This paper develops the time-delay approach to large-scale networked control systems (NCSs) with multiple local communication networks connecting sensors, controllers and actuators. The local networks operate asynchronously and independently of each other in the presence of variable sampling intervals, transmission delays and scheduling protocols (from sensors to controllers). The communication delays are allowed to be greater than the sampling intervals. A novel Lyapunov–Krasovskii method is presented for the exponential stability analysis of the closed-loop large-scale system. In the case of networked control of a single plant our results lead to simplified conditions in terms of reduced-order linear matrix inequalities (LMIs) comparatively to the recent results in the framework of time-delay systems. Polytopic type uncertainties in the system model can be easily included in the analysis. Numerical examples from the literature illustrate the efficiency of the results.

1. Introduction

Networked Control Systems are systems with spatially distributed sensors, actuators and controller nodes which exchange data over a communication data channel (Antsaklis & Baillieul, 2007). It is important to provide a stability and performance certificate that takes into account the network imperfections (such as variable sampling intervals, variable communication delays, scheduling protocols, etc.). The hybrid system approach has been applied to nonlinear NCSs under Try-Once-Discard (TOD) and Round-Robin (RR) scheduling protocols in Heemels, Teel, van de Wouw, and Nesci (2010), Nesic and Teel (2004), Walsh, Ye, and Bushnell (2002), where variable sampling intervals and small communication delays (that are smaller than the sampling intervals) have been considered. Recently the time-delay approach to NCSs (see e.g. Fridman, 2014; Fridman, Seuret, & Richard, 2004; Gao, Chen, & Lam, 2008) was extended to networked systems under TOD and RR protocols that allowed to treat large communication delays (Liu, Fridman, & Hetel, 2012, 2015).

It is common place in industry that the total plant to be controlled consists of a large number of interacting subsystems (Lunze, 1992). Usually the control of the plant is designed in a decentralized manner with local control stations allocated to individual subsystems. Most papers on NCSs assume that there is one controller and one global communication network. However, in the control of large-scale systems it is more efficient to use local controllers and local networks instead of the global ones. This leads to large-scale NCSs with independent and asynchronous local networks. Another application of NCSs with asynchronous local networks is platoons of vehicles that communicate wirelessly without timing coordination between members of the whole string (Heemels, Borgers, van de Wouw, Nesic, & Teel, 2013).

Decentralized networked control of large-scale interconnected systems with local independent networks was studied in the framework of hybrid systems (Borgers & Heemels, 2014; Heemels et al., 2013), where variable sampling or small communication delays were taken into account. Distributed estimation in the presence of synchronous sampling of local networks and RR protocol was recently analyzed in Ugrinovskii and Fridman (2014) in the framework of time-delay approach.

The goal of this paper is to extend the time-delay approach to decentralized NCS with multiple local communication networks connecting sensors, controllers and actuators. The local networks operate asynchronously and independently of each other in the presence of variable sampling intervals, transmission delays and scheduling protocols (from sensors to controllers).
communication delays are allowed to be greater than the sampling intervals. Note that direct extension of the switched system modeling under RR protocol of Liu et al. (2012) to large-scale system would lead to numerous LMIs. The Lyapunov–Krasovskii method of Liu et al. (2015) developed for hybrid time-delay models of the closed-loop systems under TOD and RR protocols involves complicated conditions on the derivative and on the jumps of Lyapunov functionals that cannot be directly extended to large-scale systems.

In the present paper a novel Lyapunov–Krasovskii method is suggested for the exponential stability analysis of the closed-loop large-scale system. In the case of networked control of a single plant our results lead to simplified conditions in terms of reduced-order LMIs comparatively to the recent results (Liu et al., 2012, 2015). Numerical examples from the literature illustrate the efficiency of the results.

**Notation:** Throughout the paper the superscript ‘′ stands for matrix transposition, $R^n$ denotes the $n$ dimensional Euclidean space with vector norm $|.|$, $R^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in R^{n \times n}$ means that $P$ is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $\ast$. The space of functions $\phi : [a, b] \rightarrow R^n$, which are absolutely continuous on $[a, b]$ and have square integrable first-order derivatives, is denoted by $W[a, b]$ with the norm $\|\phi\|_W = \|\phi\|_{W[a, b]} = \max_{(a, b)}|\phi(\theta)| + \int_a^b |\phi(s)|^2 ds$.

**Remark 1.** The assumption $\delta_{mn} \neq 0$, $n \neq m$ and $\delta_{mn} = 1 (n, m \in N)$, is the updating time instant of the Zero-Order Hold (ZOH). Communication delay is assumed to be bounded $\eta'_k \in [\eta'_{1m}, \eta'_{M}]$, where $\eta'_{1M} = \text{MAD}_j$. Differently from Borgers and Heemels (2014), we do not restrict the network delays to be small with $t'_j = s'_k + \eta'_k < s'_{k+1}$, i.e. $\eta'_k < s'_{k+1} - s'_k$. As in Naghshtabrizi, Hespanha, and Teel (2010) we allow the delay to be non-small provided that the old sample cannot get to the same destination (same controller or same actuator) after the most recent one. We suppose that the controllers and the actuators are event-driven (in the sense that they update their outputs as soon as they receive a new sample).

Assume the following assumption:

**A1** There exist $M$ gain matrices $K_j = [K_{ij} \cdots K_{nj}]$, $K_{ij} \in R^{m_i \times n_i}$ such that the matrices $A_j + B_j K_C$ are Hurwitz, where $C_j = [C_{ij} \cdots C_{nj}]^T$.

**Remark 1.** The assumption A1 means that the “nominal system” $\dot{x}_j = A_j + B_j u_j$ is stabilizable by a static output-feedback $u_j = K_j C_j x_j$. Not that in the case of only one network (from sensors to controller) in each subsystem, the presented results can be easily adapted to decentralized observer-based control of large-scale systems as shown for the case of one plant in Liu et al. (2012, 2015).

We will consider TOD and RR protocols that orchestrate the sensor data transmission to the controller. Denote $J = \{1, \ldots, M\}$, $J_{RR} = \{j \in J | \text{jth subsystem is under RR} \}$ and $J_{TOD} = \{j \in J | \text{jth subsystem is under TOD} \}$. Note that if for some $j$ there is no scheduling from the sensors ($N_j = 1$) to the controller we will refer to it as $j \in J_{RR}$, where $N_j = 1$. Thus, $J = J_{RR} \cup J_{TOD}$. Denote by $\tilde{y}_j(s'_k) = \left[\begin{array}{c} \tilde{y}_{ij}(s'_k) \\ \vdots \\ \tilde{y}_{nj}(s'_k) \end{array}\right] \in R^{n_j'}$ the most recent output information submitted to the scheduling protocol of the $j$th subsystem (i.e. the most recent information at the $j$th controller side) at the sampling instant $s'_k$. Then under A1 the resulting static output-feedbacks are given by

$$u_j(t) = \sum_{i=1}^{N_j} K_{ij} \tilde{y}_j(s'_k), \quad t \in [t'_k, t'_{k+1}),$$

$$k \in Z_+, j = 1 \ldots M.$$  

Denote

$$T \triangleq \max\{\{t'_{k+1}\}_{j \in J_{RR}}, \{t'_k\}_{j \in J_{TOD}}\}. \quad x(t) = \text{col}[x_1(t), \ldots, x_M(t)].$$

