Brief paper

Network-based $H_\infty$ filtering of parabolic systems

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A B S T R A C T

We design a network-based $H_\infty$ filter for a parabolic system governed by a vector semilinear N-D diffusion equation over a rectangular domain $\Omega$ under distributed in space measurements. The sampled in time measurements are sent to the observer over a communication network according to Round-Robin scheduling protocol (one after another in a periodic manner). The objective is to enlarge the sampling time intervals and, thus, to reduce the amount of communications, while preserving a satisfactory error system performance. We suggest to divide $\Omega$ into a finite number of rectangular sub-domains $\mathcal{N}$, where stationary or mobile sensing devices provide spatially averaged state measurements to be transmitted through communication network. Sufficient conditions in terms of Linear Matrix Inequalities (LMIs) for the internal exponential stability and $L_2$-gain analysis of the estimation error are derived via the time-delay approach to networked control systems. By solving these LMIs, the filter gain along with the upper bounds on the sampling time intervals, on the network induced time-delays, and on the diameters of the sub-domains can be found that preserve the internal stability of the error system and achieve a given $L_2$-gain. Numerical examples illustrate the efficiency of the method.

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1. Introduction

Networked Control Systems (NCSs), where the plant is controlled via communication network, is a hot topic. The introduction of communication network media brings great advantages, such as low cost, reduced weight, simple installation/maintenance and long distance control. Long distance estimation/control of chemical reactors or air polluted areas (that can be modeled by diffusion PDEs Koda & Seinfeld, 1978 and Smagina & Sheintuch, 2006) is potentially of great interest. It is important to provide a stability and performance certificate that takes into account the network imperfections: variable sampling intervals and communication delays, scheduling protocols and quantization (Heemels, Teel, van de Wouw, & Nesic, 2010). Three main approaches have been used to the NCSs: the discrete-time, the hybrid system and the time-delay approaches (Donkers, Heemels, van de Wouw, & Hetel, 2011; Gao, Chen, & Lam, 2008; Heemels et al., 2010; Liu, Fridman, & Hetel, 2012).

While there exists an extensive literature on network-based control of finite dimensional systems, there are only a few works on network-based control of PDEs. For linear parabolic systems, mobile collocated sensors and actuators were considered in Demetriou (2010). The discrete-time approach to sampled-data control of linear time-invariant distributed parameter systems was developed in Logemann (2013), Logemann, Rebarber, and Townley (2005) and Tan, Trelat, Chitour, and Nesic (2009). A model-reduction-based approach to network-based control of semilinear distributed parameter systems was introduced in Ghatasala and El-Farra (2012) and Yao and El-Farra (2012), where a finite-dimensional controller was designed on the basis of a finite-dimensional system that captures the dominant (slow) dynamics of the infinite-dimensional system. The latter approach has difficulties in the case of spatially-dependent diffusion coefficients. The above methods are not applicable to the performance (exponential decay rate or $L_2$-gain) analysis of the closed-loop infinite-dimensional systems.

Finding constructive LMI conditions for the performance analysis in terms of sampling intervals, delays and scheduling protocols for networked estimation of diffusion PDEs is of theoretical and practical importance. In the recent papers Fridman and Bar Am (2013) and Fridman and Blighovsky (2012) sampled-data control of 1-D diffusion PDEs under the spatially averaged and the point-wise measurements respectively was studied. The results of Fridman and Bar Am (2013) and Fridman and Blighovsky (2012) were...
limited to the scalar 1-D case, whereas communication constraints (scheduling protocols) were not considered.

In the present paper we study, for the first time, a network-based $H_\infty$ filtering of distributed parameter systems. We consider a vector N-D semilinear diffusion PDE over a rectangular domain $\Omega$. Similar to El-Farra and Christofides (2004), we assume that a large number of “pointwise” spatial output measurements (e.g. temperature of the rod throughout the reactor) are available so that the averaged measurements over the spatial domain $\Omega$ or over its closed sub-domains are known with sufficient accuracy. The measurements are sent over communication network to the observer in the discrete-time instances. Due to communication constraints, only one measurement can be sent per transmission. The measurements are sent according to the Round-Robin protocol in a periodic manner. The objective is to enlarge the sampling time intervals and, thus, to reduce the amount of communications, while still retaining a satisfactory estimation error performance.

We suggest to divide the spatial domain into $N_t$ rectangular sub-domains, where sensing devices provide spatially averaged measurements. Such measurements can be done either by stationary or by mobile sensors that move to the sub-domain with the measurements to be transmitted. A larger $N_t$ allows to send a more accurate approximation of the “pointwise” measurements that may improve the performance. However, due to communication constraints, an increase in $N_t$ enlarges the delays and, thus, worsens the performance. Sufficient conditions for the internal exponential stability and $L_2$-gain analysis of the error system are derived in the framework of the time-delay approach to NCSs, where the variable in time sampling intervals, network-induced delays and a Round-Robin scheduling protocol are taken into account. We show that given $N_t$, the division that minimizes the maximum diameter of the resulting sub-domains enlarges the sampling intervals.

We present below some useful inequalities.

**Lemma 1.** Let $\Omega = [0, l_1] \times \cdots \times [0, l_N]$. Assume $z : \Omega \to \mathbb{R}$ and $z \in H^1(\Omega)$.

(i) (Poincare’s inequality) If $\int_\Omega \omega(x)dx = 0$, then according to Payne and Weinberger (1960)

$$\|z\|_{L^2}^2 \leq \|\nabla z\|_{L^2}^2, \quad \mathcal{P} = \frac{\delta}{\pi} = \sqrt{\frac{l_1^2 + \cdots + l_N^2}{\pi}}. \tag{1}$$

Here $\delta$ is the diameter of $\Omega$, $\mathcal{P}$ is Poincare’s constant.

