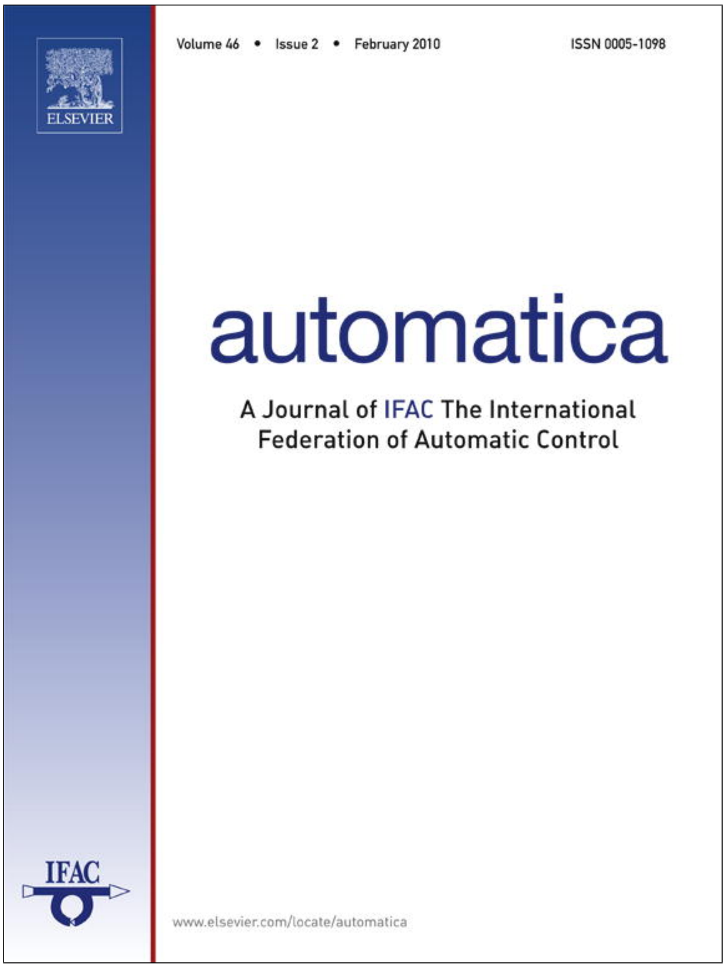
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Brief paper

A refined input delay approach to sampled-data control☆

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ABSTRACT

This paper considers sampled-data control of linear systems under uncertain sampling with the known upper bound on the sampling intervals. Recently a discontinuous Lyapunov function method was introduced by using impulsive system representation of the sampled-data systems (Naghshtabrizi, Hespanha, & Teel, 2008). The latter method improved the existing results, based on the input delay approach via time-independent Lyapunov functionals. The present paper introduces novel time-dependent Lyapunov functionals in the framework of the input delay approach, which essentially improve the existing results. These Lyapunov functionals do not grow after the sampling times. For the first time, for systems with time-varying delays, the introduced Lyapunov functionals can guarantee the stability under the sampling which may be greater than the analytical upper bound on the constant delay that preserves the stability. We show also that the term of the Lyapunov function, which was introduced in the above mentioned reference for the analysis of systems with constant sampling, is applicable to systems with variable sampling.

1. Introduction

Two main approaches have been used for the sampled-data control of linear uncertain systems leading to conditions in terms of Linear Matrix Inequalities (LMIs) (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994). The first one is the input delay approach, where the system is modeled as a continuous-time system with the delayed control input (Fridman, 1992; Mikheev, Sobolev, & Fridman, 1988). The input delay approach became popular in the networked control systems literature, being applied via *time-independent* Lyapunov–Krasovskii functionals or Lyapunov–Razumikhin functions to analysis and design of linear uncertain systems under uncertain sampling with the known upper bound on the sampling intervals (Fridman, Seuret, & Richard, 2004; Gao & Chen, 2008; Jiang, Han, Liu, & Xue, 2008; Yu, Wang, Chu, & Hao, 2005). Recently the input delay approach was revised by using the scaled small-gain theorem and a tighter upper bound on the L_2-induced norm of the uncertain term (Mirkin, 2007).

The second approach is based on the representation of the sampled-data system in the form of impulsive model (Basar & Bernard, 1995; Sivashankar & Khargonekar, 1994). The impulsive model approach was applied to sampled-data stabilization of linear uncertain systems in the case of constant sampling (Hu, Lam, Cao, & Shao, 2003), where a piecewise linear in time Lyapunov function was suggested. Recently the impulsive model approach was extended to the case of variable sampling with a known upper bound (Naghshtabrizi et al., 2008), where a discontinuous Lyapunov function method was introduced. The latter method improved the existing results, based on the input delay approach via time-independent Lyapunov functionals. Moreover, the conditions of Naghshtabrizi et al. (2008) distinguish between the cases of constant vs. variable sampling intervals and provide less conservative results for the constant sampling.

