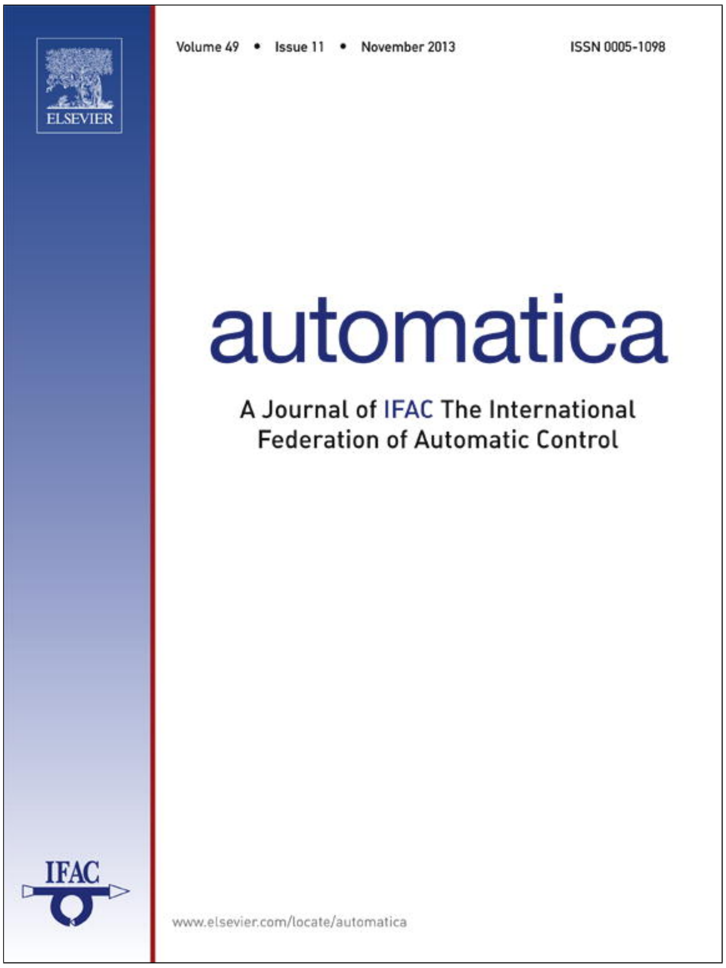

Volume 49 • Issue 11 • November 2013
ISSN 0005-1098
ELSEVIER
automatica
A Journal of IFAC The International Federation of Automatic Control
IFAC
www.elsevier.com/locate/automatica

Automatica 49 (2013) 3467–3475

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

New stability conditions for systems with distributed delays ☆

Oren Solomon, Emilia Fridman [1]

School of Electrical Engineering, Tel-Aviv University, Tel-Aviv 69978, Israel

ARTICLE INFO

Article history:
Received 13 February 2013
Received in revised form
27 May 2013
Accepted 5 August 2013
Available online 14 September 2013

Keywords:
Infinite time-delay
Gamma-distributed delay
Lyapunov–Krasovskii method

ABSTRACT

In the present paper, sufficient conditions for the exponential stability of linear systems with infinite distributed delays are presented. Such systems arise in population dynamics, in traffic flow models, in networked control systems, in PID controller design and in other engineering problems. In the early Lyapunov-based analysis of systems with distributed delays (Kolmanovskii & Myshkis, 1999), the delayed terms were treated as perturbations, where it was assumed that the system without the delayed term is asymptotically stable. Later, for the case of constant kernels and finite delays, less conservative conditions were derived under the assumption that the corresponding system with the zero-delay is stable (Chen & Zheng, 2007). We will generalize these results to the infinite delay case by extending the corresponding Jensen's integral inequalities and Lyapunov–Krasovskii constructions. Our main challenge is the stability conditions for systems with gamma-distributed delays, where the *delay is stabilizing*, i.e. the corresponding system with the zero-delay as well as the system without the delayed term are not asymptotically stable. Here the results are derived by using augmented Lyapunov functionals. Polytopic uncertainties in the system matrices can be easily included in the analysis. Numerical examples illustrate the efficiency of the method. Thus, for the traffic flow model on the ring, where the delay is stabilizing, the resulting stability region is close to the theoretical one found in Michiels, Morarescu, and Niculescu (2009) via the frequency domain analysis.

1. Introduction

Systems with infinite distributed delays

$$\dot{x}(t) = Ax(t) + A_d \int_0^h K(\theta)x(t-\theta)d\theta, \tag{1}$$

where $h = \infty$, $x(t) \in \mathbf{R}^n$, $A, A_d \in \mathbf{R}^{n\times n}$ and $K(\cdot)$ is a scalar, are present in many scientific disciplines such as population dynamics and engineering. One of the first studies devoted to population dynamics using a model with infinite distributed delay is due to Cushing (1981) (see also Kolmanovskii and Myshkis (1999) and references therein). Such delays appear in the control systems under PID controllers (Kolmanovskii & Myshkis, 1999), in the traffic flow models (Michiels et al., 2009; Morarescu, Niculescu, & Gu, 2007), in the control over communication networks (Morarescu et al., 2007) and in the machine tool vibration problem (Stepan, 1998). Particularly systems with distributed delays, whose kernel $K(\cdot)$ is a gamma-distribution with a gap, model the above mentioned applications in the population dynamics and engineering (Michiels et al., 2009; Morarescu et al., 2007).

Stability and control of systems with the constant kernel $K \equiv 1$ and *finite distributed delays* $h < \infty$ have been extensively studied in the literature (see e.g. Chen and Zheng (2007), Fridman (2001), Gu, Kharitonov, and Chen (2003), Kolmanovskii and Myshkis (1999) and Sun, Liu, and Chen (2009) and the references therein). In the early results based on the Lyapunov–Krasovskii method (Kolmanovskii & Myshkis, 1999), the delayed term was treated as the disturbance, where A was supposed to be Hurwitz. Later a less restrictive assumption on the stability of the corresponding system with the zero delay $\dot{x}(t) = (A + A_d h)x(t)$ was used (Chen & Zheng, 2007). In the case of *piecewise-constant kernels* and $h < \infty$, the discretized Lyapunov functional method is applicable to systems with distributed *stabilizing delays*, where neither A nor A_0 are Hurwitz. Simple stability conditions for systems with stabilizing discrete delays were recently derived by using augmented Lyapunov functionals and Wirtinger-based integral inequalities (Seuret & Gouaisbaut, 2013).

The case of finite distributed delay with a variable kernel is less studied in the literature. For the finite distributed delay with an exponential kernel we refer to Verriest (1999), where the system was transformed to an augmented system with a discrete delay. In Ozbay, Bonnet, and Clairambault (2008) the stability of systems

☆ This work was partially supported by the Israel Science Foundation (grant no. 754/10) and by the Kamea Fund of Israel. The material in this paper was partially presented at the 52nd IEEE Conference on Decision and Control (CDC), December 10–13, 2013, Florence, Italy. This paper was recommended for publication in revised form by Associate Editor Akira Kojima under the direction of Editor Ian R. Petersen.

E-mail addresses: orensolo@gmail.com (O. Solomon), emilia@eng.tau.ac.il (E. Fridman).

[1] Tel.: +972 36405313; fax: +972 36407095.

0005-1098/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.automatica.2013.08.025

with kernels that are finite duration impulse responses of finite dimensional systems were reduced to the stability of systems with discrete delays. Systems with polynomial kernels were studied in Gouaisbaut and Ariba (2011) by presenting them as interconnected models and using quadratic separation. For systems, where the matrix kernel K is a rational function, we refer to Goebel, Munz, and Allgöwer (2011) and the references therein.

The existing Lyapunov-based results treat the term with the infinite delay as a disturbance (Kolmanovskii & Myshkis, 1999). In the frequency domain, necessary and sufficient conditions for the stability of systems with gamma-distributed delays were found in Michiels et al. (2009) and Morarescu et al. (2007). Special attention was paid to the case of stabilizing delay, where $A = 0$ and the matrix of the corresponding zero-delay system has the zero eigenvalue. The latter situation appears e.g. in the *traffic flow* model on the ring (Michiels et al., 2009; Morarescu et al., 2007).

The objective of the present paper is to derive simple conditions for the exponential stability of systems with infinite distributed delays in terms of Linear Matrix Inequalities (LMIs). For general integrable kernels, we assume the exponential stability of the corresponding system with the zero delay or of the system without the delayed terms. Here we extend the double and triple integral terms in Lyapunov–Krasovskii functionals of Chen and Zheng (2007) and Sun et al. (2009) to the variable kernels and unbounded delays. For gamma distributions with a gap, we analyze the challenging effect of stabilizing delay. Note that for the infinite delay the discretized Lyapunov method of Gu et al. (2003) or the Wirtinger-based inequality analysis of Seuret and Gouaisbaut (2013), if applicable, would lead to complicated conditions. Here the results are derived by using augmented Lyapunov functionals. Polytopic uncertainties in the system matrices can be easily included in the analysis. Numerical examples from the literature illustrate the efficiency of the method. Thus, for the traffic flow model on the ring, the stability region which is guaranteed by the presented LMIs almost coincides with the theoretical one found in Michiels et al. (2009). Some preliminary results will be presented in Solomon and Fridman (2013).

