Automatica 106 (2019) 150–160

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Sampled-data observers for semilinear damped wave equations under spatially sampled state measurements ☆

Maria Terushkin, Emilia Fridman *

School of Electrical Engineering, Tel-Aviv University, Tel-Aviv, Israel

ARTICLE INFO

Article history:
Received 29 May 2018
Received in revised form 6 December 2018
Accepted 26 March 2019
Available online xxxx

Keywords:
Sampled-data observers
Hyperbolic systems
Linear matrix inequalities

ABSTRACT

Sampled-data observers/controllers under the sampled in space and time measurements were suggested in the past for parabolic systems. In the present paper, for the first time, a sampled-data observer is constructed for a hyperbolic system governed by 1D semilinear wave equation with either viscous or boundary damping. The measurements are sampled in space and time. Sufficient conditions for the exponential stability of the estimation error are derived by using the time-delay approach to sampled-data control and appropriate Lyapunov–Krasovskii functionals. The dual sampled-data controller problems are formulated. Numerical examples including observer design for unstable damped sine–Gordon equation illustrate the efficiency of the method.

1. Introduction

Modern control systems usually employ digital technology for controller/observer implementation (see e.g. Åström & Wittenmark, 1997 and the recent survey Hetel et al., 2017). Networked control systems, where the plant is controlled via communication network and where the signals from sensors to controllers and from controllers to actuators are transmitted in discrete-time, became another hot and related topic (Antsaklis & Baillieul, 2007; Hespanha, Naghshtabrizi, & Xu, 2007). Sampled-data control of partial differential equations (PDEs) is becoming an active research area. General results on sampled-data control of linear time-invariant PDEs were presented in Logemann (2013) and Logemann, Rebarber, and Townley (2005). A model-reduction-based approach to sampled-data control of parabolic systems was suggested in Cheng, Radisavljevic, Chang, Lin, and Su (2009) and Ghantasala and El-Farra (2012), where a finite-dimensional controller was designed on the basis of a finite-dimensional system that captures the dominant dynamics of the infinite-dimensional one.

Distributed sampled-data control of parabolic PDEs has been studied in Bar Am and Fridman (2014), Fridman and Bar Am (2013), Fridman and Blighovsky (2012), Kang and Fridman (2018) and Selivanov and Fridman (2016, 2017). In-domain (point or averaged) sampled-data measurements of the state together with control actions applied through shape functions have been considered. Sufficient conditions for the exponential convergence in terms of linear matrix inequalities (LMIs) have been derived by using the time-delay approach to sampled-data control and appropriate Lyapunov–Krasovskii functionals. Boundary sampled-data control of 1-D linear heat and transport equations were introduced recently in Karafyllis and Krstic (2017, 2018).

Distributed finite-dimensional continuous-time control of a class of damped semilinear wave equations was recently initiated in Kalantarov and Titi (2016). However, sampled-data controllers/observers have not been considered yet for semilinear hyperbolic PDEs. Note that in the case of wave equation, even arbitrarily small delays in the damping term (either boundary or viscous) may destabilize the system (Datko, 1988; Nicaise & Pignotti, 2006), but wave equations with the viscous damping are known to be robust with respect to small state-delay in the right-hand side of PDE (Fridman & Orlov, 2009). Keeping this in mind, in the present paper we introduce a sampled-data observer for a system governed by 1D semilinear wave equation either with a viscous or with a boundary damping. Such systems arise in various applications including nonlinear elasticity as a model of a vibrating string in a viscous medium, where the semilinear term corresponds to the elastic force (Pata & Zelik, 2006). The considered class of systems includes damped sine–Gordon equations that model the dynamics of a current driven coupled Josephson junctions with applications in superconducting single-electron transistors (Dickey, 1976; Levi, Hoppensteadt, & Miranker, 1978).

Similar to parabolic case (Fridman & Blighovsky, 2012), we assume that the state measurements are sampled in space and

☆ This paper was partially supported by ISF, Israel grant 1128/14 and by C. and H. Manderman Chair at Tel Aviv University. The material in this paper was partially presented at the European Control Conference, June 25–28, 2019, Naples, Italy. This paper was recommended for publication in revised form by Associate Editor Fouad Giri under the direction of Editor Miroslav Krstic..

* Corresponding author.
E-mail addresses: mariater@mail.tau.ac.il (M. Terushkin), emilia@eng.tau.ac.il (E. Fridman).

https://doi.org/10.1016/j.automatica.2019.04.044

in time, whereas the sampling intervals may be variable, but bounded. We derive sufficient LMI conditions for the exponential stability of the estimation error system by using appropriate Lyapunov–Krasovskii functionals. By solving these LMIs, upper bounds on the sampling intervals that preserve the exponential convergence and on the resulting decay rate can be found. For simplicity only we have not considered the case of additional measurement delay, but the proposed method can be easily extended to the delayed case via appropriate Lyapunov–Krasovskii functionals (see e.g. Fridman & Blighovsky, 2012). We also formulate the dual sampled-data control problems. Some preliminary results on sampled-data observer in the case of viscous damping will be presented in Terushkin and Fridman (2019).

Notation. Throughout the paper the notation $P > 0$ with $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of a symmetric matrix will be denoted by $*$. Functions, continuous (continuously differentiable) in all arguments, are referred to as of class $\mathcal{C}$ (of class $\mathcal{C}^1$). $L^2(0,\pi)$ is the Hilbert space of square integrable functions $z(\xi)$, $\xi \in [0,\pi]$ with the corresponding norm $\|z\|^2_{L^2} = \int_0^\pi z^2(\xi)d\xi$. $\mathscr{H}^1(0,\pi)$ is the Sobolev space of absolutely continuous scalar functions $z : [0,\pi] \to \mathbb{R}$ with $\frac{dz}{d\xi} \in L^2(0,\pi)$. $\mathscr{H}^2(0,\pi)$ is the Sobolev space of scalar functions $z : [0,\pi] \to \mathbb{R}$ with absolutely continuous $\frac{dz}{d\xi}$ and with $\frac{d^2z}{d\xi^2} \in L^2(0,\pi)$.

2. Mathematical preliminaries

The following inequalities will be useful:

Lemma 2.1 (*Wirtinger's Inequality Hardy, Littlewood, & Pólya, 0000*)**.** *Let $z \in \mathscr{H}^1[a,b]$ be a scalar function, with the boundary values stated below. Then*

$$c\int_a^b z^2(\xi)d\xi \le \frac{(b-a)^2}{\pi^2}\int_a^b \left[\frac{dz(\xi)}{d\xi}\right]^2 d\xi, \tag{2.1}$$

where

$$c = \begin{cases} 1, & \text{if} \quad z(a) = z(b) = 0; \\ \frac{1}{4}, & \text{if} \quad z(a) = 0 \text{ or } z(b) = 0. \end{cases}$$

Lemma 2.2 (*Halanay's Inequality Halanay, 1966 & p.138 of Fridman, 2014*)**.** *Let $0 < \alpha_1 < \alpha_0$ and let $V : [t_0 - h, \infty) \longrightarrow [0,\infty)$ be an absolutely continuous function that satisfies*

$$\dot V(t) + 2\alpha_0 V(t) - 2\alpha_1 \sup_{-h\le\theta\le 0} V(t+\theta) \le 0, \quad t \ge t_0.$$

Then

$$V(t) \le \exp\big(-2\alpha(t-t_0)\big) \sup_{-h\le\theta\le 0} V(t_0+\theta), \quad t \ge t_0, \tag{2.2}$$

where $\alpha > 0$ is a unique positive solution of

$$\alpha = \alpha_0 - \alpha_1 \exp(2\alpha h).$$

3. Sampled data observer: semilinear wave equation with viscous damping

3.1. Problem formulation

Consider the semilinear damped wave equation

$$z_{tt}(x,t) = z_{xx}(x,t) - \beta z_t(x,t) + f\big(z(x,t),x,t\big), \quad x \in (0,\pi),\ t \ge t_0 \tag{3.1}$$

under the Dirichlet

$$z(0,t) = z(\pi,t) = 0 \tag{3.2}$$

or Neumann

$$z_x(0,t) = z_x(\pi,t) = 0 \tag{3.3}$$

or mixed

$$z(0,t) = z_x(\pi,t) = 0, \quad or \quad z_x(0,t) = z(\pi,t) = 0 \tag{3.4}$$

boundary conditions. Here $z(x,t) \in \mathbb{R}$ is the state, $\beta > 0$ is the damping coefficient and f is a function of class $\mathcal{C}^1$. We assume that the derivative f_z is uniformly bounded by a constant $g_1 > 0$:

$$|f_z(z,x,t)| \le g_1 \quad \forall (z,x,t) \in \mathbb{R}\times[0,\pi]\times[t_0,\infty). \tag{3.5}$$

The initial conditions are given by

$$z(x,t_0) = z_0(x), \quad z_t(x,t_0) = z_1(x). \tag{3.6}$$

The above system with $f = g_1 \sin(z)$ is referred as damped sine–Gordon equation (Kobayashi, 2003; Levi et al., 1978). Note that the damped sine–Gordon is globally asymptotically stable for $g_1 < 1$, whereas for $g_1 > 1$ its zero solution becomes only locally stable (Dickey, 1976). In the present paper we allow $g_1 > 1$, where the system may be unstable. We would like to point out that the assumption (3.5) is restrictive, and the results of this paper are not applicable e.g. to nonlinear Klein–Gordon equation with $f = z^p$, $p = 2, 3, \ldots$. See also Remark 5.1 .

