Entrainment to subharmonic trajectories in oscillatory discrete-time systems

Rami Katz a, Michael Margaliot b,*, Emilia Fridman a

a School of Elec. Eng., Tel Aviv University, Israel
b Department of Elec. Eng.-Systems and the Sagol School of Neuroscience, Tel-Aviv University, Tel-Aviv 69978, Israel

1. Introduction

Positive dynamical systems arise naturally when the state-variables represent physical quantities that can only take non-negative values (Farina & Rinaldi, 2000; Rantzer & Valcher, 2018). For example, in compartmental systems the state-variables represent the "density" at each compartment (Sandberg, 1978), in models of traffic flow or communication networks the state-variables represent the state of queues in the system (Shorten, Wirth, & Leith, 2006), and in Markov chains the state-variables are probabilities (Margaliot, Grüne, & Kriecherbauer, 2018).

Here, we introduce and analyze a new class of positive systems called oscillatory discrete-time systems. Recall that a matrix $A \in \mathbb{R}^{n \times m}$ is called totally positive (TP) if every minor of $A$ is positive, and totally nonnegative (TN) if every minor of $A$ is non-negative.  

A matrix $A$ is called totally positive (TP) if all its minors are positive, and totally nonnegative (TN) if all its minors are nonnegative. A square matrix $A$ is called oscillatory if it is TN and some power of $A$ is TP. A linear time-varying system is called an oscillatory discrete-time system (ODTS) if the matrix defining its evolution at each time $k$ is oscillatory. We analyze the properties of $n$-dimensional time-varying nonlinear discrete-time systems whose variational system is an ODTS, and show that they have a well-ordered behavior. More precisely, if the nonlinear system is time-varying and $T$-periodic then any trajectory either leaves any compact set or converges to an $(\pi - 1)T$-periodic trajectory, that is, a subharmonic trajectory. These results hold for any dimension $n$. The analysis of such systems requires establishing that a line integral of the Jacobian of the nonlinear system is an oscillatory matrix. This is non-trivial, as the sum of two oscillatory matrices is not necessarily oscillatory, and this carries over to integrals. We derive several new sufficient conditions guaranteeing that the line integral of a matrix is oscillatory, and demonstrate how this yields interesting classes of discrete-time nonlinear systems that admit a well-ordered behavior.

Note that this implies in particular that every entry in a TP [TN] matrix is positive [non-negative]. TN and TP matrices have a remarkable variety of interesting mathematical properties (Fallat & Johnson, 2011; Pinkus, 2010). One important property is that multiplying a vector by a TP matrix cannot increase the number of sign variations in the vector. This is known as the variation diminishing property (VDP).

Oscillatory matrices are in the "middle ground" between TN and TP matrices. A matrix $A \in \mathbb{R}^{n \times m}$ is called oscillatory if $A$ is TN and there exists an integer $k > 0$ such that $A^k$ is TP. For example, it is easy to verify that all the minors of

$$A = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 9 & 11 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

are nonnegative, so $A$ is (TN) (but not TP as it has zero entries), and also that all the minors of $A^2 = \begin{bmatrix} 0.94 & 1.12 & 0.1 \\ 100.8 & 122.9 & 14 \\ 9 & 14 & 10 \end{bmatrix}$ are positive, so $A$ is oscillatory.

The product of two TP/TN/oscillatory matrices is a TP/TN/oscillatory matrix, but the sum of two TP/TN/oscillatory matrices is not necessarily a TP/TN/oscillatory matrix. For example, the matrix $A = \begin{bmatrix} 1 & 0.1 \\ 9 & 1 \end{bmatrix}$ and its transpose $A'$ are TP (and...
thus in particular TN and oscillatory), yet \( A + A' = \begin{bmatrix} 2 & 9.1 \\ 9.1 & 2 \end{bmatrix} \) is not TN (and thus not TP nor oscillatory), as \( \det(A + A') < 0 \).