Notations. The superscript 'T' stands for matrix transposition, $\mathbf{R}^n$ denotes the n-dimensional Euclidean space with the norm $|\cdot|$, and $\mathbf{R}^{n\times m}$ is the set of all $n \times m$ real matrices with the Euclidean norm $|\cdot|$. The notation $P > 0$, for $P \in \mathbf{R}^{n\times n}$, means that P is symmetric and positive definite, whereas $\lambda_{\min}(P)$ ($\lambda_{\max}(P)$) denotes its minimum (maximum) eigenvalue. In symmetric block matrices we use $*$ for terms that are induced by the symmetry. The space of continuous functions $\phi : (-\infty, 0] \to \mathbf{R}^n$ with the norm $\|\phi\|_C = \sup_{s\in(-\infty,0]} |\phi(s)| < \infty$ is denoted by $C(-\infty, 0; \mathbf{R}^n)$. The space of the continuously differentiable functions $\phi : (-\infty, 0] \to \mathbf{R}^n$ with the norm $\|\phi\|_{C^1} = \|\phi\|_C + \|\dot\phi\|_C < \infty$ is denoted by $C^1(-\infty, 0; \mathbf{R}^n)$. $L_p(a, b; \mathbf{R}^n)$ $(p = 1, 2, \ldots)$ is the Hilbert space of functions $x : (a, b) \to \mathbf{R}^n$ with $\|x\|_{L^p} = \int_a^b |x(s)|^p ds < \infty$.

2. Preliminaries and problem formulation

A linear system with infinite delay has a form

$$\dot x(t) = Ax(t) + A_d \int_0^\infty K(\theta)x(t-\theta)d\theta, \tag{2}$$

where $x(t) \in \mathbf{R}^n$, $A, A_d \in \mathbf{R}^{n\times n}$ are constant matrices. The scalar kernel function K is supposed to satisfy the following assumption:

A1. Assume that the kernel $K \in L_1(0, \infty; \mathbf{R})$ satisfies the inequality

$$\int_0^\infty |K(\theta)|e^{2\delta_0\theta}d\theta < \infty \tag{3}$$

with some $\delta_0 > 0$.

Note that e.g. gamma-distributed delays considered below satisfy A1. A solution of (2) is uniquely determined for the uniformly continuous initial function $\phi \in C(-\infty, 0; \mathbf{R}^n)$. This solution continuously depends on ϕ (see Kolmanovskii and Myshkis (1999, Theorem 3.2.3)). System (2) is said to be *exponentially stable with a decay rate* $\delta > 0$ if there exists a constant $b \ge 1$ such that the following exponential estimate holds for the solution of (2) initialized with $\phi \in C(-\infty, 0; \mathbf{R}^n)$:

$$|x(t)|^2 \le be^{-2\delta t}\|\phi\|^2 \quad \forall t \ge 0, \tag{4}$$

where $\|\phi\| = \|\phi\|_C$. The objective of the present paper is to derive sufficient conditions for the exponential stability of (2). For $\phi \in C^1(-\infty, 0; \mathbf{R}^n)$ we will find less restrictive exponential stability conditions, where $\|\phi\| = \|\phi\|_{C^1}$ in the exponential bound of (4).

2.1. Applications with gamma-distributed delays

Gamma-distributed delays can be encountered in the problem of control over communication networks, in the population dynamics (Morarescu et al., 2007) and in the machine tool vibration problem (Stepan, 1998). A research area where a distributed delay appears naturally is the traffic flow dynamics. The following system which incorporates a general memory effect represents a configuration of vehicles in a ring (Michiels et al., 2009):

$$\dot v_k(t) = \alpha_k \int_0^\infty f(\theta)(v_{k-1}(t-\theta) - v_k(t-\theta))\,d\theta \qquad k = 1, \ldots, n, \qquad v_0 \equiv v_n. \tag{5}$$

Here $v_k(t)$ is the velocity of vehicle k, n is the number of vehicles, and α_k is the coupling coefficient between the vehicles k and $k-1$. The equilibrium point of the matched speeds is referred to as a *consensus among the agents*, where no particular agent may collide with another. The asymptotic stability of (5) means that $\lim_{t\to\infty} v_k(t) = 0, k = 1, \ldots, n$ and, thus, that $\lim_{t\to\infty} \sum_{k=1}^n v_k(t)/n = 0$. The latter relation corresponds to the *consensus* as defined in Michiels et al. (2009).

Function f in (5) is a gamma distribution with a gap τ, where the gap corresponds to the minimum reaction time of the humans with respect to some external stimuli:

$$f(\xi) = \begin{cases} 0 & \xi < \tau, \\ \dfrac{(\xi-\tau)^{N-1}e^{-\frac{\xi-\tau}{T}}}{T^N(N-1)!} & \xi \ge \tau. \end{cases} \tag{6}$$

Here $N \in \mathbb{N}$ is a parameter of the distribution. Note that $\int_0^\infty f(\xi)\,d\xi = 1$. The corresponding average delay of (6) satisfies

$$\int_0^\infty \xi f(\xi)d\xi = \tau + NT.$$

The distributed delay, whose kernel is a gamma distribution with a gap, characterizes the human drivers' behavior in the average. Denoting $x(t) = [v_1(t), \ldots, v_n(t)]^T$, (5) can be represented as (2), where $K(\theta) = f(\theta)$ is piecewise-continuous and where A and A_d are constant matrices. Alternatively, taking into account that

$$\int_0^\infty f(\theta)x(t-\theta)d\theta = \int_\tau^\infty f(\theta)x(t-\theta)d\theta = \int_0^\infty f(\xi-\tau)x(t-\xi-\tau)d\xi$$

we arrive at the equivalent representation

$$\dot x(t) = Ax(t) + A_d \int_0^\infty \Gamma(\theta)x(t-\theta-\tau)d\theta, \tag{7}$$

where

$$\Gamma(\theta) = \frac{\theta^{N-1}e^{-\frac{\theta}{T}}}{T^N(N-1)!} \tag{8}$$

with a smooth kernel Γ.

2.2. Problem formulation and useful inequalities

We will first derive the exponential stability conditions for a general integrable $K \in L_1(0,\infty;\mathbf{R})$ (see Section 3). Though linear systems with integrable kernels can always be presented in the form of (2) we will derive the stability condition for a more general system with a gap $\tau \geq 0$

$$\dot{x}(t) = Ax(t) + A_d \int_0^\infty K(\theta)x(t-\theta-\tau)d\theta, \quad (9)$$

where K is subject to A1. This will allow us further to modify the results for the case of smooth kernels. Note also that our results for $K \in L_1(0,\infty;\mathbf{R})$ will be applicable to the case of finite delay $h < \infty$, where $K(\theta) = 0$, $\theta > h$.

Keeping in mind that (9) can be represented in the following form:

$$\dot{x}(t) = A_0 x(t) + A_d \int_0^\infty K(\theta)[x(t-\theta-\tau) - x(t)]\, d\theta, \quad (10)$$

we assume the exponential stability of $\dot{x}(t) = Ax(t)$ or of (10) with the zero delay

$$\dot{x}(t) = A_0 x(t), \quad A_0 \triangleq A + A_d \int_0^\infty K(\theta)\, d\theta. \quad (11)$$

Note that in many cases a system with a matrix integrable kernel $K(\theta) \in \mathbf{R}^{n\times n}$ can be presented as a system with multiple delays

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^m A_{di} \int_0^\infty K_i(\theta)x(t-\theta-\tau)d\theta \quad (12)$$

and scalar kernels $K_i \in L_1(0,\infty;\mathbf{R})$. This presentation is not unique and m may be greater than n^2 (see Example 1). Our results can be easily extended to this case (as shown in Corollary 1).

For gamma-distributed delays with a gap we will further derive LMIs for the case of "stabilizing delays", where A and A_0 may be non-Hurwitz. This is motivated e.g. by the traffic flow model on the ring (Morarescu et al., 2007), where $A = 0$ and where the zero eigenvalue of A_0 corresponds to the vehicles moving with the same velocity.

