Interval observer design and control of uncertain non-homogeneous heat equations

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A B S T R A C T

The problems of state estimation and observer-based control for heat non-homogeneous equations under distributed in space point measurements are considered. First, an interval observer is designed in the form of Partial Differential Equations (PDEs), without Galerkin projection. Second, conditions of boundedness of the interval observer solutions with non-zero boundary conditions and measurement noise are proposed. Third, the obtained interval estimates are used to design a dynamic output-feedback stabilizing controller. The advantages of the PDE-based interval observer over the approximation-based one are clearly shown in the numerical example.

1. Introduction

Due to various technical (complexity of implementation) or economic (price of solution) issues, an explicit measurement of state vector of a dynamical system may be impossible. This is especially the case, for example, in distributed parameter systems, where the system state is a function of the space and time, and only pointwise and discrete measurements are realizable by conventional sensors. Consequently, the system state in these cases has to be reconstructed using estimation algorithms (Besançon, 2007; Fossen & Nijmeijer, 1999; Meurer, Graichen, & Gilles, 2005). The most popular approaches in this domain include Luenberger observer and Kalman filter for deterministic and stochastic settings, respectively, which are developed for linear time-invariant models, that is the case where the existing theory disposes plenty of solutions. For nonlinear dynamical systems, the state estimation algorithms are often based on a partial similarity of the plant models to linear ones, or representations in various canonical forms are widely used. The same observations are also valid for control synthesis.

Many physical phenomena can be formalized in terms of PDEs (e.g. sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics), whose distributed nature introduces additional level of complexity in design. That is why control and estimation of PDEs are very popular directions of research nowadays (Bredies, Clason, Kunisch, & von Winckel, 2013; Smyshlyaev & Krstic, 2010). Frequently, for design of a state estimator or control, the finite-dimensional approximation approach is used (Alvarez & Stephanopoulos, 1982; Dochain, 2000; Hagen & Mezic, 2003; Vande Wouver & Zeitz, 2002), then the control or estimation problems are addressed in the framework of finite-dimensional systems using well-known tools. Analysis and design in the original distributed coordinates are more complicated, but also attract attention of many researchers (Ahmed-Ali, Giri, Krstic, & Lamnabhi-Lagarrigue, 2015; Fridman, 2013; Fridman & Bar Am, 2013; Fridman & Blighovsky, 2012; Hidayat, Babuska, De Schutter, & Nunzi, 2011; Schaum, Moreno, Fridman, & Alvarez, 2014; Selivanov & Fridman, 2018; Smyshlyaev & Krstic, 2010). In Pisano and Orlov (2017) a stabilizing control design with a proportional-discontinuous feedback is proposed for a parabolic PDE with pointwise colocated sensing and actuation, and with in-domain distributed disturbances. The work (Wang, Liu, & Sun, 2018) presents a Luenberger-type observer-based distributed control with non-collocated sensors and actuators.

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2. Preliminaries

The real numbers are denoted by \( \mathbb{R} \), \( \mathbb{R}^+ = \{ \tau \in \mathbb{R} : \tau \geq 0 \} \). Euclidean norm for a vector \( x \in \mathbb{R}^n \) will be denoted as \( |x| \).

If \( X \) is a normed space with the norm \( \| \cdot \|_X \), \( \Omega \subset \mathbb{R}^n \) for some \( n \geq 1 \) and \( \phi : \Omega \to X \), define
\[
\| \phi \|^2_{L^2(\Omega, X)} = \int_\Omega \| \phi(s) \|^2_X ds,
\]
\[
\| \phi \|_{L^\infty(\Omega, X)} = \text{ess sup}_{s \in \Omega} \| \phi(s) \|_X.
\]

By \( L^\infty(\Omega, X) \) and \( L^2(\Omega, X) \) denote the spaces of functions \( \Omega \to X \) with the properties \( \| \cdot \|_{L^\infty(\Omega, X)} < +\infty \) and \( \| \cdot \|_{L^2(\Omega, X)} < +\infty \), respectively. Denote \( I = [0, \ell] \) for some \( \ell > 0 \), let \( C^q(I, X) \) be the set of functions having continuous derivatives at least up to order \( k \geq 0 \) on \( I \). For any \( q > 0 \) and an interval \( I' \subseteq I \) define \( W^{q, \infty}(I', \mathbb{R}) \) as a subset of functions \( y \in C^{q-1}(I', \mathbb{R}) \) with an absolutely continuous \( y^{(q-1)} \) and essentially bounded \( y^{(q)} \) on \( I' \), \( \| y \|_{W^{q, \infty}} = \sum_{m=0}^{q} \| y^{(m)} \|_{L^\infty(I', \mathbb{R})} \). Denote by \( H^q(I, \mathbb{R}) \) with \( q \geq 0 \) the Sobolev space of functions with derivatives through order \( q \) in \( L^2(I, \mathbb{R}) \).

For two functions \( \phi_1, \phi_2 : I \to \mathbb{R} \) their relation \( \phi_1 \leq \phi_2 \) has to be understood as \( \phi_1(x) \leq \phi_2(x) \) for almost all \( x \in I \), the inner product is defined in a standard way:
\[
\langle \phi_1, \phi_2 \rangle = \int_0^\ell \phi_1(x)\phi_2(x)dx.
\]

For \( \phi \in \mathbb{R} \) define two operators \( \phi^+ \) and \( \phi^- \) as follows:
\[
\phi^+ = \max(0, \phi), \quad \phi^- = \phi - \phi^+.
\]

**Lemma 1** (Kharkovskaya et al., 2018). Let \( s, s_1, s_2 : I \to \mathbb{R} \) admit the relations \( s \leq s_1 \leq s_2 \), then for any \( \phi : I \to \mathbb{R} \) we have
\[
(s, \phi^+ - (s, \phi^-)) \leq (s, \phi) \leq (s_2, \phi^+) - (s_1, \phi^-).
\]

For later use, we need the following inequalities:

**Lemma 2** (Hardy, Littlewood, & Polya, 1988 Wirtinger’s Inequality). Let \( z \in H^1(I, \mathbb{R}) \), then
\[
\int_0^\ell z^2(\xi)d\xi \leq \frac{b^2\ell^2}{\pi^2} \int_0^\ell \left[ \frac{dz(\xi)}{d\xi} \right]^2 d\xi,
\]
and if \( z(0) = z(\ell) = 0 \), then \( b = 1 \); if only \( z(0) = 0 \) or \( z(\ell) = 0 \), then \( b = 4 \).

