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Brief paper

Exponential stability of linear distributed parameter systems with time-varying delays[☆]

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ABSTRACT

Exponential stability analysis via the Lyapunov–Krasovskii method is extended to linear time-delay systems in a Hilbert space. The operator acting on the delayed state is supposed to be bounded. The system delay is admitted to be unknown and time-varying with an *a priori* given upper bound on the delay. Sufficient delay-dependent conditions for exponential stability are derived in the form of Linear Operator Inequalities (LOIs), where the decision variables are operators in the Hilbert space. Being applied to a heat equation and to a wave equation, these conditions are reduced to standard Linear Matrix Inequalities (LMIs). The proposed method is expected to provide effective tools for stability analysis and control synthesis of distributed parameter systems.

1. Introduction

Time-delay naturally appears in many control systems, and it is frequently a source of instability (Kolmanovskii & Myshkis, 1999). In the case of distributed parameter systems, even arbitrarily small delays in the feedback may destabilize the system (see e.g. Datko (1988), Logemann, Rebarber, and Weiss (1996) and Nicaise and Pignotti (2006)). The stability issue of systems with delay is, therefore, of theoretical and practical value.

During the last decade, a considerable amount of attention has been paid to stability of Ordinary Differential Equations (ODEs) with uncertain constant or time-varying delays (see e.g. Gu, Kharitonov, and Chen (2003), Kolmanovskii and Myshkis (1999), Niculescu (2001) and Richard (2003)). Special forms of Lyapunov–Krasovskii functionals have been used for derivation of simple finite dimensional conditions in terms of LMIs (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994). These conditions are either delay-independent or delay-dependent.

The stability analysis of Partial Differential Equations (PDEs) with delay is essentially more complicated. There are only a few works on Lyapunov-based technique for PDEs with delay. The second Lyapunov method was extended to abstract nonlinear time-delay systems in the Banach spaces in Wang (1994a), and was applied to stability analysis of some scalar heat/wave equations with constant delays and with the Dirichlet boundary conditions in Wang (1994b). Stability and instability conditions for delay wave equations were found in Nicaise and Pignotti (2006).

In the present paper, we study the exponential stability of general distributed parameter systems. A class of linear systems in a Hilbert space is considered, where a bounded operator acts on the delayed state. The system delay is admitted to be unknown and time-varying. Sufficient *delay-dependent* exponential stability conditions are derived in the form of LOIs, where the decision variables are operators in the Hilbert space. General methods for solving LOI have not been developed yet. Some finite dimensional approximations were considered in Ikeda, Azuma, and Uchida (2001).

Being applied to a heat equation and to a wave equation, the derived conditions are reduced to standard finite-dimensional LMIs that appear to guarantee the exponential stability of the first order and, respectively, the second order delay-differential equations. The surprising fact is that this *reduction of infinite-dimensional LOIs to finite-dimensional LMIs is tight* in the sense that the stability of the latter delay-differential equations is necessary for the stability of the PDEs in question.

Notation and Preliminaries

The notation used throughout is fairly standard. The superscript 'T' stands for matrix transposition, $\mathbf{R}^n$ denotes the n-dimensional

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Euclidean space with the norm $|\cdot|$, $\mathbf{R}^{n\times m}$ is the set of all $n \times m$ real matrices. The notation $P > 0$, for $P \in \mathbf{R}^{n\times n}$ means that P is symmetric and positive definite, whereas $\lambda_{\min}(P)$ ($\lambda_{\max}(P)$) denotes its minimum (maximum) eigenvalue.

Let $\mathcal{H}$ be a Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $|\cdot|$. Denote by $\mathcal{L}(\mathcal{H})$ bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. Given a linear operator $P : \mathcal{H} \to \mathcal{H}$ with a dense domain $\mathcal{D}(P) \subset \mathcal{H}$, the notation P^* stands for the adjoint operator. Such an operator P is strictly positive definite, i.e., $P > 0$, iff it is self-adjoint in the sense that $P = P^*$ and there exists a constant $\beta > 0$ such that $\langle x, Px\rangle \geq \beta\langle x, x\rangle$ and for all $x \in \mathcal{D}(P)$, whereas $P \geq 0$ means that P is self-adjoint and nonnegative definite, i.e., $\langle x, Px\rangle \geq 0$ for all $x \in \mathcal{D}(P)$.

If an infinitesimal operator A generates a strongly continuous semigroup $T(t)$ on the Hilbert space $\mathcal{H}$ (see, e.g., Curtain and Zwart (1995) for details), the domain of the operator A forms another Hilbert space $\mathcal{D}(A)$ with the graph inner product $(\cdot,\cdot)_{\mathcal{D}(A)}$ defined as follows: $(x, y)_{\mathcal{D}(A)} = \langle x, y\rangle + \langle Ax, Ay\rangle$, $x, y \in \mathcal{D}(A)$. Moreover, the induced norm $\|T(t)\|$ of the semigroup $T(t)$ satisfies the inequality $\|T(t)\| \leq \kappa e^{\sigma t}$ everywhere with some constant $\kappa > 0$ and growth bound σ.

The space of the continuous $\mathcal{H}$-valued functions $x : [a, b] \to \mathcal{H}$ with the induced norm $\|x\|_{C([a,b],\mathcal{H})} = \max_{s\in[a,b]} |x(s)|$ is denoted by $C([a, b], \mathcal{H})$. The space of the continuously differentiable $\mathcal{H}$-valued functions $x : [a, b] \to \mathcal{H}$ with the induced norm $\|x\|_{C^1([a,b],\mathcal{H})} = \max(\|x\|_{C([a,b],\mathcal{H})}, \|\dot{x}\|_{C([a,b],\mathcal{H})})$ is denoted by $C^1([a, b], \mathcal{H})$. $L_2(a, b; \mathcal{H})$ is the Hilbert space of square integrable $\mathcal{H}$-valued functions on (a, b) with the corresponding norm. $W^{l,2}([a, b], R)$ is the Sobolev space of absolutely continuous scalar functions on $[a, b]$ with square integrable derivatives of the order $l \geq 1$.

Given $x(\cdot) \in L_2(a, b; \mathcal{H})$, we denote $x^t = x(t + \theta) \in L_2(-h, 0; \mathcal{H})$ for $t \in [a + h, b]$.

Lemma 1 (*Wang (1994b) (Wirtinger's Inequality and its Generalization)*). *Let $z \in W^{1,2}([a, b], R)$ be a scalar function with $z(a) = z(b) = 0$. Then*

$$\int_a^b z^2(\xi)\mathrm{d}\xi \leq \frac{(b-a)^2}{\pi^2}\int_a^b \left[\frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi}\right]^2 \mathrm{d}\xi. \tag{1}$$

If additionally $z \in W^{2,2}([a, b], R)$, then

$$\int_a^b \left[\frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi}\right]^2 \mathrm{d}\xi \leq \frac{(b-a)^2}{\pi^2}\int_a^b \left[\frac{\mathrm{d}^2z(\xi)}{\mathrm{d}\xi^2}\right]^2 \mathrm{d}\xi. \tag{2}$$

Lemma 2 (*Jensen's Inequality*). *Let $\mathcal{H}$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. For any linear bounded operator $R : \mathcal{H} \to \mathcal{H}$, $R > 0$, scalar $l > 0$ and $x \in L_2([a, b], \mathcal{H})$ the following holds:*

$$l\int_0^l \langle x(s), Rx(s)\rangle \mathrm{d}s \geq \left\langle \int_0^l x(s)\mathrm{d}s, R\int_0^l x(s)\mathrm{d}s\right\rangle. \tag{3}$$