We design an observer for (3.1) under the appropriate boundary conditions (3.2), (3.3) or (3.4) based on sampled in space and in time measurements. Similar to Fridman and Blighovsky (2012), the segment $[0,\pi]$ is divided into N sampling intervals by the points

$$0 = x_0 < x_1 < \cdots < x_N = \pi.$$

It is assumed, that N sensors are placed in the middle of each interval $[x_j, x_{j+1}]$:

$$\bar x_j = \frac{x_{j+1}+x_j}{2}, \quad j = 0,\ldots,N-1.$$

Let

$$t_0 < t_1 < \cdots < t_k, \quad \lim_{k\to\infty} t_k = \infty$$

be the sampling time instants. The sampling intervals in time and space may be variable, but have known bounds $h > 0$ and $\Delta > 0$:

$$0 \le t_{k+1} - t_k \le h, \quad x_{j+1} - x_j \le \Delta. \tag{3.7}$$

Discrete-time point measurements of the state are provided by N sensors distributed over the whole domain $[0,\pi]$:

$$y_{jk} = z(\bar x_j, t_k), \quad \bar x_j = \frac{x_{j+1}+x_j}{2}, \quad j = 0,\ldots,N-1, \quad k = 0,1,2,\ldots \tag{3.8}$$

Our objective is to construct an observer for (3.1) under the boundary conditions (3.2) or (3.3) or (3.4) by employing sampled-data measurements (3.8), and to formulate sufficient conditions for the global exponential convergence of the estimation error in terms of LMIs.

3.1.1. Well-posedness of the original system

We prove the well-posedness of (3.1) under the Dirichlet boundary conditions (3.2). For the Neumann or mixed boundary conditions, the well-posedness can be established similarly. The boundary-value problem (3.1), (3.2) can be represented as an abstract differential equation by defining the state $\zeta(t) = [\zeta_0(t)\ \zeta_1(t)]^T = [z(t)\ z_t(t)]^T$ and the operators

$$\mathscr{A} = \begin{bmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & -\beta I \end{bmatrix}, \quad F(\zeta,t) = \begin{bmatrix} 0 \\ F_1(\zeta_0,t) \end{bmatrix}. \tag{3.9}$$

Here $F_1 : \mathscr{H}^1(0,\pi)\times[t_0,\infty)\to L^2(0,\pi)$ is defined as $F_1(\zeta_0,t)=f(\zeta_0,\cdot,t)$ so that it is continuous in t for each $\zeta_0\in\mathscr{H}^1(0,\pi)$. The resulting differential equation

$$\dot\zeta(t)=\mathscr{A}\zeta(t)+F(\zeta(t),t),\quad t\ge t_0 \tag{3.10}$$

is considered in the Hilbert space $\mathscr{H}=\mathscr{H}_0^1\times L^2(0,\pi)$, where $\mathscr{H}_0^1=\left\{\zeta_0\in\mathscr{H}^1(0,\pi)\,\middle|\,\zeta_0(0)=\zeta_0(\pi)=0\right\}$, and $\|\zeta\|^2_{\mathscr{H}}=\|\zeta_{0x}\|^2_{L^2}+\|\zeta_1\|^2_{L^2}$. The operator $\mathscr{A}$ with the dense domain

$$\mathscr{D}(\mathscr{A})=\left\{\begin{bmatrix}\zeta_0\\ \zeta_1\end{bmatrix}\in\mathscr{H}^2(0,\pi)\bigcap\mathscr{H}_0^1\times\mathscr{H}_0^1\right\}$$

generates an exponentially stable semigroup (Curtain & Zwart, 1995). Due to (3.5) the following Lipschitz condition holds:

$$\|F_1(\zeta_0,t)-F_1(\bar\zeta_0,t)\|_{L^2}\le g_1\|\zeta_0-\bar\zeta_0\|_{L^2} \tag{3.11}$$

where $\zeta_0,\bar\zeta_0\in\mathscr{H}^1(0,\pi)$, $t\in\mathbb{R}$. Then by Theorem 6.1.2 of Pazy (1983), there exists a unique mild solution $\zeta\in\mathcal{C}([t_0,\infty);\mathscr{H})$ of (3.10) initialized by

$$\zeta_0(t_0)=z_0\in\mathscr{H}_0^1,\quad \zeta_1(t_0)=z_1\in L^2(0,\pi). \tag{3.12}$$

We note that $F:\mathscr{H}\times[t_0,\infty)\to\mathscr{H}$ is continuously differentiable. If $\zeta(t_0)\in\mathscr{D}(\mathscr{A})$, then this mild solution is in $\mathcal{C}^1([t_0,\infty);\mathscr{H})$ and it is a classical solution of (3.1), (3.2) with $\zeta(t)\in\mathscr{D}(\mathscr{A})$ (see Theorem 6.1.5 of Pazy, 1983).

3.2. Sampled-data observer

Consider (3.1) under the boundary conditions (3.2) and sampled in time and in space measurements (3.8). We suggest a sampled-data observer of the form

$$\begin{aligned}\hat z_{tt}(x,t)=&\hat z_{xx}(x,t)-\beta\hat z_t(x,t)+f(\hat z,x,t)\\&+L\sum_{j=0}^{N-1}\chi_j(x)[y_{jk}-\hat z(\bar x_j,t_k)],\\&x\in(0,\pi),\ t\in[t_k,t_{k+1}),\ k=0,1,2,\dots\end{aligned} \tag{3.13}$$

under the boundary conditions

$$\hat z(0,t)=\hat z(\pi,t)=0, \tag{3.14}$$

and the initial conditions $[\hat z(\cdot,t_0),\hat z_t(\cdot,t_0)]^T\in\mathscr{H}$. Here L is a scalar observer gain. The measurements are applied after multiplication by the characteristic functions $\chi_j(x)$, defined by

$$\chi_j(x)=\begin{cases}1, & \text{if } x\in[x_j,x_{j+1}];\\ 0, & \text{else.}\end{cases} \tag{3.15}$$

Note that $\sum_{j=0}^{N-1}\chi_j(x)[y(\bar x_j,t)-\hat z(\bar x_j,t)]\approx[y(x,t)-\hat z(x,t)]$ when $\Delta\to0$.

The distributed correction term in the observer aims to compensate the destabilizing effect of the nonlinearity f and to improve the convergence of the estimation error $e=z-\hat z$ governed by

$$\begin{aligned}e_{tt}(x,t)&=e_{xx}(x,t)-\beta e_t(x,t)+ge(x,t)\\&\quad-L\sum_{j=0}^{N-1}\chi_j(x)e(\bar x_j,t_k),\\&\quad x\in(0,\pi),\ t\in[t_k,t_{k+1}),\ k=0,1,2,\dots\end{aligned} \tag{3.16}$$

under the Dirichlet boundary conditions

$$e(0,t)=e(\pi,t)=0. \tag{3.17}$$

Here $ge=f(z,x,t)-f(z-e,x,t)$ is defined by

$$g=g(z,e,x,t)=\int_0^1\frac{\partial f}{\partial z}(z+(\theta-1)e,x,t)d\theta. \tag{3.18}$$

Due to (3.5)

$$|g|\le g_1\quad\forall(z,e,x,t)\in\mathbb{R}\times\mathbb{R}\times[0,\pi]\times[t_0,\infty).$$

For (3.1) under the Neumann or mixed boundary conditions, the observer (3.13) is considered under the Neumann

$$\hat z_x(0,t)=\hat z(\pi,t)=0 \tag{3.19}$$

or mixed

$$\hat z(0,t)=\hat z_x(\pi,t)=0,\quad or\quad \hat z_x(0,t)=\hat z(\pi,t)=0 \tag{3.20}$$

boundary conditions respectively. The error under the Neumann or mixed boundary conditions satisfies the corresponding boundary conditions.

The step method is applied in order to establish the well-posedness for the error system (3.16), (3.17). The error system can be presented in the form of (3.10) with $\zeta=[e\ e_t]^T$, where $\mathscr{A}$ and F are defined by (3.9) with

$$\begin{aligned}F_1(\zeta_0,t)=&f(z,x,t)-f(z-\zeta_0,x,t)\\&-L\sum_{j=0}^{N-1}\chi_j(x)\left[\zeta_0\big(x,t_0\big)-\int_{\bar x_j}^x\zeta_{0\xi}\big(\xi,t_0\big)d\xi.\right]\end{aligned} \tag{3.21}$$

By applying Theorem 6.1.2 (Theorem 6.1.5) of Pazy (1983) consecutively on each time interval $t\in[t_k,t_{k+1}]$, $k=0,1,2,\dots$ we find that a unique mild (classical) solution exists for (3.16), (3.17) initialized by $[e(\cdot,t_0),e_t(\cdot,t_0)]^T\in\mathscr{H}$ ($[e(\cdot,t_0),e_t(\cdot,t_0)]^T\in\mathscr{D}(\mathscr{A})$).

For the stability analysis, we employ the relation

$$\begin{aligned}&e\big(x,t_k\big)=e(x,t)-(t-t_k)\nu,\\&\nu(x,t)\triangleq\frac{1}{t-t_k}\int_{t_k}^t e_s(x,s)ds\end{aligned} \tag{3.22}$$

and present (3.16) as

$$\begin{aligned}e_{tt}(x,t)&=e_{xx}(x,t)-\beta e_t(x,t)+(g-L)e(x,t)\\&\quad+L\sum_{j=0}^{N-1}\chi_j(x)\left[(t-t_k)\nu+\int_{\bar x_j}^x e_\zeta\big(\zeta,t_k\big)d\zeta\right],\\&\quad x\in(0,\pi),\ t\in[t_k,t_{k+1}),\ k=0,1,2,\dots\end{aligned} \tag{3.23}$$

We will use an input delay approach to sampled-data control (Fridman, Seuret, & Richard, 2004; Mikheev, Sobolev, & Fridman, 1988), where the sampling time t_k is presented as delayed time $t-\tau(t)$ with $\tau(t)=t-t_k$ for $t\in[t_k,t_{k+1})$. In order to derive stability conditions for (3.16) we employ Lyapunov–Krasovskii functional of the form

$$V(t)=V_0(t)+V_r(t),\quad t\in[t_k,t_{k+1}), \tag{3.24}$$

where $V_0(t)$ is given by

$$V_0(t)=p_3\int_0^\pi e_x^2dx+\int_0^\pi[e\ e_t]P_0[e\ e_t]^Tdx \tag{3.25}$$

with

$$P_0\triangleq\begin{bmatrix}p_1 & p_2\\ * & p_3\end{bmatrix}>0, \tag{3.26}$$

and

$$\begin{aligned}&V_r(t)=r\int_0^\pi\big(t_{k+1}-t\big)\int_{t_k}^t\exp(2\alpha_0(s-t))e_s^2(\zeta,s)dsd\zeta,\\&r>0,\quad\alpha_0>0.\end{aligned} \tag{3.27}$$

Here V_r is the simplest Lyapunov–Krasovskii term that treats sampled-data systems as introduced for ODE systems in Fridman (2010). Note that augmented Lyapunov–Krasovskii functionals may further improve the results, but on the account of computational complexity.