**Lemma 3** (Bar Am & Fridman, 2014 Poincare’s Inequality). Let \( z \in H^1(I, \mathbb{R}) \) with \( \int_0^\ell z(\xi)d\xi = 0 \), then
\[
\int_0^\ell z^2(\xi)d\xi \leq \frac{\ell^2}{\pi^2} \int_0^\ell \left[ \frac{dz(\xi)}{d\xi} \right]^2 d\xi.
\]

3. Input-to-state stability and positivity of non-homogeneous heat equation

In this section the basic facts on heat equation and positivity of its solutions are given.

3.1. Heat equation

Consider the following PDE with associated boundary conditions:
\[
\frac{\partial z(x, t)}{\partial t} = L[x, z(x, t)] + r(x, t) + \sum_{j=0}^p b_j(x)\mu_j(t) \\
\forall (x, t) \in I \times T,
\]
\[ z(x, t_0) = z_0(x) \quad \forall x \in I, \]
\[ z(0, t) = \alpha(t), \quad z(\ell, t) = \beta(t) \quad \forall t \in T, \]
where \( I = [0, \ell] \) with \( 0 < \ell < +\infty, T = [t_0, t_0 + T] \) for \( t_0 \in \mathbb{R} \) and \( T > 0, \)
\[ L(x, z) = \frac{\partial}{\partial x} \left( a(x) \frac{\partial z}{\partial x} \right) + q(x)z, \]
\( a \in C^1(I, \mathbb{R}), q \in C(I, \mathbb{R}) \) and there exist \( a_{\min}, a_{\max} \in \mathbb{R}_+, \) such that
\[ 0 < a_{\min} \leq a(x) \leq a_{\max} \quad \forall x \in I; \]
the boundary conditions \( \alpha, \beta \in C^2(\ell, \mathbb{R}) \) and the external input \( r \in C^1(I \times T, \mathbb{R}); \) the initial conditions \( z_0 \in Z_0 = \{ z_0 \in H^2(I, \mathbb{R}) : z_0(0) = \alpha(0), \ z_0(\ell) = \beta(0) \}; \) the controls \( u_j : T \rightarrow \mathbb{R} \) are Lipschitz continuous functions. The space domain \( I \) is divided into \( p + 1 \) subdomains \( I_j \) for \( j = 0, 1, \ldots, p, \) where the control signals \( u_j(t) \) are applied through the shape functions \( b_j \in L^2(I, [0, 1]) \) such that
\[ b_j(x) = 0 \quad x \notin I_j, \]
\[ b_j(x) = 1 \quad x \in I_j. \]
(4)
The controls \( u_j \) are designed in Section 5, in Sections 3 and 4 they are assumed to be given and \( u_j \in L^\infty(T, \mathbb{R}) \) for all \( j = 0, 1, \ldots, p. \)

**Proposition 4.** Assume
\[ a_{\min} \frac{\pi^2}{\ell^2} > q_{\max}, \]
(5)
where \( q_{\max} = \sup_{\varepsilon \in I} q(x) \), then for the solutions of (3) the following estimate is satisfied for all \( t \in T: \)
\[ \frac{1}{2} \int_0^t z^2(x, t) dx \leq e^{-\chi(t-t_0)} \int_0^t u_0^2(x) dx + \frac{\ell}{2} \left[ a^2(t) + \beta^2(t) \right], \]
(6)
where \( \chi = a_{\min} \frac{\pi^2}{\ell^2} - q_{\max}, \ u_0(x) = z_0(x) - \delta(x, t_0), \delta(x, t) = \alpha(t) + \frac{\ell}{2} (\beta(t) - \alpha(t)), \) and
\[ \tilde{r}(x, t) = r(x, t) + \frac{1}{\ell} \frac{\partial a(x)}{\partial x} (\beta(t) - \alpha(t)) + q(x) \delta(x, t) - \delta_i(x, t) + \sum_{j=0}^p b_j(x) u_j(t). \]
(7)
**Proof.** Denote \( w(x, t) = z(x, t) - \delta(x, t), \) then
\[ \frac{\partial w(x, t)}{\partial t} = I[w(x, w(t)] + \tilde{r}(x, t), \]
\[ w(x, t_0) = u_0(x) \quad \forall x \in I, \]
\[ w(0, t) = w(\ell, t) = 0 \quad \forall t \in T. \]

We start with the well-posedness analysis of the system (8) under Dirichlet boundary conditions (9). The boundary-value problem (8) can be represented as an abstract differential equation
\[ \dot{z}(t) = Az(t) + F(t, \zeta(t)), \ t \geq t_0, \ \zeta(t_0) = z_0 \]
(10)
in the Hilbert space \( L^2(I, \mathbb{R}), \) where the operator \( A = \frac{\partial}{\partial x} \left( a(x) \frac{\partial}{\partial x} \right) \) has the dense domain \( \mathcal{D}(A) = \{ \zeta \in H^2(I, \mathbb{R}) : \zeta(0) = \zeta(\ell) = 0 \}; \) the nonlinear term \( F : \mathbb{R} \times L^2(I, \mathbb{R}) \rightarrow L^2(I, \mathbb{R}) \) is defined on functions \( \zeta(\cdot, t) \) according to
\[ F(t, \zeta(x, t)) = q(x) \zeta(x, t) + \tilde{r}(x, t), \]
where \( \tilde{r}(x, t) \) is given in Eq. (7). It is a well-known fact that \( A \) generates a strongly continuous exponentially stable semigroup \( \Phi, \) which satisfies the inequality \( \| \Phi(t) \| \leq ke^{-\rho t} \) for all \( t \geq 0 \) with some constant \( k \geq 1 \) and decay rate \( \rho > 0. \)

By introducing restrictions on the initial and boundary conditions \( \alpha(t), \beta(t) \) and \( \delta(x, t) \) in the PDE (8) and if \( u_i(t) \) is Lipschitz continuous in \( t, \) then \( F(t, \zeta) \) is Lipschitz continuous in both variables:
\[ \| F(t_1, \zeta_1) - F(t_2, \zeta_2) \|_{L^2(I, \mathbb{R})} \leq L_1 |t_1 - t_2| + L_2 (\zeta_1 - \zeta_2)_2 \]
for all \( t_1, t_2 \in T \) and \( \zeta_1, \zeta_2 \in L^2(I, \mathbb{R}), \) with some \( L_1 > 0 \) and \( L_2 > 0. \) Therefore, for all \( z_0 \in \mathcal{D}(A) \) there exists a strong solution of the initial value problem (10) in \( C(T, L^2(I, \mathbb{R})) \) by Pazy (1983, Theorem 6.1.6).

