



Brief paper

On global exponential stability preservation under sampling for globally Lipschitz time-delay systems[☆]



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ABSTRACT

The paper shows that the global exponential stability property is preserved, under suitably fast sampling and small input-delay, whenever the dynamics of the time-delay system at hand and the related stabilizing (in continuous-time) state feedback are described by globally Lipschitz maps. The Halanay's inequality is used in order to prove this result. Continuous-time, possibly non-affine in the control, state-delay systems are considered. The knowledge of a Lyapunov–Krasovskii functional for the continuous-time closed-loop system is not required, as long as this system is globally exponentially stable. The knowledge of a Lipschitz Lyapunov–Krasovskii functional allows for an estimation of the sampling period that preserves the exponential stability, as well as of the decay rate.

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1. Introduction

The sampled-data stabilization problem has been extensively studied in the literature, with many approaches, for nonlinear finite dimensional systems. See, for instance: Fridman and Dambrine (2009), Fridman, Seuret, and Richard (2004), Grune and Netic (2003), Kellett, Shim, and Teel (2004), Khalil (2004), Laila, Netic, and Teel (2002), Monaco, Normand-Cyrot, and Tiefense (2011), Naghshtabrizi, Hespanha, and Teel (2008), Netic and Grune (2005), Netic and Laila (2002), Netic and Teel (2004a,b), Postoyan, Ahmed-Ali, and Lamnabhi-Lagarrigue (2009), Seuret and Gomes Da Silva (2012) and Zaccarian, Teel, and Netic (2003).

In general, semi-global stability results (for instance, of practical type, with possible arbitrarily small final target ball of the origin) are provided in the literature. The global (asymptotic, exponential) stability preservation under sampling, for finite dimensional nonlinear systems, has been dealt with in: Herrmann,

Spurgeon, and Edwards (1999), Hsu and Sastry (1987), Karafyllis and Kravaris (2009), Laila, Netic, and Astolfi (2006), Mazenc, Malisoff, and Dinh (2013) and Netic, Teel, and Carnevale (2009). It is shown in Herrmann et al. (1999) and Hsu and Sastry (1987) that the global exponential stability is preserved under suitably fast sampling, for globally Lipschitz systems in control-affine form. The global asymptotic stability preservation (and the input-to-state stability preservation, with respect to external disturbances) under suitably fast sampling, is studied in Karafyllis and Kravaris (2009), where sufficient conditions, expressed by means of single and vector Lyapunov functions, are provided. The maximum allowed sampling period, by which the global (asymptotic, exponential) stability is preserved under sampling, is studied in Netic et al. (2009), by means of Lyapunov-like sufficient conditions and the hybrid systems approach. Sampled-data control of bilinear systems is investigated in Omran, Hetel, Richard, and Lamnabhi-Lagarrigue (2014), where local asymptotic stability of the (sampled state feedback) closed-loop system is proved by means of the feasibility of suitable linear matrix inequalities. In Ahmed-Ali, Fridman, Giri, Burlion, and Lamnabhi-Lagarrigue (2016), sufficient conditions, in terms of linear matrix inequalities, are given for the exponential stability, under sampling, of linear systems with globally Lipschitz perturbations, as well as of systems described by semi-linear parabolic partial differential equations. A linear time-varying state feedback is used, as standard in the generalized hold functions theory (see Briat, 2014, and the references therein). As far as sampled-data control of finite dimensional nonlinear systems, affected by small delays in the input channel, is concerned,

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sufficient conditions, in terms of Lyapunov functions, are provided for the global asymptotic stability preservation under sampling, for control affine, time-varying systems, in Mazenc et al. (2013). Sampled-data control of fully nonlinear (i.e., possibly non-affine in the control) systems, with large delays in the input/output channels, is extensively studied in Karafyllis and Krstic (2012). A few results are available in the literature, concerning the stability preservation under sampling for nonlinear systems affected by state-delays. Sampled-data control of linear systems with state-delays is studied in Suplin, Fridman, and Shaked (2009). Semi-global practical stability results, with arbitrarily small final target ball of the origin, are provided in Pepe (2014, 2016) for the class of fully nonlinear systems with state-delays, admitting suitable control Lyapunov–Krasovskii functionals and related steepest descent feedbacks.