TP matrices have important applications in the asymptotic analysis of both continuous-time and discrete-time dynamical systems. Schwarz (1970) introduced the notion of a totally positive differential system (TPDS). This is the linear time-varying (LTV) system

\[ \dot{x}(t) = A(t)x(t), \]

satisfying that the associated transition matrix \( \Phi(t_1, t_0) \) is TP for any pair \((t_1, t_0)\) with \( t_1 > t_0 \). The transition matrix is the matrix satisfying \( x(t_1) = \Phi(t_1, t_0)x(t_0) \) for all \( x(t_0) \in \mathbb{R}^n \). In the particular case where \( A(t) = A \) the transition matrix is \( \Phi(t_1, t_0) = \exp((t - t_0)A) \), and then (2) is TPDS if and only if (iff) \( A \) is tri diagonal with positive entries on the super- and sub-diagonals. Schwarz used the VDP to show that the number of sign variations in \( x(t) \) is a (integer-valued) Lyapunov function for the TPDS (2), Margaliot and Sonntag (2019) have shown that TPDSs have important applications in the stability analysis of continuous-time nonlinear cooperative dynamical systems with a tridiagonal Jacobian.

An extension to discrete-time systems, called a totally positive discrete-time system (TPDTS), has been suggested recently by Alseidi, Margaliot, and Garloff (2019). The LTV

\[ x(k + 1) = A(k)x(k), \]

with \( A : \mathbb{N} \to \mathbb{R}^{n,n} \), is called a TPDTS if \( A(k) \) is TP for all \( k \in \mathbb{N} \). It was shown that time-varying nonlinear systems, whose variational equation is a TPDTS, satisfy strong asymptotic properties including entrainment to a periodic excitation. The variational equation is an LTV with a matrix described by a line integral of the Jacobian of the nonlinear system. Since the sum of two TP matrices is not necessarily TP, it is not trivial to verify that this line integral is indeed TP.

The main contributions of this paper are two-fold. First, we introduce the new notion of an oscillatory discrete-time system (ODTS). The LTV (3) is called an ODTS if \( A(k) \) is oscillatory for all time \( k \). This is an important generalization of a TPDTs, as oscillatory matrices are much more common than TP matrices. We analyze the properties of discrete-time time-varying nonlinear systems, whose variational equation is an ODTS, and show that they satisfy useful asymptotic properties. In particular, if the \( n \)-dimensional time-varying nonlinear system is \( T \)-periodic then every solution either leaves every compact set or converges to an \((n - 1)T\)-periodic solution, i.e. a subharmonic solution.

The variational equation associated with the nonlinear system is an LTV with a matrix described by a line integral of the Jacobian of the nonlinear system. Since the sum of two oscillatory matrices is not necessarily oscillatory, it is not trivial to verify that this line integral is indeed oscillatory.

The second contribution of this paper is deriving several new sufficient conditions guaranteeing that the line integral of a matrix is oscillatory. Our first condition considers the special case of a system with scalar nonlinearities. In this case we show that the integration can be performed in closed-form. The other conditions are based on sufficient conditions for a matrix to be oscillatory or TP. We demonstrate how these conditions yield new classes of discrete-time nonlinear systems with a well-ordered behavior.

The remainder of this paper is organized as follows: Section 2 reviews known definitions and results that will be used later on including the VDPs of TN and TP matrices, and TPDTSs. The next two sections describe our main results. Section 3 defines and analyzes ODTSs. Section 4 provides several sufficient conditions verifying that the line integral of the Jacobian of a time-varying nonlinear system is oscillatory. This section also details several applications of the theoretical results. The final section concludes and describes several topics for further research.

We use standard notation. The set of nonnegative integers is \( \mathbb{N} := \{0, 1, 2, \ldots\} \). Matrices [vectors] are denoted by capital [small] letters. The transpose of a matrix \( A \) is denoted \( A' \). We use \( \text{diag}(v_1, \ldots, v_n) \) to denote the \( n \times n \) diagonal matrix with entries \( v_1, \ldots, v_n \) on the diagonal.