Now consider for (8) the following Lyapunov function
\[ V(t) = \int_0^t w^2(x, t) dx. \]
We have
\[ \dot{V}(t) = 2 \int_0^t w(x, t) \frac{\partial}{\partial x} (a(x) w(x, t)) \]
\[ + q(x) w(x, t) + \tilde{r}(x, t) dx. \]
Integrating by parts and substituting the boundary conditions of \( w(x, t) \) lead to
\[ \dot{V}(t) = 2a(x) w(x, t) w_x(x, t) \]
\[ + 2 \int_0^t q(x) w^2(x, t) dx + w(x, t) \tilde{r}(x, t) dx \]
\[ = 2 \int_0^t q(x) w^2(x, t) - a(x) w_x^2(x, t) + w(x, t) \tilde{r}(x, t) dx. \]
Using Wirtinger’s inequality (1) and Young’s inequality (Hardy et al., 1988),
\[ 2w(x, t) \tilde{r}(x, t) \leq \chi w^2(x, t) + \chi^{-1} \tilde{r}^2(x, t), \]
we obtain (recall that \( \chi = a_{\min} \frac{\pi^2}{\ell^2} - q_{\max}, \) see the formulation of the proposition):
\[ \dot{V}(t) \leq -2(a_{\min} \frac{\pi^2}{\ell^2} - q_{\max}) \int_0^t w^2(x, t) dx + 2 \int_0^t q(x) w^2(x, t) dx \]
\[ \leq -\chi V(t) + 2 \int_0^t \tilde{r}^2(x, t) dx. \]
Therefore, if \( \chi > 0 \) then the system (8) has bounded solutions:
\[ \int_0^t z^2(x, t) dx \leq 2V(t) + 2 \int_0^t \tilde{r}^2(x, t) dx \]
\[ \leq 2(e^{-\chi(t-t_0)}V(t_0) + \chi^{-1} \int_0^t \tilde{r}^2(x, t) dx + \frac{\ell}{2} \left[ a^2(t) + \beta^2(t) \right]) \]
for all \( t \in T, \) that completes the proof.

Consequently, Proposition 4 fixes the conditions under which the distributed parameter system (3) possesses the input-to-state stability (ISS) property (Dashkovskiy, Efimov, & Sonntag, 2011; Dashkovskiy & Mironchenko, 2013) with respect to the boundary conditions \( \alpha, \beta, \) the external disturbance \( r, \) and the control signals \( u_i. \) The main restriction of that proposition is (5) and can be easily validated for a sufficiently small \( \ell. \)
where
\[
\gamma(t) = \frac{1}{4}[\alpha^2(t) + \beta^2(t)] \quad \text{(weighted norm of the boundary conditions)}
\]
and
\[
\gamma'(t) = 8\pi^{-2}\gamma(t) + 2(1 + \frac{4}{T^2} + 16\pi^2)\rho(t) \quad \text{and}
\]
\[
\gamma''(t) = \frac{1}{4}[\alpha^2(t) + \beta^2(t)] \quad \text{(weighted norm of derivative of the boundary conditions)}
\]
are all bounded functions of time \( t \in T \),
\[
\delta_{\text{max}} = \sup_{x \in (0,1)} \| \partial_z \|_{L^p(0,1)}.
\]

3.2. Positivity of solutions

In general, the solution \( z(\cdot, t) \) of (3) takes its values in \( \mathbb{R} \) and it can change sign with \( x(t) \in I \times T \). For brevity of presentation of the results of this subsection we will always assume that \( u_j(t) = 0 \) for all \( t \in T \) and \( j = 0, 1, \ldots, p \).

**Definition 5.** The system (3) with \( u_j(t) = 0 \) for all \( j = 0, 1, \ldots, p \) is called nonnegative (positive) on the interval \( T \) if for
\[
a(t) \geq 0, \quad \beta(t) \geq 0, \quad r(x, t) \geq 0 \quad \forall x(t) \in I \times T,
\]
the implication \( z_0(x) \geq 0 \Rightarrow z(x, t) \geq 0 \) holds for all \( x(t) \in I \times T \) and all \( z_0 \in \mathbb{R}^p \).

A well-known example of a nonnegative system is nonhomogeneous heat equation defined over \( x \in (-\infty, +\infty) \):
\[
\frac{\partial z(x, t)}{\partial t} = a \frac{\partial^2 z(x, t)}{\partial x^2} + r(x, t) \quad \forall x(t) \in \mathbb{R} \times T,
\]
where \( a > 0 \) is a constant, \( q = 0 \) and \( \zeta_0 : \mathbb{R} \rightarrow \mathbb{R}_+ \), whose solution can be calculated analytically using Green’s function (fundamental solution or the heat kernel) (Thomée, 2006):
\[
\frac{\partial z(x, t)}{\partial t} = a \frac{\partial^2 z(x, t)}{\partial x^2} + r(x, t) \quad \forall x(t) \in \mathbb{R} \times T,
\]
where
\[
\frac{\partial z}{\partial x} (x, 0) = \zeta_0(x) \quad \forall x \in \mathbb{R},
\]
where
\[
\zeta_0(x, 0) = \zeta(x, 0) \quad \forall x \in \mathbb{R},
\]
It is straightforward to verify that for nonnegative \( \zeta_0 \) and \( r \) the expression on the right-hand side stays nonnegative for all \( x(t) \in I \times (0, +\infty) \). This conclusion is valid for the case \( x \in \mathbb{R} \).

However, if \( x \in I \), even the homogeneous heat equation (11) with \( r(x, t) = 0 \) for all \( x(t) \in I \times T \), and with the boundary condition
\[
\z_0 = \zeta(0, t) = \zeta(\ell, t) \quad \forall t \in T
\]
admits the solution in the form (Thomée, 2006):
\[
\zeta(x, t) = \sum_{n=1}^{\infty} D_n \sin \left( \frac{n\pi x}{\ell} \right) e^{-a n^2 t},
\]
where
\[
D_n = 2 \int_0^\ell \zeta_n(x) \sin \left( \frac{n\pi x}{\ell} \right) dx,
\]
whose positivity is less trivial to establish.

For this reason, using Maximum principle (Friedman, 1964) the following general result has been proven in Nguyen and Coron (2016):

**Proposition 6.** Let \( \alpha, \beta \in L^p(I \times T, \mathbb{R}_+) \), \( r \in L^2(I \times T, \mathbb{R}_+) \) and \( z_0 \in H^1(I \times \mathbb{R}_+) \), then
\[
z(x, t) \geq 0 \quad \forall x(t) \in I \times T,
\]
i.e. (3) with \( u_j(t) = 0 \) for all \( j = 0, 1, \ldots, p \) is nonnegative on the interval \( T \).

Therefore, if boundary and initial conditions, and external inputs, take only nonnegative values, then the solutions of (3) possess the same property.