Differentiating V_r and applying Jensen's inequality (Gu, Kharitonov, & Chen, 2003), we have

$$\begin{aligned} &\dot V_r + 2\alpha_0 V_r \\ &\le r(t_{k+1}-t)\int_0^\pi e_t^2 dx - re^{-2\alpha_0 h}\int_0^\pi\int_{t_k}^t e_s^2(x,s)dsdx \\ &\le r(t_{k+1}-t)\int_0^\pi e_t^2 dx - r(t-t_k)e^{-2\alpha_0 h}\int_0^\pi \nu^2 dx. \end{aligned} \tag{3.28}$$

Note that integration by parts and substitution of the boundary conditions leads to

$$\int_0^\pi e_{xx}(p_2 e + p_3 e_t)dx = -p_2\int_0^\pi e_x^2 dx - p_3\int_0^\pi e_{xt}e_x dx. \tag{3.29}$$

Then, differentiating $V(t)$ along (3.23) and employing (3.28) and (3.29) we obtain

$$\begin{aligned} \dot V + 2\alpha_0 V \le 2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}} \Big\{ &(\alpha_0 p_3 - p_2)e_x^2 + \frac{1}{2}[e\ e_t]A[e\ e_t]^T \\ &+ L(p_2 e + p_3 e_t)\Big((t-t_k)\nu + \int_{\bar x_j}^x e_\zeta(\zeta,t_k)d\zeta\Big) \\ &+ \frac{r}{2}(t_{k+1}-t)e_t^2 - \frac{r}{2}(t-t_k)\exp(-2\alpha_0 h)\nu^2 \Big\} dx, \end{aligned} \tag{3.30}$$

where A is given by

$$A \triangleq \begin{bmatrix} 2p_2(g-L)+2\alpha_0 p_1 & p_1 + p_3(g-L) + p_2(2\alpha_0-\beta) \\ * & 2p_2 + 2p_3(\alpha_0-\beta) \end{bmatrix}. \tag{3.31}$$

In order to compensate $\int_{\bar x_j}^x e_\zeta(\zeta,t_k)d\zeta$ in (3.30), we apply Halanay's inequality (2.2). For some $\alpha_1 < \alpha_0$ we have

$$\begin{aligned} &\dot V(t) + 2\alpha_0 V(t) - 2\alpha_1 \sup_{-h\le\theta\le 0} V(t+\theta) \\ &\le \dot V(t) + 2\alpha_0 V(t) - 2\alpha_1 p_3\int_0^\pi e_x^2(x,t_k)dx. \end{aligned} \tag{3.32}$$

For application of Halanay's inequality we define continuously $V(t_0+\theta) = V(t_0)$, $\theta\in[-h,0]$. By Wirtinger's inequality (2.1) with $b-a = \frac{\Delta}{2}$ and $c = \frac{1}{4}$ we find

$$\begin{aligned} &-2\alpha_1 p_3\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}} e_x^2(x,t_k)dx \\ &= -2\alpha_1 p_3\sum_{j=0}^{N-1}\left[\int_{x_j}^{\bar x_j} e_x^2(x,t_k)dx + \int_{\bar x_j}^{x_{j+1}} e_x^2(x,t_k)dx\right] \\ &\le -2\alpha_1 p_3\frac{\pi^2}{\Delta^2}\sum_{j=0}^{N-1}\left[\int_{x_j}^{\bar x_j}[e(x,t_k)-e(\bar x_j,t_k)]^2 dx \right. \\ &\left. + \int_{\bar x_j}^{x_{j+1}}[e(x,t_k)-e(\bar x_j,t_k)]^2 dx\right] \\ &\le -2\alpha_1 p_3\frac{\pi^2}{\Delta^2}\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\left[\int_{\bar x_j}^x e_\zeta(\zeta,t_k)d\zeta\right]^2 dx. \end{aligned} \tag{3.33}$$

Define

$$C = \begin{cases} 1, & \text{for Dirichlet b.c.} \\ \frac{1}{4}, & \text{for mixed b.c.} \\ 0, & \text{for Neumann b.c.} \end{cases} \tag{3.34}$$

We apply further S-procedure (Yakubovich, 1971), where the inequality (that follows from Wirtinger's inequality)

$$\lambda_1\int_0^\pi\left(e_x^2 - Ce^2\right)dx \ge 0 \tag{3.35}$$

with some $\lambda_1 \ge 0$ is added to $\dot V$. Then

$$\begin{aligned} W &\triangleq \dot V(t) + 2\alpha_0 V(t) - 2\alpha_1 p_3\int_0^\pi e_x^2(x,t_k)dx \\ &\le \sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\Big\{[e\ e_t]A[e\ e_t]^T + (2\alpha_0 p_3 - 2p_2 + \lambda_1)e_x^2 \\ &- \lambda_1 Ce^2 + 2L(p_2 e + p_3 e_t)\Big(\tau(t)\nu + \int_{\bar x_j}^x e_\zeta(\zeta,t_k)d\zeta\Big) \\ &- r\tau(t)\exp(-2\alpha_0 h)\nu^2 + r\big(h-\tau(t)\big)e_t^2 \\ &- 2\alpha_1 p_3\frac{\pi^2}{\Delta^2}\left[\int_{\bar x_j}^x e_\zeta(\zeta,t_k)d\zeta\right]^2\Big\}dx \end{aligned}$$

Denote $\eta_\tau = [e\quad e_t\quad \nu\quad \int_{\bar x_j}^x e_\zeta(\zeta,t_k)d\zeta]^T$, $\quad t-t_k = \tau(t)$.

Then

$$\begin{aligned} W \le &\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}(2\alpha_0 p_3 - 2p_2 + \lambda_1)e_x^2 dx \\ &+ \sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\eta_\tau^T\Phi_\tau\eta_\tau dx \le 0, \end{aligned}$$

if

$$2\alpha_0 p_3 - 2p_2 + \lambda_1 \le 0 \tag{3.36}$$

and

$$\begin{aligned} &\Phi_\tau \triangleq \begin{bmatrix} \phi_{11} & \phi_{12} & \tau(t)Lp_2 & Lp_2 \\ * & \phi_{\tau 22} & \tau(t)Lp_3 & Lp_3 \\ * & * & -\tau(t)r\exp(-2\alpha_0 h) & 0 \\ * & * & * & -2\alpha_1 p_3\frac{\pi^2}{\Delta^2} \end{bmatrix} \le 0, \\ &\phi_{11} = 2p_2(g-L) + 2\alpha_0 p_1 - C\lambda_1, \\ &\phi_{12} = p_1 + p_3(g-L) + p_2(2\alpha_0-\beta), \\ &\phi_{\tau 22} = 2p_2 + 2p_3(\alpha_0-\beta) + r\big(h-\tau(t)\big). \end{aligned} \tag{3.37}$$

Note that Φ_τ is affine in $g\in[-g_1,g_1]$ and in $\tau\in[0,h]$. So it is sufficient to verify $\Phi_\tau \le 0$ in the four vertices $\tau = 0$, $\tau = h$ and $g = \pm g_1$:

$$\Phi_{0|g=\pm g_1} \le 0,\quad \Phi_{h|g=\pm g_1} \le 0, \tag{3.38}$$

where

$$\begin{aligned} &\Phi_0 \triangleq \begin{bmatrix} \phi_{11} & \phi_{12} & Lp_2 \\ * & 2p_2 + 2p_3(\alpha_0-\beta) + rh & Lp_3 \\ * & * & -2\alpha_1 p_3\frac{\pi^2}{\Delta^2} \end{bmatrix}, \\ &\Phi_h \triangleq \begin{bmatrix} \phi_{11} & \phi_{12} & hLp_2 & Lp_2 \\ * & 2p_2 + 2p_3(\alpha_0-\beta) & hLp_3 & Lp_3 \\ * & * & -hr\exp(-2\alpha_0 h) & 0 \\ * & * & * & -2\alpha_1 p_3\frac{\pi^2}{\Delta^2} \end{bmatrix}. \end{aligned} \tag{3.39}$$

Moreover, the feasibility of strict inequalities (3.36) and (3.38) with $\alpha_0 = \alpha_1 > 0$ implies their feasibility with a slightly larger $\bar{\alpha}_0 = \alpha_0 + \varepsilon > 0$, where $\varepsilon > 0$ is small. Therefore, if the strict inequalities (3.36) and (3.38) hold with $\alpha_0 = \alpha_1 > 0$, then the error system (3.16) is exponentially stable with some small decay rate $\varepsilon > 0$.