Note that (3) follows from the Cauchy–Schwartz inequality

$$l\int_0^l \langle R^{\frac{1}{2}}x(s), R^{\frac{1}{2}}x(s)\rangle \mathrm{d}s \geq \left\langle \int_0^l R^{\frac{1}{2}}x(s)\mathrm{d}s, \int_0^l R^{\frac{1}{2}}x(s)\mathrm{d}s\right\rangle.$$

2. Lyapunov method for exponential stability

Consider a linear infinite-dimensional system

$$\dot{x}(t) = Ax(t) + A_1x(t - \tau(t)), \quad t \geq t_0 \tag{4}$$

evolving in a Hilbert space $\mathcal{H}$ where $x(t) \in \mathcal{H}$ is the instantaneous state of the system. Let the following assumptions be satisfied:

A1 the operator A generates a strongly continuous semigroup $T(t)$ and the domain $\mathcal{D}(A)$ of the operator A is dense in $\mathcal{H}$;

A2 the linear operator A_1 is bounded in $\mathcal{H}$;

A3 the function $\tau(t)$ is piecewise-continuous of class C^1 on the closure of each continuity subinterval and it satisfies

$$\inf_t \tau(t) > 0, \qquad \sup_t \tau(t) \leq h \tag{5}$$

with some constant $h > 0$ for all $t \geq t_0$.

Let the initial conditions

$$x^{t_0} = \varphi(\theta), \quad \theta \in [-h, 0], \phi \in W \tag{6}$$

be given in the space

$$W = C([-h, 0], \mathcal{D}(A)) \cap C^1([-h, 0], \mathcal{H}). \tag{7}$$

Definition 1. A function $x(t) \in C([t_0 - h, t_0 + \eta], \mathcal{D}(A))$ is said to be a solution of the initial-value problem (4), (6) on $[t_0 - h, t_0 + \eta]$ if $x(t)$ is initialized with (6), it is absolutely continuous for $t \in [t_0, t_0 + \eta]$, and it satisfies (4) for almost all $t \in [t_0, t_0 + \eta]$.

The initial-value problem (4), (6) turns out to be well-posed on the semi-infinite time interval $[t_0, \infty)$ and its solutions can be found as mild solutions, i.e., as those of the integral equation

$$x(t) = T(t - t_0)x(t_0) + \int_{t_0}^t T(t - s)A_1x(s - \tau(s))\mathrm{d}s, \quad t \geq t_0. \tag{8}$$

Lemma 3. *Under* A1–A3 *there exists a unique solution of the initial value problem (4), (6) on $[t_0, \infty)$. This solution is also a unique solution of the integral initial value problem (6), (8).*

Proof. To begin with, let us choose a positive η_0 small enough to ensure that $\eta_0 < \inf_t \tau(t)$ and the first discontinuity point $t_0^1 > t_0$ of $\tau(t)$ is such that the difference $t_0^1 - t_0$ is multiple to η_0, i.e., $t_0^1 = t_0 + k_0\eta_0$ for some integer $k_0 > 0$. While being viewed over the time segment $[t_0, t_0 + \eta_0]$, the initial-value problem (4) is equivalent to

$$\dot{x}(t) = Ax(t) + A_1\phi(t - t_0 - \tau(t)), \qquad x(t_0) = \phi(0) \tag{9}$$

where the inhomogeneous term $A_1\phi(t - t_0 - \tau(t))$ is of class C^1 on $[0, \eta_0]$. By Theorem 3.1.3 of Curtain and Zwart (1995), there exists a unique local solution of (9) and this solution satisfies the integral equation (8) on $[t_0, t_0 + \eta_0]$.

The same line of reasoning is step-by-step applied to the time segments $[t_{i-1}, t_{i-1} + \eta_0]$, $i = 1, \ldots, k_0$ with $t_i = t_{i-1} + \eta_0$ and $t_{k_0} = t_0^1$. Following this line, the initial-value problem is demonstrated to possess a unique solution $x(t, t_0, \phi)$ for $t \in [t_0, t_0^1]$, which satisfies the integral equation (8) on $[t_0, t_0^1]$. The assertion of Lemma 3 is then concluded by iteration on the time segments $[t_0^j, t_0^{j+1}]$, $j = 1, 2, \ldots$ where $t_0^1 < t_0^2 < \cdots$ are the successive discontinuity points of the function $\tau(t)$. □

Our aim is to derive exponential stability criteria for linear time-delay system (4). The stability concept under study is based on the initial data norm

$$\|\phi\|_W = \sqrt{|A\phi(0)|^2 + \|\phi\|^2_{C^1([-h,0],\mathcal{H})}} \tag{10}$$

in space (7). Suppose $x(t, t_0, \phi)$ denotes a solution of (4), (6) at a time instant $t \geq t_0$.

Definition 2. System (4) is said to be exponentially stable with a decay rate $\delta > 0$ if there exists a constant $K \geq 1$ such that the following exponential estimate holds:

$$|x(t, t_0, \phi)|^2 \leq Ke^{-2\delta(t-t_0)}\|\phi\|_W^2 \quad \forall t \geq t_0. \tag{11}$$

Consider Lyapunov–Krasovskii Functionals (LKFs), which depend on x and $\dot{x}$ (Kolmanovskii & Myshkis, 1999). Given a continuous functional $V : \mathbf{R} \times W \times C([-h, 0], \mathcal{H}) \to \mathbf{R}$, its upper right-hand derivative along solutions $x^t(t_0, \phi),\ t \geq t_0$ of (4), (6) is defined as follows:

$$\dot{V}(t, \phi, \dot{\phi}) = \limsup_{s \to 0^+} \frac{1}{s}[V(t+s, x^{t+s}(t, \phi), \dot{x}^{t+s}(t, \phi)) - V(t, \phi, \dot{\phi})].$$

Lemma 4. *Let* A1–A3 *be in force and let there exist positive numbers* δ, β, γ *and a continuous functional* $V : \mathbf{R} \times W \times C([-h, 0], \mathcal{H}) \to \mathbf{R}$ *such that the function* $\bar{V}(t) = V(t, x^t, \dot{x}^t)$ *is absolutely continuous for* x^t*, satisfying* (4) *and*

$$\beta|\phi(0)|^2 \leq V(t, \phi, \dot{\phi}) \leq \gamma \|\phi\|_W^2, \tag{12}$$

$$\dot{V}(t, \phi, \dot{\phi}) + 2\delta V(t, \phi, \dot{\phi}) \leq 0. \tag{13}$$

Then (4) *is exponentially stable with the decay rate* δ *and* (11) *holds with* $K = \frac{\gamma}{\beta}$.