We note that the existing methods in the framework of input delay approach are based on some Lyapunov-based analysis of systems with uncertain and bounded fast-varying delays (these are time-varying delays without any constraints on the delay derivative). Therefore, these methods cannot guarantee the stability if the delay is not smaller than the analytical upper bound on the constant delay that preserves the stability. However, it is well known (see the examples in Louisell (1999) and the discussions on *quenching* in Papachristodoulou, Peet, and Niculescu (2007), as well as Examples 1 and 2) that in many systems the upper bound on the sampling that preserves the stability may be higher than the one for the constant delay.

The objective of the present paper is to develop a novel *time-dependent Lyapunov functional-based technique for sampled-data control* in the framework of the input delay approach. We *introduce novel time-dependent Lyapunov functionals, which essentially improve the existing results* in the examples. Our results are inspired by the construction of discontinuous Lyapunov functions in Naghshtabrizi et al. (2008). Our Lyapunov functional is time-dependent and it does not grow after the sampling times. Differently from Naghshtabrizi et al. (2008), our main result for variable

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sampling is derived via continuous Lyapunov functional. The introduced time-dependent Lyapunov functionals lead to *qualitatively new results for time delay systems, allowing a superior performance under the sampling, than the one under the constant delay*. The presented approach gives efficient tools for different design problems, that can be solved in the framework of input delay approach (see e.g. Gao & Chen, 2008; Suplin, Fridman, & Shaked, 2007).

Notation. Throughout the paper the superscript 'T' stands for matrix transposition, R^n denotes the n-dimensional Euclidean space with vector norm $\|\cdot\|$, $R^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in R^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. The space of functions $\phi : [-h, 0] \to R^n$, which are absolutely continuous on $[-h, 0)$, have a finite $\lim_{\theta\to 0^-} \phi(\theta)$ and have square integrable first-order derivatives is denoted by W with the norm

$$\|\phi\|_W = \max_{\theta\in[-h,0]} |\phi(\theta)| + \left[\int_{-h}^{0} |\dot\phi(s)|^2 ds\right]^{\frac{1}{2}}.$$

We also denote $x_t(\theta) = x(t+\theta)(\theta \in [-h, 0])$.

2. Problem formulation

Consider the system

$$\begin{aligned}\dot x(t) &= Ax(t) + B_1 w(t) + B_2 u(t),\\ z(t) &= C_0 x(t) + Du(t),\end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $w(t) \in R^{n_w}$ is the disturbance, $u(t) \in R^{n_u}$ is the control input and $z(t) \in R^{n_z}$ is the controlled output, A, B_1, B_2, C_0 and D are constant matrices of appropriate dimensions.

The control signal is assumed to be generated by a zero-order hold function with a sequence of hold times $0 = t_0 < t_1 < \cdots < t_k < \cdots$

$$u(t) = u_d(t_k), \quad t_k \le t < t_{k+1}, \tag{2}$$

where $\lim_{k\to\infty} t_k = \infty$ and u_d is a discrete-time control signal.

Assume that

A1 $\quad t_{k+1} - t_k \le h \quad \forall k \ge 0.$

We define the following performance index for a prescribed scalar $\gamma > 0$:

$$J = \int_0^\infty [z^T(s)z(s) - \gamma^2 w^T(s)w(s)]ds. \tag{3}$$

We consider a state-feedback control law of the form

$$u(t) = Kx(t_k), \quad t_k \le t < t_{k+1}, \tag{4}$$

which for all samplings satisfying A1 internally stabilizes the system and leads to $J < 0$ for $x(0) = 0$ and for all nonzero $w \in L_2$.

Following Mikheev et al. (1988), we represent the digital control law as a delayed control as follows:

$$u(t) = u_d(t_k) = u_d(t - \tau(t)), \quad \tau(t) = t - t_k,\ t_k \le t < t_{k+1}. \tag{5}$$

Our objective is to analyze the exponential stability and L_2-gain of the closed-loop system (1) and (4):

$$\begin{aligned}\dot x(t) &= Ax(t) + A_1 x(t-\tau(t)) + B_1 w(t),\\ z(t) &= C_0 x(t) + C_1 x(t-\tau(t)),\\ t &\in [t_k, t_{k+1}), \quad \tau(t) = t - t_k,\end{aligned} \tag{6}$$

where

$$A_1 = B_2 K, \qquad C_1 = DK. \tag{7}$$

Under A1, $\tau(t) \in (0, h]$ and $\dot\tau(t) = 1$ for $t \ne t_k$.

In Fridman et al. (2004) h-dependent Lyapunov functional has been considered that corresponds to stability analysis of (6) in the case of fast-varying delay (i.e. in the case of no constraints on the delay derivative). In Naghshtabrizi et al. (2008) a Lyapunov function which depends on t_k has been introduced for the corresponding finite-dimensional system with jumps. In this paper we suggest a time-dependent Lyapunov functional for analysis of the time delay system (6).

3. Exponential stability and L_2-gain analysis

3.1. Time-dependent Lyapunov functional method

Lemma 1. *Let there exist positive numbers β, δ and a functional $V : \mathbf{R} \times W \times L_2[-h, 0] \to \mathbf{R}$ such that*

$$\beta|\phi(0)|^2 \le V(t, \phi, \dot\phi) \le \delta\|\phi\|_W^2. \tag{8}$$

Let the function $\bar V(t) = V(t, x_t, \dot x_t)$ is continuous from the right for x_t satisfying (6), absolutely continuous for $t \ne t_k$ and satisfies

$$\lim_{t\to t_k^-} \bar V(t) \ge \bar V(t_k). \tag{9}$$

(i) *Given $\alpha > 0$, if along (6) with $w = 0$*

$$\dot{\bar V}(t) + 2\alpha \bar V(t) \le 0, \quad \text{almost for all } t, \tag{10}$$

then (6) with $w = 0$ is exponentially stable with the decay rate α.