2. Preliminaries

We begin by reviewing the VDP of TN and TP matrices. More details and proofs can be found in the excellent monographs (Fallat & Johnson, 2011; Gantmacher & Krein, 2002; Pinkus, 2010). For a vector \( z \in \mathbb{R}^n \) with no zero entries the number of sign variations in \( z \) is

\[ \sigma(z) := |\{ i \in \{1, \ldots, n - 1\} : z_i z_{i+1} < 0 \}|. \]

For example, for \( n = 3 \) consider the vector \( z(\varepsilon) := [2 \quad \varepsilon \quad -3] \).

Then for any \( \varepsilon \in \mathbb{R} \setminus \{0\} \), \( \sigma(z(\varepsilon)) \) is well-defined and equal to one. More generally, the domain of definition of \( \sigma \) can be extended, via continuity, to the set:

\[ V := \{ z \in \mathbb{R}^n : z_1 \neq 0, z_n \neq 0 \text{ and if } z_i = 0 \text{ for some } i \in \{2, \ldots, n - 1\} \text{ then } z_{i-1}z_{i+1} < 0 \}. \]

We recall two more definitions for the number of sign variations in a vector that are well-defined for any \( y \in \mathbb{R}^n \) (Fallat & Johnson, 2011). Let

\[ s^- (y) := \sigma(\bar{y}), \]

where \( \bar{y} \) is the vector obtained from \( y \) by deleting all zero entries (with \( s(0) \) defined as zero), and let

\[ s^+(y) := \max_{x \in P(y)} \sigma(x), \]

where \( P(y) \) is the set of all vectors obtained by replacing every zero entry of \( y \) by either \(-1\) or \(+1\). For example, for \( y = [-1 \quad 0 \quad 0 \quad 4] \), \( s^- (y) = 1 \) and \( s^+(y) = 3 \). These definitions imply that

\[ 0 \leq s^- (y) \leq s^+(y) \leq n - 1 \text{ for all } y \in \mathbb{R}^n, \]

and that \( s^- (y) = s^+(y) \) iff \( y \in V \).

The following theorem states the VDPs of TP and TN matrices.

**Theorem 1** (Fallat & Johnson, 2011). Let \( A \in \mathbb{R}^{n \times m} \).

1. If \( A \) is TP then

\[ s^+(Ax) \leq s^-(x) \text{ for all } x \in \mathbb{R}^m \setminus \{0\}. \]

2. If \( A \) is TN (and in particular if it is TP) then

\[ s^-(Ax) \leq s^- (x) \text{ for all } x \in \mathbb{R}^m. \]

For example, the matrix \( A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \) is TP and for \( x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \), we have

\[ s^+(Ax) = s^+(\begin{bmatrix} -1 & -3 \end{bmatrix}) < s^- (x). \]

For square matrices (which is the relevant case when considering the transition matrices of dynamical systems) more precise results are known. Recall that a matrix is called strictly sign-regular of order \( k \) (denoted SSR\(_k\)) if its minors of order \( k \) are either all positive or all negative. For example, \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) is SSR\(_2\), because all its entries are positive, and SSR\(_1\), because its single minor of order 2 is negative. It was recently shown (Ben-Avraham, Sharon,
Assumption 1. We also assume that the trajectories of (8) evolve on a compact set $\Omega$. The VDP and (5) imply that for any $x(0) \in \mathbb{R}^n \setminus \{0\}$ we have
\[
\cdots \leq s^-(x(1)) \leq s^+(x(1)) \leq s^-(x(0)) \leq s^+(x(0)).
\] (7)
In other words, both $s^-(x(k))$ and $s^+(x(k))$ can be viewed as integer-valued Lyapunov functions for the trajectories of a TPDS. Furthermore, there can be no more than $n - 1$ strict inequalities in (7), as $s^-$ and $s^+$ take values in $\{0, 1, \ldots, n - 1\}$. This implies that there exists $m \in \mathbb{N}$ such that $s^-(x(k)) = s^+(x(k))$ for all $k \geq m$, i.e., $x(k) \in V$ for all $k \geq m$. In particular, $x_1(k) \neq 0$ (and $x_n(k) \neq 0$) for all $k \geq m$. Moreover, the following eventual monotonicity property holds: there exists $p \in \mathbb{N}$ such that either $x_1(k) > 0$ for all $k > p$ or $x_1(k) < 0$ for all $k \geq p$ (and similarly for $x_n(k)$) (Alseidi et al., 2019).