Remark 3.1. Consider now the case of continuous-time measurements

$$y_j(t) = z(\bar{x}_j, t), \quad j = 0, \ldots, N-1,$$

where the observer has the form

$$\begin{aligned}\hat{z}_{tt}(x,t) &= \hat{z}_{xx}(x,t) - \beta \hat{z}_t(x,t) + f\big(\hat{z}(x,t), x, t\big) \\ &+ L\sum_{j=0}^{N-1} \chi_j(x)[y_j(t) - \hat{z}(\bar{x}_j, t)], \quad x \in (0,\pi),\ t \geq t_0\end{aligned} \tag{3.40}$$

under the corresponding boundary conditions. The continuous-time error equation is governed by

$$\begin{aligned} e_{tt}(x,t) &= e_{xx}(x,t) - \beta e_t(x,t) + g e(x,t) \\ &- L\sum_{j=0}^{N-1}\chi_j(x)\left[e(x,t) - \int_{\bar{x}_j}^{x} e_\zeta(\zeta,t)d\zeta\right], \\ & t \geq t_0,\ x \in (0,\pi). \end{aligned} \tag{3.41}$$

Taking into account (3.36) and (3.37), for V_0 defined by (3.25) we conclude that $\dot{V}_0 + 2\alpha_0 V_0 \leq 0$ along (3.41) if the inequalities (3.36) and

$$\Psi \triangleq \begin{bmatrix} \phi_{11} & \phi_{12} & Lp_2 \\ * & 2p_2 + 2p_3(\alpha_0 - \beta) & Lp_3 \\ * & * & -\frac{\pi^2}{\Delta^2}(2p_2 - 2\alpha_0 p_3 - \lambda_1) \end{bmatrix} \leq 0 \tag{3.42}$$

are satisfied. Moreover, it is sufficient to verify (3.42) in the vertices $\pm g_1$. Since Halanay's inequality is not applied in the continuous-time case, by employing Wirtinger's inequality (2.1), the condition $P_0 > 0$ can be relaxed to

$$P_1 = \begin{bmatrix} p_1 + Cp_3 & p_2 \\ * & p_3 \end{bmatrix} > 0.$$

For $\Delta \to 0$ (3.42) holds with $\lambda_1 = 2p_2 - 2\alpha p_3$ if

$$\Psi_0 \triangleq \begin{bmatrix} \phi_{11} & \phi_{12} \\ * & 2p_2 + 2p_3(\alpha_0 - \beta) \end{bmatrix}_{|\lambda_1 = 2p_2 - 2\alpha_0 p_3} \leq 0,\ \ p_2 \geq \alpha_0 p_3.$$

The inequalities $\Psi_0 \leq 0$ and $P_1 > 0$ coincide with the Lyapunov inequalities for the exponential stability with a decay rate α_0 of the following second-order ODE

$$\dot{\zeta}(t) = \begin{bmatrix} 0 & 1 \\ -(C - g + L) & -\beta \end{bmatrix}\zeta(t), \quad \zeta \in \mathbb{R}^2. \tag{3.43}$$

In the linear case $g \equiv g_1$, (3.43) is the first mode in the modal decomposition of $z_{tt} = z_{xx} - \beta z_t + (g - L)z$ under the corresponding boundary conditions, and the choice of

$$L \geq g_1 + \frac{\beta^2}{4} - C \tag{3.44}$$

leads to the maximal possible decay rate $\alpha = \frac{\beta}{2}$. Note that the choice of minimal $L = g_1 + \frac{\beta^2}{4} - C$ subject to (3.44) enlarges the values of Δ that preserve the stability (this follows from application of Schur complements to Ψ) and corresponds to the result of Smyshlyaev, Cerpa, and Krstic (2010) (see Section 4) for the stability of the damped wave equation.

By Schur complements, LMIs (3.36) and (3.38) are feasible for $h \to 0$ and $\lambda_1 = 0$ if $p_2 \geq \alpha_0 p_3$ and

$$\Psi_{01|g=\pm g_1} + \frac{\Delta^2}{2\alpha_1 p_3 \pi^2}[Lp_2\ Lp_3]^T[Lp_2\ Lp_3] \leq 0, \tag{3.45}$$

where

$$\Psi_{01} \triangleq \begin{bmatrix} 2p_2(g - L) + 2\alpha_0 p_1 & \phi_{12} \\ * & 2p_2 + 2p_3(\alpha_0 - \beta) \end{bmatrix}.$$

The Lyapunov inequalities in the vertices $g = \pm g_1$

$$\Psi_{01|g=\pm g_1} < 0, \quad P_0 > 0 \tag{3.46}$$

guarantee the quadratic exponential stability with a decay rate α_0 of the following second-order ODE

$$\dot{\zeta}(t) = \begin{bmatrix} 0 & 1 \\ g - L & -\beta \end{bmatrix}\zeta(t), \quad \zeta \in \mathbb{R}^2. \tag{3.47}$$

Therefore, if LMIs (3.46) are feasible, then LMIs (3.36) and (3.38) are feasible for small enough Δ and h provided α_0 is not too large (to guarantee $p_2 \geq \alpha_0 p_3$). Note that in the linear case the choice of $L > g_1$ guarantees the exponential stability of (3.47).

We are in a position to summarize the main result of this section:

Theorem 3.1. *Consider the sampled-data error system (3.16) under the Dirichlet, Neumann or mixed boundary conditions with bounds Δ, h and g_1 in (3.7) and (3.5). Let C be defined by (3.34). Choose $L > g_1$ subject to (3.44).*

(i) *Given scalars $\alpha_0 > \alpha_1 > 0$ with $\alpha_0 < \frac{\beta}{2}$, assume that there exist scalars $p_1, p_2, p_3, r > 0$ and $\lambda_1 \geq 0$ that satisfy LMIs (3.26), (3.36) and (3.38) with notations given by (3.37), (3.39). Then the error system is exponentially stable with a decay rate α, meaning that (2.2) holds for solutions of the error system with V defined by (3.24)–(3.27). Here α is a unique positive solution of $\alpha = \alpha_0 - \alpha_1 \exp(2\alpha h)$. Moreover, if the strict inequalities (3.26), (3.36) and (3.38) are feasible with $\alpha_0 = \alpha_1 > 0$, then the error system is exponentially stable with a small enough decay rate.*

(ii) *If the ODE (3.47) is quadratically exponentially stable with a small enough decay rate $\alpha_0 < 0.5\beta$, i.e. if LMIs (3.46) are feasible, then LMIs (3.26), (3.36) and (3.38) are also feasible (and the error system is exponentially stable) for small enough Δ and h.*

3.3. Numerical example

Consider the damped wave equation (3.1) under the Dirichlet boundary conditions (3.2), and its corresponding observer (3.13) under the discrete-time point measurements with the following parameters:

$$\beta = 3, \quad g_1 = 2, \quad L = g_1 - 1 + \beta^2/4 = 3.25. \tag{3.48}$$

Here L is the minimal gain subject to $L > g_1$ and (3.44). As mentioned in Remark 3.1, a smaller L leads to larger Δ and h that preserve the convergence. We consider the case of either linear $f = g_1 z$, where we verify the feasibility of LMIs of Theorem 3.1 in one vertex $g = g_1$, or the general case with $|f_z| \leq g_1$, where the feasibility of LMIs is verified in both vertices $g = \pm g_1$. We use the standard LMI Toolbox of Matlab for the verification of the feasibility of LMIs. Note that for $f = g_1 z$, the system is unstable (cf. Remark 3.1). Simulations of solutions to sine–Gordon equation with $f = g_1 \sin z$ and the initial conditions

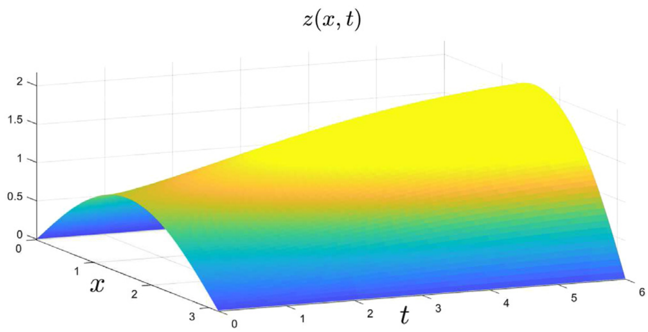

Fig. 1. The state $z(x,t)$, for $f = 2\sin z$.

Table 1
Maximal decay rate vs. number of sensors: $t_{k+1} - t_k \le 0.01$.

N	2	3	4	100	10^5
$\alpha\|_{f_z\equiv 2}$	–	0.3698	0.5098	0.8998	0.989
α_0		0.59	0.7	0.99	0.99
α_1		0.22	0.19	0.09	0.001
$\alpha\|_{\|f_z\|\le 2}$	–	0.05	0.23	0.69	0.78
α_0		0.4	0.5	0.7	0.79
α_1		0.35	0.27	0.01	0.01

Table 2
Maximal value of h vs. number of sensors and small α.

N	2	3	4	100	10^5
$h\|_{f_z\equiv 2}$	–	0.5	0.59	1.67	1.84
$\alpha_0 = \alpha_1$		0.2	0.1	0.001	0.0001
$h\|_{\|f_z\|\le 2}$	–	0.04	0.21	0.76	0.86
$\alpha_0 = \alpha_1$		0.4	0.2	0.001	0.0001

$z_0 = \sin x$, $z_1 = 0$ (see Fig. 1) show instability. For simplicity only we present in Tables 1 and 2 the LMI-based results under the uniform spatial sampling $x_{j+1} - x_j = \frac{\pi}{N}$, $j = 0, \dots, N-1$ with $\Delta = \pi/N$. The resulting maximal achievable decay rates α for small enough upper bound $h = 0.01$ on the time sampling intervals are shown in Table 1, whereas maximal values of h that preserve the exponential stability with a small enough decay rate are given in Table 2.

