1. Introduction

Systems with infinite distributed delays

\[ \dot{x}(t) = Ax(t) + A_1 \int_{0}^{h} K(\theta)x(t - \theta)d\theta, \quad (1) \]

where \( h = \infty, x(t) \in \mathbb{R}^n, A, A_1 \in \mathbb{R}^{n \times n} \) and \( K(\cdot) \) is a scalar, are present in many scientific disciplines such as population dynamics and engineering. One of the first studies devoted to population dynamics using a model with infinite distributed delay is due to Cushing (1981) (see also Kolmanovskii and Myshkis (1999) and references therein). Such delays appear in the control systems under PID controllers (Kolmanovskii & Myshkis, 1999), in the traffic flow models (Michiels et al., 2009; Morarescu, Niculescu, & Gu, 2007), in the control over communication networks (Morarescu et al., 2007) and in the machine tool vibration problem (Stepan, 1998). Particularly systems with distributed delays, whose kernel \( K(\cdot) \) is a gamma-distribution with a gap, model the exponential kernel with a gap, the above mentioned applications in the population dynamics and engineering (Michiels et al., 2009; Morarescu et al., 2007).

Stability and control of systems with the constant kernel \( K \equiv 1 \) and finite distributed delays \( h < \infty \) have been extensively studied in the literature (see e.g. Chen and Zheng (2007), Fridman (2001), Gu, Korvin, and Chen (2003), Kolmanovskii and Myshkis (1999) and Sun, Liu, and Chen (2009) and the references therein). In the early results based on the Lyapunov–Krasovskii method (Kolmanovskii & Myshkis, 1999), the delayed term was treated as the disturbance, where \( A \) was supposed to be Hurwitz. Later a less restrictive assumption on the stability of the corresponding system with the zero delay \( K(t) = (A + A_1t)x(t) \) was used (Chen & Zheng, 2007). In the case of piecewise-constant kernels and \( h < \infty \), the discretized Lyapunov functional method is applicable to systems with distributed stabilizing delays, where neither \( A \) nor \( A_1 \) are Hurwitz. Simple stability conditions for systems with stabilizing discrete delays were recently derived by using augmented Lyapunov functionals and Wirtinger-based integral inequalities (Seuret & Gouaisbaut, 2013).

The case of finite distributed delay with a variable kernel is less studied in the literature. For the finite distributed delay with an exponential kernel we refer to Verriest (1999), where the system was transformed to an augmented system with a discrete delay. In Ozban, Bonnet, and Clairambault (2008) the stability of systems...
with kernels that are finite duration impulse responses of finite dimensional systems were reduced to the stability of systems with discrete delays. Systems with polynomial kernels were studied in 
Gouaisbaut and Ariba (2011) by presenting them as interconnected models and using quadratic separation. For systems, where the matrix kernel \( K \) is a rational function, we refer to Goebel, Munz, and Allgöwer (2011) and the references therein.

The existing Lyapunov-based results treat the term with the infinite delay as a disturbance (Kolmanovskii & Myshkis, 1999). In the frequency domain, necessary and sufficient conditions for the stability of systems with gamma-distributed delays were found in Michiels et al. (2009) and Morarescu et al. (2007). Special attention was paid to the case of stabilizing delay, where a constant \( b \geq 1 \) such that the following exponential estimate holds for the solution of (2) initialized with \( \phi \in C(\mathbb{R}^n; 0, \mathbb{R}^n) \):

\[
|x(t)|^2 \leq b^{-2\gamma} \|\phi\|^2 \forall t \geq 0,
\]

where \( \|\phi\| = \|\phi\|_0 \). The objective of the present paper is to derive sufficient conditions for the exponential stability of (2). For \( \phi \in C^1(\mathbb{R}^n; 0, \mathbb{R}^n) \), we will find less restrictive exponential stability conditions, where \( \|\phi\| = \|\phi\|_1 \) in the exponential bound of (4).

2. Applications with gamma-distributed delays

Gamma-distributed delays can be encountered in the problem of control over communication networks, in the population dynamics (Morarescu et al., 2007) and in the machine tool vibration problem (Stepan, 1998). A research area where a distributed delay appears naturally is the traffic flow dynamics. The following system which incorporates a general memory effect represents a configuration of vehicles in a ring (Michiels et al., 2009):

\[
v_k(t) = a_k \int_0^\infty f(\theta) (v_{k-1}(t-\theta) - v_k(t-\theta)) d\theta,
\]

\( k = 1, \ldots, n \), \( v_0 = v_n \).

Here \( v_k(t) \) is the velocity of vehicle \( k \), \( n \) is the number of vehicles, and \( a_k \) is the coupling coefficient between the vehicles \( k \) and \( k-1 \). The equilibrium point of the matched speeds is referred to as a consensus among the agents, where no particular agent may collide with another. The asymmetric stability of (5) means that \( \lim_{t \to \infty} v_k(t) = 0 \), \( k = 1, \ldots, n \) and, thus, that \( \lim_{t \to \infty} \sum_{k=1}^n v_k(t)/n = 0 \). The latter relation corresponds to the consensus as defined in Michiels et al. (2009).

Function \( f \) in (5) is a gamma distribution with a gap \( \tau \), where the gap corresponds to the minimum reaction time of the humans with respect to some external stimuli:

\[
f(\xi) = \begin{cases} 0 & \xi < \tau, \\ \frac{\xi^{\alpha-1} e^{-\xi/\tau}}{\Gamma(N-1)!} & \xi \geq \tau. \end{cases}
\]

Here \( N \in \mathbb{N} \) is a parameter of the distribution. Note that \( \int_0^\infty f(\xi) d\xi = 1 \). The corresponding average delay of (6) satisfies

\[
\int_0^\infty f(\xi) \xi d\xi = \tau + NT.
\]

The distributed delay, whose kernel is a gamma distribution with a gap, characterizes the human drivers’ behavior in the average. Denoting \( x(t) = [v_1(t), \ldots, v_n(t)]^T \), (5) can be represented as (2), where \( K(\theta) = f(\theta) \) is piecewise-continuous and where \( A_{\gamma} \) and \( A_{\gamma} \) are constant matrices. Alternatively, taking into account that

\[
\int_0^\infty f(\xi) x(t-\theta) d\theta = \int_0^\infty f(\xi) x(t-\theta) d\theta
\]

we arrive at the equivalent representation

\[
\dot{x}(t) = A_{\gamma} x(t) + A_{\gamma} \int_0^\infty \Gamma(\theta) x(t-\theta - \tau) d\theta,
\]

where

\[
\Gamma(\theta) = \frac{\rho^{N-1} e^{-\theta/\tau}}{\Gamma(N-1)!}
\]

with a smooth kernel \( \Gamma \).
2.2. Problem formulation and useful inequalities

We will first derive the exponential stability conditions for a general integrable $K \in L_1(0, \infty; \mathbb{R})$ (see Section 3). Though linear systems with integrable kernels can always be presented in the form of (2) we will derive the stability condition for a more general system with a gap $r \geq 0$

$$\dot{x}(t) = Ax(t) + A_d \int_0^\infty K(\theta)(x(t + \theta - \tau) - x(t)) d\theta,$$  \hspace{1cm} (9)

where $K$ is subject to A1. This will allow us further to modify the results for the case of smooth kernels. Note also that our results for $K \in L_1(0, \infty; \mathbb{R})$ will be applicable to the case of finite delay $h < \infty$, where $K(\theta) = 0, \theta > h$.