Proof. As in the case of ODE, from (13) with $\phi = x^t$ we obtain

$$\frac{d}{dt}V(t, x^t, \dot{x}^t) + 2\delta V(t, x^t, \dot{x}^t) \leq 0,$$

where $V(t_0, x^{t_0}, \dot{x}^{t_0}) = V(t_0, \phi, \dot{\phi})$. Hence, by the comparison principle argument (Khalil, 1992), it follows that

$$\beta|x(t)|^2 \leq V(t, x^t, \dot{x}^t) \leq V(t_0, \phi, \dot{\phi})e^{-2\delta(t-t_0)} \leq \gamma e^{-2\delta(t-t_0)}\|\phi\|_W^2. \quad \square$$

3. Exponential stability in a Hilbert space

In this section, the delay is assumed to be either slowly-varying with $\dot{\tau} \leq d < 1$, or fast-varying (with no restrictions on the delay-derivative). Let A1–A3 be in force. We derive delay-dependent conditions by using a "simple" (as defined in Gu et al. (2003)) LKF:

$$V(t, x^t, \dot{x}^t) = \langle x(t), Px(t)\rangle + \int_{t-h}^{t} e^{2\delta(s-t)}\langle x(s), Sx(s)\rangle ds + h\int_{-h}^{0}\int_{t+\theta}^{t} e^{2\delta(s-t)}\langle \dot{x}(s), R\dot{x}(s)\rangle ds d\theta + \int_{t-\tau(t)}^{t} e^{2\delta(s-t)}\langle x(s), Qx(s)\rangle ds \tag{14}$$

where $P : \mathcal{D}(A) \to \mathcal{H}$ is a linear positive definite operator and $R, Q, S \in \mathcal{L}(\mathcal{H})$ are non negative definite operators, satisfying the following inequalities:

$$\begin{aligned} &\beta\langle x, x\rangle \leq \langle x, Px\rangle \leq \gamma_P[\langle x, x\rangle + \langle Ax, Ax\rangle], \\ &\langle x, Qx\rangle \leq \gamma_Q\langle x, x\rangle, \qquad \langle x, Rx\rangle \leq \gamma_R\langle x, x\rangle, \\ &\langle x, Sx\rangle \leq \gamma_S\langle x, x\rangle, \quad \forall x \in D(A) \end{aligned} \tag{15}$$

for some positive constants $\beta, \gamma_P, \gamma_Q, \gamma_S, \gamma_R$. Thus condition (12) of Lemma 4 is satisfied.

We note that the first inequality (15) allows one to use unbounded operators P which are upper estimated by the unbounded operator A according to (15). In the case of ODE, where A is a matrix, the above upper bound is equivalent to the standard one with $A = 0$. For ODE with delay the Lyapunov functional of the form (14) was recently introduced in He, Wang, Lin, and Wu (2007) (for $\delta = 0$), whereas this functional with $S = 0$ was introduced earlier in Fridman and Shaked (2003) (for $\delta = 0$) and in Sun, Zhao, and Hill (2006) (for $\delta > 0$).

Being viewed on solutions of (4), the LKF (14) is absolutely continuous as a function of t because the solutions are absolutely continuous in t. Differentiating V, we find

$$\begin{aligned} &\dot{V}(t, x^t, \dot{x}^t) + 2\delta V(t, x^t, \dot{x}^t) \\ &\quad \leq 2\langle x(t), P\dot{x}(t)\rangle + 2\delta\langle x(t), Px(t)\rangle + h^2\langle \dot{x}(t), R\dot{x}(t)\rangle \\ &\qquad - he^{-2\delta h}\int_{t-h}^{t}\langle \dot{x}(s), R\dot{x}(s)\rangle ds + \langle x(t), (Q+S)x(t)\rangle \\ &\qquad - (1-\dot{\tau}(t))\langle x(t-\tau(t)), Qx(t-\tau(t))\rangle e^{-2\delta h} \\ &\qquad - \langle x(t-h), Sx(t-h)\rangle e^{-2\delta h}. \end{aligned} \tag{16}$$

Following He et al. (2007), we employ the representation

$$-h\int_{t-h}^{t}\langle \dot{x}(s), R\dot{x}(s)\rangle ds = -h\int_{t-h}^{t-\tau(t)}\langle \dot{x}(s), R\dot{x}(s)\rangle ds - h\int_{t-\tau(t)}^{t}\langle \dot{x}(s), R\dot{x}(s)\rangle ds \tag{17}$$

and apply the Jensen's inequality (3)

$$\begin{aligned} &\int_{t-\tau(t)}^{t}\langle \dot{x}(s), R\dot{x}(s)\rangle ds \geq \frac{1}{h}\left\langle \int_{t-\tau(t)}^{t}\dot{x}(s)ds, R\int_{t-\tau(t)}^{t}\dot{x}(s)ds\right\rangle, \\ &\int_{t-h}^{t-\tau(t)}\langle \dot{x}(s), R\dot{x}(s)\rangle ds \geq \frac{1}{h}\left\langle \int_{t-h}^{t-\tau(t)}\dot{x}(s)ds, R\int_{t-h}^{t-\tau(t)}\dot{x}(s)ds\right\rangle. \end{aligned} \tag{18}$$

Then, taking into account that $\dot{\tau} \leq d < 1$ and following Gouaisbaut and Peaucelle (2006), we obtain

$$\begin{aligned} &\dot{V}(t, x^t, \dot{x}^t) + 2\delta V(t, x^t, \dot{x}^t) \\ &\quad \leq 2\langle x(t), P\dot{x}(t)\rangle + 2\delta\langle x(t), Px(t)\rangle + h^2\langle \dot{x}(t), R\dot{x}(t)\rangle \\ &\qquad - [\langle x(t) - x(t-\tau(t)), R(x(t) - x(t-\tau(t)))\rangle \\ &\qquad + \langle x(t-\tau(t)) - x(t-h), R(x(t-\tau(t)) - x(t-h))\rangle \\ &\qquad + (1-d)\langle x(t-\tau(t)), Qx(t-\tau(t))\rangle]\, e^{-2\delta h} \\ &\qquad + \langle x(t), (Q+S)x(t)\rangle - \langle x(t-h), Sx(t-h)\rangle e^{-2\delta h}. \end{aligned} \tag{19}$$

We will derive stability conditions in two forms. The first form will subsequently be applied to the wave equation and the second one to the heat equation. The first form is derived by substituting the right-hand side of (4) for $\dot{x}(t)$. Setting $\eta(t) = col\{x(t), x(t-h), x(t-\tau(t))\}$, we find that the condition (13) of Lemma 4

$$\dot{V}(t, x^t, \dot{x}^t) + 2\delta V(t, x^t, \dot{x}^t) \leq \langle \eta(t), \Phi_h\eta(t)\rangle \leq 0 \tag{20}$$

is satisfied if the following LOI

$$\Phi_h = \begin{bmatrix} \Phi_{11} & 0 & PA_1 \\ 0 & 0 & 0 \\ A_1^*P & 0 & 0 \end{bmatrix} + h^2\begin{bmatrix} A^*RA & 0 & A^*RA_1 \\ 0 & 0 & 0 \\ A_1^*RA & 0 & A_1^*RA_1 \end{bmatrix} - e^{-2\delta h}\begin{bmatrix} R & 0 & -R \\ 0 & (S+R) & -R \\ -R & -R & 2R + (1-d)Q \end{bmatrix} \leq 0, \tag{21}$$

holds provided that

$$\Phi_{11} = A^*P + PA + 2\delta P + Q + S. \tag{22}$$

The resulting inequality (21) is convex with respect to h: given $h_0 > 0$, it becomes feasible for all $\bar{h} \in [0, h_0]$ whenever it is feasible for h_0. The convexity follows from the fact that $\Phi_{\bar{h}} \leq \Phi_{h_0}$ since h^2 and $-e^{-2\delta h}$ multiply the non negative definite operators. Summarizing, the following result is obtained:

Theorem 1. *Let* A1–A3 *be in force. Given* $\delta > 0$*, let there exist linear operators* $P > 0$ *and* $R \geq 0, S \geq 0, Q \geq 0$ *subject to* (15) *such that the LOI* (21) *with notation* (22) *holds in the Hilbert space* $\mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A)$*. Then system* (4) *is exponentially stable with* *the decay rate δ for all differentiable delays with $\dot\tau(t) \le d < 1$. The inequality (11) is satisfied with $K = \max\{\gamma_P, h(\gamma_Q + \gamma_S + h^2\gamma_R/2)\}/\beta$. Moreover, (4) is exponentially stable for all fast-varying delays $0 \le \tau \le h$ if the LOI (21) is feasible with $Q = 0$.*

The conditions of Theorem 1 are delay-dependent, namely h-dependent, even for $\delta \to 0$. Taking in the above derivations $S = R = 0$ we obtain the following "quasi delay-independent" conditions, which become delay-independent for $\delta \to 0$ and coincide in the case of ODE with the result of Mondie and Kharitonov (2005):

Corollary 1. *Let* A1–A3 *be in force. Given $\delta > 0$, system (4) is exponentially stable with the decay rate δ for all differentiable delays with $\dot\tau(t) \le d < 1$ if there exist linear operators $P > 0$ and $Q \ge 0$ subject to (15) such that the LOI*

$$\begin{bmatrix} (A+\delta)^*P + P(A+\delta) + Q & PA_1 \\ A_1^*P & -(1-d)Q\,\mathrm{e}^{-2\delta h} \end{bmatrix} \le 0 \tag{23}$$

holds in the Hilbert space $\mathcal{D}(A) \times \mathcal{D}(A)$. The inequality (11) is satisfied with $K = \max\{\gamma_P, h\gamma_Q\}/\beta$.