As in the case of finite delays (Gu et al., 2003), a basic tool for the Lyapunov-based stability analysis will be integral inequalities. We will generalize the well-known Jensen's inequality (Gu et al., 2003) and its double integral extension of Sun et al. (2009), which have been considered on the finite intervals of integration, to the infinite ones:

Lemma 1. *Assume* A1. *Given an $n \times n$ matrix $R > 0$, a scalar function $\alpha : [0,\infty) \to (0,\infty)$, a scalar $\tau \geq 0$ and a vector function $\phi : [0,\infty) \to \mathbf{R}^n$ such that the integrations concerned are well defined. Then the following inequalities hold:*

$$\int_0^\infty \alpha(\theta)|K(\theta)|\phi^T(\theta)R\phi(\theta)\, d\theta \geq K_0^{-1} \int_0^\infty \phi^T(\theta)K(\theta)\, d\theta R \int_0^\infty K(\theta)\phi(\theta)\, d\theta, \quad (13)$$

$$K_0 = \int_0^\infty \alpha^{-1}(\theta)|K(\theta)|d\theta,$$

and

$$\int_0^\infty \int_{t-\theta-\tau}^t \alpha(\theta)|K(\theta)|\phi^T(s)R\phi(s)ds\, d\theta \geq K_1^{-1} \int_0^\infty \int_{t-\theta-\tau}^t \phi^T(s)K(\theta)dsd\theta R \times \int_0^\infty \int_{t-\theta-\tau}^t K(\theta)\phi(s)dsd\theta,$$

$$K_1 = \int_0^\infty \alpha^{-1}(\theta)|K(\theta)|(\theta+\tau)d\theta. \quad (14)$$

Proof. By Schur complements the following holds

$$\begin{bmatrix} \alpha(\theta)|K(\theta)|\phi^T(\theta)R\phi(\theta) & \phi^T(\theta)K(\theta) \\ * & R^{-1}|K(\theta)|\alpha^{-1}(\theta) \end{bmatrix} \geq 0 \quad (15)$$

for $\theta \in [0,\infty]$. Integration of (15) from 0 to ∞ yields

$$\begin{bmatrix} \int_0^\infty \alpha(\theta)|K(\theta)|\phi^T(\theta)R\phi(\theta) & \int_0^\infty \phi^T(\theta)K(\theta) \\ * & K_0 R^{-1} \end{bmatrix} \geq 0.$$

Application of Schur complements to the above matrix inequality leads to (13). Double integration of

$$\begin{bmatrix} \alpha(\theta)|K(\theta)|\phi^T(s)R\phi(s) & \phi^T(s)K(\theta) \\ * & R^{-1}|K(\theta)|\alpha^{-1}(\theta) \end{bmatrix} \geq 0$$

from $t-\theta-\tau$ to t in s and from 0 to ∞ in θ, where

$$\int_0^\infty \int_{t-\theta-\tau}^t |K(\theta)|\alpha^{-1}(\theta)\, ds\, d\theta = K_1$$

and application of Schur complements leads to (14).

Remark 1. Lemma 1 can be easily extended to the matrix K, where $\phi^T K$ in (13) and (14) should be changed by $\phi^T K^T$.

3. Stability in the case of integrable kernels

Consider (9), where $K \in L_1(0,\infty;\mathbf{R})$ is subject to A1. Assume that A or $A_0 = A + A_d \int_0^\infty K(\theta)\, d\theta$ are Hurwitz. For the exponential stability analysis of (9) with the decay rate $\delta < \delta_0$ we suggest the following Lyapunov–Krasovskii functional:

$$V(t) = V_P(t) + V_G(t) + V_H(t), \quad V_P(t) = x^T(t)Px(t) \quad (16)$$

with

$$V_G(t) = \int_0^\infty \int_{t-\theta-\tau}^t e^{-2\delta(t-s)}|K(\theta)|x^T(s)Gx(s)\, ds\, d\theta,$$

$$V_H(t) = \int_0^\infty \int_0^{\theta+\tau} \int_{t-\lambda}^t e^{-2\delta(t-s)}|K(\theta)|\dot{x}^T(s)H\dot{x}(s)dsd\lambda d\theta,$$

where P, G and H are positive $n \times n$-matrices. The term $V_G(t)$ with $\delta = 0$ extends the classical construction of Kolmanovskii and Myshkis (1999) to the vector case, where the exponential term is inserted to achieve the exponential decay rate δ. It "compensates" the delayed term in (9) provided A is Hurwitz. The term V_H extends the triple integrals of Sun et al. (2009) to the case of infinite delay and it "compensates" the integral term in (10) when A_0 is Hurwitz. The latter term also improves the results when A is Hurwitz.

Since V depends on $\dot{x}$, it is defined for differentiable initial functions. We will derive the conditions that guarantee $\dot{V}(t) + 2\delta V(t) \leq 0$ along the solutions of (9) with initial functions $\phi \in C^1(0,\infty;\mathbf{R^n})$. Then these solutions would satisfy the following inequality:

$$x^T(t)Px(t) \leq V(t) \leq e^{-2\delta t}V(0), \quad t \geq 0, \quad (17)$$

where for all $\delta \in (0,\delta_0)$

$$V(0) \leq \lambda_{\max}(P)|\phi(0)|^2 + \lambda_{\max}(G) \int_0^\infty |K(\theta)|(\theta+\tau)d\theta \|\phi\|_C + \lambda_{\max}(H) \int_0^\infty |K(\theta)|(\theta+\tau)^2/2d\theta \|\dot{\phi}\|_C. \quad (18)$$

Differentiation of V along the trajectories of (9) yields

$$\begin{aligned}\dot V(t)&+2\delta V(t)\\ &=2x^T(t)P\left[Ax(t)+A_d\int_0^\infty K(\theta)x(t-\theta-\tau)d\theta\right]\\ &\quad+2\delta x^T(t)Px(t)+\int_0^\infty |K(\theta)|\,d\theta x^T(t)Gx(t)\\ &\quad-\int_0^\infty e^{-2\delta(\theta+\tau)}|K(\theta)|x^T(t-\theta-\tau)Gx(t-\theta-\tau)\,d\theta\\ &\quad+\int_0^\infty(\theta+\tau)|K(\theta)|\,d\theta\dot x^T(t)H\dot x(t)\\ &\quad-\int_0^\infty\int_{t-\theta-\tau}^t e^{-2\delta(\theta+\tau)}|K(\theta)|\dot x^T(s)H\dot x(s)\,ds\,d\theta.\end{aligned}$$

Denote

$$\begin{aligned}K_{0\delta}&=\int_0^\infty e^{2\delta(\theta+\tau)}|K(\theta)|\,d\theta, \qquad K_{00}=K_{0\delta}|_{\delta=0},\\ K_{1\delta}&=\int_0^\infty e^{2\delta(\theta+\tau)}|K(\theta)|(\theta+\tau)\,d\theta, \qquad K_{10}=K_{1\delta}|_{\delta=0}. \end{aligned}\tag{19}$$

Applying further the extended Jensen's inequalities (13) and (14) we have

$$\begin{aligned}&-\int_0^\infty e^{-2\delta(\theta+\tau)}|K(\theta)|x^T(t-\theta-\tau)Gx(t-\theta-\tau)\,d\theta\\ &\le -K_{0\delta}^{-1}\int_0^\infty K(\theta)x^T(t-\theta-\tau)d\theta G\int_0^\infty K(\theta)x(t-\theta-\tau)d\theta\end{aligned}$$

and

$$\begin{aligned}&-\int_0^\infty\int_{t-\theta-\tau}^t e^{-2\delta(\theta+\tau)}|K(\theta)|\dot x^T(s)H\dot x(s)\,ds\,d\theta\\ &\le -K_{1\delta}^{-1}\int_0^\infty\int_{t-\theta-\tau}^t K(\theta)\dot x^T(s)dsd\theta H\\ &\quad\times\int_0^\infty\int_{t-\theta-\tau}^t K(\theta)\dot x(s)dsd\theta.\end{aligned}$$

Choose $\eta(t)=col\{x(t),\int_0^\infty K(\theta)x(t-\theta-\tau)\,d\theta\}$. Then

$$\begin{aligned}\dot V(t)+2\delta V(t)&\le\eta^T(t)\begin{bmatrix}\Phi_{00} & PA_d+K_{1\delta}^{-1}K_{00}H\\ * & -K_{0\delta}^{-1}G-K_{1\delta}^{-1}H\end{bmatrix}\eta(t)\\ &\quad+K_{10}\eta^T(t)\begin{bmatrix}A^T\\ A_d^T\end{bmatrix}H\begin{bmatrix}A^T\\ A_d^T\end{bmatrix}^T\eta(t),\end{aligned}$$

where

$$\Phi_{00}=PA+A^TP+2\delta P+K_{00}G-K_{1\delta}^{-1}K_{00}^2H. \tag{20}$$

By the Schur complements we find that if

$$\begin{bmatrix}\Phi_{00} & PA_d+K_{1\delta}^{-1}K_{00}H & K_{10}A^TH\\ * & -K_{0\delta}^{-1}G-K_{1\delta}^{-1}H & K_{10}A_d^TH\\ * & * & -K_{10}H\end{bmatrix}<0, \tag{21}$$

then $\dot V(t)+2\delta V\le 0$. We have proved the following:

Proposition 1. *Assume A1. Given $\delta\in(0,\delta_0)$ ($\delta=0$), suppose there exist positive definite matrices $P,G,H\in\mathbf{R}^{\mathbf{n}\times\mathbf{n}}$, such that LMI (21) with notations given by (19) and (20) is feasible. Then solutions of (9) initiated by $\phi\in C^1(0,\infty;\mathbf{R}^n)$ satisfy the bound (17) and (18), i.e. (9) is exponentially stable with the decay rate δ (with a small enough decay rate).*

Remark 2. Taking into account Remark 1, the LMI (21) guarantees the exponential stability of (9) with a matrix kernel K. However, numerical computation of the constants in (19) for the matrix case becomes complicated. That is why in Corollary 1 below we consider a particular but important case of matrix kernels, where the system can be presented as (12) with scalar kernels K_i.