To our best knowledge, a proof of the expected global exponential stability preservation under high frequency sampling, for fully nonlinear, globally Lipschitz time-delay systems, is missing in the literature. In this paper we provide this proof. We assume that, in continuous time, the system at hand is globally, exponentially stabilizable by a globally Lipschitz feedback. The main tools here used are the Halanay’s inequality (Halanay, 1966), as extended with the use of upper-right hand Dini derivatives (Baker & Buckwar, 2005), and the derivative in Driver’s form of Lyapunov functionals, whose existence is guaranteed by converse theorems (Karafyllis, Pepe, & Jiang, 2008; Krasovskii, 1963; Pepe & Karafyllis, 2013). The knowledge of a Lyapunov–Krasovskii functional for the closed-loop continuous-time system is not required, as long as this system is globally exponentially stable. If a globally Lipschitz Lyapunov–Krasovskii functional is known, then a precise characterization of the sufficiently small sampling period is provided. This sampling period is computed by using the involved Lipschitz constants and the lower and upper bounds related to the Lyapunov–Krasovskii functional. However, the results provided here are of the existence type, and the study of the conservativeness of the provided sampling frequency is beyond the aims of the paper. The existence results provided in the paper can be used also for delay-free, globally Lipschitz, fully nonlinear systems, which are globally exponentially stabilizable by globally Lipschitz state feedback. Small input-delays, due to computations and/or signal transmission, are also addressed. A preliminary version of this paper has been published in the conference paper (Pepe & Fridman, 2016). The main novelty of this paper, with respect to Pepe and Fridman (2016), concerns the results for the general case with memory feedback, which is the more frequent case in the control of time-delay systems. Moreover, the problem of a small input-delay is not studied in the conference paper.

Notation. The symbol R denotes the set of real numbers, R^* denotes the extended real line $[-\infty, +\infty]$, R^+ denotes the set of non-negative reals $[0, +\infty)$. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. For a positive integer n , for a non-negative real Δ (maximum involved time-delay), \mathcal{C} denotes the space of the continuous functions mapping $[-\Delta, 0]$ into R^n . The space \mathcal{C} is endowed with the supremum norm, here denoted with the symbol $\|\cdot\|_\infty$, defined, for $\phi \in \mathcal{C}$, as $\|\phi\|_\infty = \sup_{\theta \in [-\Delta, 0]} |\phi(\theta)|$. Notice that, when $\Delta = 0$, the spaces \mathcal{C} and R^n are isomorphic and, for any $\phi \in \mathcal{C}$, $\|\phi\|_\infty = |\phi(0)|$. For a continuous function $x : [-\Delta, c) \rightarrow R^n$, with $0 < c \leq +\infty$, for any real $t \in [0, c)$, x_t is the function in \mathcal{C} defined as $x_t(\tau) = x(t + \tau)$, $\tau \in [-\Delta, 0]$. For given positive integers n, m , a map $f : \mathcal{C} \times R^m \rightarrow R^n$ is said to be globally Lipschitz if there exists a positive real L such that, for any $\phi_i \in \mathcal{C}$, $u_i \in R^m$, $i = 1, 2$, the inequality holds $|f(\phi_1, u_1) - f(\phi_2, u_2)| \leq L(\|\phi_1 - \phi_2\|_\infty + |u_1 - u_2|)$. For a continuous function $z : R^+ \rightarrow R$, $D^+z : R^+ \rightarrow R^*$ denotes the upper right-hand Dini derivative of z , defined, for $t \in R^+$, as $D^+z(t) = \limsup_{h \rightarrow 0^+} \frac{z(t+h) - z(t)}{h}$. For

given positive integers n, m , continuous map $f : \mathcal{C} \times R^m \rightarrow R^n$, continuous functional $V : \mathcal{C} \rightarrow R^+$, $D^+V : \mathcal{C} \times R^m \rightarrow R^*$ denotes the derivative in Driver’s form of V , defined, for $\phi \in \mathcal{C}$, $u \in R^m$, as follows (see Driver, 1962; Karafyllis et al., 2008; Pepe & Karafyllis, 2013) $D^+V(\phi, u) = \limsup_{h \rightarrow 0^+} \frac{V(\phi_{h,u}) - V(\phi)}{h}$, where $\phi_{h,u} \in \mathcal{C}$ is given, in the case $\Delta > 0$, for $h \in [0, \Delta)$, as

$$\phi_{h,u}(\theta) = \begin{cases} \phi(\theta + h), & \theta \in [-\Delta, -h), \\ \phi(0) + (\theta + h)f(\phi, u), & \theta \in [-h, 0], \end{cases} \quad (1)$$

and, in the case $\Delta = 0$, for $h \in [0, 1)$, as

$$\phi_{h,u}(0) = \phi(0) + hf(\phi, u). \quad (2)$$

Throughout the paper, ODE stands for Ordinary Differential Equation, RFDE stands for Retarded Functional Differential Equation, GES stands for Globally Exponentially Stable or Global Exponential Stability, GAS stands for Globally Asymptotically Stable or Global Asymptotic Stability, LK stands for Lyapunov Krasovskii, MASP stands for maximum allowed sampling period, MAD stands for maximum allowed delay, ZOH stands for zero order hold, LMI stands for linear matrix inequality. A system is said to be 0-GES (0-GAS) if the origin of the state space is an equilibrium point and it is globally exponentially (asymptotically) stable.