Fig. 1. Decentralized control of systems with local networks.
We will define next a notion of solution to the closed-loop system (1), (3) and justify its existence. Monotonically increasing for each \( f = 1, \ldots, M \) sequences of updating times \( t^0_\nu < t^1_\nu < \cdots \) can be reordered in one monotonically increasing sequence \( t^0 < t^1 < \cdots \), where \( t^k = T \) for some \( k' \in \mathbb{Z}_+ \). For any initial condition \( x_T = x(T + \cdot) \in W([-T, 0]) \), by applying the step method for \( t \in [t^k, t^{k+1}) \) one can show that there exists a unique absolutely continuous function \( x : [T, \infty) \to \mathbb{R}^m + \eta_j \) satisfying (1), (3) almost for all \( t \geq T \). This function is called a solution of (1), (3) initialized by \( x_T \).

**Definition 1.** The closed-loop large-scale system (1), (3) is called exponentially stable with a decay rate \( \gamma > 0 \) if for any initial condition \( x_T = x(T + \cdot) \in W([-T, 0]) \) there exists \( c > 0 \) such that solutions of the system initiated by \( x_T \) satisfy the following inequality

\[
|x(t)| \leq ce^{-\gamma(t-T)}\|x_T\|, \quad \forall t \geq T.
\]

Our objective is to derive sufficient conditions for the exponential stability of the closed-loop system (1), (3).

### 3. NCs under scheduling protocols

As mentioned in the previous section, at each sampling instant \( s^j \), one of the system nodes \( i \in \{1, \ldots, N_j\} \) is active, that is only one of \( \tilde{y}_j(s^j) \) values is updated with the recent output \( y_j(s^j) \). Let \( t^i_j \in \{1, \ldots, N_j\} \) denote the active output node at the sampling instant \( s^j \), which will be chosen due to RR or TOD scheduling protocols (to be defined hereafter). Then

\[
\tilde{y}_j(s^j) = \begin{cases} y_j(s^j), & i = t^i_j, \\ \tilde{y}_j(s^j), & i \neq t^i_j. \end{cases}
\]

For simplicity we will omit \( j \) in \( s^j \).

#### 3.1. RR protocol and the closed-loop model

The periodic choice of \( t^i \) corresponds to **RR protocol**. Under RR scheduling the measurements are sent in a periodic manner one after another. Then the components of the most recent output on the controller side \( \tilde{y}_j(s^j) \) given by (2) can be presented as

\[
\tilde{y}_j(s^j) = y_j(s^j_{k-\Delta_j}) = y_j(s^j), \quad i = 1, \ldots, N_j
\]

with some \( \Delta_j \in \{0, \ldots, N_j - 1\} \). Following the time-delay approach to NCS denote

\[
\tau_j(t) = t - s^j_{k-\Delta_j}, \quad t \in [t^j_k, t^j_{k+1}).
\]

We have

\[
\eta_j \leq \tau_j(t) \leq t^{j+1}_{k+1} - s^j_{k-\Delta_j} \leq \hat{s}^j_{k+1} - s^j_{k-\Delta_j} + \eta_j \leq \Delta_j + 1 \cdot \text{MATI}_j + \text{MAD}_j \leq N_j \cdot \text{MATI}_j + \text{MAD}_j \triangleq \delta_j.
\]

Therefore, for \( t \geq t^j_{k-1} \) (when all the measurements are transmitted at least once) the static output-feedback (3) under RR protocol can be presented as

\[
u(t) = \sum_{i=1}^{N_j} K_j \tilde{y}_j(t - \tau_j(t)), \quad t \geq t^j_{k-1}.
\]

The resulting closed-loop model is a system with multiple delays

\[
\dot{x}_j(t) = A_j x_j(t) + \sum_{i=1}^{N_j} A_{ij} C_{ij} x_j(t - \tau_j(t))
\]

\[
+ \sum_{\nu \in A_i} F_{\nu j} x_j(t), \quad A_{ij} = B_j K_j, \quad t \geq t^j_{k-1},
\]

where \( t^j_{k} \in [\eta_{ij}^j, \eta_{ij}^j + \delta_j] \).

Note that under A1 for \( \tau_j = 0 \) and \( F_{ij} = 0 \), there exist \( K_j \) such that (7) is exponentially stable. Then for small enough \( \tau_j \) the system (7) with the same \( K_j \) is input-to-state stable (where \( x_j(t) + \nu_j \) are the inputs).

**Remark 2.** A more accurate model of the closed-loop system under RR protocol was presented in Liu et al. (2012) in the form of switched \( N_j \) subsystems with ordered multiple delays. Our simplified model (one system instead of \( N_j \), but with independent delays from the maximum delay interval \( [\eta_{ij}^j, \eta_{ij}^j + \delta_j] \)) leads to reduced-order LMI conditions.

### 3.2. TOD protocol and the closed-loop model

In TOD protocol the choice of \( t^j \) at the sampling instant \( s^j \) depends on the transmission error

\[
\epsilon_j(s^j) = \tilde{y}_j(s^j_{k-1}) - y_j(s^j), \quad i \in \{1, \ldots, N_j\}.
\]

The output node \( i \) with the greatest weighted error \( \epsilon_j(s^j) \) will be granted the access to the network.

**Definition 2 (TOD Protocol).** Let \( Q_{ij} > 0 \) (\( i = 1, \ldots, N_j \)) be some weighting matrices. At the sampling instant \( s^j \), the weighted **TOD protocol** is a protocol for which the active output node with the index \( t^i_j \) (\( k \in \mathbb{Z}_+ \)) is defined as any index that satisfies

\[
|\sqrt{Q_{ij}} \eta_j| \leq \sqrt{Q_{ij}} \epsilon_j(s^j) \leq \text{MATI}_j + \text{MAD}_j \triangleq \delta_j.
\]

Here the weighting matrices \( Q_{ij}, \ldots, Q_{ij} \) are variables to be designed. Then the feedback can be presented as

\[
u(t) = K_j y_j(s^j) + \sum_{i=1}^{N_j} K_{ij} \tilde{y}_j(s^j_{k-1}),
\]

\[
t \in [t^j_k, t^j_{k+1}), \quad k \in \mathbb{Z}_+
\]

with \( u(t) = 0 \), \( 0 \leq t < t^j_0 \).