This property can be applied to study the asymptotic properties of time-varying nonlinear discrete-time systems. Consider the system
\[
x(k + 1) = f(k, x(k)).
\] (8)
We assume that $f : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$ with respect to its second variable, and denote its Jacobian by $J_f(k, x) := \frac{\partial f}{\partial x}(k, x)$. We also assume that the trajectories of (8) evolve on a compact and convex state-space $\Omega \subset \mathbb{R}^n$. For $a \in \Omega$ and $j \in \mathbb{N}$, let $x(j, a)$ denote the solution of (8) at time $j$ with $x(0) = a$.

Fix $a, b \in \Omega$ and let $z(k) := x(k, b) - x(k, a)$. Then (see, e.g. Alseidi et al., 2019)
\[
z(k + 1) = M(k, a, b)z(k),
\] (9)
where
\[
M(k, a, b) := \int_0^1 J_f(k, r x(k, b) + (1 - r) x(k, a)) \, dr.
\] (10)
The LTV system (9) is called the variational equation associated with (8), as it describes how the variation between the two solutions $x(k, b)$ and $x(k, a)$ evolves in time.

Alseidi et al. (2019) pose two assumptions.

**Assumption 1.** The matrix
\[
F(k, a, b) := \int_0^1 J_f(k, r a + (1 - r) b) \, dr
\] (11)
is TP for all $k \in \mathbb{N}$ and all $a, b \in \Omega$.

Note that this implies that (9) is a TPDS.

**Assumption 2.** There exists $T \in \{1, 2, \ldots\}$ such that the map in (8) is $T$-periodic, that is,
\[
f(k, a) = f(k + T, a) \quad \text{for all } k \in \mathbb{N} \text{ and all } a \in \Omega.
\]

Note that in the particular case where $f$ is time-invariant this holds (vacuously) for every $T \in \mathbb{N}$.

**Theorem 2** (Alseidi et al., 2019). If Assumptions 1 and 2 hold then every solution of (8) emanating from $\Omega$ converges to a $T$-periodic solution of (8).

If the time-dependence in $f$ is due to an input (or excitation) $u$, that is, $f(k, x(k)) = g(u(k), x(k))$ for some map $g$ then Assumption 2 holds if $u$ is $T$-periodic. Theorem 2 then implies that the system *entrains* to the periodic excitation, as every solution converges to a periodic solution with the same period $T$.

In the special case where $f$ is time-invariant Theorem 2 yields the following result.

**Corollary 1** (Alseidi et al., 2019). Consider the time-invariant nonlinear system
\[
x(k + 1) = f(x(k))
\] (12)
whose trajectories evolve on a compact and convex state-space $\Omega \subset \mathbb{R}^n$. Suppose that
\[
F(a, b) := \int_0^1 J_f(r a + (1 - r) b) \, dr
\] (13)
is TP for all $a, b \in \Omega$. Then every solution of (12) emanating from $\Omega$ converges to an equilibrium point.

Note that the equilibrium point is not necessarily unique. The condition on $F(a, b)$ implies that every minor of $J_f(x)$ is positive for all $x \in \Omega$. In particular, the first-order minors, i.e., the entries of $J_f(x)$, are positive so the nonlinear system is strongly cooperative (Smith, 1995, 2017). The conditions here require more than strong cooperativity and as a consequence yield more powerful results on the asymptotic behavior of the system; see, e.g., Hirsch and Smith (2003) and Smith (1998).

In the particular case of planar systems, the conditions here require that the entries of $J_f(x)$ are positive, and that $\det(J_f(x))$ is positive. The latter condition is an orientation-preserving condition that has been used in the analysis of planar cooperative systems (Smith, 1998).