Keeping in mind that (9) can be represented in the following form:

$$\dot{x}(t) = A_0 x(t) + A_d \int_0^\infty K(\theta) x(t + \theta - \tau) d\theta,$$  \hspace{1cm} (10)

we assume the exponential stability of $\dot{x}(t) = Ax(t)$ or of (10) with the zero delay

$$\dot{x}(t) = A_0 x(t), \quad A_0 \triangleq A + A_d \int_0^\infty K(\theta) d\theta,$$  \hspace{1cm} (11)

Note that in many cases a system with a matrix integrable kernel $K(\theta) \in \mathbb{R}^{m \times n}$ can be presented as a system with multiple delays

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^m A_{0i} \int_0^\infty K_i(\theta)x(t + \theta - \tau) d\theta,$$  \hspace{1cm} (12)

and scalar kernels $K_i \in L_1(0, \infty; \mathbb{R})$. This presentation is not unique and $m$ may be greater than $n^2$ (see Example 1). Our results can be easily extended to this case (as shown in Corollary 1).

For gamma-distributed delays with a gap we will further derive LMIs for the case of “stabilizing delays,” where $A$ and $A_0$ may be non-Hurwitz. This is motivated e.g. by the traffic flow model on the ring [Moraescu et al., 2007], where $A = 0$ and where the zero eigenvalue of $A_0$ corresponds to the vehicles moving with the same velocity.

As in the case of finite delays [Gu et al., 2003], a basic tool for the Lyapunov-based stability analysis will be integral inequalities. We will generalize the well-known Jensen’s inequality (Gu et al., 2003) and its double integral extension of Sun et al. [2009], which have been considered on the finite intervals of integration, to the infinite ones:

**Lemma 1.** Assume A1. Given an $n \times n$ matrix $R > 0$, a scalar function $\alpha : [0, \infty) \to (0, \infty)$, a scalar $\tau \geq 0$ and a vector function $\phi : [0, \infty) \to \mathbb{R}^n$ such that the integrations concerned are well defined. Then the following inequalities hold:

$$\int_0^\infty \alpha(\theta) |K(\theta)\phi'(\theta)R\phi(\theta) d\theta \geq K_0 - \int_0^\infty \phi^T(\theta)K(\theta)\phi(\theta) d\theta,$$  \hspace{1cm} (13)

$$K_0 = \int_0^\infty \alpha^{-1}(\theta)|K(\theta)|d\theta,$$

and

$$\int_0^\infty \int_{t-\theta-\tau}^t \alpha(\theta) |K(\theta)\phi^T(s)\phi(\theta) ds d\theta \geq K_1 - \int_0^\infty \phi^T(s)K(\theta)\phi(\theta) d\theta R \int_0^\infty |K(\theta)| d\theta,$$  \hspace{1cm} (14)

where $K_1 = \int_0^\infty \alpha^{-1}(\theta)|K(\theta)|(\theta + \tau) d\theta$.

**Proof.** By Schur complements the following holds

$$\begin{bmatrix} \alpha(\theta) |K(\theta)\phi^T(\theta)R\phi(\theta) & \phi^T(\theta)K(\theta) R^{-1}K(\theta)\alpha^{-1}(\theta) \end{bmatrix} \geq 0$$  \hspace{1cm} (15)

for $\theta \in [0, \infty]$. Integration of (15) from 0 to $\infty$ yields

$$\int_0^\infty \alpha(\theta)|K(\theta)|\phi^T(\theta)R\phi(\theta) d\theta (\phi^T(\theta)K(\theta) R^{-1}K(\theta)\alpha^{-1}(\theta)) \geq 0.$$

Application of Schur complements to the above matrix inequality leads to (13). Double integration of

$$\int_0^\infty \int_{t-\theta-\tau}^t |K(\theta)|\alpha^{-1}(\theta) ds d\theta = K_1$$

and application of Schur complements leads to (14).

**Remark 1.** Lemma 1 can be easily extended to the matrix $K$, where $\phi^T K$ in (13) and (14) should be changed by $\phi^T K^2$.

3. Stability in the case of integrable kernels

Consider (9), where $K \in L_1(0, \infty; \mathbb{R})$ is subject to A1. Assume that $A_0 = A + A_d \int_0^\infty K(\theta) d\theta$ is Hurwitz. For the exponential stability analysis of (9) with the decay rate $\delta < \delta_0$ we suggest the following Lyapunov–Krasovskii functional:

$$V(t) = V_P(t) + V_G(t) + V_H(t), \quad V_P(t) = x^T(t)Px(t)$$  \hspace{1cm} (16)

with

$$V_P(t) = \int_0^\infty \int_{t-\theta-\tau}^t e^{-2\delta\theta} |K(\theta)|x^T(s)Gx(s) ds d\theta,$$

$$V_H(t) = \int_0^\infty \int_{t-\theta-\tau}^t \int_{t-\theta-\tau}^s e^{-2\delta\theta} |K(\theta)|x^T(s)Hx(s)ds d\theta d\theta,$$

where $P, G$ and $H$ are positive $n \times n$-matrices. The term $V_P(t)$ with $\delta = 0$ extends the classical construction of Kolmanovskii and Myshkis (1999) to the vector case, where the exponential term is inserted to achieve the exponential decay rate $\delta$. It “compensates” the delayed term in (9) provided $A$ is Hurwitz. The term $V_H$ extends the triple integrals of Sun et al. (2009) to the case of infinite delay and it “compensates” the integral term in (10) when $A_0$ is Hurwitz. The latter term also improves the results when $A$ is Hurwitz.