Remark 3. For the asymptotic stability of (9), assumption A1 is not necessary. The feasibility of LMI (21) with $\delta=0$ guarantees the asymptotic stability of (9) for $K\in L_1(0,\infty;\mathbf{R})$ such that $K_{10}<\infty$.

Remark 4. The effect of the distributed delay is reflected only as some integral in the stability conditions. Such conditions may be conservative ignoring more detailed delay distribution. For the case of gamma-distributed kernels (in Section 4 below) the derivative of the kernel will be employed to improve the results.

Remark 5. When A is Hurwitz a simpler V of (16) with $H=0$ can be applied leading to the following LMI:

$$\begin{bmatrix}PA+A^TP+2\delta P+K_{00}G & PA_d\\ * & -K_{0\delta}{}^{-1}G\end{bmatrix}<0. \tag{22}$$

For all solutions of (9) initiated by $\phi\in C(0,\infty;\mathbf{R}^{\mathbf{n}})$, the latter LMI guarantees the exponential bound (17) and (18) with $H=0$. It is easy to see that the feasibility of (22) with $\delta=0$ yields the delay-independent stability of

$$\dot x(t)=Ax(t)\pm K_{00}A_dx(t-r)$$

for all $r\ge 0$ and implies that

(1) A and $A\pm K_{00}A_d$ are Hurwitz (i.e. A_0 is Hurwitz for $K\ge 0$);
(2) the eigenvalues of $A^{-1}K_{00}A_d=A^{-1}\int_0^\infty|K(s)|dsA_d$ are inside of the unit circle (Fridman, 2002);
(3) $\|G^{0.5}(sI-A)^{-1}A_dG^{-0.5}\|_\infty<1/K_{00}$ (the scaled small gain condition), where $\|\cdot\|_\infty$ denotes the H_∞-norm.

For $\delta=0, A_d=G=I$ and a matrix K the LMI (22) is equivalent to $\|(sI-A)^{-1}\|_\infty<1/K_{00}$. The latter stability condition was derived in Ozbay et al. (2008) and Verriest (1995) for the case of finite delay.

Proposition 1 can be easily extended to systems with multiple delays and scalar kernels:

Corollary 1. *Consider (12). Assume that there exists $\delta_0>0$ such that $\int_0^\infty|K_i(\theta)|e^{2\delta_0\theta}d\theta<\infty$, $i=1,\dots,m$ and that $A_0=A+\sum_{i=1}^m A_{di}\int_0^\infty K_i(\theta)d\theta$ is Hurwitz. Given $\delta\in(0,\delta_0)$ ($\delta=0$), suppose there exist positive definite matrices $P,G_i,H_i\in\mathbf{R}^{\mathbf{n}\times\mathbf{n}}$ that satisfy the LMI*

$$\begin{bmatrix}\Phi_{00} & \Phi_{01} & \cdots & \Phi_{0i} & A^T\left(\sum_{i=1}^m K_{10}^iH_i\right)\\ * & \Phi_{11} & \cdots & 0 & A_{d1}^T\left(\sum_{i=1}^m K_{10}^iH_i\right)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ * & * & * & \Phi_{mm} & A_{dm}^T\left(\sum_{i=1}^m K_{10}^iH_i\right)\\ * & * & * & * & -\sum_{i=1}^m K_{10}^iH_i\end{bmatrix}<0, \tag{23}$$

where

$$K_{0\delta}^i = \int_0^\infty e^{2\delta(\theta+\tau)}|K_i(\theta)|\,d\theta, \qquad K_{00}^i = K_{0\delta}^i|_{\delta=0},$$
$$K_{1\delta}^i = \int_0^\infty e^{2\delta(\theta+\tau)}|K_i(\theta)|(\theta+\tau)\,d\theta, \qquad K_{10}^i = K_{1\delta}^i|_{\delta=0},$$
$$\Phi_{00} = PA + A^TP + 2\delta P + \sum_{i=1}^m [K_{00}^i G_i - (K_{1\delta}^i)^{-1}(K_{00}^i)^2 H_i],$$
$$\Phi_{0i} = PA_{di} + (K_{1\delta}^i)^{-1}K_{00}^i H_i,$$
$$\Phi_{ii} = -(K_{0\delta}^i)^{-1}G_i - (K_{1\delta}^i)^{-1}H_i.$$

Then (12) is exponentially stable with the decay rate δ (with a small enough decay rate).

Example 1 (*Gouaisbaut & Ariba, 2011*)**.** The system with non-Hurwitz A and with a matrix variable kernel

$$\dot{x}(t) = Ax(t) + \int_{-h}^0 \bar{K}(s)x(t+s)ds,$$
$$A = \begin{bmatrix} 0.2 & 0.01 \\ 0 & -2 \end{bmatrix}, \qquad \bar{K}(s) = \begin{bmatrix} -1+0.3s & 0.1 \\ 0 & -0.1 \end{bmatrix} \tag{24}$$

can be presented in the form of (12) with $m = 2$, $\tau = 0$ and $K_1 = K_2 = 0$ for $\theta > h$ in many ways. Note that $\int_{-h}^0 \bar{K}(s)x(t+s)ds = \int_0^h \bar{K}(-\theta)x(t-\theta)d\theta$. We choose the following two forms:

$$A_{d1} = \begin{bmatrix} -1 & 0.1 \\ 0 & -0.1 \end{bmatrix}, \qquad A_{d2} = -\begin{bmatrix} 0.3 & 0 \\ 0 & 0 \end{bmatrix},$$
$$K_1 \equiv 1, \qquad K_2(\theta) = \theta, \quad \theta \in [0, h] \tag{25}$$

and

$$A_{d1} = \begin{bmatrix} 0 & 0.1 \\ 0 & -0.1 \end{bmatrix}, \qquad A_{d2} = -\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$K_1 \equiv 1, \qquad K_2(\theta) = 1 + 0.3\theta, \quad \theta \in [0, h]. \tag{26}$$

Here $A_0 = A + \sum_{i=1}^2 A_{di} \int_0^\infty K_i(\theta)d\theta$ is Hurwitz for $h \geq 0.195$. It was found in Gouaisbaut and Ariba (2011) by using an analytical method that the system is asymptotically stable for $h \in [0.195, 1.71]$. The LMIs of Corollary 1 with 15 scalar variables guarantee the exponential stability of (25) for $h \in [0.207, 1.455]$ and of (26) for $h \in [0.195, 1.442]$. Thus, (24) is exponentially stable for $[0.195, 1.455]$.

The latter result essentially improves the one of Gouaisbaut and Ariba (2011), where a smaller stability interval $h \in [0.23, 1]$ was found via LMIs with 2221 decision variables. Choosing next $h = 1$ we find by using LMIs of Corollary 1 with 15 scalar variables that the maximum achievable decay rate is $\delta_{\max} = 0.433$ for (25) and $\delta_{\max} = 0.593$ for (26). Hence, (24) is exponentially stable with the decay rate 0.593.

Note that (24) has a triangular structure. Thus, it is stable if the two scalar subsystems

$$\dot{x}_1(t) = 0.2x_1(t) - \int_{-h}^0 (1-0.3s)x_1(t+s)ds,$$
$$\dot{x}_2(t) = -2x_2(t) - 0.1\int_{-h}^0 x_2(t+s)ds$$

are stable. In correspondence to (25) and (26), the system for x_1 can be presented as a system with 2 delays, where $A_{d1} = -1 = -K_1(\theta)$ and $A_{d2} = -0.3$, $K_2(\theta) = \theta$, or as a system with 1 delay, where $A_d = -1$ and $K(\theta) = 1 + 0.3\theta$ ($\theta \in [0, h]$). The LMIs of Corollary 1 with 5 scalar variables and with 3 scalar variables guarantee the exponential stability of the latter systems for $h \in [0.207, 1.455]$ and for $h \in [0.195, 1.442]$ respectively.