(ii) *For a prescribed $\gamma > 0$, if along (6)*

$$\dot{\bar V}(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0, \quad \text{almost for all } t,\ \forall w \ne 0, \tag{11}$$

then the cost function (3) achieves $J < 0$ for all nonzero $w \in L_2$ and for the zero initial condition.

Proof. (i) From (8) and (10) we have for $t \in [t_k, t_{k+1})$

$$\beta|x(t)|^2 \le \bar V(t) \le e^{-2\alpha(t-t_k)} \bar V(t_k).$$

Taking into account (9), we obtain further

$$\begin{aligned}&e^{-2\alpha(t-t_k)}\bar V(t_k) \le e^{-2\alpha(t-t_k)}\bar V(t_k^-)\\ &\quad \le e^{-2\alpha(t-t_{k-1})}\bar V(t_{k-1}) \le \cdots \le e^{-2\alpha t}\bar V(0) \le \delta e^{-2\alpha t}\|x_0\|_W^2,\end{aligned}$$

which results in $\beta|x(t)|^2 \le \delta e^{-2\alpha t}\|x_0\|_W^2$ and completes the proof of (i).

(ii) Given $N \gg 1$, we integrate (11) from 0 till t_N. Taking into account (9), we have

$$\begin{aligned}&\bar V(t_N) - \bar V(t_{N-1}) + \bar V(t_{N-1}^-) - \bar V(t_{N-2}) + \cdots + \bar V(t_1^-) - \bar V(0)\\ &\quad + \int_0^{t_N} [z^T(t)z(t) - \gamma^2 w^T(t)w(t)]dt < 0.\end{aligned}$$

Since $\bar V(t_N) \ge 0$, $\bar V(t_{k-1}^-) - \bar V(t_{k-1}) \ge 0$ for $k = 2, \ldots, N$ and $\bar V(0) = 0$, we find

$$\int_0^{t_N} [z^T(t)z(t) - \gamma^2 w^T(t)w(t)]dt < 0.$$

Thus, for $N \to \infty$ we arrive to $J < 0$. $\square$

3.2. Simple stability conditions: Variable sampling

We introduce the following simple functional for exponential stability with a given decay rate $\alpha > 0$:

$$V_s(t, x(t), \dot{x}_t) = \bar{V}(t) = x^T(t)Px(t) + V_U(t, \dot{x}_t), \tag{12}$$

where

$$V_U(t, \dot{x}_t) = (h - \tau(t)) \int_{t-\tau(t)}^{t} e^{2\alpha(s-t)} \dot{x}^T(s)U\dot{x}(s)ds, \quad \tau(t) = t - t_k, \tag{13}$$

and where $P > 0$, $U > 0$. In the existing papers in the framework of input delay approach (see e.g. Fridman et al., 2004; Gao & Chen, 2008; Jiang et al., 2008; Yue, Han, & Lam, 2005), time-independent Lyapunov functionals are usually involved. The discontinuous term V_U is different from the terms of Naghshtabrizi et al. (2008). Along the jumps V_U does not increase since $V_U \geq 0$ and V_U vanishes after the jumps because $x(t)_{|t=t_k} = x(t - \tau(t))_{|t=t_k}$. Thus, the condition $\lim_{t \to t_k^-} \bar{V}(t) \geq \bar{V}(t_k)$ holds.

Since $\frac{d}{dt}x(t - \tau(t)) = (1 - \dot{\tau}(t))\dot{x}(t - \tau(t)) = 0$, we find

$$\frac{d}{dt}V_U(t, \dot{x}_t) + 2\alpha V_U(t, \dot{x}_t) = -\int_{t-\tau(t)}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)U\dot{x}(s)ds + (h - \tau(t))\dot{x}^T(t)U\dot{x}(t) \tag{14}$$

and thus

$$\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \leq 2\dot{x}^T(t)Px(t) + 2\alpha x^T(t)Px(t) - e^{-2\alpha h}\int_{t-\tau(t)}^{t} \dot{x}^T(s)U\dot{x}(s)ds + (h - \tau(t))\dot{x}^T(t)U\dot{x}(t). \tag{15}$$

Denoting

$$v_1 = \frac{1}{\tau(t)}\int_{t-\tau(t)}^{t} \dot{x}(s)ds, \tag{16}$$

we understand by $v_{1|\tau(t)=0}$ the following: $\lim_{\tau(t) \to 0} v_1 = \dot{x}(t)$.