The next result, which seems to be new, shows that total positivity (in fact, a slightly weaker condition) implies an orientation-preserving property (with respect to a specific order) for any dimension $n$. For two vectors $x, y \in \mathbb{R}^n$, we write $x \asymp y$ if $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$. Let $D_n \in \mathbb{R}^{n \times n}$ be the diagonal matrix with $d_i = (-1)^{i+1}$ for all $i \in \{1, \ldots, n\}$. Note that $(D_n)^{-1} = D_n$.

We say that $z \in \mathbb{R}^n$ is alternating if $z_{2i-1} < 0$ for all $i \in \{1, \ldots, n - 1\}$. This implies of course that $s^-(z) = s^+(z) = n - 1$.

**Lemma 1.** Let $P \in \mathbb{R}^{n\times n}$ be TN and nonsingular. If $x, y \in \mathbb{R}^n$ are such that
\[
D_nPD_n x \asymp D_nPD_n y
\] (14)
then
\[
x \asymp y.
\]
The proof is placed in Appendix.

**Example 1.** Consider the TP matrix $P = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$. Then (14) becomes
\[
\begin{bmatrix} 1 - 3 & -2 \\ -2 & 8 \end{bmatrix} (x - y) \leq 0
\]
and
\[
\frac{1}{8} < \frac{D_n y_{2i-1}}{x_{2i-1}} < \frac{1}{2},
\]
so in particular $x \asymp y$. R. Katz, M. Margaliot and E. Fridman / Automatica 116 (2020) 108919
In the context of the LTV \( z(k + 1) = Pz(k), z(0) = z_0 \in \mathbb{R}^n \), Lemma 1 implies the following. Suppose that \( P \) is TN and non-singular and let \( y(k) := D_x z(k) \). Then it is not possible that for some \( i \geq 1 \) we have
\[
y(0) \ll y(1) \ll \cdots \ll y(i - 1) \ll y(i) \text{ and } y(i) \gg y(i + 1). \quad (15)
\]
Indeed, the last inequity here yields
\[
D_x PD_x y(i - 1) \gg D_x PD_x y(i),
\]
so Lemma 1 gives
\[
y(i - 1) \gg y(i),
\]
and this contradicts (15).

Smillie (1984) and Smith (1991) proved convergence to an equilibrium and entrainment in a certain class of continuous-time nonlinear dynamical systems. Their results are based on using the number of sign variations in the solution of the associated (continuous-time) variational system as an integer-valued Lyapunov function. It was recently shown that these results are closely related to the theory of TPDSs (Margaliot & Sontag, 2019). Theorem 2 and Corollary 1 may be regraded as discrete-time analogues of these results.

It is well-known that asymptotically stable linear systems entrain to periodic excitations. However, nonlinear systems do not necessarily entrain. This is true even for strongly monotone systems. Takáč (1992) provides interesting examples of continuous-time, strongly cooperative dynamical systems whose vector field is \( T \) periodic and admit a solution that is periodic with minimal period \( nT \), for any integer \( n \geq 2 \). Furthermore, this subharmonic solution may be asymptotically stable.

In order to apply Theorem 2 and Corollary 1 one needs to verify that the line integral of the Jacobian is TP. This is not trivial because the sum of two TP matrices is not necessarily a TP matrix, and this is naturally carried over to integrals.

\section{Oscillatory discrete-time systems}

We begin by introducing the new notion of an ODTS.

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Definition 1. The discrete-time LTV
\[ y(k + 1) = A(k)y(k), \]
with \( A : \mathbb{N} \to \mathbb{R}^{n \times n} \), is called an ODTS of order \( p \) if \( A(k) \) is oscillatory for all \( k \in \mathbb{N} \), and every product of \( p \) matrices in the form:
\[ A(k_1) \cdots A(k_p) \]
is TP. For example, if \( A(k) \) is TP for all \( k \) then (18) is an ODTS of order one.