Another simple extension of the direct Lyapunov–Krasovskii method is to input-to-state stability or to L_2-gain analysis of the perturbed systems. Consider

$$\dot{x}(t) = Ax(t) + A_d\int_0^\infty K(\theta)x(t-\theta-\tau)d\theta + Bw(t),$$
$$z(t) = Cx(t), \tag{27}$$

where $w(t) \in R^{n_w}$ is the disturbance, $z(t) \in R^{n_z}$ is the controlled output, and B and C are constant matrices. System (27) is said to have a L_2-gain less than $\gamma > 0$ if

$$J = \int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t)]dt < 0 \tag{28}$$

for all $0 \neq w \in L_2(0, \infty; R^{n_w})$ and the zero initial condition. By the standard arguments, $J < 0$ if

$$\begin{bmatrix} \Phi_{00} + C^TC & PA_d + K_{1\delta}^{-1}K_{00}H & PB & K_{10}A^TH \\ * & -K_{0\delta}^{-1}G - K_{1\delta}^{-1}H & 0 & K_{10}A_d^TH \\ * & * & -\gamma^2 I & K_{10}B^TH \\ * & * & * & -K_{10}H \end{bmatrix} < 0, \tag{29}$$

where Φ_{00} is given by (20).

Example 2 (*Goebel et al., 2011*)**.** Consider (27), where

$$A = 0.8, \qquad A_d = -41.8, \qquad B = 2, \qquad C = 1 \tag{30}$$

and $K(\theta) = 0$ for $\theta > h$, whereas $K(\theta) = \frac{3+20\theta+700\theta^2}{2-20\theta+800\theta^2}$ for $\theta \in [0, h]$. Here $A_0 < 0$ for $h > 0.011659$. For $h = 0.1$ the smallest L_2-gain achieved in Goebel et al. (2011) was 0.76, whereas by using the LMI (29) a much smaller $\gamma_{\min} = 0.3223$ is guaranteed.

Consider next (27) and (30) with infinite delay, where a modified kernel $K \in L_1(0, \infty; \mathbf{R})$ is given by $K(\theta) = \frac{3+20\theta+700\theta^2}{2-20\theta+800\theta^2}e^{-10\theta}$ for $\theta \in [0, \infty)$ and where $A_0 < 0$. By using the LMI (29), we found $\gamma_{\min} = 0.41$.

4. Systems with gamma-distributed delays

Consider now (7) and (8). Here neither A nor A_0 are supposed to be Hurwitz. The results will be derived by augmented V. Simple computation shows that for $K(\theta) = \Gamma(\theta)$ the constants in (19) have the form:

$$\Gamma_{0\delta} \triangleq K_{0\delta}|_{K=\Gamma} = \int_0^\infty e^{2\delta(\theta+\tau)}\Gamma(\theta)d\theta = \frac{e^{2\delta\tau}}{(1-2\delta T)^N},$$
$$\Gamma_{1\delta} \triangleq K_{1\delta}|_{K=\Gamma} = \int_0^\infty e^{2\delta(\theta+\tau)}\Gamma(\theta)(\theta+\tau)d\theta$$
$$= \frac{e^{2\delta\tau}}{(1-2\delta T)^N}\left(\tau + \frac{NT}{1-2\delta T}\right), \qquad \Gamma_{10} = \tau + NT, \tag{31}$$
$$\Gamma_{00} = 1.$$

Note also that

$$\int_0^\infty \Gamma(\theta)x(t-\theta-\tau)d\theta = \int_{-\infty}^t \Gamma(t-s)x(s-\tau)ds. \tag{32}$$

4.1. Gamma-distributed delay with a gap and $N = 1$

Consider first $\Gamma(\theta) = e^{-\frac{\theta}{T}}/T$, where

$$\Gamma(0) = 1/T, \qquad \dot{\Gamma}(\theta) = -1/T \cdot \Gamma(\theta). \tag{33}$$

Following Ozbay et al. (2008) and Verriest (1999) and denoting

$$y(t) = \int_{-\infty}^t \Gamma(t-s)x(s-\tau)ds, \tag{34}$$

the system (7) can be transformed to the following augmented one:

$$\dot{x}(t) = Ax(t) + A_d y(t), \qquad \dot{y}(t) = \frac{1}{T}x(t-\tau) - \frac{1}{T}y(t). \tag{35}$$

The stability of (35) implies the stability of (7), but not vice versa. This is similar to model transformations of systems with discrete delays (Gu et al., 2003), where the transformed system has additional dynamics. Thus, in the traffic flow models on the ring studied in Michiels et al. (2009), where $A = 0$ and A_d has a zero eigenvalue, the system matrix of (35) for $\tau = 0$ has the zero eigenvalue. However, it was shown in Michiels et al. (2009) that (7) may be asymptotically stable. Note that in this case A and $A_0 = A + A_d \int_0^\infty \Gamma(\theta)\, d\theta$ are not Hurwitz, implying that the results of the previous section are not applicable.

We are interested in the latter situation, where

$$V_{aug}(t) = [x^T(t)\, y^T(t)] \begin{bmatrix} P & Q \\ * & Z \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{36}$$

subject to

$$\begin{bmatrix} P & Q \\ * & Z \end{bmatrix} > 0 \tag{37}$$

cannot be used for the exponential stability analysis of (35) with $\tau = 0$.

For the exponential stability analysis of (7) we suggest the augmented Lyapunov functional:

$$\begin{aligned}
V(t) &= V_{aug}(t) + V_G(t) + V_H(t) + V_S(t) + V_R(t), \\
V_G(t) &= \int_0^\infty \int_{t-\theta-\tau}^t e^{-2\delta(t-s)} \Gamma(\theta) x^T(s) G x(s)\, ds\, d\theta, \\
V_H(t) &= \int_0^\infty \int_0^{\theta+\tau} \int_{t-\lambda}^t e^{-2\delta(t-s)} \Gamma(\theta) \dot{x}^T(s) H \dot{x}(s)\, ds\, d\lambda\, d\theta, \\
V_S(t) &= \int_{t-\tau}^t e^{-2\delta(t-s)} x^T(s) S x(s) ds, \\
V_R(t) &= \int_{-\tau}^0 \int_{t+\theta}^t e^{-2\delta(t-s)} \dot{x}^T(s) R \dot{x}(s) ds d\theta,
\end{aligned} \tag{38}$$

where $G > 0$, $H > 0$, $R > 0$, $S > 0$.

Denote $\eta(t) = col\{x(t), y(t), x(t-\tau)\}$. Differentiating V_{aug} along (7) and using (35) we obtain

$$\begin{aligned}
\dot{V}_{aug}(t) &= 2[x^T(t)\, y^T(t)] \begin{bmatrix} P & Q \\ * & Z \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} \\
&= 2\eta^T(t) \begin{bmatrix} PA & PA_d - \frac{1}{T}Q & \frac{1}{T}Q \\ Q^T A & Q^T A_d - \frac{1}{T}Z & \frac{1}{T}Z \\ 0 & 0 & 0 \end{bmatrix} \eta(t).
\end{aligned}$$

By applying Jensen's inequalities (13) and (14) and taking into account (31) with $N = 1$ we find further

$$\begin{aligned}
&\dot{V}_S(t) + \dot{V}_G(t) + 2\delta[V_S(t) + V_G(t)] \\
&\quad \le x^T(t)[S+G]x(t) - e^{-2\delta\tau} x^T(t-\tau) S x(t-\tau) \\
&\qquad - \Gamma_{0\delta}^{-1} \int_0^\infty \Gamma(\theta) x^T(t-\theta-\tau) d\theta G \\
&\qquad \times \int_0^\infty \Gamma(\theta) x(t-\theta-\tau) d\theta, \\
&\dot{V}_R(t) + \dot{V}_H(t) + 2\delta[V_R(t) + V_H(t)] \\
&\quad \le \dot{x}^T(t)[\tau R + (\tau + T)H]\dot{x}(t) \\
&\qquad - \frac{e^{-2\delta\tau}}{\tau} \int_{t-\tau}^t \dot{x}^T(s) ds R \int_{t-\tau}^t \dot{x}(s) ds \\
&\qquad - \Gamma_{1\delta}^{-1} \int_0^\infty \int_{t-\theta-\tau}^t \Gamma(\theta) \dot{x}^T(s) ds d\theta H \\
&\qquad \times \int_0^\infty \int_{t-\theta-\tau}^t \Gamma(\theta) \dot{x}(s) ds d\theta.
\end{aligned}$$