For $f = 2z$ and $f = 2\sin z$, we employ the finite-difference method and proceed with simulations of solutions to the error equation for L and β given by (3.48). The initial conditions are taken as $z_0 = \sin x$, $z_1 = 0$. We consider the uniform spatial sampling with $N = 3$ and variable time sampling with sampling intervals generated from a uniform distribution probability density function in $[0, h]$. The simulations confirm the theoretical results and illustrate their conservatism: the minimal h that leads to unstable error system from simulations is essentially larger than h in Table 2 as shown in Figs. 2 and 3 (upper lines).

4. Sampled-data observer for semilinear wave equation with boundary damping

4.1. Problem formulation

In this section we consider the wave equation

$$z_{tt}(x,t) = z_{xx}(x,t) + f\big(z(x,t), x, t\big), \quad x \in (0,\pi),\ t \ge t_0 \tag{4.1}$$

under the boundary damping

$$z(0,t) = 0, \qquad z_x(\pi,t) = -\beta z_t(\pi,t). \tag{4.2}$$

Here $z(x,t) \in \mathbb{R}$ is the state, $\beta > 0$ is the damping coefficient, and f is a function of class $\mathcal{C}^1$. We assume that the derivative f_z is uniformly bounded as follows:

$$|f_z(z,x,t) - g_1| \le \delta \quad \forall (z,x,t) \in \mathbb{R} \times [0,\pi] \times [t_0, \infty), \tag{4.3}$$

where g_1 and δ are given constants. Note that differently from the case of viscous damping, here f is close to the linear function $g_1 z$. The initial conditions are given by (3.6).

The boundary-value problem (4.1)–(4.2) can be represented as an abstract differential equation by defining the state $\zeta(t) = [\zeta_0(t)\ \zeta_1(t)]^T = [z(t)\ z_t(t)]^T$ and the operators

$$\mathscr{A} = \begin{bmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & 0 \end{bmatrix}, \quad F(\zeta, t) = \begin{bmatrix} 0 \\ F_1(\zeta_0, t) \end{bmatrix}. \tag{4.4}$$

Here $F_1 : \mathscr{H}^1(0,\pi) \times [t_0,\infty) \to L^2(0,\pi)$ is defined as $F_1(\zeta_0, t) = f(\zeta_0, \cdot, t)$ so that it is continuous in t for each $\zeta_0 \in \mathscr{H}^1(0,\pi)$. The resulting differential equation (3.10) is considered in the Hilbert space $\mathscr{H} = \mathscr{H}_L^1 \times L^2(0,\pi)$, where

$$\mathscr{H}_L^1 = \left\{ \zeta_0 \in \mathscr{H}^1(0,\pi) \;\middle|\; \zeta_0(0) = 0 \right\}.$$

and $\|\zeta\|_{\mathscr{H}}^2 = \|\zeta_{0x}\|_{L^2}^2 + \|\zeta_1\|_{L^2}^2$. The operator $\mathscr{A}$ has the dense domain

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{bmatrix} \zeta_0 \\ \zeta_1 \end{bmatrix} \in \mathscr{H}^2(0,\pi) \bigcap \mathscr{H}_L^1 \times \mathscr{H}_L^1 \middle| \zeta_{0x}(\pi) = -\beta \zeta_1(\pi) \right\}.$$

Existence of mild (classical) solutions to (4.1)–(4.2), for $\zeta(t_0) \in \mathscr{H}$ (and classical solution for $\zeta(t_0) \in \mathscr{D}(\mathscr{A})$) can be proved similar to Fridman (2013) (see Section 2).

4.2. Sampled in time and space observer

We aim to derive conditions for the case of sampled in time and in space measurements (3.8). Consider a sampled-data observer of the form

$$\begin{aligned} \hat{z}_{tt}(x,t) &= \hat{z}_{xx}(x,t) + f\big(\hat{z}, x, t\big) \\ &\quad + L \sum_{j=0}^{N-1} \chi_j(x)[y_{jk} - \hat{z}(\bar{x}_j, t_k)], \\ &\quad x \in (0,\pi),\ t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, \dots \end{aligned} \tag{4.5}$$

under the mixed boundary conditions

$$\hat{z}(0,t) = 0, \quad \hat{z}_x(\pi,t) = -\beta \hat{z}_t(\pi,t), \tag{4.6}$$

and the initial conditions $[\hat{z}(\cdot,t_0), \hat{z}_t(\cdot,t_0)]^T \in \mathscr{H}$. Here the measurements y_{jk} are given by (3.8), L is the observer gain, and the characteristic functions $\chi_j(x)$ are given by (3.15). The estimation error $e = z - \hat{z}$ satisfies the following semilinear wave equation

$$\begin{aligned} e_{tt}(x,t) &= e_{xx}(x,t) + g e(x,t) - L \sum_{j=0}^{N-1} \chi_j(x) e(\bar{x}_j, t_k), \\ &\quad x \in (0,\pi),\ t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, \dots \end{aligned} \tag{4.7}$$

under the mixed boundary conditions

$$e(0,t) = 0, \quad e_x(\pi,t) = -\beta e_t(\pi,t), \tag{4.8}$$

where g is defined by (3.18). The well-posedness of the error equation can be proved by using the step method, similarly to Section 3.2.

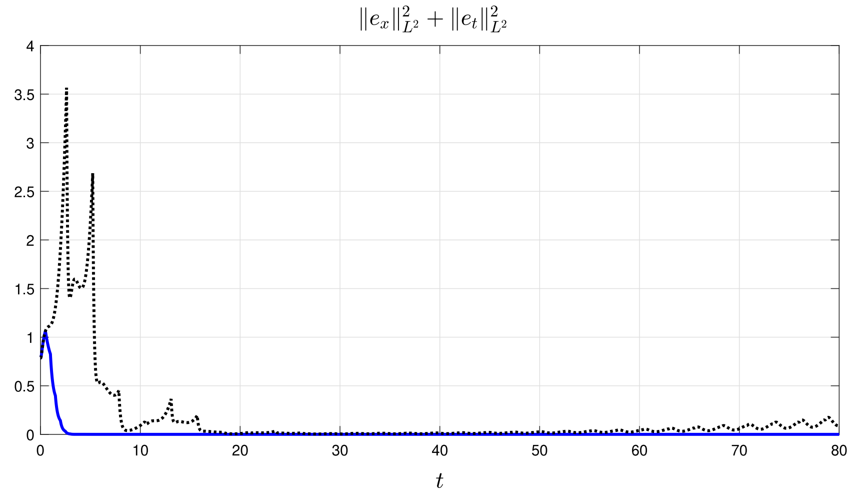

Fig. 2. Error energy for $f = 2z, N = 3$ and $h = 0.5$ (lower line) or $h = 2.62$ (upper line).

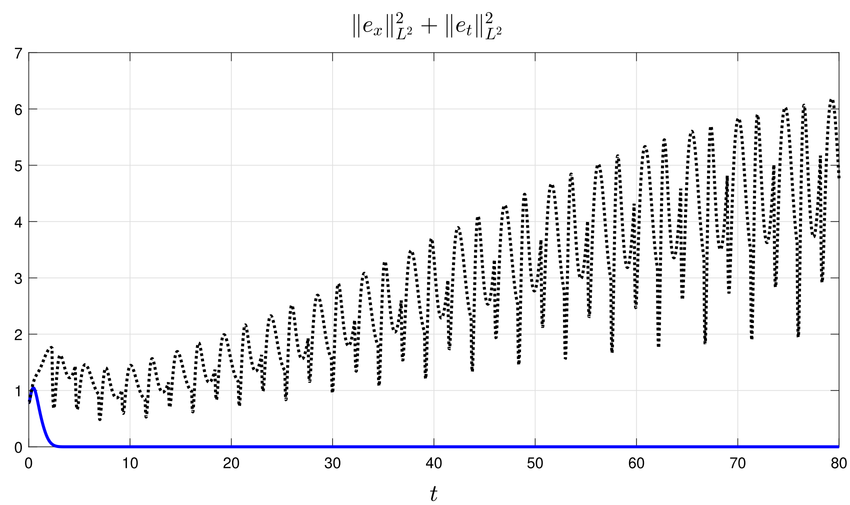

Fig. 3. Error energy for $f = 2\sin z, N = 3$, and $h = 0.04$ (lower line) or $h = 2.3$ (upper line).

By using the relation (3.22) we represent (4.7) as

$$\begin{aligned} &e_{tt}(x,t) = e_{xx}(x,t) + (g-L)e(x,t) + \sigma, \\ &\quad \sigma = L\sum_{j=0}^{N-1}\chi_j(x)\left[(t-t_k)\nu + \int_{\bar{x}_j}^{x} e_\zeta(\zeta,t_k)d\zeta\right], \\ &\quad x\in(0,\pi),\ t\in[t_k,t_{k+1}),\quad k=0,1,2,\dots \end{aligned} \tag{4.9}$$

Rewriting the error system in Riemann coordinates

$$\begin{aligned} R(x,t) &\triangleq e_t(x,t) - e_x(x,t), \\ Q(x,t) &\triangleq e_t(x,t) + e_x(x,t), \end{aligned} \tag{4.10}$$

we arrive at

$$\begin{aligned} R_t &= -R_x + (g-L)e + \sigma, \\ Q_t &= Q_x + (g-L)e + \sigma \end{aligned} \tag{4.11}$$

under the boundary conditions

$$R(0,t) = -Q(0,t), \quad Q(\pi,t) = \frac{1-\beta}{1+\beta}R(\pi,t). \tag{4.12}$$