We apply further the Jensen's inequality (Gu, Kharitonov, & Chen, 2003)

$$\int_{t-\tau(t)}^{t} \dot{x}^T(s)U\dot{x}(s)ds \geq \tau(t)v_1^T U v_1, \tag{17}$$

and descriptor method (Fridman, 2001), where the right-hand side of the expression

$$0 = 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T][(A + A_1)x(t) - \tau(t)A_1 v_1 - \dot{x}(t)], \tag{18}$$

with some $n \times n$-matrices P_2, P_3 is added into the right-hand side of (15).

Setting $\eta_1(t) = col\{x(t), \dot{x}(t), v_1\}$, we obtain that

$$\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \leq \eta_1^T(t)\Psi_s\eta_1(t) < 0, \tag{19}$$

if the following matrix inequality is feasible:

$$\Psi_s = \begin{bmatrix} \Phi_s & P - P_2^T + (A + A_1)^T P_3 & -\tau(t)P_2^T A_1 \\ * & -P_3 - P_3^T + (h - \tau(t))U & -\tau(t)P_3^T A_1 \\ * & * & -\tau(t)Ue^{-2\alpha h} \end{bmatrix} < 0, \tag{20}$$

where

$$\Phi_s = P_2^T(A + A_1) + (A + A_1)^T P_2 + 2\alpha P. \tag{21}$$

The latter matrix inequality for $\tau(t) \to 0$ and $\tau(t) \to h$ leads to the following LMIs

$$\Psi_{s0} = \begin{bmatrix} \Phi_s & P - P_2^T + (A + A_1)^T P_3 \\ * & -P_3 - P_3^T + hU \end{bmatrix} < 0, \tag{22}$$

$$\Psi_{s1} = \begin{bmatrix} \Phi_s & P - P_2^T + (A + A_1)^T P_3 & -hP_2^T A_1 \\ * & -P_3 - P_3^T & -hP_3^T A_1 \\ * & * & -hUe^{-2\alpha h} \end{bmatrix} < 0. \tag{23}$$

Denote by $\eta_0 = col\{x(t), \dot{x}(t)\}$. Then (22) and (23) imply (20) because

$$\frac{h - \tau(t)}{h}\eta_0^T\Psi_{s0}\eta_0 + \frac{\tau(t)}{h}\eta_1^T\Psi_{s1}\eta_1 = \eta_1^T\Psi_s\eta_1 < 0 \quad \forall \eta_1 \neq 0.$$

We proved the following.

Proposition 1. *Given $\alpha > 0$, let there exist $n \times n$-matrices $P > 0$, $U > 0$, P_2 and P_3 such that the LMIs (22), (23) with notation given in (21) are feasible. Then system (6) with $w = 0$ is exponentially stable with the decay rate α for all variable samplings satisfying* A1. *If LMIs (22), (23) are feasible for $\alpha = 0$, then (6) is exponentially stable with a small enough decay rate.*

Example 1. We consider the following simple and much-studied problem (see e.g. Papachristodoulou et al., 2007):

$$\dot{x}(t) = -x(t_k), \quad t_k \leq t < t_{k+1}, \ k = 0, 1, \ldots. \tag{24}$$

It is well known that the equation $\dot{x}(t) = -x(t - \tau(t))$ with constant delay $\tau(t)$ is asymptotically stable for $\tau(t) < \pi/2$ and unstable for $\tau(t) > \pi/2$, whereas for the fast-varying delay it is stable for $\tau(t) < 1.5$ and there exists a destabilizing delay with an upper bound greater than 1.5. The latter means that all the existing methods, that are based on time-independent Lyapunov functionals, corresponding to stability analysis of systems with fast-varying delays, cannot guarantee the stability for the samplings which may be greater than 1.5.

It is easy to check, that the system remains stable for all constant samplings less than 2 and becomes unstable for samplings greater than 2. Conditions of Naghshtabrizi et al. (2008) and of Mirkin (2007) guarantee asymptotic stability for all variable samplings up to 1.28 and 1.57 respectively. By applying Proposition 1 with $\alpha = 0$, we find that for all variable samplings up to 1.99 the system remains exponentially stable.

The conditions of Proposition 1 cannot be applied to (6) with A_1 from uncertain polytope, since in matrix inequality (20) A_1 is multiplied by $\tau(t)$. Moreover, additional terms in the Lyapunov functional may further improve the results.

3.3. Preliminary results: Constant sampling

We consider the constant sampling, where $t_{k+1} - t_k = h$, $k = 0, 1, \ldots$. We apply the following Lyapunov functional:

$$V(t, x_t, \dot{x}_t) = V_s(t, x(t), \dot{x}_t) + V_X(t, x_t), \tag{25}$$

where V_s is given by (12) and (13) with positive matrices P and U. The additional term V_X is defined as follows:

$$V_X = (h - \tau(t))\xi^T(t) \times \begin{bmatrix} \frac{X + X^T}{2} & -X + X_1 \\ * & -X_1 - X_1^T + \frac{X + X^T}{2} \end{bmatrix} \xi(t), \tag{26}$$

where $\xi(t) = col\{x(t), x(t - \tau(t))\}$, X and X_1 are $n \times n$-matrices. The time-dependent term V_X is similar to the one in Naghshtabrizi et al. (2008). We note that V_X vanishes before the jump (because

$\tau(t) = h$) and after the jump (because $\tau(t) = 0$ and thus $x(t-\tau(t)) = x(t)$). Thus, functional V is continuous in time.