(ii) (Wirtinger’s inequality) If $z|_{\partial \Omega} = 0$, then the following inequality holds (Hardy, Littlewood, & Polya, 1988):

$$w^2\|z\|_{L^2}^2 \leq \|\nabla z\|_{L^2}^2, \quad w^2 = \frac{\pi^2}{l_1^2} + \cdots + \frac{\pi^2}{l_N^2}. \tag{2}$$

\section{Problem formulation}

Denote by $\Omega$ the $N$-dimensional rectangle

$$\Omega = \{x = (x_1, x_2, \ldots, x_N)^T | x_k \in [0, l_k], \quad l_k > 0, \quad k = 1, 2, \ldots, N\},$$

with the boundary

$$\partial \Omega = \{x = (x_1, x_2, \ldots, x_N)^T | \exists k \in 1, 2, \ldots, N \ 	ext{s.t.} \ x_k = 0 \text{ or } x_k = l_k\}.$$ Consider the following semilinear diffusion PDE

$$z_t(x, t) = \Delta_\Omega z(x, t) - \beta \nabla_x z(x, t) + A z(x, t) + \phi(z(x, t), x, t) + B_1 w(x, t), \quad t \geq 0, \ x \in \Omega,$$

where $z_t(x, t) = [z_1^T(x, t), \ldots, z_M^T(x, t)]^T \in \mathbb{R}^M$ is the vector state, $w(x, t) \in L_2(0, \infty; L_2(\Omega))$ is the disturbance, and $A$ and $B_1$ are constant matrices, and $\beta \in \mathbb{R}^{M \times M}$ is the convection coefficients matrix. The diffusion term is given by

$$\Delta_\Omega z(x, t) = \text{div}(\Delta_\Omega z^1(x, t), \ldots, \Delta_\Omega z^M(x, t)),$$

$$\Delta_\Omega z^m(x, t) = \sum_{k=1}^N \frac{\partial}{\partial x_k} \left( \sum_{j=1}^N D_{ij}(x) \frac{\partial z^m(x, t)}{\partial x_j} \right), \quad m = 1, \ldots, M,$$

where $D^m = (D^m)^T : \Omega \to \mathbb{R}^{N \times N}$ is of class $C^1$. It is assumed that

$$0 < d_{ij}^m |_{\partial \Omega} \leq D^m(x) \in \mathbb{R}^{N \times N}, \quad m = 1, \ldots, M.$$ Then

$$D_0 \triangleq \text{diag}(d_{01}^m, \ldots, d_{0M}^m) > 0.$$ The function $\phi$ is supposed to be of class $C^1$ with a uniformly bounded $\phi_\ast$, satisfying

$$\phi_\ast^m(z, x, t) \phi_\ast(z, x, t) \leq Q \quad \forall z, x, t,$$

for some constant and positive $M \times M$-matrix $Q$. Consider (3) under the Dirichlet

$$z(x, t) |_{x \in \partial \Omega} = 0 \tag{7}$$

or under the Neumann

$$z(x, t) \cdot \hat{n} \big|_{x \in \partial \Omega} = 0 \tag{8}$$

boundary conditions, where $\hat{n}$ is a unit vector normal to the edge. The disturbance $w(x, t) \in L_2(0, \infty; L_2(\Omega))$ is said to be admissible if system (3) possesses a unique strong solution being initialized with $z(\cdot, 0) \in H^1(\Omega)$, satisfying the boundary conditions, and if this solution is globally continuably to the right. If $w$ is $C^1$ and is
uniformly bounded, then by the arguments of Fridman and Bar Am (2013), the strong solutions of (3) under the boundary conditions (7) or (8) initialized with \( z(\cdot, 0) \in \mathcal{H}^1(\Omega) \) that satisfy the corresponding boundary conditions exist, and they are continuible for \( t \geq 0 \).

The measurements are sent over communication network to the observer in the discrete-time instances. In order to construct network-based \( H_{\infty} \) filter we suggest to divide the spatial domain into \( N_s \) rectangular sub-domains \( \Omega_i \) covering the whole region \( \bigcup_{i=1}^{N_s} \Omega_i = \Omega \), that intersect only on the boundaries

\[
\Omega_i = \{ x = (x_1, x_2, \ldots, x_N)^T \in \Omega | x_k \in [x_k^{\min}(i), x_k^{\max}(i)], \\
\quad k = 1, 2, \ldots, N_s \}.
\]

We have \( N_s = n_1 \times \cdots \times n_N \), where \( n_i \) is the number of subintervals of the side \([0, l_i]\) corresponding to \( N_s \). Denote

\[
\Delta_i = \int_{x_i^{\min}(i)}^{x_i^{\max}(i)} \cdots \int_{y_i^{\min}(i)}^{y_i^{\max}(i)} dx_1 \cdots dx_N,
\]

\[
\delta_i = \sum_{k=1}^{N_s} \sum_{j=1}^{n_i} |x_k^{\max}(i) - x_k^{\min}(i)|^2, \quad \delta_i \leq \delta,
\]

where \( \Delta_i \) and \( \delta_i \) are the volume and the diameter of \( \Omega_i \) respectively, that can be variable. Here \( \delta \) is the diameters' upper bound.

Assume that sensors provide \( N_s \) averaged measurements

\[
y(t) = \int_{\Omega_i} C z(\xi, t) d\xi = \int_{\Omega_i} v'(t) d\xi, \quad i = 1, 2, \ldots, N_s, \tag{9}
\]

where \( C \) is a constant \( l \times M \)-matrix \((l \leq M)\), \( v'(t) \in \mathbb{R}^l \) is the measurement noise. The latter measurements are transmitted via a communication network to the observer. Let \( s_k \) denote the unbounded monotonously increasing sequence of sampling instances, i.e.

\[
0 = s_0 < s_1 < \cdots < s_k < \cdots, \quad k \in \mathbb{N}, \quad \lim_{k \to \infty} s_k = \infty.
\]