\section{Proofs of Theorems}

\begin{proof}
Indeed, \( e^{A(t)} \) is the solution of the autonomous system in the region of asymptotic stability, which is bounded by the stable manifold of the equilibrium point. Therefore, \( e^{A(t)} \) is bounded and continuous. Moreover, it can be shown that \( e^{A(t)} \) is a Lipschitz function of \( t \).
\end{proof}

\begin{example}
Consider the tridiagonal matrix
\[
A = \begin{bmatrix}
a_1 & b_1 & 0 & \cdots & 0 \\
c_1 & a_2 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & c_{n-1} \\
0 & \cdots & c_{n-1} & a_0 & 0
\end{bmatrix}
\]
with \( b_i, c_i \geq 0 \) for all \( i \). In this case, the dominance condition
\[
a_i \geq b_i + c_{i-1} \quad \text{for all } i \in \{1, \ldots, n\},
\]
with \( c_0 := 0 \) and \( b_0 := 0 \), guarantees that \( A \) is TN (see e.g. Fallat & Johnson, 2011, Ch. 0). If, furthermore, \( b_i, c_i > 0 \) for all \( i \) then \( A \) is irreducible. Thus, if \( A \) is also non-singular then it is oscillatory.

The next two sections describe our main result.

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is TP. For example, if \( A(k) \) is TP for all \( k \) then (18) is an ODTS of order one. Also, since the product of any \( n - 1 \) oscillatory matrices is TP (Pinkus, 2010), (18) is always an ODTS of order \( n - 1 \). In general, checking that any product of \( p \) matrices is TP is not trivial. We believe that this may be addressed using the bidiagonal factorization of TN matrices; see Fallat and Liu (2007).

We now describe the applications of ODTS to the time-varying nonlinear system:
\[ x(k + 1) = f(k, x(k)), \]
where \( f(k, x) \) satisfies Assumption 2. We assume that the trajectories of (19) evolve in a compact and convex state-space \( \Omega \in \mathbb{R}^n \). For \( k \in \mathbb{N} \) and \( a, b \in \Omega \), let
\[ F(k, a, b) := \int_0^1 f(k, ra + (1 - r)b) \, dr. \]
We pose the following assumption.

\begin{assumption}
Assumption 3. For any \( a, b \in \Omega \), the system
\[ z(k + 1) = F(k, a, b)z(k) \]
is an ODTS of order \( h \).

We can now state the main result in this section.

\begin{theorem}
Suppose that Assumptions 2 and 3 hold. Let \( u := ht \). Then every solution of (19) emanating from \( \Omega \) converges to a \( u \)-periodic solution of (19).
\end{theorem}

\begin{remark}
Remark 1. If \( F(k, a, b) \) is TP for all \( k \in \mathbb{N} \) and all \( a, b \in \Omega \) then Assumption 3 holds with \( h = 1 \) and so Theorem 4 implies that every solution of (19) emanating from \( \Omega \) converges to a \( T \)-periodic solution of (19). This recovers the TPDS case. If \( F(k, a, b) \) is oscillatory for all \( k \in \mathbb{N} \) and all \( a, b \in \Omega \) then in particular every product of \( n - 1 \) matrices is TP, so Theorem 4 implies that every solution of (19) emanating from \( \Omega \) converges to an \( (n - 1)T \)-periodic solution of (19).
\end{remark}

\begin{remark}
Remark 2. The LTV (18) is of course a special case of (19) with Jacobian \( J[f(k, x(k))] = A(k) \), and thus \( F(k, a, b) = A(k) \) for all \( a, b \in \Omega \) and all \( k \in \mathbb{N} \). We conclude that if \( A(k) = A(k + T) \) for all \( k \in \mathbb{N} \) then every solution of an ODTS of order \( h \) converges to periodic solution of (18) with period \( u := ht \).
\end{remark}
Proof of Theorem 4. Pick $\alpha, \beta \in \Omega$ with $\alpha \neq \beta$. Let $z(k) := x(k, \beta) - x(k, \alpha)$ and recall that $z$ satisfies the variational equation (9), with

$$M(k, \alpha, \beta) := \int_0^1 f(k, r x(k, \beta) + (1 - r) x(k, \alpha)) \, dr.$$ 

Assumption 3 implies that $M(k, \alpha, \beta)$ is oscillatory. Let $v(k) := z(ku)$. Then

$$v(k + 1) = M((k + 1)u - 1, \alpha, \beta) \ldots M(ku, \alpha, \beta)v(k) \quad \text{(22)}$$

The product on the right-hand side includes $\mu$ matrices, so that Assumption 3 indeed holds, and applications to a continuous-time system.