To guarantee (8) we assume that

$$\Xi(h) = \begin{bmatrix} P + h\dfrac{X+X^T}{2} & hX_1 - hX \\ * & -hX_1 - hX_1^T + h\dfrac{X+X^T}{2} \end{bmatrix} > 0. \quad (27)$$

Then for some $\beta > 0$ we have $\Xi(h) > \beta I_{2n}$, $P > \beta I_n$ and $\frac{h-\tau(t)}{h}\Xi(h) + \frac{\tau(t)}{h}\Xi(0) \geq col\{\beta I_n, 0\}$, which yields (8).

We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V_X &= -\xi^T(t)\begin{bmatrix} \dfrac{X+X^T}{2} & -X+X_1 \\ * & -X_1 - X_1^T + \dfrac{X+X^T}{2}\end{bmatrix}\xi(t) \\ &\quad + (h-\tau(t))[\dot{x}^T(t)(X+X^T)x(t) \\ &\quad + 2\dot{x}^T(t)(-X+X_1)x(t-\tau(t))]. \end{aligned} \quad (28)$$

Differentiating $\bar{V}(t) = V(t, x_t, \dot{x}_t)$ along (6) with $w = 0$, we find

$$\begin{aligned} &\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \leq 2x^T(t)P\dot{x}(t) + (h-\tau(t))\dot{x}^T(t)U\dot{x}(t) \\ &\quad - \mathrm{e}^{-2\alpha h}\int_{t-\tau(t)}^{t}\dot{x}^T(s)U\dot{x}(s)\mathrm{d}s \\ &\quad - \xi^T(t)\begin{bmatrix} \dfrac{X+X^T}{2} & -X+X_1 \\ * & -X_1 - X_1^T + \dfrac{X+X^T}{2}\end{bmatrix}\xi(t) \\ &\quad + (h-\tau(t))[\dot{x}^T(t)(X+X^T)x(t) \\ &\quad + 2\dot{x}^T(t)(-X+X_1)x(t-\tau(t))] + 2\alpha x^T(t)Px(t) \\ &\quad + 2\alpha(h-\tau(t))\xi^T(t) \\ &\quad \times \begin{bmatrix} \dfrac{X+X^T}{2} & -X+X_1 \\ * & -X_1 - X_1^T + \dfrac{X+X^T}{2}\end{bmatrix}\xi(t). \end{aligned} \quad (29)$$

We further apply the Jensen's inequality (17) and insert free-weighting $n \times n$-matrices Y_1, Y_2, T, P_2, P_3 by adding the following expressions to $\dot{\bar{V}}$:

$$\begin{aligned} 0 &= 2[x^T(t)Y_1^T + \dot{x}^T(t)Y_2^T + x^T(t-\tau(t))T^T] \\ &\quad \times [-x(t) + x(t-\tau(t)) + \tau(t)v_1], \\ 0 &= 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T][Ax(t) + A_1x(t-\tau(t)) - \dot{x}(t)]. \end{aligned}$$

Setting $\eta(t) = col\{x(t), \dot{x}(t), x(t-\tau(t)), v_1\}$, we obtain that $\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) \leq \eta^T(t)\Psi\eta(t) < 0$, if

$$\begin{bmatrix} \Phi_{11} - X_\alpha & \Phi_{12} + X_{\tau(t)} & \Phi_{13} + X_{1\alpha} & \tau(t)Y_1^T \\ * & \Phi_{22} + [h-\tau(t)]U & \Phi_{23} - X_{1\tau(t)} & \tau(t)Y_2^T \\ * & * & \Phi_{33} - X_{2\alpha} & \tau(t)T^T \\ * & * & * & -\tau(t)U\mathrm{e}^{-2\alpha h} \end{bmatrix} = \Psi < 0, \quad (30)$$

where

$$\begin{aligned} &\Phi_{11} = A^TP_2 + P_2^TA + 2\alpha P - Y_1 - Y_1^T, \\ &\Phi_{12} = P - P_2^T + A^TP_3 - Y_2, \\ &\Phi_{13} = Y_1^T + P_2^TA_1 - T, \\ &\Phi_{22} = -P_3 - P_3^T, \qquad \Phi_{23} = Y_2^T + P_3^TA_1, \\ &\Phi_{33} = T + T^T, \qquad X_{\tau(t)} = (h-\tau(t))\frac{X+X^T}{2}, \\ &X_\alpha = (1 - 2\alpha(h-\tau(t)))\frac{X+X^T}{2}, \\ &X_{1\tau(t)} = (h-\tau(t))(X - X_1), \\ &X_{1\alpha} = (1 - 2\alpha(h-\tau(t)))(X - X_1), \\ &X_{2\alpha} = (1 - 2\alpha(h-\tau(t)))\frac{X + X^T - 2X_1 - 2X_1^T}{2}. \end{aligned} \quad (31)$$