Since $V$ depends on $\dot{x}$, it is defined for differentiable initial functions. We will derive the conditions that guarantee $V(t) + 2\delta V(t) \leq 0$ along the solutions of (9) with initial functions $\phi \in C^1(0, \infty; \mathbb{R}^n)$. Then these solutions would satisfy the following inequality:

$$x^T(t)Px(t) \leq V(t) \leq e^{-2\delta t}V(0), \quad t \geq 0,$$  \hspace{1cm} (17)

where for all $\delta \in (0, \delta_0)$

$$V(0) \leq \lambda_{\max}(P)|\phi(0)|^2 + \lambda_{\max}(G) \int_0^\infty |K(\theta)|(\theta + \tau) d\theta \|\phi\|_C$$

$$+ \lambda_{\max}(H) \int_0^\infty |K(\theta)|(\theta + \tau)^2/2d\theta \|\phi\|_C.$$  \hspace{1cm} (18)
Differentiation of $V$ along the trajectories of (9) yields
\[ \dot{V}(t) + 2\delta V(t) = 2x^T(t)P[\dot{x}(t) + A_2 \int_0^\infty K(\theta)x(t - \theta - \tau) d\theta ] + 2\delta x^T(t)Px(t) + \int_0^\infty [K(\theta)] \delta x^T(t)Gx(t) d\theta - 2\int_0^\infty e^{-2\delta(\theta + \tau)} [K(\theta)] x^T(t - \theta - \tau)Gx(t - \theta - \tau) d\theta + \int_0^\infty (\theta + \tau) [K(\theta)] \delta x^T(t)H\dot{x}(t) - \int_0^\infty \int_{t-\theta-\tau}^t e^{-2\delta(\theta + \tau)} [K(\theta)] x^T(s)H\dot{x}(s) ds d\theta. \]

Denote
\[ K_{0\delta} = \int_0^\infty e^{-2\delta(\theta + \tau)} [K(\theta)] d\theta, \quad K_{00} = K_{0\delta}|_{\delta = 0}, \]
\[ K_{1\delta} = \int_0^\infty e^{-2\delta(\theta + \tau)} [K(\theta)] e^{\theta + \tau} d\theta, \quad K_{10} = K_{1\delta}|_{\delta = 0}. \]

Applying further the extended Jensen's inequalities (13) and (14) we have
\[ -\int_0^\infty e^{-2\delta(\theta + \tau)} [K(\theta)] x^T(t - \theta - \tau)Gx(t - \theta - \tau) d\theta \leq -K_{0\delta}^{-1} \int_0^\infty K(\theta)x^T(t - \theta - \tau)Gx(t - \theta - \tau) d\theta \]
and
\[ -\int_0^\infty \int_{t-\theta-\tau}^t e^{-2\delta(\theta + \tau)} [K(\theta)] x^T(s)H\dot{x}(s) ds d\theta \leq -K_{1\delta}^{-1} \int_0^\infty \int_{t-\theta-\tau}^t K(\theta)x(s)Hd\theta d\theta. \]

Choose $\eta(t) = col(x(t), \int_0^\infty K(\theta)x(t - \theta - \tau) d\theta).$ Then
\[ \dot{V}(t) + 2\delta V(t) \leq \eta^T(t) \begin{bmatrix} \Phi_{00} & PA_d + K_{10}^{-1}K_{00}H \\ * & -K_{00}^{-1}G - K_{10}^{-1}H \end{bmatrix} \eta(t) + K_{10}\eta^T(t) \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} A \\ A_d \end{bmatrix} \eta(t), \]
where
\[ \Phi_{00} = PA + A^T P + 2\delta P + K_{00}G - K_{10}^{-1}K_{00}^2H. \]

By the Schur complements we find that if
\[ \begin{bmatrix} \Phi_{00} & PA_d + K_{10}^{-1}K_{00}H \\ * & -K_{00}^{-1}G - K_{10}^{-1}H - K_{10}^{-1}H \end{bmatrix} < 0, \]
then $\dot{V}(t) + 2\delta V(t) \leq 0.$ We have proved the following:

**Proposition 1.** Assume A1. Given $\delta \in (0, \delta_0) (\delta = 0)$, suppose there exist positive definite matrices $P, G, H \in \mathbb{R}^{n \times n}$, such that LMI (21) with notations given by (19) and (20) is feasible. Then solutions of (9) initiated by $\phi \in C^1(0, \infty; \mathbb{R}^n)$ satisfy the bound (17) and (18), i.e. (9) is exponentially stable with the decay rate $\delta$ (with a small enough decay rate).

**Remark 2.** Taking into account Remark 1, the LMI (21) guarantees the exponential stability of (9) with a matrix kernel $K$. However, numerical computation of the constants in (19) for the matrix case becomes complicated. That is why in Corollary 1 below we consider a particular but important case of matrix kernels, where the system can be presented as (12) with scalar kernels $K_i$.

**Remark 3.** For the asymptotic stability of (9), assumption A1 is not necessary. The feasibility of LMI (21) with $\delta = 0$ guarantees the asymptotic stability of (9) for $K \in L_1(0, \infty; \mathbb{R})$ such that $K_{00} < \infty$.

**Remark 4.** The effect of the distributed delay is reflected only as some integral in the stability conditions. Such conditions may be conservative ignoring more detailed delay distribution. For the case of gamma-distributed kernels (in Section 4 below) the derivative of the kernel will be employed to improve the results.

**Remark 5.** When $A$ is Hurwitz a simpler $V$ of (16) with $H = 0$ can be applied leading to the following LMI:
\[ \begin{bmatrix} PA + A^T P + 2\delta P + K_{00}G \\ * \end{bmatrix} A_d = -K_{00}^{-1}G < 0. \]

For all solutions of (9) initiated by $\phi \in C(0, \infty; \mathbb{R}^n)$, the latter LMI guarantees the exponential bound (17) and (18) with $H = 0$. It is easy to see that the feasibility of (22) with $\delta = 0$ yields the delay-independent stability of
\[ \dot{x}(t) = Ax(t) + K_{00}Ax(t - r) \]
for all $r \geq 0$ and implies that
1. $A$ and $A \pm K_{00}A_d$ are Hurwitz (i.e. $A_0$ is Hurwitz for $K \geq 0$);
2. the eigenvalues of $A^{-1}K_{00}A_d = A^{-1} \int_0^\infty |K(\theta)|d\theta$ are inside of the unit circle (Fridman, 2002);
3. $\|G^{0.5} |sI - A|^{-1}A_dG^{-0.5}\|_\infty < 1/K_{00}$ (the scaled small gain condition), where $\| \cdot \|_\infty$ denotes the $H_\infty$-norm.

For $\delta = 0$, $A_d = G = I$ and a matrix $K$ the LMI (22) is equivalent to $\| (sI - A)^{-1}A_dG^{-0.5}\|_\infty < 1/K_{00}$. The latter stability condition was derived in Ozbay et al. (2008) and Verriest (1995) for the case of finite delay.