4. Interval observer design for the heat equation

Consider (3) with some uncertain boundary conditions \( \alpha, \beta \in C^2(I \times \mathbb{R}) \), an uncertain external input \( r \in C(I \times T, \mathbb{R}) \) and initial conditions \( z_0 \in \mathbb{R}^p \), and assume that the state \( z(x, t) \) is available for measurements in certain points \( 0 < z_{i-1}^n < z_i^m < \cdots < z_{n+1}^m < \ell \):
\[
y_j(t) = z(x_j, t) + v_j(t), \quad j = 1, \ldots, p.
\]
where
\[
y(t) = [y_1(t), \ldots, y_p(t)]^T \quad \text{is the measured output signal},
\]
\[
v(t) = [v_1(t), \ldots, v_p(t)] \in \mathbb{R}^p \quad \text{is the output disturbance (measurement noise).}
\]
Design of a conventional observer under similar conditions has been studied in Fridman and Bliznyuk (2012) and Schaum et al. (2014). Further, to simplify the technical presentation (to simplify the proof of well-posedness of the estimation error dynamics) we assume differentiability of the output disturbance:

**Assumption 1.** Let \( v \in C^4(T, \mathbb{R}^p) \).

A goal of the work consists in design of interval observers for the distributed parameter system (3), (13). For this purpose we need the following hypothesis:

**Assumption 2.** Let \( z_0 \leq z_0 \leq z_0, x_0 = x_0 \), let also functions \( \alpha, \overline{\alpha}, \beta, \overline{\beta} \in C^2(I \times \mathbb{R}) \), \( r \in C^4(I \times T, \mathbb{R}) \) and a constant \( v_0 > 0 \) be given such that for all \( x(t) \in I \times T \):
\[
\alpha(t) \leq \alpha(t) \leq \overline{\alpha}(t), \quad \beta(t) \leq \beta(t) \leq \overline{\beta}(t),
\]
\[
\gamma(t) \leq r(x, t) \leq \overline{\gamma}(x, t), \quad |v(t)| \leq v_0.
\]
Thus, by Assumption 2 five intervals, \([\alpha(t), \overline{\alpha}(t)], [\beta(t), \overline{\beta}(t)], [z_0, x_0], [r(x, t), \overline{r}(x, t)] \) and \([v(t), v_0] \), determine for all \( x(t) \in I \times T \) in (3), (13) the uncertainty of the values for \( \alpha(t), \beta(t), z_0, r(x, t) \) and \( v(t) \), respectively.

**Remark 7.** These imperfections can be related to various reasons, e.g. unknown parameters, external signals, nonlinearities, etc., but they have to be included in the corresponding intervals. For example, consider even more complicated case, let
\[
r(x, t) = \theta_1 r(x, t) + \theta_2 z(x, t),
\]
where
\[
\theta_1 \in [\underline{\theta}_1, \overline{\theta}_1]
\]
is an unknown parameter taking values in the given interval \([\underline{\theta}_1, \overline{\theta}_1] \), \( \theta_1 \) is a known function and \( \theta_2 : L^2(I \times \mathbb{R} \times T, \mathbb{R}^p) \) is an unknown function taking values in the given set \([\underline{\theta}_2, \overline{\theta}_2] \). Then
\[
r(x, t) = \theta_1 r(x, t) + \theta_2 z(x, t) + \theta_2 = [r(x, t), \overline{r}(x, t)],
\]
and this case also can be studied in the same way as (3).

The simplest interval observer for (3) under the introduced assumptions is as follows for \( j = 0, 1, \ldots, p :
\]
\[
\frac{\partial x_j(t)}{\partial t} = L[x, \overline{z}(x, t)] + \overline{r}(x, t) + \beta_j u_j(t)
\]
\[
\forall x(t) \in I_j \times T,
\]
\[
x_0 = z_0(x) \quad \forall x \in I_j,
\]
\[
x_j(t) = \overline{z}_j(t), \quad \overline{z}_{j+1}(t) = \overline{z}_{j+1}(t) \quad \forall t \in T;
\]

(14)
\[ \frac{\partial z(t)}{\partial t} = L(z(t)) + r(t) \]

where \( z(t) \) is the state of the system, \( r(t) \) is the external input, and \( L(z(t)) \) is the linear part of the system dynamics.

Theorem 8. Let Assumptions 1 and 2 be satisfied, then in (3), (14):
\[ e^*(x, t) \leq e(z, t) \leq \hat{e}(x, t) \quad \forall x(t) \in I \times \mathcal{T}. \]

In addition, if \( \Delta \mathbf{x}^m < \frac{1}{\sqrt{q_{\min} q_{\max}}}, \) then for all \( t \in \mathcal{T} \):
\[ ||\hat{e}(\cdot, t) - e(\cdot, t)||_{L^2(I, R)} \leq 4e^{-\frac{t}{\rho}}||z_{0}(t)||_{L^2(I, R)} + \frac{\rho}{2}e^{-\frac{t}{\rho}}||z_{0}(t)||_{L^2(I, R)} + ||r(t)||_{L^2(I, R)} \]
\[ + 8\psi(t)(r(t) - \dot{r}(t)) ||z_{0}(t)||_{L^2(I, R)} + ||\hat{r}(t)||_{L^2(I, R)} \]
\[ + 8\psi(t)(r(t) - \dot{r}(t)) ||z_{0}(t)||_{L^2(I, R)} + ||\hat{r}(t)||_{L^2(I, R)} \]

where \( \psi(t) = \frac{1}{2} ||\hat{z}(t) - Z(t)||^2 \)
\[ ||z(t)||_{L^2(I, R)} \leq \frac{1}{2} ||\hat{z}(t) - Z(t)||^2 + ||\hat{z}(t)||^2 + \frac{1}{2} ||Z(t)||^2 \]

Proof. Under Assumption 2, for all \( (x, t) \in I \times \mathcal{T} \), in (16) the external inputs
\[ f(x, t) - r(x, t) \geq 0, \quad r(x, t) - \bar{r}(x, t) \geq 0, \]
the initial conditions
\[ z(0) - z_0(x) \geq 0, \quad z_0(x) - \bar{z}(x) \geq 0, \]
the boundary conditions
\[ \mathcal{E}(x, t) = z(x, t) - \alpha(t) \geq 0, \quad \mathcal{E}(x, t) = z(x, t) - \beta(t) \geq 0, \]
and
\[ \mathcal{E}(x, t) = z(x, t) - \gamma(t) \geq 0, \]
are nonnegative. Therefore, according to Proposition 6 the PDE (16) is nonnegative on the interval \( T \), which implies the required interval estimates by the definition of \( \mathcal{E} \) and \( \mathcal{E} \).