Note that for \( K_j \) from A1 and small enough MATI, and MAD, the closed-loop system (1), (10) is input-to-state stable, where \( x_{ij}(t) \) are the inputs (cf. **Remark 6**). Denote

\[
\tau_j(t) = t - s^j, \quad t \in [t^j_k, t^j_{k+1}), \quad k \in \mathbb{Z}_+.
\]

Then

\[
\eta_j \leq \tau_j(t) \leq \text{MATI}_j + \text{MAD}_j \leq \delta_j.
\]

In order to obtain the impulsive closed-loop model we define in Liu et al. (2015) the piecewise-continuous error

\[
\epsilon_j(t) = \tilde{y}_j(t) - \tilde{y}_j(t^j), \quad t \in [t^j_k, t^j_{k+1}), \quad i = 1, \ldots, N_j.
\]

where we assume \( \tilde{y}_j(t^j) = 0 \), implying \( \epsilon_j(t^j) = -y_j(s^j) \). Then the closed-loop model has the following continuous dynamics:

\[
\dot{x}_j(t) = A_j x_j(t) + A_{ij} \tilde{C} x_j(t - \tau_j(t))
\]

\[
+ \sum_{i=1}^{N_j} B_j \tilde{y}_j(t) + \sum_{\nu \in A_i} F_{\nu j} x_j(t), \quad \tilde{y}_j(t) = 0, \quad i = 1 \ldots N_j, \quad t \geq t^j_0,
\]

\[
A_{ij} = B_j K_j, \quad B_{ij} = B_j K_{ij}.
\]
Similar to Liu et al. (2015) we obtain for $i = i_k^*$
\[ e_y(t_{k+1}^i) = \dot{y}_y(s_{k}^i) - y_y(s_{k}^i) + C_{ij}x_k(s_{k}^i) - x_i(s_{k+1}^i) \]
and for $i \neq i_k^*$
\[ e_y(t_{k+1}^i) = \dot{y}_y(s_{k}^i) - y_y(s_{k}^i) + e_y(t_{k}^i) + C_{ij}x_k(s_{k}^i) - x_i(s_{k+1}^i). \]
Thus, the delayed reset system is given by
\[ x_i(t_{k+1}^i) = x_i(t_{k}^i), \]
\[ e_y(t_{k+1}^i) = x_i(t_{k}^i) - x_i(t_{k+1}^i) + e_y(t_{k}^i) + C_{ij}x_k(s_{k}^i) - x_i(s_{k+1}^i), \]
\[ i = 1, \ldots, N_j, \quad j \in \mathbb{Z}^+, \]
where $\delta$ is Kronecker delta. Summarizing, (11)–(12) is the hybrid model of the NCS.

Note that in our model the first updating time $t_1^j$ corresponds to the time instant when the signal is received by the actuator. We define $x_i(t) = x_i(0)$ for $t < 0$. Thus the initial conditions for (11)–(12) are given by
\[ x_i(t_0^i + \cdot) \in W[-t_M^i, 0], \]
\[ e_y(t_0^i) = -C_{ij}x_i(t_0^i), \quad i = 1, \ldots, N_j, \]

### 3.3. Lyapunov-based analysis under RR protocol

Assume that the jth subsystem (1) is under RR protocol, i.e., $j \in \text{JRR}$. Consider the closed-loop model (8) and the following Lyapunov functional:
\[ V_j(t) = x_j^T(t)P_jx_j(t) + V_{0j}(t) + V_{jy}(t), \]
\[ V_{0j}(t) = \sum_{i=1}^{N_j} \left[ \int_{t-t_m}^{t} e^{2\alpha(s-t)} x_j^T(s) C_{ij}^T S_{0ij} C_{ij} x_j(s) \, ds \right], \]
\[ + \int_{t-t_m}^{t} e^{2\alpha(s-t)} x_j^T(s) C_{ij}^T R_{0ij} C_{ij} x_j(s) \, ds, \]
\[ V_{jy}(t) = \sum_{i=1}^{N_j} \left[ \int_{t-t_m}^{t} e^{2\alpha(s-t)} x_j^T(s) C_{ij}^T S_{1ij} C_{ij} x_j(s) \, ds \right], \]
\[ + h_j \int_{t-t_m}^{t} e^{2\alpha(s-t)} x_j^T(s) C_{ij}^T R_{1ij} C_{ij} x_j(s) \, ds, \]
\[ h_j \leq (\tilde{\tau}_M - \delta_{t_m}), \quad \alpha > 0, \quad m = 0, 1, \]
\[ P_j \geq 0, \quad S_{mij} \geq 0, \quad R_{mij} > 0, \quad i = 1, \ldots, N_j, \]
where we define (for simplicity) $x_j(t) = x_{0j}$, for $t < 0$. Note that differently from conventional Lyapunov functionals for the stability of systems with interval delays (see e.g. Liu et al., 2012, Park, Ko, & Jeong, 2011), the one given by (14) contains $C_{ij}$ in integral terms with the reduced-order matrices $S_{mij}$ and $R_{mij}$. The latter matrices will be decision variables of the resulting LMIs.

**Proposition 1.** Consider the jth subsystem given by (8). Given tuning parameters $\alpha > \varepsilon > 0$ and $(M - 1) n_j \times n_j$ matrices $P_j > 0$, let there exist an $n_j \times n_j$ matrix $P_j > 0$, $n_j \times n_j$ matrices $R_{0ij} > 0$, $n_j \times n_j$, $S_{1ij} > 0$, $S_{1ij} > 0$, and $W_{ij}$ ($i = 1 \ldots N_j$) that satisfy
\[ \Gamma_{ij} = \begin{bmatrix} R_{ij} & W_{ij} \\ W_{ij} & R_{ij} \end{bmatrix} \geq 0, \quad i = 1 \ldots N_j, \]
and
\[ \mathcal{S}_j = \begin{bmatrix} \Sigma_j & \Sigma_j^T H_j \\ * & \Sigma_j \end{bmatrix} < 0, \]
where
\[ \Sigma_j = \begin{bmatrix} \phi \Sigma_j & \Sigma_j H_j \\ * & \Sigma_j \end{bmatrix} \]
\[ \Phi = D_j^T (2\alpha P_j + C_j^T \hat{S}_0 C_j) D_j + (D_j^T P_j D_j + D_j^T P_j D_j) - \rho_m D_j^T(\hat{S}_0 - \hat{S}_ij) D_j - \rho_m D_j^T(\hat{S}_0 - \hat{S}_ij) D_j - \rho_m D_j^T \hat{R}_j D_j, \]
\[ H_{ij} = \begin{bmatrix} n_{ij} R_{ij} + (\tau_m - \delta_{t_m})^2 R_{ij} \end{bmatrix}, \]
\[ \rho_m = e^{-2\alpha t_m}, \quad \rho = e^{-2\alpha t_m}, \quad H_j = \text{diag}(H_{1j}, \ldots, H_{N_j}). \]
\[ \hat{S}_m = \text{diag}(S_{p1j}, \ldots, S_{pqj}), \quad p = 0, 1. \]
\[ \hat{R}_ij = \text{diag}(R_{0ij}, \ldots, R_{0ij}). \]
\[ \hat{R}_j = \text{diag}(\Gamma_{ij}, \ldots, \Gamma_{ij}), \quad D_j = [I_{n_j}^T 0_{n_j \times 3d_j}]. \]
\[ D_2 = \begin{bmatrix} A_{ij} \circ [0 1 0] \otimes A_{ij}, & \ldots & [0 1 0] \otimes A_{ij} \\ \end{bmatrix}. \]
\[ D_3 = \begin{bmatrix} 0_{n_j \times 3d} & \text{diag} \left[ [0 1 0] \otimes I_{n_j}, \ldots, [1 0 0] \otimes I_{n_j} \right] \end{bmatrix}. \]
\[ D_4 = \begin{bmatrix} C_j & \text{diag} \left[ [0 1 0] \otimes I_{n_j}, \ldots, [1 0 0] \otimes I_{n_j} \right] \end{bmatrix}. \]
\[ D_5 = \begin{bmatrix} 0_{n_j \times 3d} & \text{diag} \left[ [0 1 0] \otimes I_{n_j}, \ldots, [0 1 0] \otimes I_{n_j} \right] \end{bmatrix}. \]
\[ D_6 = \begin{bmatrix} 0_{2n_j \times 3d} & \text{diag} \left[ [1 - 1 0] \otimes I_{n_j}, \ldots, [1 - 1 0] \otimes I_{n_j} \right] \end{bmatrix}. \]
\[ \mathcal{S}_j^T \hat{F}_j^T \left[ P_j D_j C_j^T H_j \right] \mathcal{S}_j. \]
\[ \mathcal{F}_j = \text{row}_{i=1,\ldots,M}(F_j, i \neq j). \]
\[ \Pi_j = \begin{bmatrix} -2\varepsilon & \text{diag}_{i=1,\ldots,M}(P_j, i \neq j). \end{bmatrix} \]
Then the Lyapunov functional $V_j(t)$ given by (14) satisfies the following inequality along the solutions of (8):
\[ \dot{V}_j(t) + 2\alpha V_j(t) \leq \frac{2\varepsilon}{M - 1} \sum_{i=1}^{N_j} \left| x_j^T(t) P_j x_j(t) \right|, \quad t \geq t_{ij}^1 - 1. \]