Remark 1. Differently from the finite dimensional case, the feasibility of the strict LOI (21) and (23) for $h = 0$ ($\delta = 0$) does not necessarily imply the feasibility of (21) and (23) for small enough h (δ) because h^2 (δ) is multiplied by the operator, which may be unbounded.

It may be difficult to verify the feasibility of (21), if the operator that multiplies h^2 (and depends on A) in Φ_h is unbounded. To avoid this, we will derive the second form of LOI by the descriptor method (Fridman, 2001), where the right-hand sides of the expressions

$$\begin{aligned} 0 &= 2\langle x(t), P_2^*[Ax(t) + A_1x(t-\tau(t)) - \dot x(t)]\rangle, \\ 0 &= 2\langle \dot x(t), P_3^*[Ax(t) + A_1x(t-\tau(t)) - \dot x(t)]\rangle \end{aligned} \tag{24}$$

with some $P_2, P_3 \in \mathcal{L}(\mathcal{H})$ are added into the right-hand side of (19). Setting $\eta_d(t) = col\{x(t), \dot x(t), x(t-h), x(t-\tau(t))\}$, we obtain that

$$\dot V(t, x^t, \dot x^t) + 2\delta V(t, x^t, \dot x^t) \le \langle \eta_d(t), \Phi_d\eta_d(t)\rangle \le 0,$$

if the LOI

$$\Phi_d = \begin{bmatrix} \Phi_{d11} & \Phi_{d12} & 0 & P_2^*A_1 + R\mathrm{e}^{-2\delta h} \\ * & \Phi_{d22} & 0 & P_3^*A_1 \\ * & * & -(S+R)\mathrm{e}^{-2\delta h} & R\mathrm{e}^{-2\delta h} \\ * & * & * & -[2R + (1-d)Q]\mathrm{e}^{-2\delta h} \end{bmatrix} \le 0 \tag{25}$$

holds, where

$$\begin{aligned} \Phi_{d11} &= A^*P_2 + P_2^*A + 2\delta P + Q + S - R\mathrm{e}^{-2\delta h}, \\ \Phi_{d12} &= P - P_2^* + A^*P_3, \qquad \Phi_{d22} = -P_3 - P_3^* + h^2R \end{aligned} \tag{26}$$

and $*$ denotes the symmetric terms of the operator matrix. Thus, the following result is obtained.

Theorem 2. *Let* A1–A3 *be in force. Given $\delta > 0$, let there exist linear operators $P > 0$ and $R \ge 0, S \ge 0, Q \ge 0$ subject to (15) and indefinite operators $P_2, P_3 \in \mathcal{L}(\mathcal{H})$ such that the LOI (25) with notations given in (26) holds in the Hilbert space $\mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A)$. Then system (4) is exponentially stable with the decay rate δ for all differentiable delays (5) with $\dot\tau \le d < 1$. The inequality (11) is satisfied with $K = \max\{\gamma_P, h(\gamma_Q + \gamma_S + h^2\gamma_R/2)\}/\beta$. Moreover, (4) is exponentially stable for all fast-varying delays $0 \le \tau \le h$ if the LOI (25) is feasible with $Q = 0$.*

Differently from the LOI (21), the feasibility of the strict LOI (25) for $h = 0$ implies the feasibility of (25) for small enough h (h^2 is multiplied by the bounded operator R).

Remark 2. Consider now the system (4) with A and A_1 from the uncertain time-invariant polytope

$$\Omega = \sum_{j=1}^{M} f_j\Omega_j \quad \text{for some } 0 \le f_j \le 1, \sum_{j=1}^{M} f_j = 1, \tag{27}$$

where $\Omega_j = \begin{bmatrix} A^{(j)} & A_1^{(j)} \end{bmatrix}$, $A_1^{(j)} \in \mathcal{L}(\mathcal{H})$ and the operators $A^{(j)}$ have a common domain, which is dense in $\mathcal{H}$ and $A = \sum_{j=1}^{M} f_jA^{(j)}$ generates a strongly continuous semigroup for all f_j, satisfying (27). Applying conditions of Theorem 2 to the uncertain system, we conclude that (4) is exponentially stable if LOI (25) is feasible. Since LOI (25) is affine in A and A_1, by the same arguments as for LMIs (see Boyd et al. (1994)) we conclude that (25) is feasible if the LOIs (25) in M vertices are feasible for the same P_2, P_3 and for different $Q^{(j)} \ge 0, S^{(j)} > 0, R^{(j)} > 0, P^{(j)} > 0$.

4. Stability of the delay heat equation

Consider the heat equation

$$z_t(\xi, t) = az_{\xi\xi}(\xi, t) - a_0z(\xi, t) - a_1z(\xi, t-\tau(t)), \tag{28}$$

where $t \ge t_0$, $0 \le \xi \le l$, with the constant parameters $a > 0$, a_0 and a_1, with the time-varying delay $\tau(t)$, satisfying (5), and with the Dirichlet boundary condition

$$z(0, t) = z(l, t) = 0, \quad t \ge t_0. \tag{29}$$

The boundary-value problem (28) and (29) describes the propagation of heat in a homogeneous one-dimensional rod with a fixed temperature at the ends in the case of the delayed (possibly, due to actuation) heat exchange with the surroundings. Here a and a_i, $i = 0, 1$ stand for the heat conduction coefficient and for the coefficients of the heat exchange with the surroundings, respectively, $z(\xi, t)$ is the value of the temperature field of the plant at time moment t and location ξ along the rod. In the sequel, the state dependence on time t and spatial variable ξ is suppressed whenever possible.

The boundary-value problem (28) and (29) can be rewritten as the differential equation (4) in the Hilbert space $\mathcal{H} = L_2(0, l)$ with the infinitesimal operator $A = a\frac{\partial^2}{\partial\xi^2} - a_0$ with the dense domain

$$\mathcal{D}\left(\frac{\partial^2}{\partial\xi^2}\right) = \{z \in W^{2,2}([0, l], \mathbf{R}) : z(0) = z(l) = 0\}, \tag{30}$$

and with the bounded operator $A_1 = -a_1$ of the multiplication by the constant $-a_1$. The infinitesimal operator A generates an exponentially stable semigroup (see, e.g., Curtain and Zwart (1995) for details).