The latter matrix inequality for $\tau(t) \to 0$ and $\tau(t) \to h$ leads to the following LMIs

$$\begin{bmatrix} \Phi_{11} - X_{\alpha|\tau(t)=0} & \Phi_{12} + h\dfrac{X+X^T}{2} & \Phi_{13} + X_{1\alpha|\tau(t)=0} \\ * & \Phi_{22} + hU & \Phi_{23} - h(X - X_1) \\ * & * & \Phi_{33} - X_{2\alpha|\tau(t)=0} \end{bmatrix} = \Psi_0(h) < 0, \quad (32)$$

$$\begin{bmatrix} \Phi_{11} - \dfrac{X+X^T}{2} & \Phi_{12} & \Phi_{13} + X - X_1 & hY_1^T \\ * & \Phi_{22} & \Phi_{23} & hY_2^T \\ * & * & \Phi_{33} - X_{2\alpha|\tau(t)=h} & hT^T \\ * & * & * & -hU\mathrm{e}^{-2\alpha h} \end{bmatrix} = \Psi_1(h) < 0. \quad (33)$$

Lemma 2. *LMIs (27), (32) and (33) are convex in h: if they are feasible for h, then they are feasible for all $\bar{h} \in (0, h]$.*

Proof. Assume that $\Xi(h) > 0$. Then, the affinity of $\Xi(h)$ in h and the fact that $\Xi(0) \geq 0$ imply $\Xi(\bar{h}) = \frac{\bar{h}}{h}\Xi(h) + \frac{h-\bar{h}}{h}\Xi(0) > 0$ for all $\bar{h} \in (0, h]$.

Assume that $\Psi_i(h) < 0$, $i = 0, 1$. By applying Schur complements to $\Psi_1(h)$, we find that (33) is equivalent to

$$\Psi_0(0) + he^{2\alpha h}\begin{bmatrix} Y_1 & Y_2 & T \end{bmatrix}^T U^{-1}\begin{bmatrix} Y_1 & Y_2 & T \end{bmatrix} < 0.$$

Hence, $\Psi_1(h) < 0$ yields $\Psi_1(\bar{h}) < 0$ for $\bar{h} \in (0, h]$ and $\Psi_0(0) < 0$. Since $\Psi_0(h)$ is affine in h, $\Psi_0(0) < 0$ together with $\Psi_0(h) < 0$ imply $\Psi_0(\bar{h}) < 0$ for all $\bar{h} \in (0, h]$. □

For the L_2-gain analysis we consider the Lyapunov functional (25) with $\alpha = 0$. Similar to the exponential stability conditions, we find that (11) of Lemma 1 holds if

$$\left[\begin{array}{c|cc} & P_2^TB_1 & C_0^T \\ & P_3^TB_1 & 0 \\ \Psi_i|_{\alpha=0} & 0 & 0 \\ & 0 & C_1^T \\ \hline * & -\gamma^2 I & 0 \\ * & * & -I \end{array}\right] < 0, \quad i = 0, 1. \quad (34)$$

where $\Phi_{1i}, \Phi_{22}, \Phi_{i3}, \Phi_{33}$, $i = 1, 2$ are given by (31) with $\alpha = 0$. We note that if LMIs (34) hold, then (32), (33) hold for small enough α.

We summarize the results in the following

Lemma 3. *Consider (6) with a constant sampling, satisfying $t_{k+1} - t_k = \bar{h} \leq h$. Then the following holds:*

(i) *Given $\alpha > 0$, let there exist $n \times n$-matrices $P > 0$, $U > 0$, X, X_1, P_2, P_3, T, Y_1 and Y_2 such that the LMIs (27), (32) and (33)* *with notations given in (31) are feasible. Then system (6) with $w = 0$ is exponentially stable with the decay rate α.*

(ii) *Given $\gamma > 0$, if the LMIs (27) and (34) are feasible, then (6) is internally exponentially stable and the cost function (3) achieves $J < 0$ for all nonzero $w \in L_2$ and for the zero initial condition.*

Remark 1. Consider now the variable sampling satisfying A1. Choosing $X = X^T > 0$ and $X_1 = 0$ in (26), we obtain the term

$$\tilde{V}_X(t, x_t) = [h - \tau(t)][x(t) - x(t_k)]^T X[x(t) - x(t_k)], \tag{35}$$

which does not grow after the sampling times. Hence, LMIs (32)–(34), where $X = X^T > 0$ and $X_1 = 0$, give sufficient conditions for the exponential stability and for a prescribed L_2-gain of (6) under the variable sampling not greater than h.

3.4. Main result: Variable sampling

In this section we will prove our main result: LMIs of Lemma 3 guarantee the exponential stability and a prescribed L_2-*gain* for systems with variable samplings satisfying

$$0 < t_{k+1} - t_k \overset{\Delta}{=} h_k \le h. \tag{36}$$

For this purpose we consider the following time-dependent Lyapunov–Krasovskii functional:

$$\begin{aligned} V_{var}(t, x_t, \dot{x}_t) &= \bar{V}_{var}(t) = x^T(t)Px(t) \\ &+ (t_{k+1} - t)\int_{t_k}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)U\dot{x}(s)ds \\ &+ (t_{k+1} - t)\xi^T(t)\begin{bmatrix} \frac{X + X^T}{2} & -X + X_1 \\ * & -X_1 - X_1^T + \frac{X + X^T}{2} \end{bmatrix}\xi(t), \\ & t \in [t_k, t_{k+1}), \end{aligned} \tag{37}$$

where $\xi(t) = col\{x(t), x(t_k)\}$. We note that U and X, X_1-dependent terms vanish before t_k and after t_k. Therefore, $\bar{V}_{var}(t)$ is continuous since $\lim_{t\to t_k} \bar{V}_{var}(t) = \bar{V}_{var}(t_k)$. We note that for the case of constant delay $t_{k+1} - t_k = h$ we have $V(t, x_t, \dot{x}_t) = V_{var}(t, x_t, \dot{x}_t)$.