At each sampling instant \( s_k \), one of the measurements \( y_i(t) \) is transmitted via the network. The choice of the \( y_i(t) \) to be sampled is ruled by a Round-Robin scheduling protocol: \( y_i(t) \) are sampled one after another in the periodic manner, i.e. \( y_i(t) \) is transmitted only at the sampling instants \( t = s_{kN_s+1}i, \quad p \in \mathbb{N} \). After each reception, the values of \( y_i(t) \) are updated with the newly received ones, while the other values of \( y_i(t) \) for \( j \neq i \) remain the same, as no additional information is received. This leads to the constrained data exchange expressed as

\[
y_{i,k} = \int_{\Omega_i} C z(\xi, t) d\xi = \int_{\Omega_i} v'(t) d\xi + v_{i,k},
\]

\[
i = 1, 2, \ldots, N_s, \quad k = N_sp + i - 1, \quad p \in \mathbb{N}, \tag{10}
\]

\[
v(x, t) = v_{i,k}, \quad x \in \Omega_i, \quad t \in [t_k, t_{k+1}),
\]

\[
l = 1, \ldots, N_s, \quad k = N_sp + i - 1, \quad p \in \mathbb{N}.
\]

We suppose that the transmission of the information over the network is subject to a variable and bounded communication delay \( h_k \leq \text{MAD} \). Then \( t_k = s_k + h_k \) is the updating instant time of the observer. A time-delay approach to finite-dimensional network-based control under the Round-Robin scheduling was introduced in Liu et al. (2012). As in Liu et al. (2012) we do not restrict the network-induced delay to be small with \( t_k = s_k + h_k < s_{k+1}, \) i.e. \( h_k < s_{k+1} - s_k \). We assume the following:

A1. The order of the measurements \( y_i, \quad i = 1, 2, \ldots, N_s \) is not changed over the network

\[
t_{k+N_s} = s_k + h_k \geq s_k + h_k = t_k,
\]

\[
k = N_sp + i - 1, \quad p \in \mathbb{N}, \tag{11}
\]

whereas the time span between the most recent updating and the oldest sampling instant is bounded

\[
t_{k+N_s} = 0
\]

\[
\leq \text{MATI} \cdot N_s + \text{MAD} \leq \tau_M. \tag{12}
\]

A2. The measurements are sent with the time-stamps.

The assumption (11) makes the scheduling reasonable. The assumption A2 means that \( s_k = t_k - h_k \) is known on the observer side. The latter allows to use the Luenberger type observer of the form

\[
\dot{z}_i(t) = \Delta_i \dot{z}_i(t) - \beta \nabla \cdot \dot{z}_i(t) + A \dot{z}(t) + \phi(z, t, x),
\]

\[
t \in \mathbb{R}, \quad x \in \mathbb{R}^d, \quad \beta \in \mathbb{R}^d,
\]

\[
\beta \geq 0, \quad \Delta_i \geq 0, \quad \Delta_i \neq 0,
\]

\[
\phi_i(z) = \int_0^1 \phi_i(z, t) d\alpha, \quad \phi_i(z) \geq 0,
\]

\[
\phi_i(z) = \int_0^1 \phi_i(z, t) d\alpha, \quad \phi_i(z) \geq 0.
\]

\[
\beta \geq 0, \quad \Delta_i \geq 0, \quad \Delta_i \neq 0,
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\[
\beta \geq 0, \quad \Delta_i \geq 0, \quad \Delta_i \neq 0,
\]

\[
\phi_i(z) = \int_0^1 \phi_i(z, t) d\alpha, \quad \phi_i(z) \geq 0.
\]

\[
\beta \geq 0, \quad \Delta_i \geq 0, \quad \Delta_i \neq 0,
\]

\[
\phi_i(z) = \int_0^1 \phi_i(z, t) d\alpha, \quad \phi_i(z) \geq 0.
\]
for all \( \hat{z} \), \( e \), \( x \), \( t \). The boundary conditions for the error dynamics are
\[
e(x, t)|_{x \in \partial \Omega} = 0
\]  
(17)
for (3) under the Dirichlet boundary conditions (7) and
\[
e_s(x, t) \cdot \hat{n}|_{x \in \partial \Omega} = 0
\]  
(18)
for (3) under the Neumann boundary conditions (8).

Following the time-delay approach to sampled-data control (Fridman, Seuret, & Richard, 2004), denote
\[
\tau_i(t) = t - t_k + h_k, \quad t \in [t_k, t_{k+1}), \quad i = 1, 2, \ldots, N_k, \quad k = N_p + 1 - i, \quad p \in \mathcal{N}.
\]  
(19)
Then \( t_k - h_k = t - \tau_i(t) \) and, due to (13),
\[
\tau_i(t) \leq t_{k+1} - t_k + h_k = \tau_M, \quad i = 1, 2, \ldots, N_k.
\]

Similar to Fridman and Bar Am (2013), we shall use the elementary relation
\[
\int_{\Delta_i} e(\xi, t - \tau_i(t)) d\xi = e(x, t) - f_i(x, t) - \rho_i, \quad x \in \Omega_i,
\]
where \( \rho_i = \rho_i(t) \) and where
\[
f_i(x, t) = \frac{\int_{\Delta_i} e(\xi, t) - e(\xi, t) d\xi}{\Delta_i},
\]
\[
\rho_i = \frac{\int_{\Delta_i} f_i(x, t) d\xi}{\Delta_i}.
\]

Hence, the error system can be presented as
\[
e_i(x, t) = \Delta_0 e(x, t) - \beta \nabla V e(x, t) + A e(x, t) + \sum_{k=1}^{\infty} K C e(x, t) + K C \rho_i + B_k \nabla x(x, t) - K \nabla y(x, t),
\]
\[
t \geq t_{N_k-1}, \quad x \in \Omega_i, \quad i = 1, \ldots, N_k.
\]
(20)
The initial condition \( e_i(x, t) | (t \in [0, t_{N_k-1}) \) for (20) is defined as a strong solution of (15), where \( e(\cdot, 0) \in H^1(\Omega) \) satisfies the boundary conditions. Since \( f_i(x, t) = e_i(x, t) \) and \( \int_{\Delta_i} f_i(x, t) d\xi = 0 \), Poincare’s inequality (1) implies
\[
\int_{\Omega} \left[ \sum_{i=1}^{N_k} \left| \nabla e(x, t) \right|^2 \right] dx = 0, \quad t \geq t_{N_k-1}.
\]  
(21)

Note that the Lyapunov-based analysis of (20) under the corresponding boundary conditions in the case of scalar \( z \) and \( x \) with \( s_k = s_k \) and \( h_k = 0 \) was considered in Fridman and Bar Am (2013), where \( \rho_i \) was multiplied by \( \frac{1}{t_i - t_k} \).