Here

$$e(x,t) = 0.5\int_0^x (Q(\xi,t) - R(\xi,t))\,d\xi.$$

Following Bastin and Coron (2016), we choose the Lyapunov function

$$V_q(t) = \int_0^\pi \left[q_1R^2(x,t)\exp(-\mu x) + q_2Q^2(x,t)\exp(\mu x)\right]dx \tag{4.13}$$

with positive scalars $q_{1,2}$ and μ. Differentiating (4.13) along (4.11) we have

$$\begin{aligned} \dot{V}_q &= 2\int_0^\pi [q_1RR_t\exp(-\mu x) + q_2QQ_t\exp(\mu x)]\,dx \\ &= \int_0^\pi \Big[-q_1\frac{\partial}{\partial x}R^2\exp(-\mu x) + q_2\frac{\partial}{\partial x}Q^2\exp(\mu x) \\ &+2\big(q_1R\exp(-\mu x) + q_2Q\exp(\mu x)\big)(g-L)e\ \Big]\,dx \\ &+2L\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\big(q_1R\exp(-\mu x) + q_2Q\exp(\mu x)\big) \\ &\qquad \times\left((t-t_k)\nu + \int_{\bar{x}_j}^{x} e_\zeta(\zeta,t_k)d\zeta\right)dx. \end{aligned}$$

Integrating by parts and substituting the boundary conditions we obtain

$$\begin{aligned}\dot V_q &\le -\mu V_q - \Big(q_2-q_1\Big)R^2(0,t)\\ &-\Big(q_1\exp(-\mu\pi)-q_2(\frac{1-\beta}{1+\beta})^2\exp(\mu\pi)\Big)Q^2(\pi,t)\\ &+2\int_0^\pi\big(q_1R\exp(-\mu x)+q_2Q\exp(\mu x)\big)(g-L)e dx\\ &+2L\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\big(q_1R\exp(-\mu x)+q_2Q\exp(\mu x)\big)\\ &\times\left((t-t_k)\nu+\int_{\bar x_j}^{x}e_\zeta(\zeta,t_k)d\zeta\right)dx.\end{aligned}\tag{4.14}$$

For the stability analysis of (4.9), we employ the Lyapunov–Krasovskii functional $V_b(t) = V_q(t)+V_r(t)$ with $V_r(t)$ given by (3.27) (to compensate ν) and Halanay's inequality (2.2) (to compensate $\int_{\bar x_j}^x e_\zeta(\zeta,t_k)d\zeta$). For some $\alpha_1<\alpha_0$ we have

$$\begin{aligned}&\dot V_b(t)+2\alpha_0V_b(t)-2\alpha_1\sup\nolimits_{-h\le\theta\le0}V_b(t+\theta)\\ &\le \dot V_b(t)+2\alpha_0V_b(t)-2\alpha_1\int_0^\pi\big[q_1R^2(x,t_k)\exp(-\mu x)\\ &+q_2Q^2(x,t_k)\exp(\mu x)\big]dx.\end{aligned}\tag{4.15}$$

Note that the diagonal Lyapunov function V_q leads to non-restrictive conditions for (4.11) with $e=\sigma=0$ (see Theorem 2.4 of Bastin & Coron, 2016). So, the term $(L-g)e$ will be treated in our stability analysis as the disturbance, and it is clear from (4.14) that the best choice for the observer gain is $L=g_1$ leading to the minimal interval $g-L\in[-\delta,\delta]$. By Wirtinger's inequality (2.1)

$$\int_0^\pi(g-L)^2e^2dx\le\int_0^\pi\delta^2e^2dx\le4\delta^2\int_0^\pi e_x^2dx,$$

where $e_x = 0.5[Q-R]$. Then, by S-procedure, the following non-negative term

$$\lambda_2\int_0^\pi\left[\delta^2(Q-R)^2-(g-L)^2e^2\right]dx\ge0,\quad \lambda_2\ge0,\tag{4.16}$$

can be added to the right-hand side of (4.15). Taking into account (3.33), we add to the right-hand side of (4.15) one more non-negative term

$$\begin{aligned}&\lambda_3\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\Bigg[\frac14(Q(x,t_k)-R(x,t_k))^2\\ &-\frac{\pi^2}{\Delta^2}\left(\int_{\bar x_j}^x e_\zeta(\zeta,t_k)d\zeta\right)^2\Bigg]dx,\quad \lambda_3\ge0.\end{aligned}\tag{4.17}$$

Denote

$$\eta_1=\begin{bmatrix}R\exp(-\mu x) & Q\exp(\mu x) & \nu & \int_{\bar x_j}^x e_\zeta(\zeta,t_k)d\zeta & (g-L)e\end{bmatrix}^T$$

and $\eta_2 = \begin{bmatrix}R(x,t_k) & Q(x,t_k)\end{bmatrix}^T$. Then from (4.14), (4.16)–(4.17) and (3.28) we have

$$\begin{aligned}&\dot V_b(t)+2\alpha_0V_b(t)-2\alpha_1\sup_{-h\le\theta\le0}V_b(t+\theta)\\ &\le -\left(q_1\exp(-\mu\pi)-q_2\left(\frac{1-\beta}{1+\beta}\right)^2\exp(\mu\pi)\right)Q^2(\pi,t)\\ &-\Big(q_2-q_1\Big)R^2(0,t)+\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}(\eta_1\Xi\eta_1^T+\eta_2\Upsilon\eta_2^T)dx\le0,\end{aligned}$$

if

$$q_2\ge q_1,\ \ q_1(1+\beta)^2-q_2(1-\beta)^2\exp(2\mu\pi)\ge0,\tag{4.18}$$

$$\Upsilon\triangleq\begin{bmatrix}-2\alpha_1q_1\exp(-\mu x)+\frac{\lambda_3}{4} & -\frac{\lambda_3}{4}\\ * & -2\alpha_1q_2\exp(\mu x)+\frac{\lambda_3}{4}\end{bmatrix}\le0$$

and

$$\Xi\triangleq\tag{4.19}$$
$$\begin{bmatrix}\xi_{11} & -\lambda_2\delta^2+\frac{r(h-\tau)}{4} & \tau Lq_1 & Lq_1 & q_1\\ * & \xi_{22} & \tau Lq_2 & Lq_2 & q_2\\ * & * & -r\tau\exp(-2\alpha_0h) & 0 & 0\\ * & * & * & -\frac{\pi^2}{\Delta^2}\lambda_3 & 0\\ * & * & * & * & -\lambda_2\end{bmatrix}\le0,$$

where

$$\begin{aligned}\xi_{11}&=\left(\lambda_2\delta^2+\frac{r(h-\tau)}{4}\right)\exp(2\mu x)-(\mu-2\alpha_0)q_1\exp(\mu x),\\ \xi_{22}&=\left(\lambda_2\delta^2+\frac{r(h-\tau)}{4}\right)\exp(-2\mu x)-(\mu-2\alpha_0)q_2\exp(-\mu x).\end{aligned}$$

By upper bounding the diagonal elements of Υ and Ξ and employing the affinity of Ξ in $\tau\in[0,h]$, we arrive at

$$\Upsilon\le\begin{bmatrix}-2\alpha_1q_1\exp(-\mu\pi)+\frac{\lambda_3}{4} & -\frac{\lambda_3}{4}\\ * & -2\alpha_1q_2+\frac{\lambda_3}{4}\end{bmatrix}\le0,\tag{4.20}$$

and

$$\Xi\Big|_{\tau=0,h}\le0,\tag{4.21}$$

where

$$\begin{aligned}\xi_{11}&=-(\mu-2\alpha_0)q_1+\left(\lambda_2\delta^2+\frac{r(h-\tau)}{4}\right)\exp(2\mu\pi),\\ \xi_{22}&=-(\mu-2\alpha_0)q_2\exp(-\mu\pi)+\lambda_2\delta^2+\frac{r(h-\tau)}{4}.\end{aligned}\tag{4.22}$$

Note that in LMI conditions of Theorem 4.1, the damping β appears only in (4.18). So, $\beta = 1$ leads to less restrictive conditions. Note also that by Schur complements, for all $\mu > 2\alpha_0 > 0$ and small enough δ,Δ and h the inequalities (4.18), (4.20) and (4.21) are always feasible for $\beta = 1$. We are in a position to formulate our main result for the case of boundary damping:

Theorem 4.1. *Consider the sampled-data error system (4.9) under the boundary conditions (4.8) with bounds Δ, h and δ in (3.7) and (4.3). Choose $L=g_1$.*

(i) *Given positive scalars μ, α_0 and α_1 such that $\alpha_1<\alpha_0<0.5\mu$, assume that there exist scalars $q_{1,2}>0$, $r>0$ and $\lambda_{2,3}>0$ that satisfy LMIs (4.18), (4.20) and (4.21) with the notations (4.19) and (4.22). Then, the error system is exponentially stable with a decay rate α, meaning that the following inequality holds:*

$$V_b(t)\le\exp\big(-2\alpha(t-t_0)\big)V_b(t_0),\ \ t\ge t_0,$$

where α is a unique positive solution of $\alpha=\alpha_0-\alpha_1\exp(2\alpha h)$. Moreover, if the strict inequalities (4.18), (4.20) and (4.21) are feasible with $\alpha_0=\alpha_1>0$, then the sampled-data error system is exponentially stable with a small enough decay rate.