Theorem 9. Following the idea from Fridman (2013), the well-posedness of (14) can be established by showing the well-posedness of the estimation errors (15), which satisfy the Eqs. (16). By the introduced constraints on the system parameters, \( r(x, t), z(x, t), \bar{r}(x, t), \) initial conditions \( z_0(x), z_0(x), \) and \( \bar{z}(x), \) and boundary conditions for the error dynamics (16) (recall (19) for \( \alpha, \beta, \gamma, \delta, \epsilon, \rho \in C^2(\mathcal{T}, \mathbb{R}) \) and \( v \in C^2(\mathcal{T}, \mathbb{R}) \) by Assumptions 1 and 2), and for \( z_0 - z_0, z_0 - z_0 \in \mathcal{P}(\mathcal{A}) \) there exists a strong solution \( e \in C(\mathcal{T}, \mathcal{L}^2(I, \mathbb{R})) \) of initial value problem (16) with \( e(x, t), \bar{e}(x, t) \in \mathcal{P}(\mathcal{A}) \) by Pazy (1983, Corollary 4.2.5). Therefore, if \( e(x, t), \bar{e}(x, t) \in \mathcal{P}(\mathcal{A}) \) and \( \bar{e}(x, t) \in \mathcal{P}(\mathcal{A}) \), then there exists a unique solution \( e, \bar{e} \in C(\mathcal{T}, \mathcal{L}^2(I, \mathbb{R})) \) to the interval observer system (14) with \( \bar{e}(x, t), \bar{e}(x, t) \in \mathcal{P}(\mathcal{A}) \) for all \( t \in \mathcal{T} \).

It is a well-known fact that the system (16) can be unstable if the function \( q \) takes sufficiently big values (Curtain & Zwart, 1995). In Fridman and Blighovsky (2012) it has been proven, for \( \alpha(t) = \beta(t) = 0 \) and \( \gamma(t) = 0 \), that the observer (14) is asymptotically stable if the difference \( \Delta \mathbf{x}^m \) is sufficiently small (i.e. there are sufficient quantity of sensors uniformly distributed in \( I \)). The presented Theorem 8 ensures positivity of the interval estimation errors and boundedness of the interval estimates \( \bar{e} \) and \( \bar{e} \) in the presence of non-zero boundary conditions \( \alpha(t), \beta(t) \) and measurement noise \( \nu(t) \).
this restriction, let us consider together the system (3) and the interval observer (14), designed in Section 4, both endowed with control input \( u_0(t) \in H^1(\mathbb{T}, \mathbb{R}) \) through the shape functions \( b_j(x) \in L^2(I, \mathbb{R}) \) on each space subdomain \( I_j \), where the control is chosen as an interval observer state feedback:

\[
u_0(t) = -\frac{K_j}{\Delta x_j} \int_0^{\Delta x_j} (\xi(t) + \Xi_0(t)) \, dx + j = 0, \ldots, p,
\]

where \( K_j \) are the sequential feedback gains to be designed on each \( I_j \), \( K_j > 0 \) and

\[\Delta x_j^m = (x_j^{m+1} - x_j^m) \quad \forall j \in \{0, 1, \ldots, p\}.
\]

**Remark 10.** For brevity we consider the same number of sensors and actuators with collocated subintervals \( I_j \). It is not difficult to extend our results to the non-collocated case by modifying arguments of Selivanov and Fridman (2018). This is because our design is based on separation of the controller and the observer designs. While the observer part of this paper is completely new, the controller part is based on a modification of the existing controller method from Fridman and Bar Am (2013). Our modification of the existing controller design is as follows: we use transformation to move boundary disturbances into the right-hand side of PDE and employ a special structure of the controller based on the interval observer. Then the ISS analysis of the closed-loop system follows the existing method for controller design. Thus, by modifying arguments of Section 2 of Selivanov and Fridman (2018), it is possible to achieve ISS by using a boundary controller at \( x = \ell \) via the backstepping.

Thus, the control is applied in order to ensure boundedness of the observer estimates \( Z(x, t) \), \( Z_0(x, t) \), that in its turn (since \( z(x, t) \leq Z(x, t) \leq \Xi_0(x, t) \) for all \( x(t) \in I \times T \), see Theorem 8) will provide boundedness of \( Z(x, t) \) as in Efimov, Raisi, and Zolghadri (2013). Recall the shape functions (4) \( b_j(x) = 1 \) on \( I_j \) and \( b_j(x) = 0 \) if \( x \notin I_j \) and substitute the control (20) in (3) on interval \( I_j \) for all \( j = 0, \ldots, p \):

\[
\begin{align*}
\frac{\partial z(x, t)}{\partial t} & = \frac{\partial}{\partial x} (a(x)z_0(x, t)) + q(x)z(x, t) + \bar{r}(x, t) \\
& - \frac{K_j}{\Delta x_j} \int_0^{\Delta x_j} (\xi(t) + \Xi_0(t)) \, dx, \quad \forall (x, t) \in I_j \times T.
\end{align*}
\]

We consider the same shift for the system as before \( \delta(x, t) = \alpha(t) + \frac{1}{2} (\beta(t) - \alpha(t)) \), then the new state variable (as in the proof of Proposition 4) \( u(x, t) = z(x, t) - \delta(x, t) \), and it satisfies the following PDE with zero boundary conditions:

\[
\frac{\partial u(x, t)}{\partial t} = \frac{\partial}{\partial x} (a(x)u(x, t)) + q(x)u(x, t) + \bar{r}(x, t) - \sum_{j=0}^{p} b_j(x) \frac{K_j}{\Delta x_j} \int_0^{\Delta x_j} (\xi(t) + \Xi_0(t)) \, dx \quad \forall (x, t) \in I \times T,
\]

\[
u_0(t) = u_0(x(t)) \quad \forall x(t) \in I,
\]

\[
u(0, t) = u(0, t) = 0 \quad \forall t \in T,
\]

where \( \bar{r}(x, t) = \bar{r}(x, t) + \frac{1}{\Delta x_j} (\beta(t) - \alpha(t)) \).

Consider the interval observer error dynamics (16), which is nonnegative by Theorem 8 and bounded if the condition (18) is satisfied. Recall the relations \( z(x, t) = z(x, t) - \xi(x, t) \) and \( z_0(x, t) = z(x, t) + \Xi_0(x, t) \) and substitute them into the dynamics of \( u(x, t) \):

\[
\frac{\partial u(x, t)}{\partial t} = \frac{\partial}{\partial x} (a(x)u(x, t)) + q(x)u(x, t) + \bar{r}(x, t) - \sum_{j=0}^{p} b_j(x) \frac{K_j}{\Delta x_j} \int_0^{\Delta x_j} (\xi(t) + \Xi_0(t)) \, dx
\]

\[
+ \sum_{j=0}^{p} b_j(x) \frac{K_j}{\Delta x_j} \int_0^{\Delta x_j} (\xi(t) - \Xi_0(t)) \, dx
\]