Moreover, in the case where the jth subsystem (8) is independent of other subsystems (i.e. $F_{ij}(\cdot) = 0, l = 1, \ldots, M$), if $\Sigma_j < 0$, then (8) is exponentially stable with a decay rate $\alpha$.

**Proof.** We follow the standard arguments for the exponential stability analysis via Krasovskii method (see e.g. Fridman, 2014, Park et al., 2011). Differentiating $V_j(t)$ we have
\[ \dot{V}_j(t) + 2\alpha V_j(t) \leq 2\alpha x_j^T(t)P_jx_j(t) + 2x_j^T(t)P_jx_j(t) \]
\[ \leq \sum_{i=1}^{N_j} \left| x_{ij}(t) C_{ij} x_j(t) \right|^2 - \left| x_{ij}(t) C_{ij} x_j(t) \right|^2 \]
\[ + \rho_m \left| x_{ij}(t) C_{ij} x_j(t) \right|^2 - \left| x_{ij}(t) C_{ij} x_j(t) \right|^2 \]
\[ - \left| \int_{t-t_m}^{t} e^{2\alpha(s-t)} C_{ij} x_j(s) \, ds \right|^2 \]
\[ - h_j \int_{t-t_m}^{t} e^{2\alpha(s-t)} C_{ij} x_j(s) \, ds \]
\[ \leq \left| \int_{t-t_m}^{t} e^{2\alpha(s-t)} C_{ij} x_j(s) \, ds \right|^2 \]
\[ \leq \rho_m \left| x_{ij}(t) C_{ij} x_j(t) \right|^2. \]
By Jensen’s inequality
\[ \rho_m \left( \int_{t-t_m}^{t} e^{2\alpha(s-t)} C_{ij} x_j(s) \, ds \right)^2 \]
\[ \geq \rho_m \left| x_{ij}(t) C_{ij} x_j(t) \right|^2, \]

## 204

D. Freirich, E. Fridman / Automatica 69 (2016) 201–209
whereas under (15) by arguments of Park et al. (2011)

\[ e^{2\alpha t} x_0 \int_{t-\tau_M^+}^{t-\tau_M^-} e^{2\alpha \tau_M^+} \sqrt{\det R_{\tau_M}} e^{-(\alpha-\delta) s} C_{\tau_M} \hat{y}(s)^2 \, ds \]

\[ \geq C_{\tau_M} (x_0 (t - \eta_M^+) - x_l (t - \eta_M^-)) \]

\[ \times \Gamma_{ij} \left[ C_{\tau_M} (x_0 (t - \tau_M^+) - x_l (t - \tau_M^-)) \right] \]

\[ \leq \left[ \xi_j^T(t) \right] \left[ \begin{array}{c} \Phi \delta_j^+ \xi_j^T(t) \\ \xi_j(t) \end{array} \right] \]

\[ + [D_2 \xi_j(t) + \mathcal{F}_j X(t)]^T C_{\tau_M} [D_2 \xi_j(t) + \mathcal{F}_j X(t)]. \]

Then, by Schur’s complement, (16) implies (17).

For the case of the single jth subsystem, \( \Sigma_j \prec 0 \) implies \( V_j(t) + 2\alpha V_j(t) \leq 0 \), i.e. by comparison principle

\[ x_j^2(t) p_j x_j(t) \leq V_j(t) \leq e^{-2\alpha(t-t_{j-1})} V_j(t_{j-1}). \]

The latter guarantees the exponential stability since \( V_j(t_{j-1}) \leq y_j^2 x_{j-1}^{1/2} \) for some \( y_j > 0 \). \( \Box \)

**Remark 3.** Under A1 there exists \( P_j > 0 \) such that

\[ P_j \left( A_j + \sum_{i=1}^{N_j} A_i C_j \right) + \left( A_j + \sum_{i=1}^{N_j} A_i C_j \right)^T P_j < 0. \]

Then, by standard arguments for delay-dependent conditions (Fridman, 2014), for small enough \( r_j \) and \( \alpha > 0 \) there exist \( R_{\tau_M^+} > 0, R_{\tau_M^-} > 0, S_{\tau_M^+} > 0, S_{\tau_M^-} > 0 \) and \( W_j (i = 1, \ldots, N_j) \) that satisfy (15) and \( \Sigma_j \prec 0 \) with the same \( P_j \). Therefore, by Schur complements, (16) is feasible for given \( \epsilon > 0 \) and small enough \( F_j \).

**Remark 4.** The LMI’s of Proposition 1 and of Theorem 1 (see Section 4) are affine in the system matrices. Therefore, in the case of system matrices from an uncertain time-varying polytope one have to solve these LMI’s simultaneously for all the vertices of the polytope applying the same decision matrices.