For the system (20), we choose the Lyapunov–Krasovskii functional of the form
\[
V(t) = V_P + V_S + V_R,
\]
\[
V_P = \int_{\Omega} e^T(x, t) P e(x, t) dx
\]
\[
+ \sum_{k=1}^{\infty} \int_{\Omega} P_k \nabla e(x, t) D M(x) \nabla e(x, t) dx,
\]  
(22)
\[
V_S = \int_{\Omega} \int_{t_{N_k}}^{t} e^{2 \alpha (t-s)} e^T(x, s) S e(x, s) ds dx,
\]
\[
V_R = \int_{\Omega} \int_{t_{N_k}}^{t} e^{2 \alpha (t-s)} e^T(x, s) R e(x, s) ds dx,
\]
with positive matrices \( P_i, S, R \in R^{M \times M} \) and positive diagonal \( M \times M \)-matrix \( P_1 = \text{diag}(p_1, \ldots, p_M) \). This \( V \) extends the construction of Fridman and Blighovsky (2012) to N-D case. Following Liu and

Fridman (2014), in order to guarantee that \( V(t) \) is defined for all \( t \geq t_{N_k-1} \) we set \( e(x, t) \equiv e(0, 0) \) for \( t < 0 \).

Our objective is to find a constant gain \( K \) that internally exponentially stabilizes the error system, i.e. exponentially stabilizes the disturbance–free system in the sense that the following holds:
\[
V(t) \leq e^{-2 \alpha (t-t_{N_k-1})} V(t_{N_k-1}), \quad t \geq t_{N_k-1},
\]
where \( \alpha > 0 \) is the decay rate. While internally stabilizing the parabolic error process, the influence of the admissible external disturbances on the controlled output
\[
\xi(x, t) = C e(x, t)
\]  
(23)
with a constant matrix \( C_i \in R^{d \times M} \) is to be attenuated.

The following \( H_\infty \) filtering problem is thus under study. Given \( \gamma > 0 \), it is required to find an observer (14) that internally exponentially stabilizes the estimation error dynamics (20) and leads to a negative performance index
\[
J(T) = \int_{t_{N_k-1}}^{T} \int_{\Omega} \left[ \left| \xi(x, t) \right|^2 - \gamma^2 \left[ |w(x, t)|^2 + |v(x, t)|^2 \right] \right] dx dt
\]
\[
< V(t_{N_k-1}) \quad \forall T > t_{N_k-1}
\]  
(24)
for all admissible disturbances such that
\[
\int_{\Omega} \left[ |w(x, t)|^2 + |v(x, t)|^2 \right] dx > 0, \quad t > t_{N_k-1}.
\]  
(25)

Then it is said that the error dynamics (20) has an \( L_2 \)-gain less than \( \gamma \).

In the design process, the geometrical properties of the sensors (leading to a certain choice of convenient sub-domains \( \Omega_i \)) should be taken into account. We will show that given \( N_k \), the division with the minimum diameters’ bound \( \delta \) enforces \( n \) leading to a larger maximum sampling interval \( \tau_M = (t_M - M AD)/N_k \). In the 2-D case as shown in Fig. 2(a), where \( N_k = 2 \), the choice of the right rectangular with the smallest maximum diagonal leads to a larger \( \tau_M \). Increasing \( N_k \), the accuracy of the measurements to be transmitted enlarges \( \tau_M \) as well as the amount of communications. Our objective is to maximize \( \tau_M \) by verifying the feasibility of the LMIs of Theorem 1 below, we will optimize the choice of \( N_k \).

For example, in the case of Fig. 2(b), for \( MAD \rightarrow 0 \) the left division may be preferable, but for \( MAD \rightarrow \tau_M \) it may happen that only the right one guarantees the desired performance of the error system.

3. Main results: \( L_2 \)-gain analysis and design

In order to solve the problem we will derive sufficient conditions for the following dissipative inequality
\[
W(t) = \dot{V}(t) + 2 \alpha V(t) + \sum_{i=1}^{N_k} \int_{\Omega_i} \left[ \left| |\xi(x, t)|^2 - \gamma^2 |w(x, t)|^2 \right| \right] dx < 0, \quad \alpha > 0, \quad t \geq t_{N_k-1}
\]  
(26)
\[\text{Fig. 2. Spatial design dilemmas for 2-D problem.}\]
to hold along the trajectories of (20) with the corresponding boundary conditions provided (25) is valid. The integration of (26) in \( t \) from \( t_0 \) to \( T \) would yield (24) since \( V(t) \geq 0 \). For the unperturbed system (20), (26) implies \( V(t) + 2\alpha V(t) \leq 0 \) and, thus, the exponential stability of (20) with the decay rate \( \alpha \).