2. Preliminaries

Let us consider the system described by the following fully nonlinear (i.e., non-affine in the control) RFDE

$$\dot{x}(t) = f(x_t, u(t)), \quad x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0], \quad (3)$$

where $\Delta \geq 0$ is the maximum involved state time-delay, $x(t) \in R^n$, $x_0, x_t \in \mathcal{C}$, $u(t) \in R^m$, $t \geq 0$, n, m are positive integers, f is a map from $\mathcal{C} \times R^m$ to R^n , satisfying $f(0, 0) = 0$ (regularity of the map f will be established in forthcoming Assumption 2). The following lemma establishes necessary and sufficient conditions for the global exponential stability of the continuous-time system described by (4), with f globally Lipschitz, in closed-loop with a globally Lipschitz state feedback.

Lemma 1 (See Krasovskii, 1963; Karafyllis et al., 2008; Pepe & Karafyllis, 2013). *Let the map f in (3) be globally Lipschitz. Let $k : \mathcal{C} \rightarrow R^m$ be a globally Lipschitz map, satisfying $k(0) = 0$. Then, the continuous-time closed-loop system described by (3), with $u(t) = k(x_t)$, $t \geq 0$, is 0-GES if and only if there exist a globally Lipschitz functional $V : \mathcal{C} \rightarrow R^+$, with L_V as Lipschitz constant, and positive reals α_i , $i = 1, 2, 3$, such that the following inequalities hold for any $\phi \in \mathcal{C}$: (i) $\alpha_1 \|\phi\|_\infty \leq V(\phi) \leq \alpha_2 \|\phi\|_\infty$; (ii) $D^+V(\phi, k(\phi)) \leq -\alpha_3 \|\phi\|_\infty$.*

We introduce here the following assumption for the system described by (3).

Assumption 2. The map $f : \mathcal{C} \times R^m \rightarrow R^n$ is globally Lipschitz in $\mathcal{C} \times R^m$ with Lipschitz constant L_f ; there exists a globally Lipschitz feedback $k : \mathcal{C} \rightarrow R^m$, with Lipschitz constant L_k , satisfying $k(0) = 0$, such that the continuous-time, closed-loop system described by the RFDE

$$\dot{x}(t) = f(x_t, k(x_t)) \quad (4)$$

is 0-GES.

The following lemma (Baker & Buckwar, 2005) is a key issue for the results of the paper. It extends the Halanay’s inequality (see Halanay, 1966) to the case of continuous functions and related upper right-hand Dini derivative (the original Halanay’s inequality is given for the case of continuous functions and related lower left-hand Dini derivative, see Halanay, 1966). Actually, in forthcoming study with an involved Lyapunov function or LK functional, the upper right-hand Dini derivative is needed.

Lemma 3 (Baker & Buckwar, 2005, Lemma 6 and Theorem 7). Let a, b, r be positive reals, $a > b$. Let $z : [-r, +\infty) \rightarrow R^+$ be a continuous function satisfying the inequality

$$D^+z(t) \leq -az(t) + b \sup_{\theta \in [-r, 0]} z(t + \theta), \quad t \geq 0. \quad (5)$$

Let λ be the positive real solution of the equation

$$a - \lambda = be^{\lambda r}. \quad (6)$$

Then, the inequality $z(t) \leq \sup_{\theta \in [-r, 0]} z(\theta)e^{-\lambda t}$ holds for any $t \geq 0$.

Definition 4 (See Clarke, Ledyaev, Sontag, & Subbotin, 1997). A partition $\pi = \{t_i, i = 0, 1, \dots\}$ of $[0, +\infty)$ is a countable, strictly increasing sequence t_i , with $t_0 \geq 0$, such that $t_i \rightarrow +\infty$ as $i \rightarrow +\infty$. The diameter of π , denoted $\text{diam}(\pi)$, is defined as $\sup_{i \geq 0} t_{i+1} - t_i$.

We denote with δ_u (the acronym MAD is used in Fridman, 2014) an upper bound on the delay induced by data transmission and/or by computations. We denote with δ_S (the acronym MASP is used in Nesic & Teel, 2004b; Nesic et al., 2009) an upper bound for the time elapsed between any two sensor updates (i.e., successive sensor updates are separated by at most δ_S s). We denote with $\pi_{\text{ZOH}} = \{t_0, t_1, \dots\}$ the partition induced by the update times of the ZOH device, assumed to be co-located with the controlled system. That is, the value of the piece-wise constant control law acting on the system, previously computed by the controller and sent to the ZOH device after receiving the sensor data, is updated at $t_k, k = 0, 1, \dots$. We denote with $\pi_S = \{s_0 = 0, s_1, \dots\}$ the partition induced by the update times of the sensor device, assumed to be co-located with the controlled system. That is, the sensor output is updated at times $s_k, k = 0, 1, \dots$. Notice that we assume $s_0 = 0$. The plant sampler is time-driven, whereas the controller, which may not be co-located with the controlled system, and the ZOH device are event-driven. That is, the controller starts computing a new value for the piece-wise control law as soon as it receives a new sample, as well as the ZOH device updates its output as soon as it receives the new data from the controller. Finally, we denote with $\eta_k, k = 0, 1, \dots$, the time-delay due to data transmission and/or to computations, elapsed from time s_k , at which the sensor's output is updated, and the time the related new value of the piece-wise constant control signal is received by the ZOH device. We assume that, for any $k = 0, 1, \dots$, the relation holds $t_k = s_k + \eta_k$. So, the piece-wise control law acting on the system, related to the sampled data x_{s_k} , is updated at $t_k = s_k + \eta_k$. We assume here that successive sensor's messages, sent to controller at times s_k and s_{k+1} , with $s_k < s_{k+1}, k = 0, 1, \dots$, reach the ZOH device at successive times, that is at times t_k and t_{k+1} , respectively, with $t_k < t_{k+1}, k = 0, 1, \dots$. Moreover, without any loss of generality (no finite time escape phenomenon arises, since the systems considered here are globally Lipschitz), we assume that in the interval $[0, t_0] = [0, \eta_0]$, the control signal, as provided by the ZOH device to the system, is constant and equal to \bar{u} , with $\bar{u} \in R^m$ of given value. The reader can refer to Fridman (2014, Chapter 7.5, pp. 309–314) (see, in particular, Figure 7.5), for more detailed explanations on the above described control system.