\[
-2 \sum_{j=0}^{p} b_j(x) \frac{K_j}{\Delta x_j} \int_0^{\Delta x_j} u_0(x, t) \, dx \quad \forall (x, t) \in I \times T,
\]

where \( \bar{r}(x, t) = \bar{r}(x, t) - 2 \sum_{j=0}^{p} b_j(x) \frac{K_j}{\Delta x_j} \int_0^{\Delta x_j} \delta(x, t) \, dx \). Since \( \delta(x, t) \geq 0, \Xi_0(x, t) \geq 0 \) and bounded under the condition (18), the terms \( \int_0^{\Delta x_j} (\xi(t) - \Xi_0(t)) \, dx \) can be made a part of a new disturbance

\[
R(x, t) = \bar{r}(x, t) + \sum_{j=0}^{p} b_j(x) \frac{K_j}{\Delta x_j} \int_0^{\Delta x_j} (\xi(t) - \Xi_0(t)) \, dx,
\]

then

\[
\frac{\partial u(x, t)}{\partial t} = \frac{\partial}{\partial x} (a(x)u(x, t)) + q(x)u(x, t) + R(x, t)
\]

\[
-2 \sum_{j=0}^{p} b_j(x) \frac{K_j}{\Delta x_j} \int_0^{\Delta x_j} u_0(x, t) \, dx \quad \forall (x, t) \in I \times T.
\]

In order to analyze the influence of the interval feedback, let us use the relation

\[
\frac{1}{\Delta x_j} \int_0^{\Delta x_j} u_0(x, t) \, dx = u(x, t) - f(x, t) \quad x \in I_j,
\]

proposed in Fridman and Bar Am (2013), where

\[
f(x, t) \equiv \frac{1}{\Delta x_j} \int_0^{\Delta x_j} [u_0(x, t) - w(x, t)] \, dx
\]

is a piecewise continuous function and \( \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} \). Finally, the following closed-loop system has been obtained:

\[
\frac{\partial u(x, t)}{\partial t} = \frac{\partial}{\partial x} (a(x)u(x, t)) + q(x)u(x, t) + R(x, t)
\]

\[
-2 \sum_{j=0}^{p} b_j(x) u(x, t) + 2 \sum_{j=0}^{p} b_j(x) f(x, t).
\]

Validity of the interval inclusion (17) can be proven repeating the same arguments as in Theorem 8 since the observer design is independent on the form of control. To analyze stability of the closed-loop system (22) let us consider a Lyapunov function:

\[
V(t) = \int_0^\ell u^2(x, t) \, dx,
\]

whose derivative takes the form for any \( \gamma > 0 \) and \( \kappa > 0 \):

\[
\dot{V}(t) + 2\kappa V(t) - \gamma^2 \int_0^\ell R(x, t)^2 \, dx =
\]

\[
= 2 \int_0^\ell u(x, t) \frac{\partial}{\partial x} (a(x)u(x, t))
\]

\[
+ q(x)u(x, t) + R(x, t)] \, dx + 2\kappa \int_0^\ell u^2(x, t) \, dx
\]

\[
-\gamma^2 \int_0^\ell R^2(x, t) \, dx - 4 \int_0^\ell \left[ \sum_{j=0}^{p} b_j(x) u(x, t) \right] u(x, t) \, dx
\]

\[
+ 4 \int_0^\ell \left[ \sum_{j=0}^{p} b_j(x) f(x, t) \right] u(x, t) \, dx.
\]
Integration by parts and substitution of the boundary conditions for \(w(x, t)\) lead to
\[
2 \int_0^\ell w(x, t) \frac{\partial}{\partial x} (a(x)w(x, t)) \, dx \leq -2a_{\min} \int_0^\ell w_0^2(x, t) \, dx.
\]

The function \(f(x, t)\) has the zero average \(\int_{\eta_j} f(x, t) \, dx = 0\) and \(f_j = w_x\), and by applying Poincaré's inequality (2) on subdomains \(\eta_j\) the following upper estimate is obtained:
\[
-2a_{\min} \int_{\eta_j}^{\eta_{j+1}} w_0^2(x, t) \, dx \leq -2a_{\min} \frac{\pi^2}{(\Delta x_m^2)^2} \int_{\eta_j}^{\eta_{j+1}} f^2(x, t) \, dx,
\]
then
\[
-2a_{\min} \int_0^\ell w_0^2(x, t) \, dx = -2a_{\min} \sum_{j=0}^p \int_{\eta_j}^{\eta_{j+1}} w_0^2(x, t) \, dx
\]
\[
\leq -2a_{\min} \frac{\pi^2}{(\Delta x_m^2)^2} \sum_{j=0}^p \int_{\eta_j}^{\eta_{j+1}} f^2(x, t) \, dx.
\]

The next term of (23) can be rewritten using the fact that \(b_j(x) = 1\) on \(\eta_j\) in (4) and under a mild simplifying restriction that \(K_j = K\) for all \(j = 0, \ldots, p\):
\[
-4 \int_0^\ell \left( \sum_{j=0}^p K_j b_j(x)w^2(x, t) \right) \, dx = -4K \sum_{j=0}^p \int_{\eta_j}^{\eta_{j+1}} w^2(x, t) \, dx.
\]
And the cross term of (23) can be treated in the same way:
\[
4 \int_0^\ell \left[ \sum_{j=0}^p K_j b_j(x)f(x, t)w(x, t) \right] \, dx = 4K \sum_{j=0}^p \int_{\eta_j}^{\eta_{j+1}} w(x, t)f(x, t) \, dx.
\]

Therefore, using an upper bound \(\int_0^\ell q(x)w^2(x, t) \leq q_{\max} \int_0^\ell w^2(x, t) \, dx\) and denoting \(\eta^T = [w(x, t)f(x, t)R(x, t)]\), we get
\[
\dot{V}(t) + 2\kappa \dot{V}(t) - \gamma^2 \int_0^\ell R(x, t)^2 \, dx \leq \sum_{j=0}^p \int_{\eta_j}^{\eta_{j+1}} \eta^T \Phi \eta \, dx \leq 0
\]
provided that
\[
\Phi \leq \begin{bmatrix}
2(\kappa + q_{\max} - 2K) & 2K & 0 \\
2K & \frac{2a_{\min}q_{\max}^2}{(\Delta x_m)^2} & 0 \\
0 & 2 & -\gamma^2
\end{bmatrix} \leq 0.
\]

for \(\Delta x_m = \max_{0 \leq j \leq p} |\Delta x_j|\). Using the Schur complement the above inequality is satisfied if
\[
\begin{bmatrix}
2a_{\min}q_{\max}^2 & 0 \\
0 & 2 - 2a_{\min}q_{\max}^{-2}
\end{bmatrix} \leq 0,
\]
where the first property is valid by proposed construction and the last one is a quadratic inequality with respect to \(K\). Using the imposed restriction (18) there exists \(K > 0\) such that
\[
\frac{(\Delta x_m)^2}{a_{\min}q_{\max}^2} = \frac{1}{q_{\max} + \varphi},
\]
then the needed inequality holds if
\[
2K - \frac{1}{q_{\max} + \varphi} K^2 - \kappa - q_{\max} - \frac{1}{2} \gamma^{-2} \geq 0,
\]
that always has a solution for
\[
\kappa + \frac{1}{2} \gamma^{-2} \leq \varphi.
\]