**Proposition 1** can be easily extended to systems with multiple delays and scalar kernels:

**Corollary 1.** Consider (12). Assume that there exists $\delta_0 > 0$ such that $\int_0^\infty |K(\theta)|e^{\delta_0t} d\theta < \infty$, $i = 1, \ldots, m$ and that $A_0 = A + \sum_{i=1}^m A_d \int_0^\delta K_i(\theta)d\theta$ is Hurwitz. Given $\delta \in [0, \delta_0)$ ($\delta = 0$), suppose there exist positive definite matrices $P, G, H \in \mathbb{R}^{n \times n}$ such that the LMI
\[ \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \Phi_{0i} & \Phi_{0i+1} & \cdots & \Phi_{0m} \\ \Phi_{0i+1} & \Phi_{11} & \cdots & \Phi_{1i} & \Phi_{1i+1} & \cdots & \Phi_{1m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Phi_{0m} & \Phi_{1m} & \cdots & \Phi_{mm} & \Phi_{0m+1} & \cdots & \Phi_{02m} \\ \Phi_{0m+1} & \Phi_{1m+1} & \cdots & \Phi_{1m+1} & \Phi_{1m+2} & \cdots & \Phi_{12m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Phi_{02m} & \Phi_{12m} & \cdots & \Phi_{12m} & \Phi_{22m} & \cdots & \Phi_{22m} \end{bmatrix} < 0, \]
\[ \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \Phi_{0i} & \Phi_{0i+1} & \cdots & \Phi_{0m} \\ \Phi_{0i+1} & \Phi_{11} & \cdots & \Phi_{1i} & \Phi_{1i+1} & \cdots & \Phi_{1m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Phi_{0m} & \Phi_{1m} & \cdots & \Phi_{mm} & \Phi_{0m+1} & \cdots & \Phi_{02m} \\ \Phi_{0m+1} & \Phi_{1m+1} & \cdots & \Phi_{1m+1} & \Phi_{1m+2} & \cdots & \Phi_{12m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Phi_{02m} & \Phi_{12m} & \cdots & \Phi_{12m} & \Phi_{22m} & \cdots & \Phi_{22m} \end{bmatrix} < 0, \]
where
\[ K_{00}^i = \int_{0}^{\infty} e^{2\delta(\theta + t)} |K_0(\theta)| d\theta, \quad K_{00} = K_{00}^i|_{\delta = 0}, \]
\[ K_{11}^i = \int_{0}^{\infty} e^{2\delta(\theta + t)} |K_1(\theta)| (\theta + t) d\theta, \quad K_{11} = K_{11}^i|_{\delta = 0}, \]
\[ \Phi_{00} = PA + A^T P + 2\delta P + \sum_{i=1}^{m} \left( K_{0i}^H G_i - (K_{1i}^H)^{-1} (K_{00}^i)^H H_i \right), \]
\[ \Phi_{0i} = PA_i + (K_{1i}^H)^{-1} K_{0i} H_i, \]
\[ \Phi_{ii} = - (K_{0i}^H)^{-1} G_i - (K_{1i}^H)^{-1} H_i. \]

Then (12) is exponentially stable with the decay rate $\delta$ (with a small enough decay rate).

**Example 1** (Gouaisbaut & Ariba, 2011). The system with non-Hurwitz $A$ and with a matrix variable kernel
\[ \dot{x}(t) = Ax(t) + \int_{-\infty}^{\infty} \tilde{K}(s)x(t + s) ds, \]
\[ A = \begin{bmatrix} 0.2 & 0.01 \\ 0 & -2 \end{bmatrix}, \quad \tilde{K}(s) = \begin{bmatrix} -1 + 0.3s & 0.1 \\ 0 & -1 \end{bmatrix}, \]
can be presented in the form of (12) with $m = 2$, $\tau = 0$ and $K_1 = K_2 = 0$ for $\theta > h$ in many ways. Note that \[ \int_{-\infty}^{\infty} \tilde{K}(s)(t + s) ds = \int_{0}^{h} \tilde{K}(-\theta)(t - \theta) d\theta. \] We choose the following two forms:
\[ A_{01} = \begin{bmatrix} -1 & 0.1 \\ 0 & -1 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0 \end{bmatrix}, \]
\[ K_1(t) = \begin{bmatrix} \theta \\ \theta \end{bmatrix}, \quad \theta \in [0, h], \]
and
\[ A_{21} = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \]
\[ K_1(t) = \begin{bmatrix} \theta \\ \theta \end{bmatrix}, \quad \theta \in [0, h]. \]

Here $A_0 = A + \sum_{i=0}^{\infty} A_i \int_{0}^{\infty} K_0(\theta) d\theta$ is Hurwitz for $h > 0.195$. It was found in Gouaisbaut and Ariba (2011) using an analytical method that the system is asymptotically stable for $h \in [0.195, 1.71]$. The LMI of Corollary 1 with 15 scalar variables guarantee the exponential stability of (25) for $h \in [0.207, 1.455]$ and of (26) for $h \in [0.195, 1.442]$. Thus, (24) is exponentially stable for [0.195, 1.455].

The latter essentially improves the one of Gouaisbaut and Ariba (2011), where a smaller stability interval $h \in [0.195, 1.25]$ was found via LMI with 2221 decision variables. Choosing next $h = 1$ we find by using LMI of Corollary 1 with 15 scalar variables that the maximum achievable decay rate is $\delta_{\text{max}} = 0.433$ for (25) and $\delta_{\text{max}} = 0.593$ for (26). Hence, (24) is exponentially stable with the decay rate 0.593.

Note that (24) has a triangular structure. Thus, it is stable if the two scalar subsystems
\[ \dot{x}_1(t) = 0.2x_1(t) - \int_{-h}^{0} (1 - 0.3s)x_1(t + s) ds, \]
\[ \dot{x}_2(t) = -2x_2(t) - 0.1 \int_{-h}^{0} x_2(t + s) ds \]
are stable. In correspondence to (25) and (26), the system for $x_1$ can be presented as a system with 2 delays, where $A_{01} = -1 = -K_0(\theta)$ and $A_{21} = -0.3$, $K_1(\theta) = \theta$, or as a system with 1 delay, where $A_d = -1$ and $K(\theta) = 1 + 0.3\theta (\theta \in [0, h])$. The LMI of Corollary 1 with 5 scalar variables and with 3 scalar variables guarantee the exponential stability of the latter systems for $h \in [0.195, 1.455]$ and for $h \in [0.195, 1.442]$ respectively.