By applying arguments of the previous subsection to the case of variable samplings satisfying (36), we find that $\dot{\bar{V}}_{var} + 2\alpha\bar{V}_{var} \le 0$ for $t \in [t_k, t_{k+1})$ if $\Psi_i(h_k) < 0$, $i = 0, 1$, where the matrices Ψ_i are defined in (32), (33). Assume now that LMIs (32), (33) are feasible. Then the convexity of $\Psi_i(h) < 0$ in h (see Lemma 2) and the fact that $h_k \le h$ yield $\Psi_i(h_k) < 0$. Moreover, the feasibility of (27) and the convexity of $\Xi(h)$ in h implies that $\Xi(h_k) > 0$ and thus $V_{var}(t, x_t, \dot{x}_t) \ge \beta|x(t)|^2$ for some $\beta > 0$. We arrive to our main result:

Theorem 1. *Consider (6) with variable sampling, satisfying $t_{k+1} - t_k \le h$. Then the following holds:*

(i) *Given $\alpha > 0$, let there exist $n \times n$-matrices $P > 0$, $U > 0$, X, X_1, P_2, P_3, T, Y_1 and Y_2 such that the LMIs (27), (32), (33) with notations given in (31) are feasible. Then system (6) with $w = 0$ is exponentially stable with the decay rate α.*

(ii) *Given $\gamma > 0$, let there exist $n \times n$-matrices $P > 0$, $U > 0$, X, X_1, P_2, P_3, T, Y_1 and Y_2 such that the LMIs (27) and (34) are feasible. Then (6) is internally exponentially stable and the cost function (3) achieves $J < 0$ for all nonzero $w \in L_2$ and for the zero initial condition.*

Remark 2. We note that the LMIs of Theorem 1 are affine in the system matrices. Therefore, in the case of polytopic type uncertainty with A, B_1, B_2, C_0 and D from the uncertain time-varying polytope

$$\begin{aligned} &\Omega = \sum_{j=1}^{M} f_j\Omega_j \quad 0 \le f_j(t) \le 1, \qquad \sum_{j=1}^{M} f_j = 1, \\ &\Omega_j = \begin{bmatrix} A^{(j)} & B_1^{(j)} & B_2^{(j)} & C_0^{(j)} & D^{(j)} \end{bmatrix}, \end{aligned} \tag{38}$$

one have to solve these LMIs simultaneously for all the M vertices Ω_j, applying the same decision matrices.

Remark 3. The new term $V_U(t, \dot{x}_t)$ (with $\alpha = 0$) replaces the following standard time-independent integral term (see e.g. Fridman et al., 2004; Fridman & Shaked, 2003; He, Wang, Xie, & Lin, 2007; Park & Ko, 2007)

$$V_R(\dot{x}_t) = \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta, \quad R > 0, \tag{39}$$

which was introduced for systems with fast-varying delays in Fridman and Shaked (2003). The term (39) was modified in Naghshtabrizi et al. (2008) as follows:

$$\tilde{V}_R(t, \dot{x}_t) = \int_{t_k}^{t} (h - t + s)\dot{x}^T(s)R\dot{x}(s)ds, \quad R > 0. \tag{40}$$

Differentiating $V_R(\dot{x}_t)$ and $\tilde{V}_R(t, \dot{x}_t)$, we obtain

$$\begin{aligned} \frac{d}{dt}V_R(\dot{x}_t) &= -\int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)ds + h\dot{x}^T(t)R\dot{x}(t) \\ &= -\int_{t-\tau(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds + h\dot{x}^T(t)R\dot{x}(t) \\ &\quad -\int_{t-h}^{t-\tau(t)} \dot{x}^T(s)R\dot{x}(s)ds \end{aligned} \tag{41}$$

and

$$\frac{d}{dt}\tilde{V}_R(t, \dot{x}_t) = -\int_{t-\tau(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds + h\dot{x}^T(t)R\dot{x}(t). \tag{42}$$

Comparing now the right-hand sides of (14) and of (41), (42), we see that all of them possess the same (for $U = R$) negative integral term from $t - \tau(t)$ to t, which allows to derive LMIs, and different positive terms. In (41) we have one more (not necessary) integral term. The main advantage of the new method that the positive term is multiplied by $[h - \tau(t)]$ and, thus, appears only in one vertex (where $\tau(t) \to 0$) of the resulting LMIs. This is different from Park and Ko (2007) and Naghshtabrizi et al. (2008), where the same term is multiplied by h and appears in the both vertices of the resulting LMIs.