In particular, for \(\kappa + \frac{1}{2} \gamma^{-2} = \varphi\) we obtain:
\[
K = q_{\max} + \varphi = \frac{a_{\min}q_{\max}^2}{(\Delta x_m)^2}.
\]

The inequality
\[
\dot{V}(t) + 2\kappa \dot{V}(t) - \gamma^2 \int_0^\ell R(x, t)^2 \, dx \leq 0
\]
implies boundedness of the solutions \(w(x, t)\) as in the proof of Proposition 4. We have proved the following theorem.

**Theorem 11.** Let Assumptions 1 and 2 be satisfied. Let there exist \(\kappa > 0, K > 0, \gamma > 0\) and \(\Delta x_m < \pi \sqrt{\frac{a_{\min}}{q_{\max}}}\) that satisfy the LMI
\[
\Phi \leq 0.
\]

Then for the solutions of the closed-loop system (21), the interval inclusion (17) and the estimates on \(\|z(\cdot, t) - \bar{z}(\cdot, t)\|_{L^2(I, R)}\), \(\|\bar{z}(\cdot, t) - z(\cdot, t)\|_{L^2(I, R)}\) from Theorem 8 are valid and
\[
\frac{1}{2} \int_0^\ell z^2(x, t) \, dx \leq e^{-2\kappa(t-t_0)} \int_0^\ell w_0^2(x) \, dx + \frac{\gamma^2}{2K} \int_0^\ell R(x, t)^2 \, dx
\]
\[
+ \frac{\pi}{\varphi} \left( \varphi^2(t) + \beta^2(t) \right) \quad \forall(x, t) \in I \times T.
\]

**Remark 12.** Note that qualitatively the above \(L^2\) boundedness estimate for \(z\) can also be obtained using static output feedback, however it can be rather conservative, and using the on-line calculated upper and lower observer bounds \(\underline{z}\) and \(\bar{z}\) we can deduce a tighter interval estimate on the state. This can be an important advantage for applications dedicated to state constrained problems (e.g. in reactors).

### 6. Example

In this section we will consider two applications of the proposed interval observer in order to compare the obtained results with the interval observer from Kharkovskaia et al. (2018) and the control from Fridman and Blizhovskiy (2012).

#### 6.1. Controller based on the interval observer

Consider an academic example of (3) for
\[
a(x) = \frac{1}{4}(1 + \frac{3}{4}\sin(2\pi x)), \quad q(x) = 5 + \frac{1}{2}\cos(\pi x),
\]
\[
r(x, t) = \sin(\pi x)\cos(2t) + \epsilon(t), \quad |\epsilon(t)| \leq 1,
\]
with \(T = 2\) and \(\ell = 1\), then \(\epsilon\) is an uncertain part of the input \(r\) (for simulation \(\epsilon(t) = \cos(10t)\)), and
\[
r(x, t) = \sin(\pi x)\cos(2t - 1), \quad \bar{r}(x, t) = \sin(\pi x)\cos(2t + 1).
\]

The uncertainty of initial conditions is given by the interval
\[
\Delta x_0 = \Delta(0, 1) \cdot \Delta x_m,
\]
where \(\Delta x_0 = 5\sin(\pi x)\), and for boundary initial conditions
\[
\Delta x_0 = 5\sin(\pi x)\cos(2t - 1), \quad \bar{r}(x, t) = \sin(\pi x)\cos(2t + 1).
\]

The uncertainty of initial conditions is given by the interval
\[
\Delta x_0 = \Delta(0, 1) \cdot \Delta x_m,
\]
where \(\Delta x_0 = 5\sin(\pi x)\), and for boundary initial conditions
\[
\Delta x_0 = 5\sin(\pi x)\cos(2t - 1), \quad \bar{r}(x, t) = \sin(\pi x)\cos(2t + 1).
\]

The uncertainty of initial conditions is given by the interval
\[
\Delta x_0 = \Delta(0, 1) \cdot \Delta x_m, \quad \bar{r}(x, t) = \sin(\pi x)\cos(2t + 1).
\]

The uncertainty of initial conditions is given by the interval
\[
\Delta x_0 = \Delta(0, 1) \cdot \Delta x_m, \quad \bar{r}(x, t) = \sin(\pi x)\cos(2t + 1).
\]

The uncertainty of initial conditions is given by the interval
\[
\Delta x_0 = \Delta(0, 1) \cdot \Delta x_m, \quad \bar{r}(x, t) = \sin(\pi x)\cos(2t + 1).
\]

The uncertainty of initial conditions is given by the interval
\[
\Delta x_0 = \Delta(0, 1) \cdot \Delta x_m, \quad \bar{r}(x, t) = \sin(\pi x)\cos(2t + 1).
\]
the system is unstable (the conditions of Proposition 4 fail to satisfy). The maximum distance between sensors is \( \Delta x^0 = 0.3 \), and the restriction (18) for the interval observer is still verified. Therefore, Theorem 8 can be used to construct an observer for the unstable system (3). Then, to stabilize it, following the conditions of Theorem 11, the control gain \( K = 3.2865 \) was calculated, and the controls \( u(t) \) on each interval \( I_i = [x^m_i, x^M_i] \), \( i = 0, p \) with \( x^0_0 = 0 \) and \( x^{p+1}_1 = T \) were computed by (20).

For calculation of scalar product in space and for simulation of the discretized PDE in time, the implicit Euler method has been used with the step size \( dt = 0.01 \). The results of a simultaneous interval estimation and control are shown in Fig. 1, where the red surface corresponds to \( z(x, t) \), while green and blue ones represent \( z(x, t) \) and \( \overline{z}(x, t) \), respectively (20 and 100 points are used for plotting in space and in time).

In order to compare the proposed interval observer based control (20) with a static output feedback control

\[
u_j(t) = -K^* y_j(t)\]  \(\text{(24)}\)

the feedback gain \( K^* = 4.8832 \) is calculated following the result of the work (Fridman & Blighovsky, 2012). Since the system (3) contains uncertainties in disturbances \( r(x, t), v(t) \) and boundary conditions \( a(t), b(t) \), the static output feedback can guarantee only input-to-state stability in the sense of Proposition 4 with respect to the input \( f(x, t) \), which contains all this uncertainty.

