are introduced. These delay partitioning-based LKFs lead to stability conditions that depend on both the upper and lower bounds on the delay derivative. Two examples illustrate the efficiency of the new method and the improvement that can be achieved by using the lower bound on the delay derivative.

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References