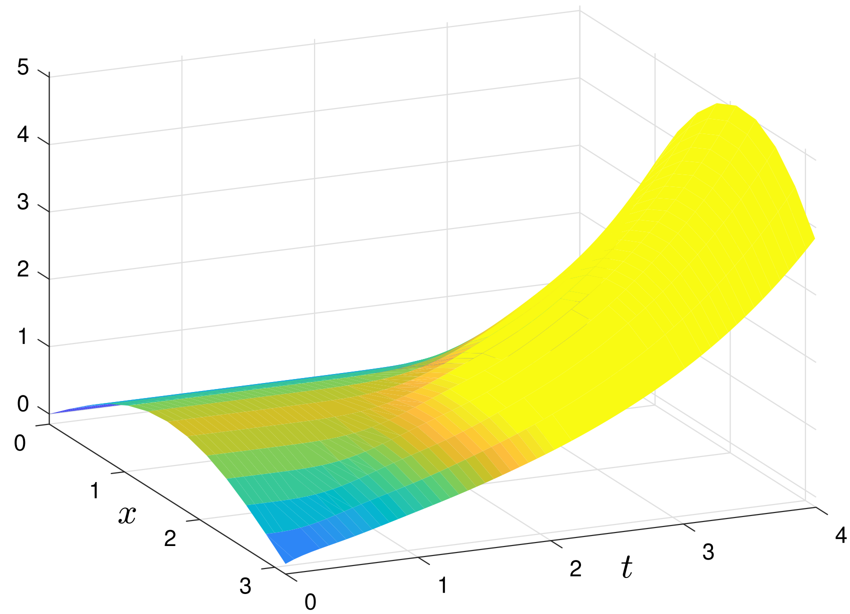

Fig. 4. The state $z(x,t)$ for $f=g_1z$.

Table 3
$\max\{\alpha\}$ vs. N, for $t_{k+1}-t_k\le 10^{-5}$ and $\mu=0.4$.

N	4	5	100	1e5
$\alpha\big\vert_{f_z\equiv g_1}$	0.013	0.046	0.1881	0.1989
α_0	0.13	0.1	0.19	0.199
α_1	0.108	0.054	0.002	0.0001
$\alpha\big\vert_{\vert f_z-g_1\vert\le 0.01}$	–	0.006	0.133	0.1471
α_0	–	0.14	0.147	0.1472
α_1	–	0.0801	0.014	0.0001

Table 4
$\max\{h\}$ vs. N, for small α and $\mu=0.4$.

N	4	5	100	10^5
$h\big\vert_{f_z\equiv g_1}$	0.013	0.032	0.089	0.089
$\alpha_0=\alpha_1$	0.09	0.09	0.03	0.03
$h\big\vert_{\vert f_z-g_1\vert\le 0.01}$	–	0.004	0.063	0.063
$\alpha_0=\alpha_1$	–	0.09	0.03	0.03

(ii) *For $\beta=1$, LMIs (4.18), (4.20) and (4.21) are always feasible (hence, the error system is exponentially stable) for small enough δ, Δ and h.*

4.3. Numerical example

Consider the boundary-damped wave system (4.1)–(4.2), and its corresponding observer (4.5)–(4.6) with

$$\beta=1,\quad g_1=1,\quad L=g_1,\quad \delta=0.01. \tag{4.23}$$

We consider the case of either linear $f=g_1z$, or the general case with $|f_z-g_1|\le\delta$. Note that simulation of solution to the original system with the initial condition $z_0=\sin x$, $z_1=0$ for $f=g_1z$ shows instability (see Fig. 4). For simplicity, the results are presented under the uniform spatial sampling $x_{j+1}-x_j=\frac{\pi}{N}$, $j=0,\dots,N-1$ with $\Delta=\pi/N$. We verify the LMI conditions of Theorem 4.1. See Table 3 for maximal decay rates under the time sampling $t_{k+1}-t_k\le 10^{-5}$ and Table 4 for maximal time sampling and small enough decay rate.

By employing the finite-difference method, we proceed with simulations of solutions to the error equation for $f=g_1z$ and $f=g_1z+\delta\sin z$, with the initial conditions $z_0=\sin x$, $z_1=0$. In simulations we choose $L=g_1$ and β,δ that are given by (4.23) and consider the uniform spatial sampling. The variable time sampling is chosen with sampling intervals that are generated from a uniform distribution probability density function in $[0,h]$. Similar to the case of viscous damping, the minimal h that leads to unstable error system from simulation is essentially larger than h obtained in Table 4 and is shown in Fig. 5 for $f_z\equiv g_1$, and in Fig. 6 for $f=g_1z+\delta\sin z$.

5. Dual sampled-data control problems

We formulate in this section the dual distributed sampled-data controller problems that can be solved by the methods of the previous sections. The controlled semilinear wave equation with the viscous damping has the following form:

$$\begin{aligned} z_{tt}(x,t)&=z_{xx}(x,t)-\beta z_t(x,t)+f\big(z(x,t),x,t\big)z(x,t)\\ &\quad+\sum_{j=0}^{N-1}\chi_j(x)u_j(t_k),\\ &\quad x\in(0,\pi),\ t\in[t_k,t_{k+1}),\ k=0,1,2\dots, \end{aligned} \tag{5.1}$$

whereas the boundary conditions are either Dirichlet (3.2) or Neumann (3.3) or mixed (3.4). The controlled boundary-damped wave equation has the form

$$\begin{aligned} z_{tt}(x,t)=&z_{xx}(x,t)+f\big(z(x,t),x,t\big)z(x,t)+\sum_{j=0}^{N-1}\chi_j(x)u_j(t_k),\\ &x\in(0,\pi),\ t\in[t_k,t_{k+1}),\ k=0,1,2\dots \end{aligned} \tag{5.2}$$

under the mixed boundary conditions

$$z(0,t)=0,\qquad z_x(\pi,t)=-\beta z_t(\pi,t),\quad \beta>0. \tag{5.3}$$

In (5.1) and (5.2)–(5.3), $\beta>0$ is a damping coefficient, $u_j(t_k)$ is the control input, $f(z,x,t)$ is a function of class $\mathcal{C}^1$, satisfying

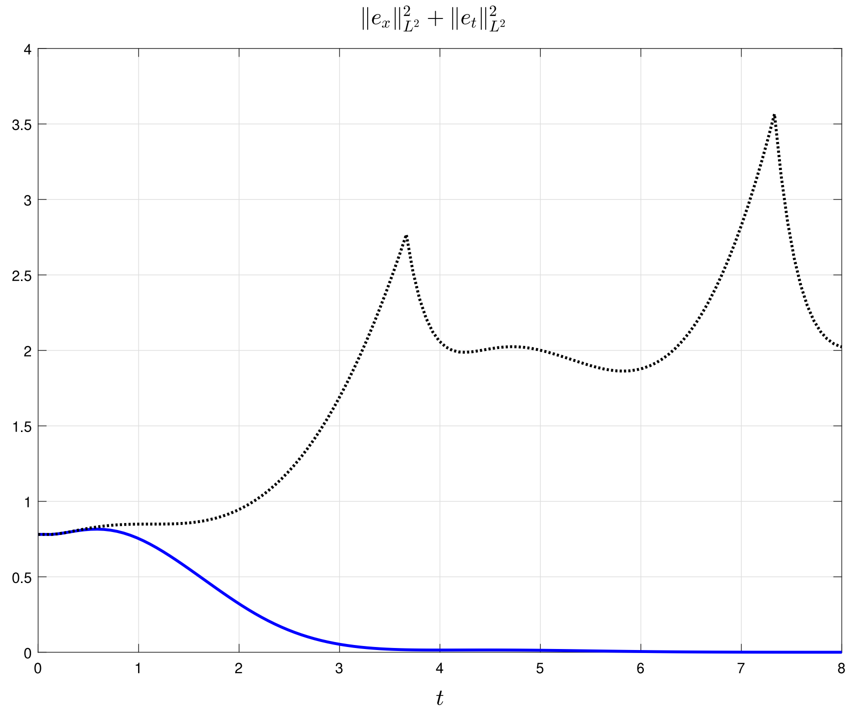

Fig. 5. Error energy for $f = g_1 z$, $N = 4$, and $h = 0.013$ (lower line) or $h = 3.7$ (upper line).

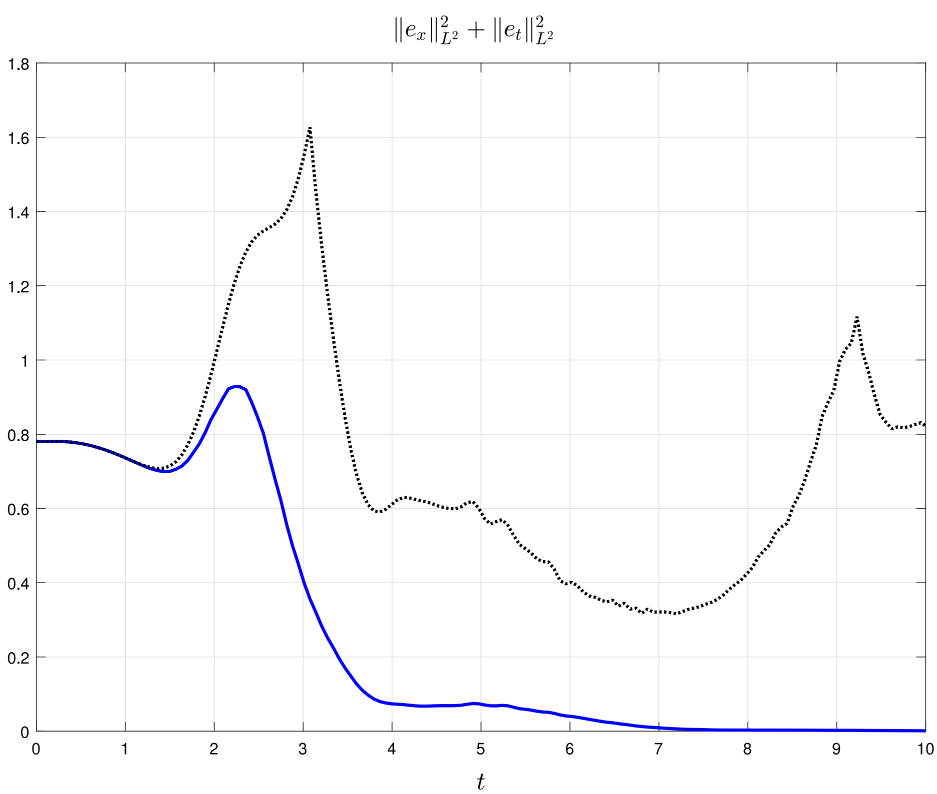

Fig. 6. Error energy for $f = g_1 z + \delta \sin z$, $N = 5$, and $h = 0.004$ (lower line) or $h = 3.1$ (upper line).