**Theorem 1.** Given positive scalars \( N_0, \delta, \gamma, \alpha, \tau_M \) and a matrix \( K_0 \in \mathbb{R}^{M \times l} \), let there exist a matrix \( G \in \mathbb{R}^{M \times M} \), positive \( M \times M \)-matrices \( P_1, S \) and \( R \), diagonal \( M \times M \)-matrices \( P_2 > 0 \) and \( P_3 \), and scalars \( \lambda_0 > 0, \lambda_j > 0 \) \( (j = 1, 2) \) and \( \lambda_0 > 0 \) for the Dirichlet or \( \lambda_0 = 0 \) for the Neumann boundary conditions such that the following LMI s are feasible:

\[
\mathcal{S} \triangleq \begin{bmatrix} R & G \\ G & R \end{bmatrix} > 0, \quad \Phi_{\gamma} < 0, \tag{27}
\]

where

\[
\Phi_{\gamma} = \begin{bmatrix} \psi_i \\ \Phi_{11} \end{bmatrix} = \begin{bmatrix} \psi_i & \Phi_{11} \\ \Phi_{11} & \Phi_{12} \end{bmatrix} \begin{bmatrix} \Phi_{12} \\ \Phi_{13} \\ \Phi_{14} \\ \Phi_{15} \\ \Phi_{16} \\ \Phi_{17} \\ \Phi_{18} \\ \Phi_{19} \\ \Phi_{20} \\ \Phi_{21} \\ \Phi_{22} \\ \Phi_{23} \\ \Phi_{24} \\ \Phi_{25} \\ \Phi_{26} \\ \Phi_{27} \\ \Phi_{28} \end{bmatrix} 
\]

and

\[
\Phi_{11} = 2\alpha P_1 + P_2 (A - K_0 C) + (A - K_0 C)^T P_2 + S
\]

\[-\text{Re}^{-2\alpha t_M} + \lambda_2 I_M - \lambda_0 \sum_{k=1}^{M} \frac{I_k^2}{I_k^2} I_M + \lambda_0 Q + C_1^T C_1, \tag{28}\]

\[
\Phi_{12} = P_1 - P_2 + \lambda_0 I_M, \quad \Phi_{13} = \Phi_{14} = \Phi_{23} = \Phi_{24} = P_2 K_0 C,
\]

\[
\Phi_{15} = -\lambda_1 I_M, \quad \Phi_{16} = (R - G) e^{-2\alpha t_M} - \lambda_2 I_M, \quad \Phi_{17} = \text{Ge}^{-2\alpha t_M},
\]

\[
\Phi_{22} = R t_M^2 - 2P_2, \quad \Phi_{23} = \Phi_{24} = P_2 K_0 C, \quad \Phi_{25} = -P_2 \beta, \quad \Phi_{28} = P_3,
\]

\[
\Phi_{25} = -2D_0 (P_2 - \alpha P_3) \otimes I_N + [\lambda_0 + \lambda_1] I_M N_M, \quad \Phi_{66} = (G + C^T - 2R) e^{-2\alpha t_M} + \lambda_2 I_M, \quad \Phi_{07} = (R - G) e^{-2\alpha t_M}, \quad \Phi_{77} = -\text{Re}^{-2\alpha t_M} - 2e^{-2\alpha t_M}.
\]

Then the error system (20) with the observer gain \( K_0 = P_2^{-1} Y \) under the Dirichlet (17) or under the Neumann (18) boundary conditions is internally exponentially stable with the decay rate \( \alpha \) and has \( L_2 \)-gain less than \( \gamma \). Moreover, if the above conditions are feasible with \( \alpha = 0 \), then (20) is internally exponentially stable with a small enough decay rate and has \( L_2 \)-gain less than \( \gamma \).

**Remark 1.** Note that \( \delta \) appears only in \( \Phi_{33} = -\frac{\lambda_1 \gamma^2}{\delta} I_M \) of \( \Phi_{\gamma} \), meaning that a smaller \( \delta \) enlarges \( \tau_M \) that preserves the \( H_\infty \) performance. Therefore, given \( N_0 \) one has to choose such a division of \( \Omega \) that minimizes the maximum diameter of the resulting subdomains. This choice enlarges \( \text{MATL} = \frac{\tau_M - \text{MAD}}{N_0} \).

**Remark 2.** Compared to Fridman and Bar Am (2013) and Fridman and Blighovsky (2012), an improved technique (based on S-procedure) is presented (see Appendix) leading to less restrictive LMIs under the Dirichlet than under the Neumann boundary conditions. The LMIs are less restrictive due to the negative term \( -\lambda_0 \sum_{k=1}^{M} \frac{I_k^2}{I_k^2} I_M \) in \( \Phi_{11} \) of (28).

If \( K_0 \) is unknown then the matrix inequalities of Theorem 1 are nonlinear. In order to linearize these inequalities we follow the method of Suplin, Fridman, and Shaked (2007): assume \( P_3 = \varepsilon P_2 \) and denote \( Y = P_2 K_0 \). We obtain the following result for the \( H_\infty \) filter design:

**Corollary 2.** Given positive scalars \( N_0, \delta, \gamma, \alpha, \tau_M \) and a tuning parameter \( \epsilon \), let there exist matrices \( G \in \mathbb{R}^{M \times l} \) and \( Y \in \mathbb{R}^{l \times l} \), positive \( M \times M \)-matrices \( P_1, S \) and \( R \), diagonal \( M \times M \)-matrices \( P_2 > 0 \) and \( P_3 \), and scalars \( \lambda_0 > 0, \lambda_j > 0 \) \( (j = 1, 2) \) and \( \lambda_0 > 0 \) for the Dirichlet or \( \lambda_0 = 0 \) for the Neumann boundary conditions such that the LMIs (27) are feasible, where \( \Phi_{\gamma} \) is given by (28) with \( P_2 B_1, P_3 K_0 \) and \( P_2 K_0 \) replaced by \( \varepsilon P_2 B_1, Y \) and \( \varepsilon Y \) respectively, and where \( \psi_i \) is given by (29) with

\[
\Phi_{11} = 2\alpha P_1 + P_2 A + A^T P_2 - Y C - C^T Y^T + S
\]

\[-\text{Re}^{-2\alpha t_M} + \lambda_2 I_M - \lambda_0 \sum_{k=1}^{M} \frac{I_k^2}{I_k^2} I_M + \lambda_0 Q + C_1^T C_1, \tag{29}\]

\[
\Phi_{12} = P_1 - P_2 + \varepsilon P_2 A - \varepsilon Y C, \quad \Phi_{13} = \Phi_{14} = \Phi_{15} = \Phi_{23} = \Phi_{24} = P_2 A - Y C, \quad \Phi_{25} = -\varepsilon P_2 \beta, \quad \Phi_{28} = \varepsilon P_2,
\]

\[
\Phi_{55} = -2D_0 (P_2 - \alpha P_3) \otimes I_N + [\lambda_0 + \lambda_1] I_M N_M, \quad \Phi_{66} = (G + C^T - 2R) e^{-2\alpha t_M} + \lambda_2 I_M, \quad \Phi_{07} = (R - G) e^{-2\alpha t_M}, \quad \Phi_{77} = -\text{Re}^{-2\alpha t_M} - 2e^{-2\alpha t_M}.
\]