**Example 1** (Geromel, Karogui, & Bernussou, 2007, Liu et al., 2015), Consider an inverted pendulum mounted on a small cart. The linearized model can be written as (1) with one subsystem \( F_j = 0, M = 1 \), where \( A_1 = E^{-1} A_1, B_1 = E^{-1} B_0 \) and where

\[ E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/4 & 1/6 \end{bmatrix}. \]

\[ A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -f_c/\epsilon & f_b/\epsilon \end{bmatrix}. \]

\[ \eta_j = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}. \]

Here \( f_c (t) \in [0.15, 0.2] \) and \( f_b (t) \in [0.25, 0.25] \) are uncertain parameters. Thus \( A_1 \) belongs to uncertain polytope, defined by four vertices \( A_k^j \) (\( p = 1, \ldots, 4 \)) corresponding to \( f_c / f_b = 0.15 \) and \( f_c / f_b = 0.25 \). The pendulum can be stabilized by a state feedback \( u_1(t) = K_1 x_1(t) \) with

\[ K_1 = [11.2062 - 128.8597 10.7823 - 22.2629]. \]

Suppose that the state variables are not accessible simultaneously. Consider the case of \( N_1 = 2 \) measures, where \( C_{11} = [1 0 0 0, 0 1 0 0] \), \( C_{21} = [0 0 1 0, 0 0 0 1] \).

For the values of \( \eta_j \) given in Table 1, we apply Remark 4, where LMI (15) and LMI’s (16) corresponding to 4 vertices \( A_k^j \) (with \( A_k^j \) substituted by \( A_k^j \)) are solved with the same decision variables and with \( \alpha = 0.015 \). Table 2 presents the MATI for a given MAD = 0.204 (the case of large communication delay). It is observed that under RR protocol the LMI conditions of Proposition 1 possess essentially less decision variables and are given in terms of smaller LMI’s than Liu et al. (2012, 2015) though in some cases guarantee the exponential stability for larger MATI.

### 3.4. Lyapunov-based analysis under TOD protocol

In this section we assume that the jth subsystem (1) is under TOD scheduling protocol, i.e. \( j \in J_{\text{TOD}} \). Consider the closed-loop model (11)–(12) and the following Lyapunov functional:

\[ V_j^T(t) = V_j(t) + \sum_{i=1}^{N_j} e^{T}(t) Q_{ij} e_{ij}(t) + \omega_j^T(t). \]

(18)

where

\[ \omega_j^T(t) = 2\alpha(t_k - t) e^{T}(t_k) Q_{ij} e_{ij}(t) \]

\[ + \sum_{i=1, i \neq j}^{N_j} \frac{t_k - t}{t_{k+1} - t_k} e^{T}(t_k) U_{ij} e_{ij}(t). \]

\[ V_j(t) = \dot{V}_j(t) + V_j^T(t), \]

\[ V_j^C = \sum_{i=1}^{N_j} h_i \int_{t_k}^{t} e^{2\alpha(t-s)} \sqrt{\det R_{\tau_M}} C_{ij} \hat{y}(s)^2 \, ds; \]
\[ \dot{V}_f(t) = x_f^T(t)P_f x_f(t) \]
\[ + \int_{t_{i-1}}^{t_i} e^{2\alpha(s-t_i)}x_f^T(s)C_f^T S_0 C_f x_f(s)ds \]
\[ + \int_{t_{i-1}}^{t_i} e^{2\alpha(s-t_i)}x_f^T(s)C_f S_f C_f x_f(s)ds \]
\[ + \eta_m \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} e^{2\alpha(s-t_{i-1})}x_f^T(s)C_f R_0 C_f x_f(s)dsd\theta \]
\[ + h_j \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} e^{2\alpha(s-t_{i-1})}x_f^T(s)C_f R_0 C_f x_f(s)dsd\theta, \]

In Liu et al. (2015) the stability of (11)–(12) with \( \tilde{V}_f(t) \) is crucial for the extension of the results to large-scale systems. Indeed, denoting \( \dot{V}_f(t) = V_f(t) |_{w_f=0} \) we have

\[ \dot{V}_f(t) + 2\alpha V_f(t) \leq \dot{V}_f(t) + 2\alpha \dot{V}_f(t) - \frac{1}{\tau_m - \eta_m} \sum_{i=1}^{N_j} |\sqrt{U_j}e_j(t)|^2 \]
\[ - 2\alpha |\sqrt{Q_j}e_j(t)|^2, \quad t \in [t_i, t_{i+1}) \]

along the \( j \)-th hybrid system with \( F_{ij} \). The inequality (20) immediately implies the exponential stability of the \( j \)-th hybrid subsystem, that essentially simplifies the proof of the stability (which is crucial for the extension of the results to large-scale hybrid systems).

The terms
\[ e_j^T(t)Q_j e_j(t) \equiv e_j^T(t_i)Q_j e_j(t_i), \quad t \in [t_i, t_{i+1}) \]

are piecewise-constant, \( \dot{V}(t) \) presents a Lyapunov functional (with reduced-order decision matrices) for systems with interval delays \( \tau(t) \in [\tau_{i-1}, \tau_{i+1}] \). The piecewise-continuous in time term \( \dot{V}_f(t) \) has been introduced in Liu et al. (2015) to cope with the delays in the reset conditions:

\[ V_f(t_{k+1}^-) - V_f(t_{k+1}^+) \]
\[ = \sum_{i=1}^{N_j} h_j \int_{t_{k+1}^-}^{t_{k+1}^+} e^{2\alpha(t_{k+1}^- - s)} |\sqrt{G_{ij}}C_f x_f(s)|^2 ds \]
\[ - \sum_{i=1}^{N_j} h_j \int_{t_{k+1}^-}^{t_{k+1}^+} e^{2\alpha(t_{k+1}^- - s)} |\sqrt{G_{ij}}C_f x_f(s)|^2 ds \]

where we applied Jensen’s inequality (see e.g., Gu, Kharitonov, & Chen, 2003). The function \( V_f(t) \) is continuous and differentiable over \([t_{i+1}^-, t_{i+1}^+]\). The following lemma gives sufficient conditions for the positivity of \( V_f(t) \) and for the fact that it does not grow in the jumps \( t_i^+ \).

**Remark 5.** Differently from Liu et al. (2015), the Lyapunov functional (18) contains novel negative terms \( \tilde{W}_f(t) \) that essentially simplifies the exponential stability analysis of the hybrid system. Indeed, denoting \( \tilde{V}_f(t) = V_f(t) |_{w_f=0} \) we have

\[ \dot{V}_f(t) + 2\alpha \tilde{V}_f(t) \leq \dot{V}_f(t) + 2\alpha \dot{V}_f(t) - \frac{1}{\tau_m - \eta_m} \sum_{i=1}^{N_j} |\sqrt{U_j}e_j(t)|^2 \]
\[ - 2\alpha |\sqrt{Q_j}e_j(t)|^2, \quad t \in [t_i, t_{i+1}) \]

In Liu et al. (2015) the stability of (11)–(12) with \( F_{ij} = 0 \) is guaranteed if the right-hand sides of (19) are non-positive along the system for some \( \alpha > 0 \). By using the novel functional (18), under the same LMIs as in Liu et al. (2015) up to the order reduction due to \( C_f \) in \( \dot{V}_f(t) \) (see LMIs of Proposition 2) we will guarantee that \( V_f \) is positive, does not grow at \( t_i \) and satisfies

\[ \dot{V}_f(t) + 2\alpha V_f(t) \leq 0, \quad t \in [t_i, t_{i+1}) \]

along the \( j \)-th hybrid system with \( F_{ij} \). The inequality (20) immediately implies the exponential stability of the \( j \)-th hybrid subsystem, that essentially simplifies the proof of the stability (which is crucial for the extension of the results to large-scale hybrid systems).