We will first derive simple delay-independent conditions, based on LOI (23). Consider the LKF of the form

$$V = p\int_0^l z^2(\xi, t)\mathrm{d}\xi + q\int_{t-\tau(t)}^{t}\int_0^l \mathrm{e}^{2\delta(s-t)}z^2(\xi, s)\mathrm{d}\xi\mathrm{d}s \tag{31}$$

with some positive constants p and q. Then the operators P and Q in (23) take the form $P = p$, $Q = q$ of the bounded operators of the multiplication by positive constants p and q, respectively. Integrating by parts and taking into account (29), we find that for $x \in \mathcal{D}(A)$

$$\begin{aligned} \langle x, (A^*P + PA)x\rangle &= 2a\int_0^l pzz_{\xi\xi}\mathrm{d}\xi - 2a_0\int_0^l pz^2\mathrm{d}\xi \\ &= -2\int_0^l (apz_\xi^2 + a_0pz^2)\mathrm{d}\xi \le -2\left(\frac{\pi^2}{l^2}a + a_0\right)\int_0^l pz^2\mathrm{d}\xi \end{aligned} \tag{32}$$

where the last inequality follows from the Wirtinger's inequality (1). We thus obtain that (23) is satisfied if

$$\Psi_\delta \triangleq \begin{bmatrix} q-2\left(\frac{\pi^2}{l^2}a+a_0-\delta\right)p & -a_1p \\ -a_1p & -(1-d)qe^{-2\delta h}\end{bmatrix} < 0. \quad (33)$$

From $\Psi_0 < 0$ it follows that $\Psi_\delta < 0$ for small enough δ, since $\Psi_\delta = \Psi_0 + \mathrm{diag}\{2\delta p, (1-d)q(1-e^{-2\delta h})\}$. We proved

Theorem 3. *Given $\delta > 0$, let the LMI (33) holds for some scalars $p > 0$ and $q > 0$. Then the Dirichlet boundary-value problem (28), (29) is exponentially stable for all differentiable delays (5) with $\dot\tau(t) \le d < 1$, and the inequality*

$$\int_0^l z^2(\xi,t)\mathrm{d}\xi \le Ke^{-2\delta(t-t_0)} \max_{s\in[t_0-h,t_0]} \int_0^l z^2(\xi,s)\mathrm{d}\xi \quad (34)$$

is satisfied for all $t \ge t_0$ with $K = 1 + hq/p$. If (33) holds for $\delta = 0$, then the inequality (34) is satisfied with $K = 1 + hq/p$ and small enough δ.

Remark 3. By Schur complements formula, LMI (33) with $\delta = 0$ is feasible iff for some $p > 0$ and $q > 0$ the following holds: $q^2 - 2(\frac{\pi^2}{l^2}a + a_0)pq + a_1^2p^2/(1-d) < 0$. The left part of the latter inequality achieves its minimum at $q = (\frac{\pi^2}{l^2}a + a_0)p$ and, thus, the inequality holds iff

$$\frac{\pi^2}{l^2}a + a_0 > 0, \quad a_1^2 < \left(\frac{\pi^2}{l^2}a + a_0\right)^2 (1-d). \quad (35)$$

To derive delay-dependent conditions, we apply Theorem 2 (it is difficult to find operators that satisfy Theorem 1 since h^2 is multiplied by the unbounded operator in (21)). For simplicity we consider $l = \pi$. We choose V of the form

$$V(t,z^t,z_s^t) = (p_1 - p_3a)\int_0^\pi z^2(\xi,t)\mathrm{d}\xi + p_3a\int_0^\pi z_\xi^2(\xi,t)\mathrm{d}\xi$$
$$+ \int_0^\pi \Bigg[r\int_{-h}^0\int_{t+\theta}^t e^{2\delta(s-t)}z_s^2(\xi,s)\mathrm{d}s\mathrm{d}\theta$$
$$+ s\int_{t-h}^t e^{2\delta(s-t)}z^2(\xi,s)\mathrm{d}s + q\int_{t-\tau(t)}^t e^{2\delta(s-t)}z^2(\xi,s)\mathrm{d}s\Bigg]\mathrm{d}\xi$$

with some constants $p_1 > 0$, $p_3 > 0$, $s > 0$, $r > 0$ and $q \ge 0$. Then the operators in (14) take the form $P = -p_3(a\frac{\partial^2}{\partial\xi^2} + a) + p_1$, $R = r$, $Q = q$, $S = s$. We choose $P_2 = p_2$ and $P_3 = p_3$, where $p_2 > 0$ and

$$p_2 - \delta p_3 \ge 0. \quad (36)$$

Here P is unbounded operator and all the others are bounded operators in $L_2(0,\pi)$. We note that the above choice of P, depending on the slack variable P_3, is different to that of the ODEs (where these matrices are independent). Thus, for the first time, the slack variable allows one to construct an appropriate LKF.

Integrating by parts and utilizing the Wirtinger's inequality (1), we find that for $x \in D(A)$

$$\langle x, Px\rangle = \int_0^\pi \left[-p_3az_{\xi\xi}z - p_3az^2 + p_1z^2\right]\mathrm{d}\xi$$
$$= \int_0^\pi \left[ap_3[z_\xi^2 - z^2] + p_1z^2\right]\mathrm{d}\xi \ge p_1\int_0^\pi z^2\mathrm{d}\xi > 0.$$

Moreover, (15) is satisfied, since by the generalized Wirtinger's inequality (2) the following holds

$$\langle x, Px\rangle \le \int_0^\pi [ap_3(z_{\xi\xi})^2 + (ap_3+p_1)z^2]\mathrm{d}\xi \le \gamma_P[|Ax|^2 + |x|^2]$$

for some $\gamma_P > 0$. We obtain that

$$\langle \dot x, (P - P_2^* + A^*P_3)x\rangle = \langle \dot x, (p_1 - p_2 - (a+a_0)p_3)x\rangle;$$
$$\langle x, A^*P_2x\rangle + \langle x, P_2^*Ax\rangle + 2\delta\langle x, Px\rangle$$
$$= 2a(p_2 - \delta p_3)\int_0^\pi z_{\xi\xi}z\mathrm{d}\xi + 2[\delta(p_1 - ap_3) - a_0p_2]\int_0^\pi z^2\mathrm{d}\xi$$
$$= -2a(p_2 - \delta p_3)\int_0^\pi z_\xi^2\mathrm{d}\xi + 2[\delta(p_1 - ap_3) - a_0p_2]\int_0^\pi z^2\mathrm{d}\xi$$
$$\le [-2(a+a_0)p_2 + 2\delta p_1]\int_0^\pi z^2\mathrm{d}\xi,$$

where the latter inequality follows from (36) and the Wirtinger's inequality (1). Therefore, (25) holds if

$$\begin{bmatrix} \phi_{11} & \phi_{12} & 0 & \phi_{14} \\ * & -2p_3 + h^2r & 0 & -p_3a_1 \\ * & * & -(s+r)e^{-2\delta h} & re^{-2\delta h} \\ * & * & * & \phi_{44}\end{bmatrix} < 0,$$
$$\phi_{11} = -2(a+a_0)p_2 + 2\delta p_1 + q + s - re^{-2\delta h},$$
$$\phi_{12} = p_1 - p_2 - (a+a_0)p_3, \qquad \phi_{14} = -p_2a_1 + re^{-2\delta h},$$
$$\phi_{44} = -[2r + (1-d)q]e^{-2\delta h}. \quad (37)$$

Summarizing the following result is obtained

Theorem 4. *Given $\delta > 0$, let there exist scalars $p_1 > 0, p_2 > 0, p_3 > 0, s > 0, r > 0$ and $q \ge 0$ such that LMIs (36) and (37) hold. Then the boundary-value problem (28) and (29), where $l = \pi$, is exponentially stable with the decay rate δ for all differentiable delays (5) with $\dot\tau \le d < 1$ and the inequality*

$$p_1\int_0^\pi z^2(\xi,t)\mathrm{d}\xi \le e^{-2\delta(t-t_0)}\Bigg\{ap_3\int_0^\pi z_\xi^2(\xi,t_0)\mathrm{d}\xi$$
$$+ \max\{p_1 - p_3a + hq + hs, h^3r/2\}$$
$$\times \max_{s\in[t_0-h,t_0]}\int_0^\pi [z^2(\xi,s) + z_t^2(\xi,s)]\mathrm{d}\xi\Bigg\} \quad (38)$$

is satisfied for all $t \ge t_0$. Moreover, (28), (29) is exponentially stable with the decay rate δ for all fast varying delays (5) with no restrictions on $\dot\tau$ if (36), (37) are feasible with $q = 0$. If (37) holds for $\delta = 0$, then (28), (29) is exponentially stable with a sufficiently small decay rate.