Another simple extension of the direct Lyapunov–Krasovskii method is to input-to-state stability or to $L_2$-gain analysis of the perturbed systems. Consider
\[ \dot{x}(t) = Ax(t) + A_d \int_{0}^{\infty} K(\theta)x(t - \theta - t) d\theta + Bw(t), \]
\[ z(t) = Cx(t), \]
where $w(t) \in R^{2\nu}$ is the disturbance, $z(t) \in R^{2\nu}$ is the controlled output, and $B$ and $C$ are constant matrices. System (27) is said to have an $L_2$-gain less than $\gamma > 0$ if
\[ J = \int_{0}^{\infty} \left( z(t)^T z(t) - \gamma^2 w(t)^T w(t) \right) dt < 0 \]
for all $0 \neq w \in L_2(0, \infty; R^{2\nu})$ and the zero initial condition. By the standard arguments, $J < 0$ if
\[ \begin{bmatrix} \Phi_{00} & C^T C & P A_d + K_{10}^{-1} G_0 H & P B & K_{10} A_d^T H \\ \ast & -K_{00} G - K_{10}^{-1} H & 0 & K_{10} A_d^T H & \ast \\ \ast & \ast & -\gamma^2 I & K_{10} B^T H & \ast \\ \ast & \ast & \ast & -K_{10} H & \ast \end{bmatrix} < 0, \]
where $\Phi_{00}$ is given by (20).

**Example 2** (Goebel et al., 2011). Consider (27), where
\[ A = 0.8, \quad A_d = -41.8, \quad B = 2, \quad C = 1 \]
and $K(\theta) = 0$ for $\theta > h$, whereas $K(\theta) = \frac{3 + 2\theta + 7\theta^2}{2 - 2\theta + 8\theta^2} e^{-10\theta}$ for $\theta \in [0, h]$. Here $A_0 < 0$ for $h > 0.011659$. For $h = 0.1$ the smallest $L_2$-gain achieved in Goebel et al. (2011) was 0.76, whereas by using the LMI (29) a much smaller $\gamma_{\text{min}} = 0.3223$ is guaranteed.

Consider next (27) and (30) with infinite delay, where a modified kernel $K \in L_1(0, \infty; R)$ is given by $K(\theta) = \frac{3 + 2\theta + 7\theta^2}{2 - 2\theta + 8\theta^2} e^{-10\theta}$ for $\theta \in [0, \infty)$ and where $A_0 < 0$. By using the LMI (29), we found $\gamma_{\text{min}} = 0.41$.

4. **Systems with gamma-distributed delays**

Consider now (7) and (8). Here neither $A$ nor $A_d$ are supposed to be Hurwitz. The results will be derived by augmented $V$. Simple computation shows that for $K(\theta) = \Gamma(\theta)$ the constants in (19) have the form:
\[ \Gamma_{00} \leq K_{00}|_{\delta = 0} = \int_{0}^{\infty} e^{2\delta(\theta + t)} \Gamma(\theta) d\theta \leq \frac{e^{2\delta t}}{(1 - 2\delta T)^{\delta}}, \]
\[ \Gamma_{ii} \leq K_{ii}|_{\delta = 0} = \int_{0}^{\infty} e^{2\delta(\theta + t)} \Gamma(\theta)(\theta + \tau) d\theta \leq \frac{e^{2\delta t}}{(1 - 2\delta T)^{\delta}}, \quad \Gamma_{10} = \Gamma = T, \]
\[ \Gamma_{00} = 1. \]

Note also that
\[ \int_{0}^{\infty} \Gamma(\theta)x(t - \theta - t) d\theta = \int_{-\infty}^{t} \Gamma(t - s)x(s - t) ds. \]

4.1. Gamma-distributed delay with a gap and $N = 1$

Consider first $\Gamma(\theta) = e^{-\frac{\theta}{T}}/T$, where
\[ \Gamma(0) = 1/T, \quad \Gamma'(\theta) = -1/T \cdot \Gamma(\theta). \]
Following Özbay et al. (2008) and Verriest (1999) and denoting
\[ y(t) = \int_{-\infty}^{t} \Gamma(t - s)x(s - t) ds, \]
the system (7) can be transformed to the following augmented one:
\[
\dot{x}(t) = Ax(t) + A_0 y(t), \quad \dot{y}(t) = \frac{1}{\tau} x(t - \tau) - \frac{1}{\tau} y(t).
\] (35)

The stability of (35) implies the stability of (7), but not vice versa. This is similar to model transformations of systems with discrete delays (Gu et al., 2003), where the transformed system has additional dynamics. Thus, in the traffic flow models on the ring studied in Michaels et al. (2009), where \( A = 0 \) and \( A_0 \) has a zero eigenvalue, the system matrix of (35) for \( \tau = 0 \) has the zero eigenvalue. However, it was shown in Michaels et al. (2009) that (7) may be asymptotically stable. Note that in this case \( A \) and \( A_0 = A + A_0 \int_0^\infty \Gamma(\theta)d\theta \) are not Hurwitz, implying that the results of the previous section are not applicable.

We are interested in the latter situation, where \( V_{\text{aug}}(t) = [x^T(t) \ y^T(t)] \left[ \begin{array}{cc} P & Q \\ Z & \gamma(t) \end{array} \right] \) (36) subject to
\[
\begin{array}{cc}
P & Q \\ Z & \gamma(t)
\end{array} > 0
\] (37)
cannot be used for the exponential stability analysis of (35) with \( \tau = 0 \).

For the exponential stability analysis of (7) we suggest the augmented Lyapunov functional:
\[
V(t) = V_{\text{aug}}(t) + V_C(t) + V_{\text{hc}}(t) + V_{\text{fc}}(t)
\]
\[
V_C(t) = \int_0^t \int_{t-\tau-\tau}^{t-\tau} e^{-2\delta(t-\tau)} \Gamma(\theta) x^T(s) G x(s) ds d\theta,
\]
\[
V_{\text{hc}}(t) = \int_0^t \int_{t-\tau}^{t-\tau} e^{-2\delta(t-\tau)} \Gamma(\theta) \dot{k}^T(s) H \dot{k}(s) ds d\lambda d\theta,
\]
\[
V_{\text{fc}}(t) = \int_0^t \int_{t-\tau}^{t} e^{-2\delta(t-\tau)} \dot{x}(s) R \dot{x}(s) ds d\theta,
\]
where \( G > 0, H > 0, R > 0, S > 0 \).