The terms
\[ e_j^T(t)Q_j e_j(t) \equiv e_j^T(t_i)Q_j e_j(t_i), \quad t \in [t_i, t_{i+1}) \]

are piecewise-constant, \( \dot{V}(t) \) presents a Lyapunov functional (with reduced-order decision matrices) for systems with interval delays \( \tau(t) \in [\tau_{i-1}, \tau_{i+1}] \). The piecewise-continuous in time term \( \dot{V}_f(t) \) has been introduced in Liu et al. (2015) to cope with the delays in the reset conditions:

\[ V_f(t_{k+1}^-) - V_f(t_{k+1}^+) \]
\[ = \sum_{i=1}^{N_j} h_j \int_{t_{k+1}^-}^{t_{k+1}^+} e^{2\alpha(t_{k+1}^- - s)} |\sqrt{G_{ij}}C_f x_f(s)|^2 ds \]
\[ - \sum_{i=1}^{N_j} h_j \int_{t_{k+1}^-}^{t_{k+1}^+} e^{2\alpha(t_{k+1}^- - s)} |\sqrt{G_{ij}}C_f x_f(s)|^2 ds \]

where we applied Jensen’s inequality (see e.g., Gu, Kharitonov, & Chen, 2003). The function \( V_f(t) \) is continuous and differentiable over \([t_{i+1}^-, t_{i+1}^+]\). The following lemma gives sufficient conditions for the positivity of \( V_f(t) \) and for the fact that it does not grow in the jumps \( t_i^+ \).

**Lemma 1.** Given a tuning parameter \( \alpha > 0 \), let there exist matrices

\[ 0 < Q_j \in \mathbb{R}^{d_j \times d_j}, 0 < U_j \in \mathbb{R}^{d_i \times d_i} \text{ and } 0 < G_{ij} \in \mathbb{R}^{d_j \times d_i}, i = 1 \ldots N_j \text{ that satisfy the LMIs} \]
\[ \Omega_{ij} \equiv \left[ -\frac{1 - 2\alpha(\tau_j - \eta_m)}{\eta_m} Q_j + U_j \begin{array}{cc} Q_j & \text{ } \\ \\ Q_j & -G_{ij}e^{-2\alpha\tau_j} \end{array} \right] < 0, \quad i = 1 \ldots N_j. \]

Then \( V_f(t) \) of (18) is positive in the sense that

\[ V_f(t) \geq \beta \left( |x_i(t)|^2 + |e_j(t)|^2 \right), \quad t \geq t_0, \quad e_j(t) \equiv \text{col}(e_1(t), \ldots, e_{N_j}(t)) \]

for some \( \beta > 0 \). Moreover, \( V_f(t) \) does not grow in the jumps along (11)–(12):

\[ \theta \equiv V_f(t_{k+1}^-) - V_f(t_{k+1}^+) \leq 0. \]

**Proof.** It can be seen that (22) implies

\[ \alpha(\tau_j - \eta_m) < 0.5 \quad \text{and} \quad U_j < Q_j \]

yielding the positivity of \( V_f(t) \).

We show next that \( V_f(t) \) does not grow in the jumps. Since
\[ \dot{V}_f(t_{k+1}^-) = \dot{V}_f(t_{k+1}^+) \quad \text{and} \quad e_j(t_{k+1}^-) = e_j(t_{k+1}^+) \],

we obtain by taking into account (21)

\[ \theta \equiv \sum_{i=1}^{N_j} \left[ |\sqrt{Q_j}e_j(t_{k+1}^-)|^2 - |\sqrt{Q_j}e_j(t_{k+1}^+)|^2 \right] \]
\[ + 2\alpha(t_{k+1}^- - t_{k+1}^+) |\sqrt{Q_j}e_j(t_{k+1}^-)|^2 \]
\[ + \sum_{i \neq j} |\sqrt{U_j}e_j(t_{k+1}^-)|^2 + V_f(t_{k+1}^-) - V_f(t_{k+1}^+) \]
\[ \leq |\sqrt{Q_j}e_j(t_{k+1}^-)|^2 + \sum_{i=1}^{N_j} \left[ |\sqrt{Q_j}e_j(t_{k+1}^+)|^2 \right] - |\sqrt{Q_j} - U_j^T | \left| \sqrt{Q_j}e_j(t_{k+1}^-) \right|^2 \]
\[ - \sum_{i=1}^{N_j} e^{-2\alpha\tau_j} |\sqrt{G_{ij}}C_f x_f(s_{k+1})|^2 \]

Under TOD
\[ - |\sqrt{Q_j}e_j(t_{k+1}^-)|^2 \leq - \frac{1}{N_j} \sum_{i \neq j} |\sqrt{Q_j}e_j(t_{k+1}^-)|^2. \]
Denote \( \zeta_i = \text{col} \{ e_i(t_i^0), C_i [x_i(t_i^0) - x_i(s^1_{i+1})] \} \). Then, employing (12) we arrive at
\[
\Theta \leq -\left[ G_i^T e^{-2\alpha t_i^0} Q_i - Q_i G_i^T[C_i x_i(s^1_i) - x_i(s^1_{i+1})] \right]^2
+ \sum_{i \neq j} \left[ S_i^T \left[ C_i x_i(s^1_i) - x_i(s^1_{i+1}) \right] + e_j(t_i^0) \right]^2
- \left[ Q_i - \frac{1 - 2\alpha h_j}{N_i - 1} \right] Q_j - U_i e_j(t_i^0)^2
- e^{-2\alpha t_i^0} \left[ C_i x_i(s^1_i) - x_i(s^1_{i+1}) \right]^2
\]
\[
= -\left[ G_i^T e^{-2\alpha t_i^0} Q_i - Q_i G_i^T[C_i x_i(s^1_i) - x_i(s^1_{i+1})] \right]^2
+ \sum_{i \neq j} \left[ \zeta_i^T \Omega_{ij} \zeta_i \right].
\]
Therefore, under (22) \( \Theta \leq 0 \). \( \square \)

By applying Lemma 1 and by modifying derivations of Liu et al. (2015) for \( F_j \neq 0 \) we arrive at.