Remark 4. The same LMIs (33) (with $l = \pi$) and (37) guarantee the exponentially stability of the scalar ODE

$$\dot y(t) + (a+a_0)y(t) + a_1y(t-\tau(t)) = 0. \quad (39)$$

System (39) corresponds to the first modal dynamics (with $k = 1$) in the modal representation of the Dirichlet boundary-value problem (28), (29) with $l = \pi$

$$\dot y_k(t) + (ak^2 + a_0)y_k(t) + a_1y_k(t-\tau(t)) = 0, \quad k = 1, 2, \ldots \quad (40)$$

projected on the eigenfunctions of the operator $\frac{\partial^2}{\partial\xi^2}$ (this operator has eigenvalues $-k^2$, see e.g. Wu (1996)). The stability of (28), (29) implies the stability of (40). Thus the reduction of infinite-dimensional LOI of Corollary 1 (Theorem 2) to finite-dimensional LMI of Theorem 3 (Theorem 4) is tight, since the stability of (39) is necessary for the stability of (28), (29).

Remark 5. Consider now the Dirichlet boundary-value problem (28), (29) with the uncertain coefficients from the uncertain time-invariant polytope Ω given by (27) with $\Omega_j = \begin{bmatrix} a^{(j)} & a_0^{(j)} & a_1^{(j)}\end{bmatrix}$. Here $M = 2^k$ and k is the number of uncertain parameters and it may take values from the finite set $\{1, 2, 3\}$. The uncertain infinitesimal operator $A = \sum_{j=1}^M f_ja^{(j)}\frac{\partial^2}{\partial\xi^2} - a_0^{(j)}$ with the dense domain (30) generates strongly continuous semigroup, whereas the uncertain operator $A_1 = \sum_{j=1}^{M} f_j a^{(j)}$ is bounded. By applying Remark 2 and Theorem 4, we conclude that (28), (29) is exponentially stable if (36) holds and LMIs (37) in the vertices are feasible for the same p_2, p_3 and for different $q^{(j)} \geq 0, s^{(j)} > 0, r^{(j)} > 0, p^{(j)} > 0,\ j = 1, \dots, M$. By Theorem 3, (28), (29) is exponentially stable if LMIs (33) in the vertices are feasible for the same $p > 0$ and for different $q^{(j)},\ j = 1, \dots, M$.

Example 1. Consider the controlled heat equation

$$z_t(\xi, t) = z_{\xi\xi}(\xi, t) + rz(\xi, t) + u, \qquad z(0, t) = z(l, t) = 0, \quad (41)$$

where $\xi \in (0, l)$, $t > 0$ and where r is uncertain parameter satisfying $|r| \leq \beta$ with given β. It was shown in Rebiai and Zinober (1993) that for $l = 1$ the state-feedback $u = -\gamma z(\xi, t)$ with $\gamma > (\frac{\beta}{2\pi})^2$ exponentially stabilizes (41). By verifying LMI (33), we conclude that the closed-loop system is exponentially stable if there exists $p > 0$ such that $-2(\pi^2 - r + \gamma)p < 0$ for all $|r| \leq \beta$, i.e. if $\gamma > \beta - \pi^2$. Since $\beta - \pi^2 \leq (\frac{\beta}{2\pi})^2$, our method guarantees exponential stabilization of (41) via a lower gain, which becomes essentially lower for big β.

Noting that a time-delay often appears in the feedback, we consider next the case of $l = \pi$, $\beta = 1.9$ and the delayed feedback $u = -z(\xi, t - \tau(t))$ with uncertain delay satisfying A3. This is a polytopic system reached by choosing $r = \pm 1.9$. Applying Theorem 4 (with $\delta = 0$) and Remark 5 to the resulting closed-loop system, we verify the feasibility of LMI (37) in the two vertices corresponding to $r = \pm 1.9$. We use LMI toolbox of Matlab and let d to be 0 and unknown, respectively. We find the maximum values of h for which the system remains exponentially stable: $d = 0,\ h = 1.025$; unknown d, $h = 1.021$.

As shown before, the latter results correspond also to the stability of $\dot{y} = (-1 + r)y(t) - y(t - \tau(t))$, with $|r| \leq 1.9$.

5. Stability of the delay wave equation

Consider the wave equation

$$\begin{aligned} z_{tt}(\xi, t) &= az_{\xi\xi} - \mu_0 z_t(\xi, t) - \mu_1 z_t(\xi, t - \tau(t)) \\ &\quad -a_0 z(\xi, t) - a_1 z(\xi, t - \tau(t)), \quad t \geq t_0, 0 \leq \xi \leq \pi \end{aligned} \quad (42)$$

with the Dirichlet boundary condition (29), where $l = \pi$ and with the constant parameters $a > 0$, $\mu_0 > 0$, μ_1, a_0, and a_1, with the time-varying delay $\tau(t)$, satisfying (5). The boundary-value problem (29), (42) describes the oscillations of a homogeneous string with fixed ends in the case of the delayed (possibly, due to actuation) stiffness restoration and dissipation. Here a stands for the elasticity coefficient, μ_0 and μ_1 stand for the dissipation coefficients, and a_0, a_1 stand for the restoring stiffness coefficients, the state vector $x = col\{z, z_t\}$ consists of the deflection $z(\xi, t)$ of the string and its velocity $z_t(\xi, t)$ at time moment t and location ξ along the string.

Let us introduce the operators

$$A = \begin{bmatrix} 0 & 1 \\ a\dfrac{\partial^2}{\partial \xi^2} - a_0 & -\mu_0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 \\ -a_1 & -\mu_1 \end{bmatrix} \quad (43)$$

where A_1 is a bounded operator of multiplication by the constant matrix and where the domain $\mathcal{D}(\frac{\partial^2}{\partial \xi^2})$ of the double differentiation operator is determined by (30). Then the boundary-value problem (29), (42) can be represented as the differential equation (4) in the Hilbert space $\mathcal{H} = L_2(0, \pi) \times L_2(0, \pi)$ with the infinitesimal operator A, possessing the domain $\mathcal{D}(A) = \mathcal{D}(\frac{\partial^2}{\partial \xi^2}) \times L_2(0, \pi)$ and generating an exponentially stable semigroup (see, e.g., Curtain and Zwart (1995) for details).