Denote \( \eta(t) = \text{col}(x(t), y(t), x(t - \tau)) \). Differentiating \( V_{\text{aug}} \) along (7) and using (35) we obtain
\[
V_{\text{aug}}(t) = 2[x^T(t) \ y^T(t)] \left[ \begin{array}{cc} P & Q \\ Z & \gamma(t) \end{array} \right] \dot{x}(t)
\]
\[
= 2x^T(t) \left[ \begin{array}{ccc} PA & P A_0 - \frac{1}{\tau} Q & \frac{1}{\tau} Q \\ Q^T A & Q^T A_0 - \frac{1}{\tau} Z & -\frac{1}{\tau} Z \\ 0 & 0 & 0 \end{array} \right] \eta(t).
\]

By applying Jensen’s inequalities (13) and (14) and taking into account (31) with \( N = 1 \) we find further
\[
\dot{V}_C(t) + \dot{V}_{\text{hc}}(t) + 2\delta \dot{V}_{\text{fc}}(t) + V_{\text{fc}}(t) \\
\leq x^T(t)[S + G]x(t) - e^{-2\delta t} x^T(t - \tau) SX(t - \tau) \\
- \frac{1}{\tau} \int_0^\infty \Gamma(\theta) x^T(t - \theta - \tau) d\theta G \\
- \frac{1}{\tau} \int_0^\infty \Gamma(\theta) x^T(t - \theta - \tau) d\theta,
\]
\[
\dot{V}_{\text{fc}}(t) + \dot{V}_{\text{hc}}(t) + 2\delta \dot{V}_{\text{fc}}(t) + V_{\text{fc}}(t) \\
\leq x^T(t)[ \Gamma R + (\tau + T) H] \dot{k}(t) \\
- \frac{1}{\tau} \int_0^\infty \dot{k}(s) ds R \int_0^\infty \dot{k}(s) ds.
\]

Hence \( \dot{V}(t) + 2\delta V(t) \leq 0 \) if
\[
\begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} \\
\Psi_{12}^T & (-\frac{D}{\tau})^T H & 0 \\
\Psi_{13} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13}
\end{bmatrix}
\leq 0,
\]
(39)

where
\[
\begin{array}{c}
\Psi_{11} = PA + A_0 R + 2\delta P + S + G - \frac{e^{2\delta t}}{\tau} R - \Gamma x^T \Gamma^T, \\
\Psi_{12} = PA_0 - \frac{1}{\tau} Q + A_0 Q + \Gamma x^T \Gamma^T + 12 Q, \\
\Psi_{13} = Q^T A_0 + A_0^T R + 2 - \frac{1}{\tau} Z - \Gamma x^T \Gamma^T + 12 Q + 2\delta Z.
\end{array}
\]

For the case of \( \tau = 0 \), we use \( V(t) = V_{\text{aug}}(t) + V_C(t) + V_{\text{fc}}(t) \), where \( V_{\text{aug}}(t), V_C(t) \) and \( V_{\text{fc}}(t) \) are as in (38), but with \( \tau = 0 \). The corresponding LMI has a form
\[
\begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} \\
\Psi_{12}^T & (-\frac{D}{\tau})^T H & 0 \\
\Psi_{13} & 0 & 0
\end{bmatrix}
\leq 0,
\]
(41)

where
\[
\begin{array}{c}
\Psi_{11} = PA + A_0 R + 2\delta P + S + G - \frac{1}{\tau} Q + \Gamma x^T \Gamma^T, \\
\Psi_{12} = PA_0 - \frac{1}{\tau} Q + A_0 Q + \Gamma x^T \Gamma^T + 12 Q, \\
\Psi_{13} = Q^T A_0 + A_0^T R + 2 - \frac{1}{\tau} Z - \Gamma x^T \Gamma^T + 12 Q + 2\delta Z.
\end{array}
\]

We have proved the following:

**Proposition 2.** System (7) with gamma-distributed kernel (8), where \( N = 1 \), is exponentially stable with the decay rate \( \delta \in (0, 1/T) \) if
\( \frac{r}{(x > 0)} \) there exist positive definite matrices \( P, R, S, G, H, Z \in \mathbb{R}^{n, n} \), and a matrix \( Q \in \mathbb{R}^{n, T} \) such that LMIs (37) and (39) with notations given by (31) and (40) are feasible;
\( \frac{r}{(x = 0)} \) there exist positive definite matrices \( P, R, S, G, H, Z \in \mathbb{R}^{n, n} \), and a matrix \( Q \in \mathbb{R}^{n, T} \) such that LMIs (37) and (41) with notations given by (31) and (42) are feasible.

Moreover, if the above LMIs are feasible with \( \delta = 0 \), then (7) is exponentially stable with a small enough decay rate.

4.2. Gamma-distributed delay with a gap and \( N \geq 2 \)

We have
\[
\Gamma(0) = 0, \quad \Gamma(\theta) = -1/T \cdot \Gamma(\theta) + g(\theta).
\]
\[
g(\theta) = \frac{g^{n-2}e^{-\rho \theta}}{T^N(N-2)!}.
\]
(43)

Choose \( V(t) = V_{\text{aug}}(t) + V_C(t) + V_{\text{hc}}(t) + V_{\text{fc}}(t) \), where \( V_{\text{aug}} \) is given by (36), and \( V_C(t) \) and \( V_{\text{fc}}(t) \) are defined in (38). Here the last two terms
\[
V_{\text{hc}}(t) = \int_0^\infty \int_{t-\tau}^{t} e^{-2\delta(t-\tau)} g(\theta) x^T(s) E x(s) ds d\theta,
\]
\[
V_{\text{fc}}(t) = \int_0^\infty \int_{t-\tau}^{t} e^{-2\delta(t-\tau)} g(\theta) x^T(s) F x(s) ds d\theta.
\]
with $E > 0$ and $F > 0$ are added to “compensate” the integral term $\int^t_\infty g(t-\theta)x^2(\theta-\tau)\,d\theta$ in $V_{aug}$. Similar to (31) we find

$$G_{01} = K_{01}\|x-g\| = \int^\infty_0 e^{2\delta(t-\tau)}g(\theta)\,d\theta = 1$$

$$G_{11} = K_{11}\|x-g\| = e^{2\delta t}\frac{N}{(1 + \frac{1}{2}(1 - 2\delta T))},$$

$$G_{00} = \frac{0}{1}, \quad G_{10} = N - 1 + \frac{\tau}{T}.$$  (44)

By arguments of Proposition 2 we arrive at

**Proposition 3.** Given $\delta \in (0, 1/T)$ ($\delta = 0$), let there exist positive definite matrices $P, G, H, E, F, Z \in \mathbb{R}^{n \times n}$ and a matrix $Q \in \mathbb{R}^{n \times n}$ such that LMs (37) and

$$\begin{bmatrix}
\mathcal{E}_{11} & \mathcal{E}_{12} & Q + G_{00}\Gamma_{10}F + \Gamma_{10}H \\
\mathcal{E}_{21} & Z & A_f^T(G_{10}F + \Gamma_{10}H)
\end{bmatrix}
\begin{bmatrix}
\mathcal{E}_{22}
\end{bmatrix}
< 0,$$

with notations given by (31) and (44) and

$$\mathcal{E}_{11} = PA + A^TP + 2\delta P + \frac{1}{T}E + G - \frac{G_{00}}{G_{10}}F - \frac{1}{G_{10}}H,$$

$$\mathcal{E}_{12} = PA_d - \frac{1}{T}Q + A_d^TQ + 2\delta Q + \frac{1}{G_{10}}H,$$

$$\mathcal{E}_{22} = Q^TA_d + A_d^TQ - \frac{2}{T}Z + 2\delta Z - \frac{1}{G_{10}}G - \frac{1}{G_{10}}H$$

are feasible. Then (7) is exponentially stable with the decay rate $\delta$ (with a small enough decay rate).