Then the error system (20) with the observer gain \( K_0 = P_2^{-1} Y \) under the Dirichlet (17) or under the Neumann (18) boundary conditions is internally exponentially stable with the decay rate \( \alpha \) and has \( L_2 \)-gain less than \( \gamma \). Moreover, if the above conditions are feasible with \( \alpha = 0 \), then (20) is internally exponentially stable with a small enough decay rate and has \( L_2 \)-gain less than \( \gamma \).

**Remark 3.** Practical implementation of the above \( H_\infty \) filter involves a synchronization method for the sensors and observer nodes that may lead to errors in \( s_k \). For simplicity, in the present paper we ignore the latter errors. See e.g. Seuret and Richard (2008) for the corresponding stability analysis (not including \( H_\infty \) performance and scheduling protocols) in the case of finite-dimensional systems.

**Remark 4.** Different from the distributed estimation considered e.g. in Zhang, Feng, and Yu (2012), where each node has a local estimate of the state by employing his measurement and estimates from his neighbors, we construct a global (centralized) observer that is based on all (but delayed) sensor measurements. Thus we have no problem of disagreements among the various local estimates.

### 3.1: Discussions: the case of “pointwise” measurements

For large-scale plants (e.g. air polluted areas), a more realistic situation arises when there are some subregions in \( \Omega \) without measurements. In the \( H_\infty \) filtering framework, our results can be extended to this case provided the subregions without
measurements are non-disturbed. Consider (3). Denote by $\Omega$ the non-disturbed sub-domain of $\Omega$ (where $w \equiv 0$), and by $\Omega_i$ ($i = 1, \ldots, N$) the disturbed small rectangular sub-domains with diagonals $\delta_i$. Assume that $\bigcup_{i=1}^{N_{i}} \Omega_i = \Omega$ and that $\Omega_i$ ($i = 0, \ldots, N$) intersect only on the boundaries. Suppose that $N_i$ sensors provide “pointwise” measurements defined by (9).

Under the same assumptions on $s_{i}$, $\eta_{i}$ as above and under the Round Robin scheduling protocol, let the measurements be defined by (10). Consider the observer given by (14) and by "pointwise" measurements defined by (9).

Then the error system can be presented as (20) and by (10). Consider the observer given by (14) and by "pointwise" measurements defined by (9).

Condition $f(T)$ defined by (24). We are looking for conditions that guarantee $f(T) < 0$ for all $T > t_{N_{i}}$ and for all admissible $w \in L_{2}(0, \infty; L_{2}(\Omega)) : w|_{w = \partial \Omega} \equiv 0$ subject to (25).

By modifying the proof of Theorem 1, we arrive at the following conditions that guarantee $L_{2}$-gain less than $\gamma$ for the error system: (27) and $\Phi|_{K_{0}=0} < 0$. Note that the latter LMI guarantees the exponential stability of the unperturbed error dynamics with $K_{0} = 0$. Therefore, the observer with $K_{0} = 0$ (that does not use the measurements) leads to a finite $\gamma$ of the error system, but $K_{0} \neq 0$ may significantly enhance the performance (see Example 1 below).

**Remark 5.** In the case of non-collocated disturbances and sensors, an extension of the presented results to the case of “pointwise” measurements is not clear. Thus, in 1-D case and the “pointwise” measurements, the Lyapunov analysis may be based on a combination of the Lyapunov–Krasovskii method with Halanay’s inequality (Fridman & Blighovsky, 2012). However, Halanay’s inequality is not applicable to $L_{2}$-gain analysis. The presented results can be extended to networked Luengoer type observer of the unperturbed N-D diffusion PDEs under the “pointwise” measurements. Such an extension may be a topic for the future research.

4. Examples

**Example 1.** Consider $H_{\infty}$ filtering of the scalar 2-D PDE (3) with the controlled output (23) and the measurements (10), where $d_{0} = 10^{-4}, A = \beta = \phi = 0, \Omega = [0, 0.1] \times [0, 0.06]$ and $C = B_1 = C = 1$. This PDE with $B_1 = 0$ was considered in Demetriou (2010). We take $N_i$ as in the Table 1 and choose such a division of $\Omega$ into $N_i$ equal rectangles that corresponds to the minimum diameter $\delta$ (see Table 1). Thus, for $N_i = 2$ the division of the side $[0, 0.1]$ into two subintervals (i.e. $N_i = 2 \times 1$) leads to a smaller $\delta^2 = 0.05^2 + 0.06^2 = 0.0122$ than the division of the side $[0, 0.06]$ into two subintervals ($N_i = 1 \times 2$) with $\delta^2 = 0.1^2 + 0.03^2 = 0.019$. Consider the Neumann boundary conditions. Using (4) with $\lambda_{0} = 0$, $K_{0} = 0.2$, $\alpha = 0$ and $\gamma = 10.1$ we find the maximum values of $r_{ms}$ that guarantee $f(T) < 0$ (see Table 1). For MAD = 0.4, the strategy of dividing the whole region $\Omega$ into two parts results in the maximum sampling interval bound $MATI$. For large MAD = 3.5, only the division into $N_i = 6$ subdomains preserves the error performance provided the sampling is fast enough with $MATI = 0.04$.