Hence $\dot{V}(t) + 2\delta V \le 0$ if

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \frac{1}{T}Q + \frac{e^{-2\delta\tau}}{\tau}R & A^T[\tau R + (\tau+T)H] \\ * & \Psi_{22} & \frac{1}{T}Z & A_d^T[\tau R + (\tau+T)H] \\ * & * & -e^{-2\delta\tau}S - \frac{e^{-2\delta\tau}}{\tau}R & 0 \\ * & * & * & -\tau R - (\tau+T)H \end{bmatrix} < 0, \tag{39}$$

where

$$\begin{aligned}
\Psi_{11} &= PA + A^T P + 2\delta P + S + G - \frac{e^{-2\delta\tau}}{\tau}R - {\Gamma_{1\delta}}^{-1}H, \\
\Psi_{12} &= PA_d - \frac{1}{T}Q + A^T Q + {\Gamma_{1\delta}}^{-1}H + 2\delta Q, \\
\Psi_{22} &= Q^T A_d + A_d^T Q - \frac{2}{T}Z - \Gamma_{0\delta}^{-1}G - {\Gamma_{1\delta}}^{-1}H + 2\delta Z.
\end{aligned} \tag{40}$$

For the case of $\tau = 0$, we use $V(t) = V_{aug}(t) + V_G(t) + V_H(t)$, where $V_{aug}(t)$, $V_H(t)$ and $V_G(t)$ are as in (38), but with $\tau = 0$. The corresponding LMI has a form

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & TA^T H \\ * & \Psi_{22} & TA_d^T H \\ * & * & -TH \end{bmatrix} < 0, \tag{41}$$

where

$$\begin{aligned}
\Psi_{11} &= PA + A^T P + 2\delta P + G + \frac{1}{T}(Q + Q^T) - {\Gamma_{1\delta}}^{-1}H, \\
\Psi_{12} &= PA_d - \frac{1}{T}Q + A^T Q + \frac{1}{T}Z + 2\delta Q + {\Gamma_{1\delta}}^{-1}H, \\
\Psi_{22} &= Q^T A_d + A_d^T Q - \frac{2}{T}Z + 2\delta Z - \Gamma_{0\delta}^{-1}G - {\Gamma_{1\delta}}^{-1}H.
\end{aligned} \tag{42}$$

We have proved the following:

Proposition 2. *System (7) with gamma-distributed kernel (8), where $N = 1$, is exponentially stable with the decay rate $\delta \in (0, 1/T)$ if*

(i) *(for $\tau > 0$) there exist positive definite matrices $P, R, S, G, H, Z \in \mathbf{R}^{n\times n}$, and a matrix $Q \in \mathbf{R}^{n\times n}$ such that LMIs (37) and (39) with notations given by (31) and (40) are feasible;*

(ii) *(for $\tau = 0$) there exist positive definite matrices $P, G, H, Z \in \mathbf{R}^{n\times n}$, and a matrix $Q \in \mathbf{R}^{n\times n}$ such that LMIs (37) and (41) with notations given by (31) and (42) are feasible.*

Moreover, if the above LMIs are feasible with $\delta = 0$, then (7) is exponentially stable with a small enough decay rate.

4.2. Gamma-distributed delay with a gap and $N \ge 2$

We have

$$\Gamma(0) = 0, \qquad \dot{\Gamma}(\theta) = -1/T \cdot \Gamma(\theta) + g(\theta),$$

$$g(\theta) = \frac{\theta^{N-2} e^{-\frac{\theta}{T}}}{T^N (N-2)!}. \tag{43}$$

Choose $V(t) = V_{aug}(t) + V_G + V_H + V_E + V_F$, where V_{aug} is given by (36), and V_G and V_H are defined in (38). Here the last two terms

$$\begin{aligned}
V_E(t) &= \int_0^\infty \int_{t-\theta-\tau}^t e^{-2\delta(t-s)} g(\theta) x^T(s) E x(s)\, ds\, d\theta, \\
V_F(t) &= \int_0^\infty \int_0^{\theta+\tau} \int_{t-\lambda}^t e^{-2\delta(t-s)} g(\theta) \dot{x}^T(s) F \dot{x}(s) ds d\lambda d\theta
\end{aligned}$$

with $E > 0$ and $F > 0$ are added to "compensate" the integral term $\int_{-\infty}^{t} g(t-\theta)x^T(\theta-\tau)d\theta$ in $\dot{V}_{aug}$. Similar to (31) we find

$$\begin{aligned} G_{0\delta} &= K_{0\delta}|_{K=g} = \int_0^\infty e^{2\delta(\theta+\tau)} g(\theta)d\theta \\ &= e^{2\delta\tau}\frac{1}{T(1-2\delta T)^{N-1}}, \\ G_{1\delta} &= K_{1\delta}|_{K=g} = e^{2\delta\tau}\frac{N-1+\frac{\tau}{T}(1-2\delta T)}{(1-2\delta T)^N}, \\ G_{00} &= \frac{1}{T}, \qquad G_{10} = N-1+\frac{\tau}{T}. \end{aligned} \tag{44}$$

By arguments of Proposition 2 we arrive at

Proposition 3. *Given $\delta \in (0, 1/T)$ ($\delta = 0$), let there exist positive definite matrices $P, G, H, E, F, Z \in \mathbf{R}^{n\times n}$ and a matrix $Q \in \mathbf{R}^{n\times n}$ such that LMIs (37) and*

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & Q+\frac{G_{00}}{G_{1\delta}}F & A^T(G_{10}F+\Gamma_{10}H) \\ * & \Xi_{22} & Z & A_d^T(G_{10}F+\Gamma_{10}H) \\ * & * & -G_{0\delta}^{-1}E - G_{1\delta}^{-1}F & 0 \\ * & * & * & -G_{10}F-\Gamma_{10}H \end{bmatrix} < 0,$$

with notations given by (31) and (44) and

$$\Xi_{11} = PA + A^TP + 2\delta P + \frac{1}{T}E + G - \frac{G_{00}^2}{G_{1\delta}}F - \frac{1}{\Gamma_{1\delta}}H,$$

$$\Xi_{12} = PA_d - \frac{1}{T}Q + A^TQ + 2\delta Q + \frac{1}{\Gamma_{1\delta}}H,$$

$$\Xi_{22} = Q^TA_d + A_d^TQ - \frac{2}{T}Z + 2\delta Z - \frac{1}{\Gamma_{0\delta}}G - \frac{1}{\Gamma_{1\delta}}H$$

are feasible. Then (7) is exponentially stable with the decay rate δ (with a small enough decay rate).

Remark 6. Note that also for $N \geq 2$, systems with gamma-distributed delay can be transformed to augmented systems with the discrete delay τ. Denoting the gamma-distributed kernel as $\Gamma_N(\theta)$ and employing the fact that $\dot{\Gamma}_N(\theta) = \frac{1}{T}\Gamma_{N-1}(\theta) - \frac{1}{T}\Gamma_N(\theta)$ one can arrive at the augmented system with respect to the state $\eta_N(t) = col\{x(t), \int_0^\infty \Gamma_N(\theta)x(t-\theta-\tau)d\theta, \ldots, \int_0^\infty \Gamma_1(\theta)x(t-\theta-\tau)d\theta\}$, and can formulate less conservative LMIs via correspondingly augmented V (but on account of computational complexity). Thus, for $N = 2$ denote $\xi(t) = \int_{-\infty}^{t} g(t-\theta)x(\theta-\tau)d\theta$. Then solutions of (7) satisfy the following system

$$\dot{x}(t) = Ax(t) + A_dy(t), \qquad \dot{y}(t) = -\frac{1}{T}y(t) + \xi(t),$$

$$\dot{\xi}(t) = \frac{1}{T^2}x(t-\tau) - \frac{1}{T}\xi(t)$$

that can be not asymptotically stable for singular A and A_d. Here V with $V_{aug} = \eta_2^T(t)\mathcal{P}\eta_2(t)$ leads to the following LMIs:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} & \Lambda_{15} \\ * & \Lambda_{22} & \Lambda_{23} & \frac{1}{T^2}Q_3 & \Lambda_{25} \\ * & * & \Lambda_{33} & \frac{1}{T^2}P_3 & 0 \\ * & * & * & \Lambda_{44} & 0 \\ * & * & * & * & \Lambda_{55} \end{bmatrix} < 0, \tag{45}$$

$$\mathcal{P} = \begin{bmatrix} P_1 & Q_1 & Q_2 \\ * & P_2 & Q_3 \\ * & * & P_3 \end{bmatrix} > 0$$