To compare the precision ensured by both controllers in our example, first, the \( L^2 \) upper estimate of \( z(x, t) \) for this feedback control is calculated as follows. Note that

\[
V(t) = \| z \|_{L^2(I, R)} \leq \| \overline{z} \|_{(t)}.
\]

where

\[
\overline{z}(t) = \max_{x \in I} |z(x, t)|
\]

Clearly,

\[
z(t, x) \in [-\overline{z}(t), \overline{z}(t)] \ \forall (x, t) \in I \times T.
\]

For another side, the obtained \( L^2 \) estimates can be presented as

\[
V(t) \leq e^{-2\delta(t-t_0)} \sqrt{\gamma} + \gamma \int_0^t \| f(x, t) \|^2 \, dx = \overline{V}(t).
\]

where \( \overline{V}(t) \) can be calculated on-line for the given gain \( K^* \) (it determines the values of parameters \( \delta > 0 \) and \( \gamma > 0 \) and the imposed upper bounds on \( f(x, t) \)). Second, for illustration we assume that \( \overline{V}(t) = \ell \overline{z}(t) \), then the obtained bounds \( [-\overline{z}(t), \overline{z}(t)] = [-\sqrt{\ell}^{-1} \overline{V}(t), \sqrt{\ell}^{-1} \overline{V}(t)] \) are shown in Fig. 2 (black solid lines) together with the interval estimates of the proposed observer (14) (green and blue ones) for different instances of time. Red curves in Fig. 2 represent the simulation of the stabilized heat equation (3) state using the interval observer, while the black dashed curves represent the state of (3) stabilized by output feedback (24). As we can conclude from this evaluation, the guaranteed bounds given by the interval observer based control are almost always more accurate than provided by the static feedback from \( L^2 \) estimates.

Remark 13. Note that since for calculation of solutions the finite-element discretization/approximation methods are used, then their error of approximation has to be taken into account in the final estimates in order to ensure the desired interval inclusion property for all \( x \in I \) and \( t \in T \), see Kharkovskaya et al. (2018) where the result from Wheeler (1973) was applied for an evaluation of this error.

Remark 14. As mentioned in Fridman and Blighovsky (2012), there are no advantages of the Luenberger observer-based controller in the case of colocated sensors and actuators over the corresponding static output-feedback. However, as it is shown in this example, interval observer allows to achieve essentially lower state bounds than the corresponding static output-feedback.
6.2. The interval observer comparison

Consider a heat equation (3) with:
\[ a(x) = 2 + 0.7 \sin(\pi x), \quad q(x) = 0.5 \sin(0.5x), \]
\[ r(x, t) = r_1(x)r_2(t), \quad r_2(t) = 2 \cos(3\pi x), \quad |r_2(t)| \leq 1, \]
\[ T = 10 \text{ and } \ell = 1. \] Here \( r_2 \) is an uncertain part of the input \( r \) (for simulation \( r_2(t) = \cos(15t) \)), and
\[ \beta(x, t) = -|r_1(x)|, \quad \theta(x, t) = |r_1(x)|. \]

The uncertainty of initial conditions is given by the interval
\[ z_0(x) = z_0(x) - 1, \quad z_0(x) = z_0(x) + 1, \]
where \( z_0(x) = \cos(5\pi x) \), and the boundary conditions \( \alpha(t) \) and \( \beta(t) \) are assumed to be 0, since the approach from Kharkovskaya et al. (2018) does not employ nonzero conditions. Let \( p = 3 \) with \( x^1_0 = 0.3, x^2_0 = 0.5, x^3_0 = 0.8, \) and
\[ \nu(t) = 0.2|\sin(20t)| \cos(15t) \cos(25t)^T, \]
then \( v_0 = 0.2. \) In this case \( \Delta(x) = 0.3, \alpha_{\min} = 1.3, \alpha_{\max} = 0.5 \) and the restriction (18) is satisfied. Take \( \Delta = [0, h, 2h, \ldots, 1 - h, 1] \) with \( h = 1/N' \), and a pyramidal basis
\[ \Phi_i(x) = \begin{cases} 0 & \text{if } x \leq x_{i-1}, \\ x-x_{i-1} & \text{if } x = x_i, \\ x_{i-1} & \text{if } x < x_i, \\ x_{i+1} - x & \text{if } x_i < x \leq x_{i+1}, \\ 0 & \text{if } x \geq x_{i+1} \end{cases} \]
for \( i = 0, \ldots, N = N' \) (it is assumed \( x_{-1} = -h \) and \( x_{N+1} = 1 + h \)). For simulation we took \( N = 20 \), then the approximated dynamics from Kharkovskaya et al. (2018) is an observable system, and assume that the error of approximation for both approaches \( g(h^{\alpha + 1}(l_1 + l_2) = 0.1. \) For the Galerkin approximation approach (Kharkovskaya et al., 2018) the matrix \( L \) has been chosen to ensure distinct eigenvalues of the matrix \( A - LC \) in the interval \([-30.9, -0.67]\), then \( S^{-1} \) has been composed by eigenvectors of the matrix \( A - LC \) and the matrix \( D \) has been selected diagonal (all these matrices are defined in Kharkovskaya et al., 2018).

As before, for the calculation of scalar product in space and for simulation of the discretized PDE in time, the implicit Euler method has been used with the step size \( dt = 0.01 \) for the PDE interval observer, and the explicit one with the same step for the approximation approach. The results of comparison of the two approaches, the present and the approximation one from Kharkovskaya et al. (2018), are shown in Fig. 3, where the red lines corresponds to \( z(x, \cdot) \), while green and blue ones represent \( z(x, \cdot) \) and \( z(x, \cdot) \), respectively, at the instances \( t = 0, 1, 5, 10 \). From this figure one can clearly notice that the obtained interval for the state is more precise with the PDE interval observer approach (14).

Fig. 3. The results of (1) the PDE interval observer (14) and (2) the approximation approach interval observer from Kharkovskaya et al. (2016), \( N = 20 \). Here the lower bound is \( z(x, \cdot) \), the state is \( z(x, \cdot) \) and the upper bound is \( z(x, \cdot) \). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

7. Conclusion

Taking a heat equation with Dirichlet boundary conditions, a method of design of interval observers is proposed, which is not based on a finite-element approximation. The design employs the positivity of solutions of the heat equation proposed in Nguyen and Coron (2016). The proposed interval observer is used for stabilization of an uncertain PDE system. The efficiency of the approach is demonstrated through numerical experiments.

For future developments, more complex uncertainty of PDE equation can also be incorporated in the design procedure and the approach can be extended to PDEs with Neumann, Robin, or mixed boundary conditions. Possibility of averaged measurements and the corresponding positivity conditions can also be considered as a direction of future research.

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References


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