Note that under the Dirichlet boundary conditions, where the LMIs of Theorem 1 are used with $\lambda_{0} \neq 0$, the resulting values of $\gamma$ (for the same values of $r_{ms}$ and $\delta$) are essentially smaller: e.g. for $N_i = 2 \times 1$ and $r_{ms} = 25$ the resulting $\gamma = 5.3$ (instead of 10.1 in Table 1). Simulation of $f(T)$ under the Dirichlet boundary conditions corresponding to the error system with the zero initial

<table>
<thead>
<tr>
<th>$N_i$</th>
<th>$\delta$</th>
<th>$r_{ms}$</th>
<th>MATI (MAD = 0.4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 = 1 x 1</td>
<td>0.1166</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2 = 1 x 2</td>
<td>0.0781</td>
<td>2.5</td>
<td>1.05</td>
</tr>
<tr>
<td>4 = 2 x 2</td>
<td>0.0583</td>
<td>3.37</td>
<td>0.7425</td>
</tr>
<tr>
<td>6 = 3 x 2</td>
<td>0.0448</td>
<td>3.75</td>
<td>0.5583</td>
</tr>
</tbody>
</table>

**Table 1** Example 1: $\gamma = 10.1$ and the Neumann boundary conditions.

Consider next the case of “pointwise” collocated disturbances and sensors under the Dirichlet boundary conditions. Note that for $K_{0} = 0$ the conditions of Theorem 1 guarantee that $f(T) < 0$ for $\gamma = 2.7$. For $K_{0} = 0.5$ and $\delta = 0.001$ Theorem 1 guarantees a smaller $\gamma = 1.5$ for $r_{ms} = 0.6$. Thus, in the case of $N_i = 5$ “pointwise” disturbances and sensors inside of rectangles with the diagonals not greater than 0.001, MAD = 0.1, MATI = 0.1 and under Round Robin scheduling protocol, $K_{0} = 0.5$ leads to error dynamics with a reduced $L_{2}$-gain $\gamma = 1.5$.

**Example 2.** Consider the chemical reactor model from Smagina and Sheintuch (2006) governed by (3) under the Neumann boundary conditions with the controlled output (23) and the measurements (10), where $M = 2, N = 1, \Omega = [0, 10], D_0 = \text{diag}(0.01, 0.005), \beta = \text{diag}(0.011, 1.1)$, $C_1 = [1, 0], \quad C = I_2$.

The nonlinearity $\phi = \text{col}(\phi_1(z^1), 0)$ is assumed to satisfy $\frac{d\phi}{dz} \in [0, 0.01]$. This model accounts for an activator temperature $z^1$, that undergoes reaction (expressed as 0.01$z^2 + \phi$), advection and diffusion, and for a fast inhibitor concentration $z^2$, which may be advected by the flow. The elements of $\beta$ are convective velocities.

By dividing the interval $\Omega = [0, 10]$ into $N_i = 20$ equal subintervals of the length $\delta$ and applying Corollary 2 with $r_{ms} = 0.25$ and $\gamma = 10$ we arrive at $K_{0} = \begin{bmatrix} 0.5777 & -0.0542 \\ -0.4080 & 0.1216 \end{bmatrix}$. Applying

**Fig. 3.** Ex. 1: $N_i = 2, h_{0} = 0.4, s_{k+1} - s_{k} = 1.05, \gamma = 5.3$.
Table 2

<table>
<thead>
<tr>
<th>( N_0 )</th>
<th>( \delta = \frac{\tau_m}{\tau} )</th>
<th>( \tau_m )</th>
<th>MATI (MAD = 0.1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.5</td>
<td>0.68</td>
<td>0.029</td>
</tr>
<tr>
<td>30</td>
<td>0.333</td>
<td>1.26</td>
<td>0.0387</td>
</tr>
<tr>
<td>50</td>
<td>0.2</td>
<td>1.47</td>
<td>0.0274</td>
</tr>
</tbody>
</table>

further Theorem 1 to the resulting error dynamics with the latter \( K_\nu \) and \( \gamma = 10 \) we find the maximum value of \( \tau_m \) that guarantees \( J(T) < 0 \) (see Table 2). As seen from Table 2, the strategy of dividing the whole interval \([0, 10]\) into \( N_0 = 30 \) equal parts results in the maximum sampling interval MATI = 0.0387. Moreover, for \( N_0 = 20 \) the resulting \( \tau_m = 0.68 \) is essentially larger than \( \tau_m = 0.25 \) achieved by Corollary 2. This mirrors the conservatism of Corollary 2. Also in this example simulations confirm the theoretical results.

5. Conclusion

\( H_\infty \) filter has been designed for convection–diffusion PDEs over rectangular N-D domain \( \Omega \) in the situation, where distributed in space measurements are sent to observer through communication network. The objective is to enlarge the sampling time intervals, while preserving a satisfactory error system performance in spite of variable sampling, network-induced delays and Round Robin scheduling of communication protocols. We have suggested to divide \( \Omega \) into a finite number \( N_0 \) of rectangular sub-domains, where stationary or mobile sensing devices provide spatially averaged measurements. Given \( N_0 \), we have found that the division that minimizes the maximum diameter of the resulting sub-domains is advantageous leading to larger sampling intervals.