3. Main results

In the following we show that, if the time-delay system at hand is globally Lipschitz, and is globally exponentially stabilizable by a globally Lipschitz state feedback, when applied in continuous time, then there exists a positive real δ_{\max} such that, if $\delta_S + \delta_u < \delta_{\max}$, the global exponential stability is preserved. Moreover, we provide also a method for the computation of δ_{\max} , based on LK functionals.

Theorem 5. Let Assumption 2 hold. Let $\alpha_1, \alpha_2, \alpha_3, L_V, L_k, L_f$ be the positive constants provided in Lemma 1 and Assumption 2. Let δ_{\max} be the positive real

$$\delta_{\max} = \frac{\alpha_3 \alpha_1}{\alpha_2 L_V L_f^2 L_k (1 + L_k)}. \quad (7)$$

Let $\delta_S + \delta_u < \delta_{\max}$. Let λ be the real positive solution of the equation

$$\frac{\alpha_3}{\alpha_2} - \lambda = \frac{L_V L_f^2 L_k (1 + L_k)}{\alpha_1} (\delta_S + \delta_u) e^{\lambda(\Delta + 2(\delta_S + \delta_u))}. \quad (8)$$

Then, the solution of the RFDE

$$\begin{aligned} \dot{x}(t) &= f(x_t, u(t)), \\ x(\tau) &= x_0(\tau), \quad \tau \in [-\Delta, 0], \quad x_0 \in \mathcal{C}, \\ u(t) &= \begin{cases} \bar{u}, & t_0 > 0, \quad t \in [0, t_0), \\ k(x_{s_j}), & t \in [t_j, t_{j+1}), \quad s_j \in \pi_S, \quad t_j \in \pi_{\text{ZOH}}, \\ & j = 0, 1, \dots, \end{cases} \end{aligned} \quad (9)$$

exists for all $t \geq 0$, and, furthermore, satisfies the inequality

$$|x(t)| \leq Re^{-\lambda t}, \quad t \geq 0, \quad (10)$$

with

$$R = \frac{\alpha_2}{\alpha_1} (\|x_0\|_\infty + L_f(\Delta + 3(\delta_S + \delta_u))|\bar{u}|) \cdot e^{L_f(1+L_k)(\Delta+3(\delta_S+\delta_u))}. \quad (11)$$

Proof. From the global Lipschitz property of the map f and the properties of the partition π_{ZOH} , it follows that the system described by (9) admits a (unique) locally absolutely continuous solution in R^+ . Let $x_t \in \mathcal{C}$ be the solution of (9). We show first that, for any $t \in [0, \Delta + 3(\delta_S + \delta_u)]$, the inequality holds

$$\|x_t\|_\infty \leq (\|x_0\|_\infty + L_f(\Delta + 3(\delta_S + \delta_u))|\bar{u}|) \cdot e^{L_f(1+L_k)(\Delta+3(\delta_S+\delta_u))}. \quad (12)$$

We have, for $t \in [0, \Delta + 3(\delta_S + \delta_u)]$,

$$\begin{aligned} \sup_{\theta \in [0, t]} \|x_\theta\|_\infty &\leq \|x_0\|_\infty + \int_0^t |f(x_\tau, u(\tau))| d\tau \\ &\leq \|x_0\|_\infty + \int_0^t L_f (\|x_\tau\|_\infty + |u(\tau)|) d\tau \\ &\leq \|x_0\|_\infty + \int_0^t L_f \left((1 + L_k) \sup_{\alpha \in [0, \tau]} \|x_\alpha\|_\infty + |\bar{u}| \right) d\tau \\ &\leq \|x_0\|_\infty + L_f(\Delta + 3(\delta_S + \delta_u))|\bar{u}| \\ &\quad + \int_0^t L_f(1 + L_k) \sup_{\alpha \in [0, \tau]} \|x_\alpha\|_\infty d\tau. \end{aligned} \quad (13)$$