We first derive quasi delay-independent conditions by choosing V in the form

$$\begin{aligned} V &= ap_{03} \int_0^{\pi} z_\xi^2(\xi, t) \mathrm{d}\xi + \int_0^{\pi} v^{\mathrm{T}}(\xi, t) P_0 v(\xi, t) \mathrm{d}\xi \\ &\quad + \int_{t-\tau(t)}^{t} \int_0^{\pi} v^{\mathrm{T}}(\xi, s) \mathrm{e}^{2\delta(s-t)} Q\, v(\xi, s) \mathrm{d}\xi \mathrm{d}s, \end{aligned} \quad (44)$$

$$P_0 = \begin{bmatrix} p_{01} & p_{02} \\ p_{02} & p_{03} \end{bmatrix}, \qquad P_w \triangleq P_0 + \mathrm{diag}\{ap_{03}, 0\} > 0, \quad Q \geq 0,$$

where $v^{\mathrm{T}}(\xi, t) = [z(\xi, t)\ \ z_t(\xi, t)]$. Then the operators P (unbounded) and Q (bounded) in (23) are given by $P = \mathrm{diag}\{-ap_{03}\frac{\partial^2}{\partial \xi^2}, 0\} + P_0$, $Q \geq 0$. Now, integrating by parts and taking into account the inequality $p_{03} > 0$ (extracted from $P_w > 0$) and Wirtinger's inequality (1) we obtain that

$$\begin{aligned} \langle x, Px \rangle &= \int_0^{\pi} \left[-ap_{03} z_{\xi\xi} z + z^{\mathrm{T}} P_0 z \right] \mathrm{d}\xi \\ &= a \int_0^{\pi} p_{03} z_\xi^2 \mathrm{d}\xi + \langle x, P_0 x \rangle \\ &\geq \langle x, P_w x \rangle \geq \lambda_{\min}(P_w)|x|^2 > 0 \end{aligned} \quad (45)$$

for all $x \in \mathcal{D}(A) \times L_2(0, \pi)$. Moreover, by the generalized Wirtinger's inequality (2) the following holds

$$\langle x, Px \rangle \leq \int_0^{\pi} ap_{03}(z_{\xi\xi})^2 \mathrm{d}\xi + \langle x, P_0 x \rangle \leq \gamma_P(|Ax|^2 + |x|^2) \quad (46)$$

with some constant $\gamma_P > 0$ and thus (15) is satisfied.

Finally, integration by parts and application of Wirtinger's inequality (1) under $p_{02} - \delta p_{03} \geq 0$ yield

$$\begin{aligned} &\langle x, P(A + \delta)x \rangle + \langle x, (A^* + \delta)Px \rangle \\ &= \int_0^{\pi} [z\ z_t] \left\{ \begin{bmatrix} p_{01} - ap_{03}\dfrac{\partial^2}{\partial \xi^2} & p_{02} \\ p_{02} & p_{03} \end{bmatrix} \begin{bmatrix} \delta & 1 \\ a\dfrac{\partial^2}{\partial \xi^2} - a_0 & -\mu_0 + \delta \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} \delta & a\dfrac{\partial^2}{\partial \xi^2} - a_0 \\ 1 & -\mu_0 + \delta \end{bmatrix} \begin{bmatrix} p_{01} - ap_{03}\dfrac{\partial^2}{\partial \xi^2} & p_{02} \\ p_{02} & p_{03} \end{bmatrix} \right\} \begin{bmatrix} z \\ z_t \end{bmatrix} \mathrm{d}\xi \\ &= -2a(p_{02} - p_{03}\delta) \int_0^{\pi} (z_\xi)^2 \mathrm{d}\xi - \int_0^{\pi} [z\ z_t] \\ &\quad \times \begin{bmatrix} -2p_{02}a_0 + 2p_{01}\delta & p_{01} - (\mu_0 - 2\delta)p_{02} - p_{03}a_0 \\ p_{01} - (\mu_0 - 2\delta)p_{02} - p_{03}a_0 & 2p_{02} - 2(\mu_0 - \delta)p_{03} \end{bmatrix} \\ &\quad \times \begin{bmatrix} z \\ z_t \end{bmatrix} \mathrm{d}\xi \leq \int_0^{\pi} [z\ z_t](P_w C_\delta + C_\delta^{\mathrm{T}} P_w) \begin{bmatrix} z \\ z_t \end{bmatrix} \mathrm{d}\xi, \end{aligned} \quad (47)$$

where

$$C_\delta = \begin{bmatrix} \delta & 1 \\ -a - a_0 & -\mu_0 + \delta \end{bmatrix}. \quad (48)$$

Therefore (23) is feasible if the following LMI

$$\Omega_\delta \triangleq \begin{bmatrix} C_\delta^{\mathrm{T}} P_w + P_w C_\delta + Q & P_w A_1 \\ A_1^{\mathrm{T}} P_w & -(1-d)\mathrm{e}^{-2\delta h} Q \end{bmatrix} \leq 0 \quad (49)$$

is feasible. Since the LMI (49) ensures that the (1,1)-term of its left-hand side satisfies

$$-2p_{02}(a_0 + a) + 2(p_{01} + ap_{03})\delta + q_1 \leq 0,$$

whereas $q_1 + 2\delta(p_{01} + ap_{03}) \geq 0$, it follows that $p_{02} \geq 0$. From $\Omega_0 < 0$ it follows that $p_{02} > 0$ and, thus, $p_{02} - \delta p_{03} \geq 0$ for small enough δ. Moreover, $\Omega_0 < 0$ yields $\Omega_\delta < 0$ for small enough δ, since $\Omega_\delta = \Omega_0 + \mathrm{diag}\{2\delta P_w, (1-d)Q(1 - \mathrm{e}^{-2\delta h})\}$. Summarizing, we arrive at the following.

Theorem 5. *Given $\delta > 0$, let the LMIs $p_{02} - \delta p_{03} \geq 0$ and (49) hold for some symmetric 2×2-matrices $P_w > 0$ and $Q \geq 0$, where p_{02} and p_{03} are respectively $(1, 2)$ and $(2, 2)$ terms of P_w. Then the Dirichlet boundary-value problem (42), (29), where $l = \pi$, is exponentially stable with the decay rate δ for all differentiable delays (5) with $\dot{\tau}(t) \leq d < 1$, and the inequality*

$$\lambda_{\min}(P_w) \int_0^{\pi} [z^2(\xi, t) + z_t^2(\xi, t)] \mathrm{d}\xi \leq \mathrm{e}^{-2\delta(t-t_0)} \left\{ K_w \max_{s \in [t_0 - h, t_0]} \int_0^{\pi} [z^2(\xi, s) + z_t^2(\xi, s)] \mathrm{d}\xi + a p_{03} \int_0^{\pi} z_\xi^2(\xi, t_0) \mathrm{d}\xi \right\} \tag{50}$$

is satisfied with

$$K_w = \lambda_{\max}(P_w - \mathrm{diag}\{a p_{03}, 0\}) + h \lambda_{\max}(Q)$$

for all $t \geq t_0$. If $\Omega_0 < 0$ holds for $\delta = 0$, then (42), (29) is exponentially stable with a sufficiently small decay rate.

Corollary 2. *In a particular case where $a = 1$ and $a_0 = a_1 = 0$, the Dirichlet boundary-value problem (29), (42), where $l = \pi$, is exponentially stable for all differentiable delays with $\dot{\tau} \leq d < 1$ if $\mu_1^2 < (1 - d)\mu_0^2$.*

Proof. Since the delay appears only in z_t, we choose the decision variables of LMI (49) in the form $P_w = \begin{bmatrix} 1 & 2\delta \\ 2\delta & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 \\ 0 & \mu_0 \end{bmatrix}$, and thus $P_w A_1 = \begin{bmatrix} 0 & -2\mu_1\delta \\ 0 & -\mu_1 \end{bmatrix}$. Deleting from Ω_δ the column and the row, consisting of zero elements, we obtain the matrix

$$\begin{bmatrix} -2\delta & 4\delta^2 - 2\mu_0\delta & -2\mu_1\delta \\ * & -\mu_0 + 6\delta & -\mu_1 \\ * & * & -(1-d)\mathrm{e}^{-2\delta h}\mu_0 \end{bmatrix}.$$