The next example demonstrates Remark 3 in a simple case.

Example 4. Consider the continuous-time system: $\dot{x} = f(t, x)$ for some $f$. We may assume that $f$ is nonnegative and satisfies the dominance condition described in Example 3 for all $k \in \mathbb{N}$ and all $a \in \Omega$. Then the system (24) satisfies Assumption 3 with $h = n - 1$ for any $\alpha > 0$ sufficiently small.

Note that since $f(k, a)$ is triangular, it is not TP, so the TPDS framework cannot be used to analyze this case.

Proof. Pick $k \in \mathbb{N}$ and $a \in \Omega$. The assumptions on $\frac{df}{dx}(k, a)$ imply that $f(k, a)$ is irreducible for all $\alpha > 0$. Also, $f(k, a)$ is nonsingular and satisfies the dominance condition described in Example 3 for all $k \in \mathbb{N}$ and all $a \in \Omega$. Then the system (24) satisfies Assumption 3 with $h = n - 1$ for any $\alpha > 0$ sufficiently small, and is thus TN. Furthermore, all these properties carry over to the matrix $F$ defined in (20).

The next example describes an application of Lemma 2 to an important nonlinear model from systems biology.

Example 5. Cells often sense and respond to various stimuli by modification of proteins. One mechanism for this is phosphorelay (also called phosphotransfer), in which a phosphate group is transferred through a serial 1D chain of proteins from an initial histidine kinase (HK) down to a final response regulator (RR). The nonlinear compartmental system:

$$\dot{x}_1 = (p_1 - x_1)x - \eta_1x_1(p_2 - x_2) - \xi_1x_1,$$

$$\dot{x}_2 = \eta_1x_1(p_2 - x_2) - \eta_1x_2(p_3 - x_3) - \xi_2x_2,$$

...,

$$\dot{x}_{n-1} = \eta_{n-2}x_{n-2}(p_{n-1} - x_{n-1}) - \eta_{n-1}x_{n-1}(p_n - x_n) - \xi_{n-1}x_{n-1},$$

$$\dot{x}_n = \eta_{n-1}x_{n-1}(p_n - x_n) - \eta_nx_n,$$

has been suggested as a model for phosphorelay by Csikasz-Nagy, Cardelli, and Soyer (2011); see also Bar-Shalom, Ovseevich, and Margaliot (2020) for an application of a similar result to mRNA translation and for rigorous analysis. Here $c(t) \geq 0$ is the strength at time $t$ of the stimulus activating the HK, $x(t) \in [0, p_1]$ is the concentration of the phosphorylated form of the protein at the $i$th layer at time $t$, the parameter $p_i > 0$ denotes the total protein concentration at that layer, and $\eta_i > 0, \xi_i \geq 0$ are reaction rates. Note that $\eta_nx_n(t)$ is the flow at time $t$ of the phosphate group to an external receptor molecule.
In the particular case where \( p_i = 1 \) and \( \xi_i = 0 \) for all \( i \) Eq. (26) becomes the ribosome flow model (RFM) (Reuveni, Melijson, Kupec, Ruppin, & Tuller, 2011). This is the dynamic mean-field approximation of a fundamental model from non-equilibrium statistical physics called the totally asymmetric simple exclusion process (TASEP); see Blythe and Evans (2007). The RFM describes the unidirectional flow along a chain of \( n \) sites. The state-variable \( x_i \in [0, 1] \) describes the normalized occupancy at site \( i \), where \( x_i = 0 \) \([x_i = 1]\) means that site \( i \) is completely free (full), and \( \eta_i \) is the capacity of the link that connects site \( i \) to site \( i+1 \). This has been used to model and analyze mRNA translation (see, e.g., Nanikashvili, Zarai, Ovseeich, Tuller, & Margaliot, 2019; Poker, Zarai, Margaliot, & Tuller, 2014; Raveh, Margaliot, Sonntag, & Tuller, 2016; Zarai, Margaliot, & Tuller, 2016), where every site corresponds to a group of codons on the mRNA strand, \( x_i(t) \) is the normalized occupancy of ribosomes at site \( i \) at time \( t \), and \( \eta_i \) is the elongation rate from site \( i \) to site \( i+1 \).