Let the function $g : [0, \Delta + 3(\delta_S + \delta_u)] \rightarrow R^+$ be defined, for $t \in [0, \Delta + 3(\delta_S + \delta_u)]$, as $g(t) = \sup_{\theta \in [0, t]} \|x_\theta\|_\infty$. Let

$$\begin{aligned} \mu_1 &= \|x_0\|_\infty + L_f(\Delta + 3(\delta_S + \delta_u))|\bar{u}|, \\ \mu_2 &= L_f(1 + L_k). \end{aligned} \quad (14)$$

Then, for $t \in [0, \Delta + 3(\delta_S + \delta_u)]$, the inequality holds $g(t) \leq \mu_1 + \int_0^t \mu_2 g(\tau) d\tau$. By the Gronwall–Bellman Lemma (see Lemma A.1, pp. 651–652, in Khalil, 2000), for $t \in [0, \Delta + 3(\delta_S + \delta_u)]$, the inequality follows $g(t) \leq \mu_1 e^{\mu_2 t}$. Therefore, we have, for $t \in [0, \Delta + 3(\delta_S + \delta_u)]$,

$$\|x_t\|_\infty \leq g(t) \leq \mu_1 e^{\mu_2 t} \leq \mu_1 e^{\mu_2(\Delta+3(\delta_S+\delta_u))} = (\|x_0\|_\infty + L_f(\Delta + 3(\delta_S + \delta_u))|\bar{u}|) \cdot e^{L_f(1+L_k)(\Delta+3(\delta_S+\delta_u))}. \quad (15)$$

The inequality (12) is proved. Now, let $w : R^+ \rightarrow R^+$ be the continuous function defined, for $t \in R^+$, as $w(t) = V(x_t)$, with V the functional provided in Lemma 1 (for the system described by (4)). Let p be the smallest integer such that $s_p \geq \Delta + 2(\delta_S + \delta_u)$. It follows that $t_p \in [\Delta + 2(\delta_S + \delta_u), \Delta + 3(\delta_S + \delta_u)]$. The following equalities hold for $t_j \leq t < t_{j+1}, j = p, p + 1, \dots$,

$$\begin{aligned} D^+ w(t) &= \limsup_{h \rightarrow 0^+} \frac{1}{h} (V(x_{t+h}) - V(x_t)) \\ &= \limsup_{h \rightarrow 0^+} \frac{1}{h} \left(V(x_{t+h}) - V\left((x_t)_{h,k(x_{s_j})}\right) \right. \\ &\quad \left. + V\left((x_t)_{h,k(x_{s_j})}\right) - V(x_t) \right), \end{aligned} \tag{16}$$

where, for $t \geq 0$, and any $v \in R^m$, $(x_t)_{h,v} \in \mathcal{C}$ is defined (see (1), (2) in Notation section) as, in the case $\Delta > 0$, for $h \in [0, \Delta]$,

$$(x_t)_{h,v}(\theta) = \begin{cases} x_t(\theta + h), & \theta \in [-\Delta, -h), \\ x_t(0) + (\theta + h)f(x_t, v), & \theta \in [-h, 0], \end{cases} \tag{17}$$

and as, in the case $\Delta = 0$, for $h \in [0, 1]$,

$$(x_t)_{h,v}(0) = x_t(0) + hf(x_t, v). \tag{18}$$

Now, the following equalities/inequality hold for any positive real $h < \min\{t_{j+1} - t_j, \Delta\}$, in the case $\Delta > 0$, and for any positive real $h < \min\{t_{j+1} - t_j, 1\}$, in the case $\Delta = 0$ (see Yoshizawa, 1966; Driver, 1962),

$$\begin{aligned} &\left| \frac{1}{h} \left(V(x_{t+h}) - V\left((x_t)_{h,k(x_{s_j})}\right) \right) \right| \\ &\leq \frac{L_V}{h} \left\| x_{t+h} - (x_t)_{h,k(x_{s_j})} \right\|_\infty \\ &= \frac{L_V}{h} \sup_{\theta \in [-\Delta, 0]} \left| x_{t+h}(\theta) - (x_t)_{h,k(x_{s_j})}(\theta) \right| \leq \frac{L_V}{h} \\ &\quad \cdot \sup_{\theta \in [-h, 0]} \left| x(t+h+\theta) - x(t) - (\theta+h)f(x_t, k(x_{s_j})) \right|. \end{aligned} \tag{19}$$

From (19), we obtain

$$\begin{aligned} &\left| \frac{1}{h} \left(V(x_{t+h}) - V\left((x_t)_{h,k(x_{s_j})}\right) \right) \right| \\ &\leq \frac{L_V}{h} \sup_{\theta \in [-h, 0]} \left| x(t) + \int_t^{t+h+\theta} f(x_\tau, k(x_{s_j})) d\tau \right. \\ &\quad \left. - x(t) - (\theta+h)f(x_t, k(x_{s_j})) \right| \\ &= \frac{L_V}{h} \sup_{\theta \in (-h, 0]} \left| \int_t^{t+h+\theta} f(x_\tau, k(x_{s_j})) d\tau \right. \\ &\quad \left. - (\theta+h)f(x_t, k(x_{s_j})) \right| \\ &= \frac{L_V}{h} \sup_{\theta \in (-h, 0]} \left| (\theta+h) \left(\frac{1}{\theta+h} \int_t^{t+h+\theta} f(x_\tau, k(x_{s_j})) d\tau \right. \right. \\ &\quad \left. \left. - f(x_t, k(x_{s_j})) \right) \right|. \end{aligned} \tag{20}$$