with notations (31) and (44) for $N = 2$ and

$$\begin{aligned} \Lambda_{11} &= P_1A + A^TP_1 + S - \frac{e^{-2\delta\tau}}{\tau}R + 2\delta P_1 \\ &\quad + G_{00}E + \Gamma_{00}G - \frac{G_{00}^2}{G_{1\delta}}F - \frac{\Gamma_{00}^2}{\Gamma_{1\delta}}H, \end{aligned}$$

$$\Lambda_{12} = P_1A_d + A^TQ_1 - \frac{1}{T}Q_1 + 2\delta Q_1 + \frac{\Gamma_{00}}{\Gamma_{1\delta}}H,$$

$$\Lambda_{13} = A^TQ_2 + Q_1 - \frac{1}{T}Q_2 + \frac{G_{00}}{G_{1\delta}}F + 2\delta Q_2,$$

$$\Lambda_{22} = Q_1^TA_d + A_d^TQ_1 - \frac{2}{T}P_2 + 2\delta P_2 - \frac{1}{\Gamma_{0\delta}}G - \frac{1}{\Gamma_{1\delta}}H,$$

$$\Lambda_{23} = A_d^TQ_2 + P_2 - \frac{2}{T}Q_3 + 2\delta Q_3,$$

$$\Lambda_{33} = -\frac{2}{T}P_3 + Q_3 + Q_3^T - \frac{1}{G_{0\delta}}E - \frac{1}{G_{1\delta}}F + 2\delta P_3,$$

$$\Lambda_{14} = \frac{1}{T^2}Q_2 + \frac{e^{-2\delta\tau}}{\tau}R, \qquad \Lambda_{44} = -e^{-2\delta\tau}\left(S + \frac{1}{\tau}R\right),$$

$$\Lambda_{55} = -(\tau R + \Gamma_{10}H + G_{10}F),$$

$$\Lambda_{15} = -A^T\Lambda_{55}, \qquad \Lambda_{25} = -A_d^T\Lambda_{55}.$$

Remark 7. LMIs of Propositions 1–3 (of Corollary 1) are affine in A and A_d ($A_{d1}, \ldots, A_{dm}$). Therefore, if the matrices reside in the uncertain polytope, one has to solve the LMIs in the vertices only.

4.3. About finite delays with the exponential kernels

Also for finite delays augmented functionals may treat the case where both A and A_0 are not Hurwitz. Consider (7) with $h < \infty$ and (for simplicity) with $\tau = 0$:

$$\dot{x}(t) = Ax(t) + A_d\int_0^h \Gamma(\theta)x(t-\theta)d\theta, \quad \Gamma(\theta) = \frac{1}{T}e^{-\frac{\theta}{T}}. \tag{46}$$

It is well-known (Ozbay et al., 2008; Verriest, 1999) that with the notation

$$y(t) = \int_0^h \Gamma(\theta)x(t-\theta)d\theta = \int_{t-h}^{t} \Gamma(t-s)x(s)ds$$

(46) is transformed to the following augmented system with a discrete delay:

$$\dot{x}(t) = Ax(t) + A_dy(t),$$

$$\dot{y}(t) = \frac{1}{T}[x(t) - y(t) - e^{-\frac{h}{T}}x(t-h)].$$

As for $h = \infty$, in the case of singular A and A_0 the transformed system may be not asymptotically stable, whereas (46) is exponentially stable.

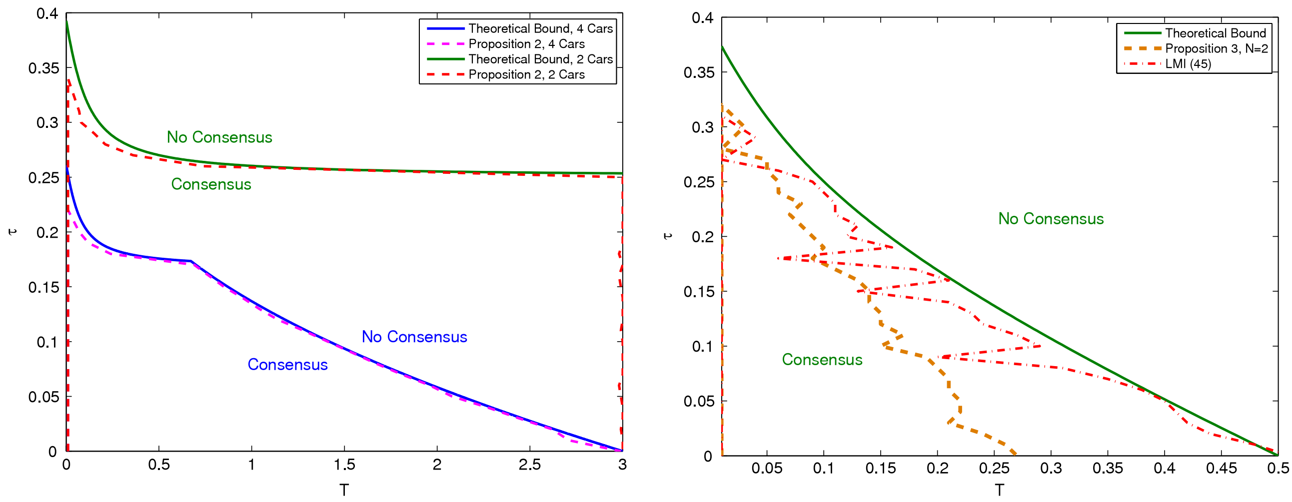
Here we suggest to choose $\bar{V}(t) = V_{aug}(t) + \bar{V}_G(t) + \bar{V}_H(t) + \bar{V}_R(t) + \bar{V}_S(t)$, where $V_{aug}(t)$ is given by (36) and

$$\bar{V}_G(t) = \int_0^h \int_{t-\theta}^{t} e^{-2\delta(t-s)}\Gamma(\theta)x^T(s)Gx(s)\,ds\,d\theta,$$

$$\bar{V}_H(t) = \int_0^h \int_0^\theta \int_{t-\lambda}^{t} e^{-2\delta(t-s)}\Gamma(\theta)\dot{x}^T(s)H\dot{x}(s)\,ds\,d\lambda\,d\theta,$$

$$\bar{V}_S(t) = \int_{t-h}^{t} e^{-2\delta(t-s)}x^T(s)Sx(s)ds,$$

$$\bar{V}_R(t) = \int_{-h}^{0}\int_{t+\theta}^{t} e^{-2\delta(t-s)}\dot{x}^T(s)R\dot{x}(s)dsd\theta \tag{47}$$



(a) 4 and 2 cars: $N = 1$.

(b) 2 cars: $N = 2$.

Fig. 1. Consensus region for the ring configuration.

with positive G, H, R and S. Denote

$$\Gamma^h_{00} = \int_0^h \Gamma(\theta)d\theta = 1 - e^{-\frac{h}{T}},$$

$$\Gamma^h_{0\delta} = \int_0^h e^{2\delta\theta}\Gamma(\theta)d\theta = \frac{e^{(2\delta-\frac{1}{T})h} - 1}{2\delta T - 1}$$

$$\Gamma^h_{10} = \int_0^h \Gamma(\theta)\theta d\theta = T - (h+T)e^{-\frac{h}{T}},$$

$$\Gamma^h_{1\delta} = \int_0^h e^{2\delta\theta}\Gamma(\theta)\theta d\theta = \frac{T}{(2\delta T - 1)^2} + \left(h - \frac{T}{2\delta T - 1}\right)\frac{e^{\left(2\delta-\frac{1}{T}\right)h}}{2\delta T - 1}.$$

By the above arguments we arrive at

Corollary 2. *Given $\delta > 0$ ($\delta = 0$), let there exist positive definite matrices $P, R, S, G, H, Z \in \mathbf{R}^{n\times n}$, and a matrix $Q \in \mathbf{R}^{n\times n}$ such that LMIs (37) and*

$$\begin{bmatrix} \bar\Psi_{11} & \bar\Psi_{12} & -\Gamma(h)Q + \frac{e^{-2\delta h}}{h}R & A^T[hR + \Gamma^h_{10}H] \\ * & \bar\Psi_{22} & -\Gamma(h)Z & A_d^T[hR + \Gamma^h_{10}H] \\ * & * & -e^{-2\delta h}S - \frac{e^{-2\delta h}}{h}R & 0 \\ * & * & * & -hR - \Gamma^h_{10}H \end{bmatrix} < 0,$$

where

$$\bar\Psi_{11} = PA + A^TP + \frac{1}{T}(Q + Q^T) + S + 2\delta P + \Gamma^h_{00}G - \frac{e^{-2\delta h}}{h}R - {\Gamma^h_{1\delta}}^{-1}{\Gamma^h_{00}}^2 H,$$

$$\bar\Psi_{12} = PA_d - \frac{1}{T}Q + A^TQ + \frac{1}{T}Z + 2\delta Q + {\Gamma^h_{1\delta}}^{-1}\Gamma^h_{00}H,$$

$$\bar\Psi_{22} = Q^TA_d + A_d^TQ - \frac{2}{T}Z + 2\delta Z - K_{0\delta}^{-1}G - {\Gamma^h_{1\delta}}^{-1}H$$

are feasible. Then (46) is exponentially stable with the decay rate δ (with a small enough decay rate).