**Proposition 2.** Consider the jth hybrid subsystem (11)-(12). Given tuning parameters \( \alpha > \varepsilon > 0 \) and \( (M - 1) N_i \times n_i \) matrices \( P_i \succ 0 \), let there exist a \( n_i \times n_i \) matrix \( P_i \succ 0 \), \( n_i \times n_i \) matrices \( R_{0j} \succ 0 \), \( R_{ij} \succ 0 \), \( S_{ij} \succ 0 \), \( W_{ij} \) and \( n_i \times n_i \) matrices \( Q_{ij} \succ 0 \), \( U_{ij} \succ 0 \), \( G_{ij} \succ 0 \) \((i = 1 \ldots N_j)\) that satisfy the LMIs (22) and
\[
\Gamma_j = \begin{bmatrix} R_{ij} & W_j \\ W_j^T & R_{ij} \end{bmatrix} \succ 0, \quad \Sigma_j \preceq 0, \quad i = 1 \ldots N_j, \quad (25)
\]
where
\[
\Sigma_j = \begin{bmatrix} \phi_j & D_j^T C_j^T H_j \\ 0 & -H_j \end{bmatrix}, \quad \phi_j = D_j^T (2\alpha P_j + C_j^T S_j C_j) D_j + (D_j^T P_j D_j + D_j^T P_j D_j)
- \rho_m D_j^T (S_j - S_j) D_j - \rho_m D_j^T R_0 j D_j - \rho D_j^T S_j D_j
- \rho D_j^T \Omega_{ij} D_j + D_j^T \Omega_{ij} D_j + D_j^T \Omega_{ij} D_j \succ 0,
\]
\[
H_j = n_i^2 R_{0j} + (\tau_m - \eta_m)^2 R_{ij} + h_j \cdot \text{diag}(G_{1j}, \ldots, G_{Nj}),
\]
\[
\Psi_j = \text{diag}(\psi_1, \ldots, \psi_{j-1}, \ldots), \quad \psi_i \preceq 2\alpha Q_j - \frac{1}{h_j} U_{ij}, \quad 0.
\]

We are in a position to formulate the main result:

**Theorem 1.** Given tuning parameters \( \alpha > \varepsilon > 0 \), let there exist \( n_i \times n_i \) matrices \( P_i \succ 0 \) \((i \in J)\), \( n_i \times n_i \) matrices \( R_{0i} \succ 0 \), \( R_{ij} \succ 0 \), \( S_{ij} \succ 0 \), \( W_{ij} \) and \( n_i \times n_i \) matrices \( Q_{ij} \succ 0 \), \( U_{ij} \succ 0 \), \( G_{ij} \succ 0 \) \((i = 1 \ldots N_j, j \in J_{\text{req}})\) that satisfy the LMIs (15) and (16) for all \( j \in J_{\text{req}} \) and \( n_i \times n_i \) matrices \( R_{0j} \succ 0 \), \( R_{ij} \succ 0 \), \( S_{ij} \succ 0 \), \( W_{ij} \) and \( n_i \times n_i \) matrices \( Q_{ij} \succ 0 \), \( U_{ij} \succ 0 \), \( G_{ij} \succ 0 \) \((i = 1 \ldots N_j, j \in J_{\text{req}})\) that satisfy the LMIs (22) and (25) for all \( j \in J_{\text{req}} \). Then the closed-loop large-scale system (1), (3) is exponentially stable with a decay rate \( \alpha_0 = \alpha - \varepsilon \).

**Proof.** Let the LMIs of the theorem be feasible. We choose the following Lyapunov functional for the large-scale system (1), (3):
\[
V(t) = \sum_{j \in J_{\text{req}}} V_j(t) + \sum_{j \in J_{\text{fod}}} V_j^0(t), \quad t \geq 0,
\]
where \( V_j(t)_{l \in J_{\text{fod}}} \) is given by (14) and \( V_j^0(t)_{l \in J_{\text{fod}}} \) is given by (18). Define \( x(t) = x(0) \) for \( t < 0 \) and denote
\[
\Delta_{\text{fod}} = \{ t \geq 0 \mid t = t_k, j \in J_{\text{fod}}, k \in \mathbb{Z}_+ \}.
\]
Let \( T \) be given by (4).

We apply further Propositions 1 and 2. Then for some constants \( 0 < \beta_m < \beta_M \) \( V \) satisfies the following bounds:
\[
\beta_m \left[ x(t)^2 \right] + \sum_{j \in J_{\text{fod}}} \left[ \xi_j(t)^2 \right] \leq V(t) \leq \beta_M \left[ \| x(t) \|_{W([-\tau_m, 0])} + \sum_{j \in J_{\text{fod}}} \| \xi_j(t) \|_{W([-\tau_m, 0])} \right],
\]
(27)
where $\tau_m = \max_{j \in J} \tau_m^j$. Moreover, by summing in $j = 1, \ldots, M$ the inequalities (17) and (26) we obtain that
\[
\dot{V}(t) + 2\alpha V(t) \leq 2\varepsilon \sum_{l=1}^{M} e_l^j(t) P_l x_l(t)
\]
for all $t \geq T$ and $t \notin \Delta_{\text{TOD}}$, implying
\[
\dot{V}(t) + 2(\alpha - \varepsilon) V(t) \leq 0, \quad \forall t \geq T, \quad t \notin \Delta_{\text{TOD}}.
\] (28)
Additionally we have
\[
V(t) - V(t^-) \leq 0, \quad \forall t \geq T, \quad t \in \Delta_{\text{TOD}}
\] (29)
along (1), (3). The inequalities (28) and (29) yield
\[
V(t) \leq e^{-2\alpha(t-T)} V(T), \quad t \geq T.
\] (30)
Then from (27), (30) for some $\gamma > 0$ we have
\[
|\dot{x}(t)|^2 + \sum_{j \in J_{\text{TOD}}} |e_j(t)|^2 \leq \gamma e^{-2\alpha(t-T)} \left[ \|x_T\|_{W[-T,0]}^2 + \sum_{j \in J_{\text{TOD}}} |e_j(T)|^2 \right], \quad t \geq T.
\] (31)

We will show next that for some $\gamma_0 > 0$
\[
\sum_{j \in J_{\text{TOD}}} |e_j(T)|^2 \leq \gamma_0 \|x_T\|_{W[-T,0]}^2.
\] (32)
Indeed, from (24) and (26) we obtain that for some $\gamma_1 > 0$ the following holds for $t \in [t_k^j, t_{k+1}^j)$:
\[
V_j^\infty(t) \leq e^{-2\alpha(t-t^-)} V_j^\infty(t^-) + \gamma_1 \int_{t^-}^{t} e^{-2\alpha(s-t^-)} |x(s)|^2 ds
\leq \cdots \leq e^{-2\alpha(t-t^-)} V_j^\infty(t^-) + \gamma_1 \int_{t^-}^{t} e^{-2\alpha(s-t^-)} |x(s)|^2 ds.
\] (33)
Taking into account the initial conditions (13) and $x(t) = x(0)$ for $t < 0$ we arrive at $V_j^\infty(t^-) \leq \beta_{j\infty} \|x(t^-)\|_{W[-t^-,0]}$ with some $\beta_{j\infty} > 0$. Moreover, $\beta_{j\infty} |e_j(t^-)|^2 \leq V_j^\infty(t^-)$ for some $\beta_{j\infty} > 0$ that together with (33) yield (32). The inequalities (31) and (32) imply (5) with $c = \sqrt{\gamma} + \gamma_0$. □

**Remark 7.** The inequalities (31), (32) imply the exponentially converging bound on the errors $e_j(t), j \in J_{\text{TOD}}$ meaning the exponential stability of the large-scale hybrid system given by (11)–(12) for $j \in J_{\text{TOD}}$ and by (8) for $j \in J_{\text{RR}}$.