Appendix. Proof of Theorem 1

We will derive LMI conditions for \( W(t) < 0 \) (\( t \geq t_{N_0-1} \)) via the descriptor method (Fridman, 2001), where

\[
0 \equiv 2 \int_\Omega \left[ e^T(t, x) P_2 + e^T(t, x) P_3 \right] \left[ -e(t, x) + \Delta_\sigma e(t, x) A e(t, x) - \beta \nabla e(t, x) + (\phi' - K_\nu C) e(t, x) \right. \\
\left. + B_1 w(t, x) \right] dx + 2 \sum_{i=1}^{N_0} \int_{\Omega} \left[ e^T(t, x) P_2 \\
+ e^T(t, x) P_3 K_\nu C f_i(x, t) + \rho_i - \nu(x, t) \right] dx
\]

(A.1)

with some free diagonal matrix \( P_2 = \text{diag}(p_{21}, \ldots, p_{2M}) \) is added to \( V + 2\alpha V \). Taking into account the boundary conditions, by Green’s formula we obtain

\[
2 \int_\Omega e^T(t, x) P_3 \Delta_\sigma e(t, x) dx = 2 \sum_{m=1}^{M} p_{3m}^T \int_\Omega e^T(t, x) \Delta_\sigma^m e^m(t, x) dx \\
= -2 \sum_{m=1}^{M} p_{3m}^T \int_\Omega e^T(t, x) D^m(t) \nabla e^m(t, x) dx,
\]

(A.2)

\[
2 \int_\Omega e^T(t, x) P_2 \Delta_\sigma e(t, x) dx \\
= -2 \sum_{m=1}^{M} p_{2m}^T \int_\Omega e^T(t, x) D^m(t) \nabla e^m(t, x) dx.
\]

We have

\[
\dot{V}_f(t) + 2\alpha V_f(t) = 2 \int_\Omega e^T(t, x) P_1 e(t, x) dx \\
\]

Further we find

\[
\dot{V}_S(t) + 2\alpha V_S = \int_\Omega [e^T(t, x) S e(t, x) \\
- e^{-2\tau M} e^T(t, x - \tau M) S e(t, x - \tau M)] dx,
\]

Denote \( \alpha_1 = \frac{\tau(t)}{\tau_m}, \alpha_2 = \frac{\tau(t) - \tau M}{\tau M}, \xi^T(t, x) = [e^T(t, x) - e^T(t, x - \tau(t)) e^T(t, x - \tau(t)) - e^T(t, x - \tau M)], \) Applying Jensen’s inequality and convex analysis of Park, Ko, and Jeong (2011), we obtain

\[
-\tau M e^{-2\tau M} \sum_{i=1}^{N_0} \int_\Omega \int_{-\tau M}^{\tau M} \xi^T(t, x) \xi(x, s) ds dx \\
\leq -\xi^T(t, x) \xi(x, t) \leq -\xi^T(t, x) \Xi \xi(x, t),
\]

where \( \Xi > 0 \) due to (27). Then

\[
\dot{V}_R(t) + 2\alpha V_R \leq \tau M \int_\Omega \int_{-\tau M}^{\tau M} \xi^T(t, x) \xi(x, t) dx e^{-2\tau M} \\
\]

\[
\eta_0 \eta_0^T = \begin{bmatrix} R & G - R & -G \\ R & 2R - G & C^T \\ G + R & -C & R \end{bmatrix} \eta_0 dx.
\]
By Jensen’s inequality for $i = 1, 2, \ldots, N_i$,
\[ \int_{D_i} \left| e(x, t) - e(x, t - \tau_i(t)) \right|^2 dx \geq \frac{1}{\Delta_i} \left\{ \int_{D_i} \left[ e(x, t) - e(x, t - \tau_i(t)) \right] dx \right\}^2 = \Delta_i |\alpha|^2. \]

Therefore,
\[ \int_{D_i} \left[ e(x, t) - e(x, t - \tau_i(t)) \right]^2 dx \geq |\alpha|^2 \]
where $i = 1, 2, \ldots, N_i$. Summation in (21) leads to
\[ \sum_{i=1}^{N_i} \int_{D_i} \left[ \nabla_t e(x, t) \right]^2 - \frac{\pi^2}{\delta^2} \left[ f_i(x, t) \right]^2 dx \geq 0. \]  

Similarly, under the Dirichlet boundary conditions, the Wirtinger inequality (2) implies
\[ \sum_{i=1}^{N_i} \int_{D_i} \left[ \nabla_t e(x, t) \right]^2 - \frac{\pi^2}{\delta^2} \left[ f_i(x, t) \right]^2 dx \geq 0. \]  

Taking into account (A.5)–(A.7) and (16) and applying the S-procedure, we add to $V + 2\alpha V$ the left-hand sides of
\[ \lambda_0 \sum_{i=1}^{N_i} \int_{D_i} \left[ \nabla_t e(x, t) \right]^2 - \frac{\pi^2}{\delta^2} \left[ f_i(x, t) \right]^2 dx \geq 0, \]
\[ \lambda_1 \sum_{i=1}^{N_i} \int_{D_i} \left[ \nabla_t e(x, t) \right]^2 - \frac{\pi^2}{\delta^2} \left[ f_i(x, t) \right]^2 dx \geq 0, \]
\[ \lambda_2 \sum_{i=1}^{N_i} \int_{D_i} \left[ e(x, t) - e(x, t - \tau_i(t)) \right]^2 - |\alpha|^2 dx \geq 0, \]
\[ \lambda_3 \sum_{i=1}^{N_i} \int_{D_i} \left[ e(x, t) - e(x, t - \tau_i(t)) \right]^2 - |\alpha|^2 dx \geq 0, \]  
\[ \lambda_4 \sum_{i=1}^{N_i} \int_{D_i} \left[ e(x, t) - e(x, t - \tau_i(t)) \right]^2 - |\alpha|^2 dx \geq 0, \]  

where $\lambda_1 \geq 0 (j = 0, 1, 2)$ and $\lambda_0 > 0$ are some constants. Finally, from (A.4)–(A.8) it follows that (27) yields (26) provided (25) holds, where
\[ W(t) \leq \sum_{j=1}^{N_i} \int_{D_i} \eta_i^T \Phi_{ij} \eta_i dx < 0, \quad t \geq t_{N_i-1}, \]
\[ \eta_i \coloneqq \text{col} \left[ e(x, t), e_i(x, t), \dot{e}(x, t), d_i(x, t), \nabla_t e(x, t), \right. \]
\[ e(x, t - \tau_i(t)) \), e(x, t - \tau_M(t)) \), \phi(x, t) \), w(x, t), v(x, t) \). \]

References