**Remark 6.** Note that also for $N \geq 2$, systems with gamma-distributed delay can be transformed to augmented systems with the discrete delay $\tau$. Denoting the gamma-distributed kernel as $\Gamma_{\tau}(\theta)$ and employing the fact that $\Gamma_{\tau}(\theta) = \frac{1}{\Gamma_{\tau} \cdot (\theta) - \frac{1}{\Gamma_{\tau} \cdot (\theta)}}$ one can arrive at the augmented system with respect to the state $\eta_{\tau}(t) = \text{col}(\bar{\theta}(t), \int_0^t \bar{\theta}(\theta)\,d\theta, \ldots, \int_0^t \bar{\theta}(\theta)\,d\theta, \ldots, \int_0^t \bar{\theta}(\theta)\,d\theta)$, and can formulate less conservative LMs via correspondingly augmented $V$ (but on account of computational complexity). Thus, for $N = 2$ denote $\xi(t) = \int^\infty_0 g(t-\theta)x(t-\theta)\,d\theta$. Then solutions of (7) satisfy the following system

$$\dot{x}(t) = Ax(t) + Ay(t), \quad \dot{y}(t) = -\frac{1}{T}y(t) + \xi(t),$$

$$\dot{\xi}(t) = \frac{1}{T^2}x(t-\tau) - \frac{1}{T}\xi(t)$$

that can be not asymptotically stable for singular $A$ and $A_d$. Here $V$ with $V_{aug} = \eta_{\tau}^T(t)^TP\eta_{\tau}(t)$ leads to the following LMs:

$$\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\
* & A_{22} & A_{23} & 1 & 0 \\
* & 1 & 0 & 0 & 0 \\
* & * & * & 0 & 0 \\
* & * & 0 & 0 & 0 \\
\end{bmatrix}
< 0,$$

$$\mathcal{S} = \begin{bmatrix}
P_1 & Q_1 & Q_2 \\
P_2 & P_2 & Q_3 \\
P_3 & * & P_3 \\
\end{bmatrix}
> 0$$  (45)

with notations $\mathcal{S}$ and (44) for $N = 2$ and

$$A_{11} = P_1A + A^T P_1 + S - e^{2\delta T}R + 2\delta P_1,$$

$$A_{12} = P_1A_d + A_d^T Q_1 - \frac{1}{T}Q_0 + 2\delta Q_1 + \frac{1}{G_{10}}H,$$

$$A_{13} = A_d^T Q_d + Q_d^T + \frac{1}{G_{10}}F + 2\delta Q_d,$$

$$A_{22} = Q_d^TA_d + A_d^T Q_1 - \frac{2}{T}P_2 + 2\delta P_2 - \frac{1}{G_{00}}G - \frac{1}{G_{10}}H,$$

$$A_{23} = A_d^T Q_d + P_2 - \frac{2}{T}Q_3 + 2\delta Q_3,$$

$$A_{33} = -\frac{2}{T}P_3 + Q_3 + Q_4^T + \frac{1}{G_{00}}E - \frac{1}{G_{10}}F + 2\delta P_3,$$

$$A_{34} = \frac{1}{T}Q_2 + e^{-2\delta T}R, \quad A_{44} = -e^{-2\delta T}\left(S + \frac{1}{T}R\right).$$

**Remark 7.** LMs of Propositions 1–3 (of Corollary 1) are affine in $A$ and $A_d$ ($A_{a1}, \ldots, A_{an}$). Therefore, if the matrices reside in the uncertain polytope, one has to solve the LMs in the vertices only.

**4.3. About finite delays with the exponential kernels**

Also for finite delays augmented functionals may treat the case where both $A$ and $A_d$ are Hurwitz. Consider (7) with $h < \infty$ and (for simplicity) with $\tau = 0$:

$$\dot{x}(t) = Ax(t) + Ay(t), \quad \dot{y}(t) = -\frac{1}{T}y(t) + \xi(t).$$

It is well-known (Ozbay et al., 2008; Verriest, 1999) that with the notation

$$y(t) = \int^t_0 \Gamma(t-\theta)x(t-\theta)\,d\theta = \int^t_0 \Gamma(t-\theta)x(t-\theta)\,d\theta$$

(46) is transformed to the following augmented system with a discrete delay:

$$\dot{x}(t) = Ax(t) + Ay(t), \quad \dot{y}(t) = \frac{1}{T}[x(t) - y(t) - e^{\frac{1}{T}}x(-h)].$$

As for $h = \infty$, in the case of singular $A$ and $A_d$ the transformed system may be not asymptotically stable, whereas (46) is exponentially stable.

Here we suggest to choose $\bar{V}(t) = V_{aug}(t) + \bar{V}_c(t) = \bar{V}_h(t) + \bar{V}_k(t)$, where $V_{aug}(t)$ is given by (36) and

$$\bar{V}_c(t) = \int^t_0 \int^t_0 e^{-2\delta(t-s)}\Gamma(t-x)\bar{G}(s)\mathbf{x}(s)\,ds\,dt,$$

$$\bar{V}_h(t) = \int^t_0 \int^t_0 e^{-2\delta(t-s)}\Gamma(t-x)\bar{Q}(s)\mathbf{x}(s)\,ds\,dt,$$

$$\bar{V}_k(t) = \int^t_0 \int^t_0 e^{-2\delta(t-s)}X^T(s)\mathbf{x}(s)\,ds\,dt$$

(47)
with positive $G$, $H$, $R$ and $S$. Denote

$$R^h_{00} = \int_0^h \Gamma^h(\theta) d\theta = 1 - e^{-\frac{\theta}{T}},$$

$$R^h_{0\delta} = \int_0^h e^{2\delta \theta} \Gamma^h(\theta) d\theta = \frac{e^{(2\theta - 1)h} - 1}{2\delta - 1},$$

$$R^h_{10} = \int_0^h \Gamma^h(\theta) d\theta = T - (h + T)e^{-\frac{h}{T}},$$

$$R^h_{1\delta} = \int_0^h e^{2\delta \theta} \Gamma^h(\theta) d\theta = \frac{T}{(2\delta - 1)^2} + \left( h - \frac{T}{2\delta - 1} \right) e^{(2\theta - 1)h}.$$