$|f(z, x, t)| \leq g_1$ with a known $g_1 > 0$. The discrete sampling points are given by (3.8).

The stabilizing sampled-data controller has a form

$$u_j(t_k) = -Kz(\bar{x}_j, t_k), \quad \bar{x}_j = \frac{x_{j+1} - x_j}{2}, \quad k = 0, 1, 2, \ldots, \quad j = 0, \ldots, N-1 \tag{5.4}$$

with the gain $K > 0$. Then, conditions of Theorems 3.1 and 4.1 with $K = L$ guarantee the exponential stability of the closed-loop systems (5.1), (5.4) and (5.2), (5.4), respectively.

Remark 5.1. As already mentioned, the global boundedness assumption (3.5) is restrictive. This assumption can be relaxed to regional boundedness as considered in Section 4 of Fridman and Terushkin (2016) that, for the case of control, should lead to regional stabilization and should be applicable to Klein–Gordon equation. Regional stabilization under regional boundedness in (3.5) may be a topic for future research.

6. Conclusions

In this paper we introduced sampled-data observers for 1D damped semilinear wave equations under sampled in time and in space measurements. Sufficient LMI-based conditions for the exponential stability of the estimation error were formulated in terms of LMIs. The dual distributed sampled-data control problem was presented. The presented method can be developed for event-triggered control under discrete time measurements. Extension of the method to other classes of hyperbolic systems may be a topic for the future research.

References

Antsaklis, P., & Baillieul, J. (2007). Special issue on technology of networked control systems. *Proceedings of the IEEE*, *95*(1), 5–8.

Åström, K. J., & Wittenmark, B. (1997). *Computer-controlled systems*. Prentice Hall.

Bar Am, N., & Fridman, E. (2014). Network-based H_∞ filtering of parabolic systems. *Automatica*, *50*, 3139–3146.

Bastin, G., & Coron, J.-M. (2016). *Stability and boundary stabilization of 1-d hyperbolic systems, Vol. 88*. Springer.

Cheng, M.-B., Radisavljevic, V., Chang, C.-C., Lin, C.-F., & Su, W.-C. (2009). A sampled-data singularly perturbed boundary control for a heat conduction system with noncollocated observation. *IEEE Transactions on Automatic Control*, *54*(6), 1305–1310.

Curtain, R., & Zwart, H. (1995). *An introduction to infinite-dimensional linear systems theory, Vol. 21*. Springer.

Datko, R. (1988). Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks. *SIAM Journal on Control and Optimization*, *26*(3), 697–713.

Dickey, R. W. (1976). Stability theory for the damped sine-gordon equation. *SIAM Journal of Applied Mathematics*, *30*(2), 248–262.

Fridman, E. (2010). A refined input delay approach to sampled-data control. *Automatica*, *46*, 421–427.

Fridman, E. (2013). Observers and initial state recovering for a class of hyperbolic systems via Lyapunov method. *Automatica*, *49*(7), 2250–2260.

Fridman, E. (2014). *Introduction to time-delay systems: analysis and control*. Birkhauser, Systems and Control: Foundations and Applications.

Fridman, E., & Bar Am, N. (2013). Sampled-data distributed H_∞ control of transport reaction systems. *SIAM Journal on Control and Optimization*, *51*(2), 1500–1527.

Fridman, E., & Blighovsky, A. (2012). Robust sampled-data control of a class of semilinear parabolic systems. *Automatica*, *48*, 826–836.

Fridman, E., & Orlov, Y. (2009). Exponential stability of linear distributed parameter systems with time-varying delays. *Automatica*, *45*(2), 194–201.

Fridman, E., Seuret, A., & Richard, J. P. (2004). Robust sampled-data stabilization of linear systems: an input delay approach. *Automatica*, *40*(8), 1441–1446.

Fridman, E., & Terushkin, M. (2016). New stability and exact observability conditions for semilinear wave equations. *Automatica*, *63*, 1–10.

Ghantasala, S., & El-Farra, N. (2012). Active fault-tolerant control of sampled-data nonlinear distributed parameter systems. *International Journal of Robust and Nonlinear Control*, *22*(1), 24–42.

Gu, K., Kharitonov, V., & Chen, J. (2003). *Stability of time-delay systems*. Boston: Birkhauser.

Halanay, A. (1966). *Differential Equation: Stability, Oscillations, Time Lags*. New york, USA: Academic Press.

Hardy, G. H., Littlewood, J. E., & Pólya, G. (0000). Inequalities, Mathematical Library, Cambridge.

Hespanha, J., Naghshtabrizi, P., & Xu, Y. (2007). A survey of recent results in networked control systems. *Proceedings of the IEEE*, *95*(1), 138–162.

Hetel, L., Fiter, C., Omran, H., Seuret, A., Fridman, E., Richard, J.-P., et al. (2017). Recent developments on the stability of systems with aperiodic sampling: An overview. *Automatica*, *76*, 309–335.

Kalantarov, V. K., & Titi, E. S. (2016). Finite-parameters feedback control for stabilizing damped nonlinear wave equations. In *Contemp. Math*: *vol. 659*, *Nonlinear analysis and optimization* (pp. 115–133).

Kang, W., & Fridman, E. (2018). Distributed sampled-data control of Kuramoto-Sivashinsky equation. *Automatica*.

Karafyllis, I., & Krstic, M. (2017). Sampled-data boundary feedback control of 1-D linear transport PDEs with non-local terms. *Systems & Control Letters*, *107*, 68–75.

Karafyllis, I., & Krstic, M. (2018). Sampled-data boundary feedback control of 1-D parabolic PDEs. *Automatica*, *81*, 226–237.

Kobayashi, T. (2003). Boundary feedback stabilization of the sine-gordon equation without velocity feedback. *Journal of Sound and Vibration*, *266*(4), 775–784.

Levi, M., Hoppensteadt, F. C., & Miranker, W. L. (1978). Dynamics of the Josephson junction. *Quarterly of Applied Mathematics*, *36*(2), 167–198.

Logemann, H. (2013). Stabilization of well-posed infinite-dimensional systems by dynamic sampled-data feedback. *SIAM Journal on Control and Optimization*, *51*(2), 1203–1231.

Logemann, H., Rebarber, R., & Townley, S. (2005). Generalized sampled-data stabilization of well-posed linear infinite-dimensional systems. *SIAM Journal on Control and Optimization*, *44*(4), 1345–1369.

Mikheev, Y., Sobolev, V., & Fridman, E. (1988). Asymptotic analysis of digital control systems. *Automation and Remote Control*, *49*, 1175–1180.

Nicaise, S., & Pignotti, C. (2006). Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM Journal on Control and Optimization*, *45*(5), 1561–1585.

Pata, V., & Zelik, S. V. (2006). *Attractors and their regularity for 2-D wave equations with nonlinear damping*. WIAS.

Pazy, A. (1983). *Semigroups of linear operators and applications to partial differential equations, Vol. 44*. Springer New York.

Selivanov, A., & Fridman, E. (2016). Distributed event-triggered control of diffusion semilinear PDEs. *Automatica*, *68*, 344–351.

Selivanov, A., & Fridman, E. (2017). Sampled-data relay control of diffusion PDEs. *Automatica*, *82*, 59–68.

Smyshlyaev, A., Cerpa, E., & Krstic, M. (2010). Boundary stabilization of a 1-D wave equation with in-domain antidamping. *SIAM Journal on Control and Optimization*, *48*(6), 4014–4031.

Terushkin, M., & Fridman, E. (2019). Sampled-data observers for semilinear damped wave equations under point measurements. In *The european control conference*.

Yakubovich, V. (1971). S-procedure in nonlinear control theory. *Vestnik Leningrad University*, *1*, 62–77.

Maria Terushkin received her M.Sc. degree in 2016, in Electrical Engineering at Tel-Aviv University, Israel, where she is currently a Ph.D. student. Her research interests include distributed-parameter systems, networked-control systems, and time-delay systems.

Emilia Fridman received the M.Sc. degree from Kuibyshev State University, USSR, in 1981 and the Ph.D. degree from Voronezh State University, USSR, in 1986, all in mathematics. From 1986 to 1992 she was an Assistant and Associate Professor in the Department of Mathematics at Kuibyshev Institute of Railway Engineers, USSR. Since 1993 she has been at Tel Aviv University, where she is currently Professor of Electrical Engineering-Systems. She has held visiting positions at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin (Germany), INRIA in Rocquencourt (France), Ecole Centrale de Lille (France), Valenciennes University (France), Leicester University (UK), Kent University (UK), CINVESTAV (Mexico), Zhejiang University (China), St. Petersburg IPM (Russia), Melbourne University (Australia), Supelec (France), KTH (Sweden).

Her research interests include time-delay systems, networked control systems, distributed parameter systems, robust control, singular perturbations and nonlinear control. She has published more than 150 articles in international scientific journals. She is the author of the monograph "Introduction to Time-Delay Systems: Analysis and Control" (Birkhauser, 2014). She serves/served as Associate Editor in Automatica, SIAM Journal on Control and Optimization and IMA Journal of Mathematical Control and Information. In 2014 she was Nominated as a Highly Cited Researcher by Thomson ISI. Since 2018, she has been the incumbent for Chana and Heinrich Manderman Chair on System Control at Tel Aviv University. She is IEEE Fellow since 2019. She is currently a member of the IFAC Council.