From (20), it follows

$$\begin{aligned} &\left| \frac{1}{h} \left(V(x_{t+h}) - V\left((x_t)_{h,k(x_{s_j})}\right) \right) \right| \\ &\leq L_V \sup_{\theta \in (-h, 0]} \left| \left(\frac{1}{\theta+h} \int_t^{t+h+\theta} f(x_\tau, k(x_{s_j})) d\tau \right. \right. \\ &\quad \left. \left. - f(x_t, k(x_{s_j})) \right) \right|. \end{aligned} \tag{21}$$

From (21), taking into account of the continuity of the map f and of the solution $x_\tau \in \mathcal{C}, \tau \in R^+$ (see Lemma 2.1, p. 40, in Hale & Verduyn Lunel, 1993) the limit follows

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \left(V(x_{t+h}) - V\left((x_t)_{h,k(x_{s_j})}\right) \right) = 0. \tag{22}$$

From (16) and (22) it follows that

$$\begin{aligned} D^+ w(t) &= \limsup_{h \rightarrow 0^+} \frac{1}{h} \left(V\left((x_t)_{h,k(x_{s_j})}\right) - V(x_t) \right) \\ &= \limsup_{h \rightarrow 0^+} \frac{1}{h} \left(V\left((x_t)_{h,k(x_{s_j})}\right) - V\left((x_t)_{h,k(x_t)}\right) \right. \\ &\quad \left. + V\left((x_t)_{h,k(x_t)}\right) - V(x_t) \right). \end{aligned} \tag{23}$$

From (23) we obtain, taking into account Lemma 1,

$$\begin{aligned} D^+ w(t) &\leq \limsup_{h \rightarrow 0^+} \frac{1}{h} \left(V\left((x_t)_{h,k(x_t)}\right) - V(x_t) \right) \\ &\quad + \limsup_{h \rightarrow 0^+} \frac{1}{h} \left(V\left((x_t)_{h,k(x_{s_j})}\right) - V\left((x_t)_{h,k(x_t)}\right) \right) \\ &= D^+ V(x_t, k(x_t)) \\ &\quad + \limsup_{h \rightarrow 0^+} \frac{1}{h} \left(V\left((x_t)_{h,k(x_{s_j})}\right) - V\left((x_t)_{h,k(x_t)}\right) \right) \\ &\leq -\alpha_3 \|x_t\|_\infty \\ &\quad + \limsup_{h \rightarrow 0^+} \frac{1}{h} \left| V\left((x_t)_{h,k(x_{s_j})}\right) - V\left((x_t)_{h,k(x_t)}\right) \right|. \end{aligned} \tag{24}$$

Moreover, we have

$$\begin{aligned} &\limsup_{h \rightarrow 0^+} \frac{1}{h} \left| V\left((x_t)_{h,k(x_{s_j})}\right) - V\left((x_t)_{h,k(x_t)}\right) \right| \\ &\leq \limsup_{h \rightarrow 0^+} \frac{L_V}{h} \left\| (x_t)_{h,k(x_{s_j})} - (x_t)_{h,k(x_t)} \right\|_\infty \\ &\leq \limsup_{h \rightarrow 0^+} \frac{L_V}{h} \sup_{\theta \in [-h, 0]} (\theta + h) \\ &\quad \cdot \left| f(x_t, k(x_{s_j})) - f(x_t, k(x_t)) \right| \\ &\leq L_V L_f L_k \|x_t - x_{s_j}\|_\infty. \end{aligned} \tag{25}$$

From (24), (25), we obtain

$$\begin{aligned} D^+ w(t) &\leq -\alpha_3 \|x_t\|_\infty + L_V L_f L_k \|x_{s_j} - x_t\|_\infty, \\ &\quad t_j \leq t < t_{j+1}, j = p, p + 1, \dots \end{aligned} \tag{26}$$