By the above arguments we arrive at

**Corollary 2.** Given $\delta > 0$ ($\delta = 0$), let there exist positive definite matrices $P$, $R$, $S$, $G$, $H$, $Z$ in $\mathbb{R}^{n \times n}$, and a matrix $Q$ in $\mathbb{R}^{m \times n}$ such that LMIs (37) and

$$\begin{bmatrix}
\hat{\psi}_{11} & \hat{\psi}_{12} & -\Gamma(h)Q + \frac{e^{-2\delta h}}{R} A^T_1[1 + \Gamma^h_{10} - 1] \\
\hat{\psi}_{21} & \hat{\psi}_{22} & -\Gamma(h)Z - \frac{e^{-2\delta h}}{R} A^T_2[1 + \Gamma^h_{10} - 1] \\
\hat{\psi}_{31} & \hat{\psi}_{32} & -e^{-2\delta h} S - \frac{e^{-2\delta h}}{R} A^T_3[1 + \Gamma^h_{10} - 1] \\
\hat{\psi}_{41} & \hat{\psi}_{42} & -e^{-2\delta h} R - \frac{e^{-2\delta h}}{R} \Gamma^h_{10} [1 + \Gamma^h_{10} - 1]
\end{bmatrix} < 0,$$

where

$$\hat{\psi}_{11} = PA + A^T P + \frac{1}{T}(Q + Q^T) + S + 2\delta P + \Gamma^h_{00} G - \frac{e^{-2\delta h}}{h} R - \frac{e^{-2\delta h}}{R} \Gamma^h_{10} H,$$

$$\hat{\psi}_{12} = PA_d - \frac{1}{T} Q + A^T Q + \frac{1}{T} Z + 2\delta Q + \Gamma^h_{10} - \Gamma^h_{10} H,$$

$$\hat{\psi}_{22} = Q^T A_d + A^T d - \frac{2}{T} Z + 2\delta Z - \Gamma^h_{00} G + \Gamma^h_{10} H,$$

are feasible. Then (46) is exponentially stable with the decay rate $\delta$ (with a small enough decay rate).

### 4.4. Examples: traffic flow models on the ring

(a) We start with the first example in Michiels et al. (2009), where (5) was studied with $n = 4$, $N = 1$ and $a_1 = a_4 = 5$.

$$\alpha_2 = \alpha_3 = 1.$$ This leads to (7) with $x = col[v_1, \ldots, v_4], A = 0$ and

$$A_d = \begin{bmatrix}
-5 & 0 & 0 & 5 \\
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 5 & -5
\end{bmatrix}.$$ (48)

Note that here $A_d$ defined by (11) coincides with $A_d$ and it possesses the zero eigenvalue, i.e. we have the case of stabilizing delay. Using Proposition 2 with $\delta = 0$, we construct a curve which defines a region in the $(T, \tau)$ plane, where consensus is guaranteed. This is done by creating a grid in the $(T, \tau)$ plane with a resolution of 0.01 in each direction. Fig. 1(a) illustrates the resulting consensus region (below the dashed purple curve) guaranteed by Proposition 2, while the solid blue line represents the theoretical bound of Michiels et al. (2009) obtained in the frequency domain. It is clear that the two regions almost coincide.

Consider now (46) with a finite delay, where $A = 0$ and $A_d$ is given by (48). Also here $A_d$ is not Hurwitz. Application of Corollary 2 leads to a larger stability region than the one for $h = \infty$. Therefore, for $h < \infty$ and for $h = \infty$, the delay is stabilizing.

(b) Following Morarescu et al. (2007), consider two cars on the ring with $A = 0$ and $A_d = \begin{bmatrix}
-1 & -2 \\
2 & -2
\end{bmatrix}$. Also here the delay is stabilizing since $A_d = A_d$ has the zero eigenvalue. Using Propositions 2 and 3 with $\delta = 0$, we construct curves which define regions in the $(T, \tau)$ plane, where consensus is guaranteed. For $N = 1$ Fig. 1(a) illustrates the consensus curve created using Proposition 2 (dashed red) in comparison with the theoretical bound found in Morarescu et al. (2007) (solid green). For $N = 2$ Fig. 1(b) illustrates the consensus curves created by using Proposition 3 (dashed orange) and the LMI (45) (dashed–dot red) in comparison with the theoretical bounds found in Morarescu et al. (2007) (solid green). Also here our results for $N = 1$ are very close to analytical ones, whereas for $N = 2$ the results obtained by Proposition 2 are more conservative in comparison with the results obtained by (45).

(c) Consider now a modified traffic flow model on the ring

$$\dot{v}_k(t) = -a_k v_k(t) + a_k \int_0^\infty \Gamma(\theta) v_{k-1}(t - \theta - \tau) d\theta,$$

$k = 1, \ldots, n, \quad v_0 \equiv v_n,$

where each driver has an instantaneous access to his state $v_k(t).$ The latter system can be presented as (7) with Hurwitz $A$ and with
\( A_0 = A + A_\tau \) having a zero eigenvalue. For the case of 2 cars on the ring we have

\[
A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.
\]  (49)

Note that here the transformed system (35) is not asymptotically stable for \( \tau = 0 \). For example, for \( T = 0.01 \) and \( \tau = 0 \) the eigenvalues of (35) are \(-0.4085, -97914\) and \(-102\). For \( N = 1 \) and \( N = 2 \), by Proposition 1 the system (7) with matrices (49) is exponentially stable for \((T, \tau) \in [0, 1000] \times [0, 1000] \). The same stability region is found for \( N = 1 \) and \( N = 2 \) in the modified model of 4 cars on the ring. Note that for \( N = 2 \) the augmented Lyapunov functionals slightly improve the results.

5. Conclusions

In the present paper, simple LMI conditions for the exponential stability of linear systems with infinite distributed delays have been presented. These systems are motivated by various applications in biology and engineering. Particularly in the traffic flow models on the ring, the gamma-distributed delay with a gap (which characterizes the human drivers’ behavior on the average) is stabilizing. The latter means that the corresponding system with the zero-delay as well as the system without the delayed term are not asymptotically stable. In the numerical examples, our LMIs are feasible in the case of stabilizing delay leading to the results close to the theoretical ones (Michiels et al., 2009). Polytopic uncertainties in the system matrices can be easily included in the analysis. Extension of the presented direct Lyapunov method to nonlinear systems as well as different applications may be topics for future research.

References


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