Applying the Schur complements formula to the last column and the last row of this matrix, we conclude that (49) is feasible if the following LMI holds

$$\begin{bmatrix} -2\delta + O(\delta^2) & O(\delta) \\ O(\delta) & -\mu_0 + \dfrac{\mu_1^2}{(1-d)\mu_0} \mathrm{e}^{2\delta h} + 6\delta \end{bmatrix} < 0, \tag{51}$$

where $|O(\delta^k)| \leq c\delta^k$ $(k = 1, 2)$ for some constant $c > 0$ and for all sufficiently small δ. If $\mu_1^2 < (1 - d)\mu_0^2$, then the (2,2)-term of the left-hand side of (51) is negative for small enough δ. Finally, applying the Schur complements formula to the second column and the second row of the matrix in (51), we obtain the expression of the form $-2\delta + O(\delta^2)$, which is negative for small δ. Therefore, (51) and, thus (49) are feasible for small δ. □

Remark 6. The condition $0 \leq \mu_1 < \mu_0$ for the stability of the wave equation with constant delay and $a = 1$, $a_0 = a_1 = 0$ and with mixed Dirichlet–Neumann boundary condition was obtained by Nicaise and Pignotti (2006), where it was shown that if $\mu_1 \geq \mu_0$, there exists a sequence of arbitrary small delays that destabilize the system.

We will further derive delay-dependent stability conditions for (42), (29) with $\mu_1 = 0$. We apply the conditions of Theorem 1. We choose V as follows:

$$V = a p_{03} \int_0^{\pi} z_\xi^2(\xi, t) \mathrm{d}\xi + \int_0^{\pi} [z(\xi, t)\; z_t(\xi, t)] P_0 \times \begin{bmatrix} z(\xi, t) \\ z_t(\xi, t) \end{bmatrix} \mathrm{d}\xi + \int_0^{\pi} \left[hr \int_{-h}^{0} \int_{t+\theta}^{t} z_t^2(\xi, s) \mathrm{e}^{2\delta(s-t)} \mathrm{d}s \mathrm{d}\theta + s \int_{t-h}^{t} z^2(\xi, s) \mathrm{e}^{2\delta(s-t)} \mathrm{d}s + q \int_{t-\tau}^{t} z^2(\xi, s) \mathrm{e}^{2\delta(s-t)} \mathrm{d}s \right] \mathrm{d}\xi,$$

where P_0 and p_{03} satisfy (44) and where $r > 0$, $s > 0$, $q \geq 0$. Then the operators P, Q, R in (21) are given by

$$P = \mathrm{diag}\left\{ -a p_{03} \frac{\partial^2}{\partial \xi^2}, 0 \right\} + P_0 > 0, \qquad Q = \mathrm{diag}\{q, 0\} \geq 0,$$
$$R = \mathrm{diag}\{r, 0\} \geq 0, \qquad S = \mathrm{diag}\{s, 0\} \geq 0.$$

We have $h^2 A^* R A = \mathrm{diag}\{0, h^2 r\}$, $h^2 A^* R A_1 = 0$, $h^2 A_1^* R A_1 = 0$. From (47) it follows that (21) is feasible if $p_{02} - \delta p_{03} \geq 0$ and the following LMI are satisfied:

$$\begin{bmatrix} \phi_w & 0 & P_w \begin{bmatrix} 0 \\ -a_1 \end{bmatrix} + \begin{bmatrix} r\mathrm{e}^{-2\delta h} \\ 0 \end{bmatrix} \\ * & -(s + r)\mathrm{e}^{-2\delta h} & r\mathrm{e}^{-2\delta h} \\ * & * & -(2r + (1-d)q)\mathrm{e}^{-2\delta h} \end{bmatrix} < 0, \tag{52}$$

where $\phi_w = C_\delta^{\mathrm{T}} P_w + P_w C_\delta + \mathrm{diag}\{q + s - r\mathrm{e}^{-2\delta h}, h^2 r\}$. Summarizing the following result is obtained

Theorem 6. *Given $\delta > 0$, let there exist a 2×2-matrix $P_w > 0$ and scalars $q \geq 0$, $r > 0$, $s > 0$ such that satisfy LMIs $p_{02} - \delta p_{03} \geq 0$ and (52), where p_{02} and p_{03} are respectively $(1, 2)$ and $(2, 2)$ terms of P_w. Then the wave time-delay equation (42) with $\mu_1 = 0$ and with the Dirichlet boundary condition (29), where $l = \pi$, is exponentially stable with the decay rate δ for all differentiable delays (5) with $\dot{\tau} \leq d < 1$, and the inequality (50) is satisfied with*

$$K_w = \lambda_{\max}(P_w - \mathrm{diag}\{a p_{03}, 0\}) + \max\{hq + hs, h^3 r/2\}$$

for all $t \geq t_0$. Moreover, if LMIs (36) and (52) are feasible with $q = 0$, then (29), (42) with $\mu_1 = 0$ is exponentially stable with the decay rate δ for all fast varying delays $0 \leq \tau \leq h$. If the LMI (52) holds for $\delta = 0$, then (29), (42) with $\mu_1 = 0$ is exponentially stable with a sufficiently small decay rate.

Remark 7. The same LMIs (49) and (52) appear to guarantee the exponential stability of ODE with delay $\dot{\bar{z}}(t) = C_0 \bar{z}(t) + A_1 \bar{z}(t - \tau(t))$, $\bar{z}(t) \in \mathbf{R}^2$ or, equivalently, of the first modal dynamics (with $k = 1$) of the modal representation of the Dirichlet boundary-value problem (29), (42) with $l = \pi$

$$\ddot{y}_k(t) + \mu_0 \dot{y}_k(t) + \mu_1 \dot{y}_k(t - \tau(t)) + (ak^2 + a_0) y_k(t) + a_1 y_k(t - \tau(t)) = 0, \quad k = 1, 2, \ldots$$

on the eigenfunctions of the operator $\frac{\partial^2}{\partial \xi^2}$. Hence, the results of Theorem 5 and of Theorem 6 are tight.

Remark 8. One can derive "mixed" stability conditions for the wave equation (42) with $\mu_1 \neq 0$: delay-dependent (with respect to delay in z)/ delay-independent (with respect to delay in z_t). This is similar to neutral systems, where the delay in the state derivative is treated in the delay-independent manner (Niculescu, 2001).

Example 2. Consider the controlled wave equation

$$z_{tt}(\xi, t) = 0.1 z_{\xi\xi}(\xi, t) - 2 z_t(\xi, t) + u, \tag{53}$$

with boundary condition (29) and $l = \pi$, $t \geq t_0$, $0 \leq \xi \leq \pi$, $0 \leq \tau \leq h$, $\dot{\tau} \leq d < 1$. Applying Theorem 6 to the open-loop system we find that (53) with $u = 0$ is exponentially stable with the decay rate $\delta = 0.05$. Considering next a delayed feedback $u = -z(\xi, t - \tau(t))$ and verifying conditions of Theorem 6, we find that the closed-loop system is exponentially stable with a greater decay rate $\delta = 0.8$ for all $0 \leq \tau(t) \leq 0.31$.

6. Conclusions

A general framework is given for the stability analysis of linear time-delay systems in a Hilbert space with a bounded operator acting on the delayed state. The exponential stability conditions are derived in terms of linear operator inequalities in the Hilbert space. In the case of a heat/wave scalar equation with the Dirichlet boundary conditions, these LOIs are reduced to finite-dimensional LMIs by applying new Lyapunov–Krasovskii functionals. The reduced-order LMIs coincide with the stability conditions for appropriate ODEs with delay, whereas the stability of the latter ODEs are necessary for the stability of the original boundary value problems. LOIs are expected to provide effective tools for robust control of distributed parameter systems.

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