Now, notice that, for any $t \in [\delta_S + \delta_u, +\infty)$, there exists a real $\theta \in [t - \delta_S - \delta_u, t]$ such that $u(t) = k(x_\theta)$. Let us pick any $j \geq p$. We have, for $t \in [t_j, t_{j+1})$,

$$\begin{aligned} \|x_{s_j} - x_t\|_\infty &= \sup_{\theta \in [-\Delta, 0]} |x(s_j + \theta) - x(t + \theta)| \\ &\leq \sup_{\theta \in [-\Delta, 0]} \left| x(s_j + \theta) - x(s_j + \theta) - \int_{s_j + \theta}^{t + \theta} f(x_\beta, u(\beta)) d\beta \right| \\ &\leq \sup_{\theta \in [-\Delta, 0]} \int_{s_j + \theta}^{t + \theta} L_f (\|x_\beta\|_\infty + |u(\beta)|) d\beta \\ &\leq \sup_{\theta \in [-\Delta, 0]} \int_{s_j + \theta}^{t + \theta} L_f \cdot \left(\sup_{\alpha \in [s_j - \Delta, t]} \|x_\alpha\|_\infty + \sup_{\alpha \in [s_j - \Delta, t]} |u(\alpha)| \right) d\beta \\ &\leq \sup_{\theta \in [-\Delta, 0]} \int_{s_j + \theta}^{t + \theta} L_f (1 + L_k) \sup_{\alpha \in [s_j - \Delta - \delta_S - \delta_u, t]} \|x_\alpha\|_\infty d\beta. \end{aligned} \tag{27}$$

Now, the following equalities/inequalities hold

$$\begin{aligned}
 s_j - \Delta - \delta_S - \delta_u &= -(s_{j+1} - s_j) + s_{j+1} - \Delta - \delta_S - \delta_u \\
 &\geq s_{j+1} - \Delta - 2\delta_S - \delta_u \\
 &= t_{j+1} - \eta_{j+1} - \Delta - 2\delta_S - \delta_u \geq t - \Delta - 2(\delta_S + \delta_u). \tag{28}
 \end{aligned}$$

From (27), (28), we obtain

$$\begin{aligned}
 \|x_{s_j} - x_t\|_\infty &\leq \sup_{\theta \in [-\Delta, 0]} \int_{s_j+\theta}^{t+\theta} L_f(1 + L_k) \\
 &\quad \cdot \sup_{\alpha \in [t-\Delta-2(\delta_S+\delta_u), t]} \|x_\alpha\|_\infty d\beta \\
 &\leq L_f(1 + L_k)(t - s_j) \sup_{\alpha \in [t-\Delta-2(\delta_S+\delta_u), t]} \|x_\alpha\|_\infty \\
 &\leq L_f(1 + L_k)(\delta_S + \delta_u) \sup_{\alpha \in [t-\Delta-2(\delta_S+\delta_u), t]} \|x_\alpha\|_\infty \\
 &\leq \frac{(\delta_S + \delta_u)L_f(1 + L_k)}{\alpha_1} \sup_{\alpha \in [t-\Delta-2(\delta_S+\delta_u), t]} w(\alpha). \tag{29}
 \end{aligned}$$

By (26), (29), the following inequality holds for all $t \geq t_p$:

$$\begin{aligned}
 D^+ w(t) &\leq -\frac{\alpha_3}{\alpha_2} w(t) + \frac{L_V L_f^2 L_k (1 + L_k)}{\alpha_1} (\delta_S + \delta_u) \\
 &\quad \cdot \sup_{\theta \in [-\Delta-2(\delta_S+\delta_u), 0]} w(t + \theta). \tag{30}
 \end{aligned}$$

By (30), taking the bound on the maximum interval $\delta_S + \delta_u$ into account, the result of the theorem follows from the application of Lemma 3 in the interval $[t_p, +\infty)$, and from (12).

Next corollary readily follows from Theorem 5, and therefore the proof is omitted.

Corollary 6. *Let Assumption 2 hold. Then, there exist positive reals M, λ, δ such that, if $\delta_S + \delta_u < \delta$, the solution of the RFDE (9) exists for all $t \geq 0$, and, furthermore, satisfies the inequality*

$$|x(t)| \leq M(\|x_0\|_\infty + |\bar{u}|)e^{-\lambda t}, \quad t \geq 0. \tag{31}$$

4. Illustrative example

Let us consider the system described by the following scalar RFDE

$$\begin{aligned}
 \dot{x}(t) &= -x(t) + \tanh(x(t) + x(t - \Delta) + u(t)), \\
 x(\tau) &= x_0(\tau), \quad \tau \in [-\Delta, 0], \tag{32}
 \end{aligned}$$

where $x(t) \in \mathbb{R}$, Δ is a positive real, $u(t) \in \mathbb{R}$ is the control input, $x_0 \in \mathcal{C}$ is the initial state. In this case we have, for $\phi \in \mathcal{C}$, $u \in \mathbb{R}$, $f(\phi, u) = -\phi(0) + \tanh(\phi(0) + \phi(-\Delta) + u)$ and $L_f = 3$. Let us choose as state feedback the map $k : \mathcal{C} \rightarrow \mathbb{R}$ defined, for $\phi \in \mathcal{C}$, as $k(\phi) = -\phi(0) - \phi(-\Delta)$ (i.e., on the solution, $k(x_t) = -x_t(0) - x_t(-\Delta) = -x(t) - x(t - \Delta)$). Then the closed-loop continuous-time system is described by the delay-free equation $\dot{x}(t) = -x(t)$, and is globally exponentially stable. Thus, by Corollary 6, we can readily conclude that there exist suitable positive reals M, λ and δ such that, if $\delta_S + \delta_u < \delta$, the solution of the RFDE