**Example 2 (Borgers & Heemels, 2014).** Consider two coupled inverted pendulums under the scenario of decentralized networked control, where $M = 2$, $N_j = 2$ or $N_j = 4$ ($j = 1, 2$). The system matrices are given by

\[
A_1 = A_2 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
2.9156 & 0 & -0.0005 & 0 \\
0 & 0 & 0 & 1 \\
-1.6663 & 0 & 0.0002 & 0
\end{bmatrix},
\]

\[
B_1 = B_2 = \begin{bmatrix}
0 & -0.0042 & 0 & 0.0167 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
F_{12} = F_{21} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0.0011 & 0 & 0.0005 & 0 \\
0 & 0 & 0 & 0 \\
-0.0003 & 0 & -0.0002 & 0
\end{bmatrix}.
\]

**Table 3**

<table>
<thead>
<tr>
<th>$N$</th>
<th>$2\tau_1^\infty$</th>
<th>$\gamma_1^\infty$</th>
<th>$4\tau_1^\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 1 (RR)</td>
<td>0.0079</td>
<td>0.0012</td>
<td>0.0038</td>
</tr>
<tr>
<td>Theorem 1 (TOD)</td>
<td>0.0079</td>
<td>0.0012</td>
<td>0.0038</td>
</tr>
</tbody>
</table>

**Table 4**

<table>
<thead>
<tr>
<th>$N$</th>
<th>$2\tau_1^\infty$</th>
<th>$\gamma_1^\infty$</th>
<th>$4\tau_1^\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 1 (RR)</td>
<td>0.0104</td>
<td>0.0037</td>
<td>0.0005</td>
</tr>
<tr>
<td>Theorem 1 (TOD)</td>
<td>0.0079</td>
<td>0.0012</td>
<td>0.0038</td>
</tr>
</tbody>
</table>

**Table 5**

<table>
<thead>
<tr>
<th>$N$</th>
<th>$2\tau_1^\infty$</th>
<th>$\gamma_1^\infty$</th>
<th>$4\tau_1^\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 1 (RR)</td>
<td>0.0079</td>
<td>0.0012</td>
<td>0.0038</td>
</tr>
<tr>
<td>Theorem 1 (TOD)</td>
<td>0.0079</td>
<td>0.0012</td>
<td>0.0038</td>
</tr>
</tbody>
</table>

K_1 = [k_{11} k_{21} k_{31} k_{41}] = [11 396 7196.2 573.96 1199.0],

K_2 = [k_{12} k_{22} k_{32} k_{42}] = [129 241 18 135 2875.3 3693.9].

In the case of $N_j = 2$ we consider

C_{1j} = [1 0 0 0],

C_{2j} = [1 0 1 0],

K_{1j} = [k_{1j} k_{2j}],

K_{2j} = [k_{3j} k_{4j}], \quad j = 1, 2.

In the case of $N_j = 4$, $C_{1j}, \ldots, C_{4j}$ are the rows of $I_4$ and $K_{1j}, \ldots, K_{4j}$ are the entries of $K_j$.

We analyze the exponential stability for $\eta_m^j = 0$ by applying LMI conditions of Theorem 1 with $\alpha = 0.015$ and $\varepsilon = 0.002$ for the case where both pendulums are either under RR or under TOD protocols (the resulting decay rate $\alpha_0$ is 0.013). Maximum values of $\tau_m^j$ that preserve the stability are given in Table 3. Then for $MAD_j = 0$ and $MAD_j = 0.005$ ($MAD_j = 0.005$ is larger than max $MAD_j$ achieved in Borgers and Heemels (2014)) the resulting maximum $MAD_j$ that preserve the stability are given in Tables 4 and 5 respectively.

It is seen that the presented method leads to essentially larger values of maximum $MAD_j$ comparatively to Borgers and Heemels (2014) and allows large values of $MAD_j$. Moreover, our method is applicable in this example with a much stronger coupling. Thus, for $F_{12} = F_{21} = 40 \cdot A_{12}$ by Theorem 1 the stability is preserved e.g. for $MAD_j = MADM_j = 0.001(j = 1, 2)$ (either under RR or under TOD protocols).

**5. Conclusions**

In this paper, a time-delay approach has been developed for the decentralized exponential stabilization of large-scale NCSS with local networks, where asynchronous variable sampling intervals, large bounded variable communication delays and RR/TOD scheduling protocols are taken into account. The presented novel Lyapunov–Krasovskii method leads to LMI conditions for the exponential stability of the closed-loop large-scale system. Being applied to the example of two coupled pendulums with local networks, our results are favorably compared to the existing ones. The presented new technique may be useful for decentralized control of microgrids with islanded generators. Future work may
involve consideration of stochastic communication delays and observer-based networked control. The presented approach may be useful for asynchronous decentralized control in microgrids (Vasquez et al., 2010).

References


Dror Freirich received his B.Sc. in Computers Engineering from the Technion (IIT). He is finishing his studies towards an M.Sc. degree in Electrical Engineering at Tel Aviv University, Israel.

His research interests include stability and control of time-delay and networked control systems, as well as machine-learning applications.

Emilia Fridman received the M.Sc. degree from Kuibyshev State University, USSR, in 1981 and the Ph.D. degree from Voronezh State University, USSR, in 1986, all in mathematics.

From 1986 to 1992 she was an assistant and associate professor in the Department of Mathematics at Kuibyshev Institute of Railway Engineers, USSR. Since 1993 she has been at Tel Aviv University, where she is currently Professor of Electrical Engineering-Systems. She has held visiting positions at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin (Germany), INRIA in Rocquencourt (France), Ecole Centrale de Lille (France), Valenciennes University (France), Leicester University (UK), Kent University (UK), CINVESTAV (Mexico), Zhejiang University (China), St. Petersburg IPM (Russia), Melbourne University (Australia), Sup-elec (France), and KTH (Sweden).

Her research interests include time-delay systems, networked control systems, distributed parameter systems, robust control, singular perturbations and nonlinear control.

She has published more than 100 articles in international scientific journals. She is the author of the monograph “Introduction to Time-Delay Systems: Analysis and Control” (Birkhauser, 2014).

In 2014 she was nominated as a Highly Cited Researcher by Thomson ISI. Currently she serves as an associate editor in Automatica and SIAM Journal on Control and Optimization.