4.4. Examples: traffic flow models on the ring

(a) We start with the first example in Michiels et al. (2009), where (5) was studied with $n = 4$, $N = 1$ and $\alpha_1 = \alpha_4 = 5$, $\alpha_2 = \alpha_3 = 1$. This leads to (7) with $x = col\{v_1, \ldots, v_4\}$, $A = 0$ and

$$A_d = \begin{bmatrix} -5 & 0 & 0 & 5 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & -5 \end{bmatrix}. \qquad (48)$$

Note that here A_0 defined by (11) coincides with A_d and it possesses the zero eigenvalue, i.e. we have the case of stabilizing delay. Using Proposition 2 with $\delta = 0$, we construct a curve which defines a region in the (T, τ) plane, where consensus is guaranteed. This is done by creating a grid in the (T, τ) plane with a resolution of 0.01 in each direction. Fig. 1(a) illustrates the resulting consensus region (below the dashed purple curve) guaranteed by Proposition 2, while the solid blue line represents the theoretical bound of Michiels et al. (2009) obtained in the frequency domain. It is clear that the two regions almost coincide.

Consider now (46) with a finite delay, where $A = 0$ and A_d is given by (48). Also here A_0 is not Hurwitz. Application of Corollary 2 leads to a larger stability region than the one for $h = \infty$. Therefore, for $h < \infty$ and for $h = \infty$, the delay is stabilizing.

(b) Following Morarescu et al. (2007), consider two cars on the ring with $A = 0$ and $A_d = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$. Also here the delay is stabilizing since $A_0 = A_d$ has the zero eigenvalue. Using Propositions 2 and 3 with $\delta = 0$, we construct curves which define regions in the (T, τ) plane, where consensus is guaranteed. For $N = 1$ Fig. 1(a) illustrates the consensus curve created using Proposition 2 (dashed red) in comparison with the theoretical bound found in Morarescu et al. (2007) (solid green). For $N = 2$ Fig. 1(b) illustrates the consensus curves created by using Proposition 3 (dashed orange) and the LMI (45) (dash–dot red) in comparison with the theoretical bounds found in Morarescu et al. (2007) (solid green). Also here our results for $N = 1$ are very close to analytical ones, whereas for $N = 2$ the results obtained by Proposition 2 are more conservative in comparison with the results obtained by (45).

(c) Consider now a modified traffic flow model on the ring

$$\dot v_k(t) = -\alpha_k v_k(t) + \alpha_k \int_0^\infty \Gamma(\theta)v_{k-1}(t-\theta-\tau)\,d\theta,$$

$$k = 1, \ldots, n, \qquad v_0 \equiv v_n,$$

where each driver has an instantaneous access to his state $v_k(t)$. The latter system can be presented as (7) with Hurwitz A and with

$A_0 = A + A_d$ having a zero eigenvalue. For the case of 2 cars on the ring we have

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}. \tag{49}$$

Note that here the transformed system (35) is not asymptotically stable for $\tau = 0$. For example, for $T = 0.01$ and $\tau = 0$ the eigenvalues of (35) are $0, -4.085, -97.914$ and -102. For $N = 1$ and $N = 2$, by Proposition 1 the system (7) with matrices (49) is exponentially stable for $(T, \tau) \in [0, 1000] \times [0, 1000]$. The same stability region is found for $N = 1$ and $N = 2$ in the modified model of 4 cars on the ring. Note that for $N = 2$ the augmented Lyapunov functionals slightly improve the results.

5. Conclusions

In the present paper, simple LMI conditions for the exponential stability of linear systems with infinite distributed delays have been presented. These systems are motivated by various applications in biology and engineering. Particularly in the traffic flow models on the ring, the gamma-distributed delay with a gap (which characterizes the human drivers' behavior on the average) is *stabilizing*. The latter means that the corresponding system with the zero-delay as well as the system without the delayed term are not asymptotically stable. In the numerical examples, our LMIs are feasible in the case of *stabilizing delay* leading to the results close to the theoretical ones (Michiels et al., 2009). Polytopic uncertainties in the system matrices can be easily included in the analysis. Extension of the presented direct Lyapunov method to nonlinear systems as well as different applications may be topics for future research.

References

Chen, W. H., & Zheng, W. X. (2007). Delay-dependent robust stabilization for uncertain neutral systems with distributed delays. *Automatica*, *43*, 95–104.

Cushing, J. M. (1981). In M. Iannalli (Ed.), *Volterra integro-differential equations in population dynamics, in mathematics of biology* (pp. 81–148). Naples: Ligouri Editore.

Fridman, E. (2001). New Lyapunov–Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems & Control Letters*, *43*, 309–319.

Fridman, E. (2002). Stability of linear descriptor systems with delay: a Lyapunov-based approach. *Journal of Mathematical Analysis and Applications*, *273*, 24–44.

Goebel, G., Munz, U., & Allgöwer, F. (2011). L_2-gain-based controller design for linear systems with distributed input delay. *IMA Journal of Mathematical Control and Information*, *28*(2), 225–237.

Gouaisbaut, F., & Ariba, Y. (2011). Delay range stability of a class of distributed time delay systems. *Systems & Control Letters*, *60*(3), 211–217.

Gu, K., Kharitonov, V., & Chen, J. (2003). *Stability of time-delay systems*. Boston: Birkhauser.

Kolmanovskii, V., & Myshkis, A. (1999). *Applied theory of functional differential equations*. Kluwer.

Michiels, W., Morarescu, C. I., & Niculescu, S. I. (2009). Consensus problems with distributed delays, with application to traffic flow models. *SIAM Journal on Control and Optimization*, *48*, 77–101.

Morarescu, C. I., Niculescu, S. I., & Gu, K. (2007). Stability crossing curves of shifted gamma-distributed delay systems. *SIAM Journal on Applied Dynamical Systems*, *6*(2), 475–493.

Ozbay, H., Bonnet, C., & Clairambault, J. (2008). Stability analysis of systems with distributed delays and application to hematopoietic cell maturation dynamics. In *47th IEEE conference on decision and control* (pp. 2050–2055). Cancun. 9–11 Dec.

Seuret, A., & Gouaisbaut, F. (2013). Wirtinger-based integral inequality: application to time-delay systems. *Automatica*, *49*.

Solomon, O., & Fridman, E. (2013). Systems with gamma-distributed delays: a Lyapunov-based analysis. In *52nd IEEE conference on decision and control*. Firenze. Dec.

Stepan, G. (1998). Delay-differential equation models for machine tool chatter. In F. C. Moon (Ed.), *Dynamics and chaos in manufacturing process* (pp. 165–192). New York: Wiley.

Sun, J., Liu, G. P., & Chen, J. (2009). Delay-dependent stability and stabilization of neutral time-delay systems. *International Journal of Robust and Nonlinear Control*, *19*, 1364–1375.

Verriest, E. (1995). Stability of systems with distributed delays. In *Proceedings of the IFAC workshop on system structure and control*. Nantes, France. July 5–7.

Verriest, E. (1999). Linear systems with rational distributed delay: reduction and stability, In *Proceedings of the European control conference, DA-1*, Karlsruhe, Germany. Sept.

Oren Solomon is finishing his M.Sc. studies at Electrical Engineering, Tel Aviv University, Israel. His research interests include stability and robust control of infinite-dimensional systems.

Emilia Fridman received the M.Sc. degree from Kuibyshev State University, USSR, in 1981 and the Ph.D. degree from Voronezh State University, USSR, in 1986, both in mathematics. From 1986 to 1992 she was an Assistant and Associate Professor in the Department of Mathematics at Kuibyshev Institute of Railway Engineers, USSR. Since 1993 she has been at Tel Aviv University, where she is currently Professor of Electrical Engineering-Systems. She has held visiting positions at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin (Germany), INRIA in Rocquencourt (France), Ecole Centrale de Lille (France), Valenciennes University (France), Leicester University (UK), Kent University (UK), CINVESTAV (Mexico), Zhejiang University (China), St. Petersburg IPM (Russia), Melbourne University (Australia).

Her research interests include time-delay systems, networked control systems, distributed parameter systems, robust control, singular perturbations and nonlinear control. She has published about 100 articles in international scientific journals. Seven of her papers have appeared in the list of Highly Cited Papers in ISI Web of Knowledge. Currently she serves as Associate Editor for Automatica, SIAM Journal on Control and Optimization and IMA Journal of Mathematical Control & Information.