$$\begin{aligned}
 \dot{x}(t) &= -x(t) + \tanh(x(t) + x(t - \Delta) + u(t)), \\
 x(\tau) &= x_0(\tau), \quad \tau \in [-\Delta, 0], \quad x_0 \in \mathcal{C}, \\
 u(t) &= \begin{cases} \bar{u}, & t_0 > 0, \quad t \in [0, t_0), \\ -x(s_j) - x(s_j - \Delta), & t \in [t_j, t_{j+1}), \\ s_j \in \pi_S, & t_j \in \pi_{2OH}, \quad j = 0, 1, \dots, \end{cases} \tag{33}
 \end{aligned}$$

exists for all $t \geq 0$, and, furthermore, satisfies the inequality (31). In this case, no knowledge of any LK functional is needed for establishing the existence-type result as stated in Corollary 6. In order to

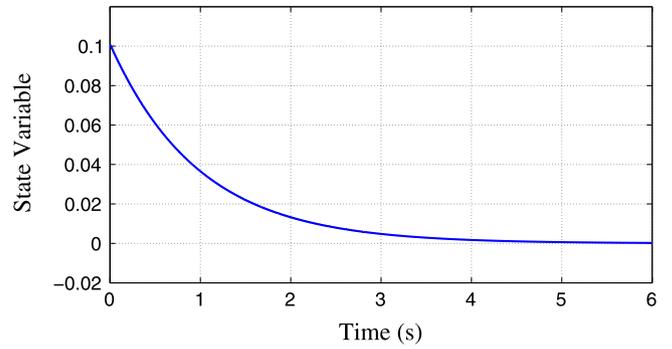


Fig. 1. State variable, $\Delta = 0.1$, sampling period and input-delay equal to 5 ms.

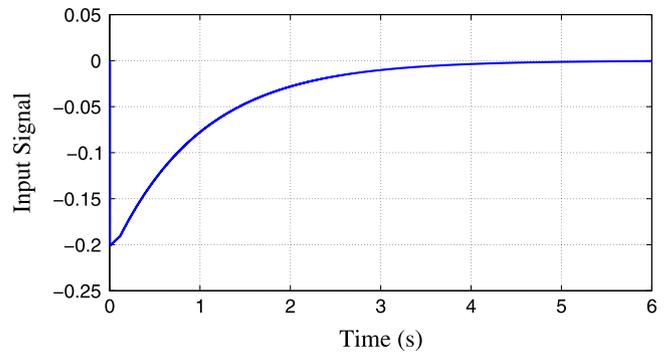


Fig. 2. Input signal, $\Delta = 0.1$, sampling period and input-delay equal to 5 ms.

provide an upper bound for $\delta_S + \delta_u$, according to Theorem 5, let us choose $V : \mathcal{C} \rightarrow \mathbb{R}^+$ defined, for $\phi \in \mathcal{C}$, as $V(\phi) = |\phi(0)|$. Then we have $\alpha_1 = \alpha_2 = \alpha_3 = L_V = 1$. Moreover, $L_k = 2$. We obtain $\delta_{\max} = 18.5$ ms, according to (7). For $\Delta = 0.1$, $\delta_S + \delta_u = 10$ ms, we obtain $\lambda = 0.43$, according to (8). Simulations have been performed with $\delta_S = \delta_u = 5$ ms. Uniform sampling is used, and the sampling period has been chosen equal to δ_S . The input-delay, induced by transmission and/or computations, has been chosen constant and equal to δ_u . The initial state is chosen constant in $[-0.1, 0]$ and equal to 0.1. The initial input \bar{u} is chosen equal to 0. In Fig. 1, the behavior of the state variable is reported. In Fig. 2, the control signal is reported. The simulation fully validates the theoretical results. The performance of the controller is similar with $\delta_S = \delta_u = 100$ ms. Oscillations are observed with $\delta_S = \delta_u = 350$ ms. In-stability is observed with $\delta_S = \delta_u = 450$ ms.

5. Discussions and conclusions

In this paper it is shown that the global exponential stability is preserved, under suitable fast sampling, for globally Lipschitz, fully nonlinear time-delay systems, which, in continuous-time, are globally exponentially stabilizable by globally Lipschitz state feedbacks. The result here stated is of the existence type, and concerns the proof of an expected result, though so far just conjectured. As far as the maximum allowed sampling period is concerned, we believe that the provided result is rather conservative. Anyway, the provision of a non conservative sampling frequency is beyond the aims of this paper. The reader can refer to the paper (Nesic et al., 2009), for a deep analysis of the maximum allowed sampling period, in the delay-free case. Whether the globally Lipschitz hypothesis may be weakened, in order to obtain the same kind of results (i.e., the continuous-time 0-GAS property is preserved, without any further conditions, under suitable fast sampling), is an interesting open problem, which is left for future investigations. More general Lyapunov converse theorems (see Teel & Praly, 2000, and the references therein) may be instrumental for this interesting research topic.

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