Another advantage of our method over time-independent functionals for fast-varying delays is in the use of X, X_1-dependent term, which is not possible in the case of fast-varying delays since $\frac{d}{dt}x(t - \tau(t)) = [1 - \dot{\tau}(t)]\dot{x}(t - \tau(t))$. Lyapunov functional with such a term is a kind of augmented functional (He et al., 2007). Comparatively to Naghshtabrizi et al. (2008), where a similar to V_X term was introduced for constant sampling, we use X, X_1-dependent term for variable sampling, which strengthens the result.

Remark 4. An advantage of the direct Lyapunov method considered in the present paper over the small-gain theorem-based results (see e.g. Fujioka, 2009; Mirkin, 2007) is in its wider applications: to exponential bounds on the solutions of the initial value problems, to systems with polytopic type uncertainties, to finding

Table 1
Example 2: Maximum upper bound on the variable sampling.

Method	Park and Ko (2007)	Naghshtabrizi et al. (2008)	Fujioka (2009), Mirkin (2007)	Proposition 1	Remark 1	Theorem 1
h	1.04	1.11	1.36	1.62	1.68	1.69

Table 2
Example 3: Maximum upper bound on the variable sampling.

Method	Park and Ko (2007)	Naghshtabrizi et al. (2008)	Fujioka (2009), Mirkin (2007)	Proposition 1	Remark 1	Theorem 1
h	1.06	0.73	0.87	0.57	0.83	1.64

domains of attraction of some nonlinear systems (Fridman et al., 2004), where the small-gain approach is not applicable. Moreover, our stability analysis is based on the convex in $\tau(t)$ approach, which has been introduced in Park and Ko (2007) and which seems to be not applicable in the framework of Mirkin (2007) and Fujioka (2009).

Remark 5. Compare now the number of scalar decision variables in LMIs of different methods. LMIs of Fujioka (2009) have the minimal number $0.5(n^2 + n) + n_u^2 + n_u$ of variables. We note that in all numerical examples of the present paper LMIs of Fujioka (2009) lead to the same results as conditions of Mirkin (2007). LMIs of Proposition 1 have the same number $3n^2 + n$ of scalar decision variables as in Mirkin (2007), whereas in Naghshtabrizi et al. (2008) there are $3.5n^2 + 1.5n$ and $5n^2 + n$ variables for variable and for constant samplings respectively. In conditions of Remark 1 and of Theorem 1 (for stability) there are $6.5n^2 + 1.5n$ and $8n^2 + n$ variables respectively. In Park and Ko (2007) there are $11.5n^2 + 1.5n$ variables for fast-varying delays.

3.5. Examples

Example 2. Consider the following system (Yue et al., 2005):

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} Kx(t_k) + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w(t),$$
$$z(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + 0.1Kx(t_k), \quad K = -\begin{bmatrix} 3.75 & 11.5 \end{bmatrix}. \tag{43}$$

It was found in Naghshtabrizi et al. (2008) by applying the standard techniques from digital control that the system (with $w = 0$) remains stable for all constant samplings less than 1.7 and becomes unstable for samplings greater than 1.7. Moreover, the above system with $w = 0$ and with constant delay h, where $x(t_k)$ is changed by $x(t-h)$, is asymptotically stable for $h < 1.36$ and becomes unstable for $h > 1.36$. Therefore, all the existing methods, that are based on time-independent Lyapunov functionals, cannot guarantee the stability for the samplings greater than 1.36.

The results obtained (by various methods in the literature and by Proposition 1, Remark 1 and Theorem 1 with $\alpha = 0$) for the upper bounds on the samplings, which preserve the stability, are listed in Table 1.

Choosing further $h = 1.11$ and applying LMIs (32), (33), we find that the system is exponentially stable with the decay rate $\alpha = 0.29$ for the variable samplings up to 1.11.

Consider next $w \neq 0$. By choosing $h = 0.8695$, it is found in Jiang et al. (2008) that the system achieves $\gamma = 1.0005$, which improves the result $\gamma = 6.82$ of Yue et al. (2005). By Theorem 1 we achieve essentially smaller $\gamma = 0.2101$ (for all varying samplings up to 0.8695). In this example, the continuous-time controller $u(t) = Kx(t)$ achieves the same minimum value of $\gamma = 0.2101$. Moreover, the same value of γ is achieved for all variable samplings up to $h = 0.98$. Thus, the obtained value of L_2-gain for variable samplings up to 0.98 cannot be improved.

Example 3. Consider another much-studied system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x(t - \tau(t)).$$

For constant delays this system is delay-independently stable. The results obtained (by various methods in the literature and by Proposition 1, Remark 1 and Theorem 1 with $\alpha = 0$) for the admissible upper bounds on the samplings, which preserve the stability, are listed in Table 2.

4. Conclusions

A time-dependent Lyapunov functional method has been introduced for analysis of linear systems under uncertain sampling with a given upper bound on the sampling intervals. This method has been developed in the framework of input delay approach. In some well-studied numerical examples our method approaches analytical values of minimum L_2-gain and of maximum sampling, preserving the stability.

The presented method gives insight into new constructions of Lyapunov functionals for systems with time-varying delays. It can be extended to networked control systems, where the network-induced delay is taken into account. The new method gives efficient tools for different design problems.

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