Write (26) as \( \dot{x} = f(x) \). Then \( \frac{\partial f}{\partial y}(x) \) is tridiagonal, with entries \( \eta_i x_i \) on the super-diagonal, and \( \eta_i(p_{i+1} - x_{i+1}) \), \( i = 1, \ldots, n \), \( n-1 \), on the sub-diagonal.

Consider the corresponding discretized system (24). It is not difficult to show that \( \Omega := [0, p_1] \times \cdots \times [0, p_n] \) is an invariant set of (24) for any \( \epsilon > 0 \) sufficiently small. Furthermore, for any \( a \in \Omega \) we have that \( x(k, a) \in \text{int}(\Omega) \) for all \( k \geq 1 \) and then the conditions in Lemma 2 on \( \Omega \) defined in (25) hold. Fig. 1 depicts the trajectories of the discretized system with \( n = 4, \epsilon = 0.1, \xi_i = 3, \eta_i = 1, p_1 = 0.8, p_2 = p_3 = p_4 = 2 \), initial condition \( x(0) = [0.5 \ 0.1 \ 0.6 \ 0.3]^T \), and the periodic stimulus \( c(k) = 3 + \sin(k\pi/4) \). Note that this means that the map is \( T \)-periodic with (minimal) period \( T = 8 \). Combining Theorem 4 and Lemma 2, we conclude that any solution of the discretized system converges to a periodic solution with period \( (n-1)T = 24 \). It may be seen that the specific solution depicted in Fig. 1 converges to a periodic solution with period 8.

In general, our approach is to find sufficient conditions guaranteeing that the line integral of a matrix is oscillatory without actually calculating the integral. However, there is an important special case where the integral can be computed explicitly.

4.2. Strictly monotone scalar nonlinearities

Let \( f_i : \mathbb{R} \to \mathbb{R}, i = 1, \ldots, n \), be \( C^1 \) functions such that

\[
\frac{\partial f_i}{\partial y}(y) > 0 \quad \text{for all } i \text{ and all } y \in \mathbb{R}. \tag{27}
\]

Consider the time- varying nonlinear system:

\[
\begin{bmatrix}
\frac{d}{dt}x_1(k)
\frac{d}{dt}x_2(k)
\vdots
\frac{d}{dt}x_n(k)
\end{bmatrix} = C(k)
\begin{bmatrix}
f_1(x_1(k))
f_2(x_2(k))
\vdots
f_n(x_n(k))
\end{bmatrix}, \tag{28}
\]

with \( C : \mathbb{N} \to \mathbb{R}^{n \times n} \).

Theorem 5. Suppose that the trajectories of (28) evolve on a compact and convex state-space \( \Omega \), and that \( C(k) \) is \( T \)-periodic. If \( z(k+1) = C(k)z(k) \) is an ODTS of order \( h \) then every solution of (28) emanating from \( \Omega \) converges to an \( (ht) \)-periodic solution of (28).

Proof. The Jacobian of (28) is

\[
J(k,x) = C(k) \text{diag}(f_1'(x_1), \ldots, f_n'(x_n)).
\]

Substituting this in (11) and integrating yields

\[
F(k, a, b) = C(k) \text{diag}(g_1(a_1, b_1), \ldots, g_n(a_n, b_n)). \tag{29}
\]
The trajectory of the system for Theorem 5, with $T$-periodic solution with $\bar{P}$.

Consider a measurable and essentially bounded matrix function $A := A(t)$.

The checkerboard partial order on $\mathbb{R}^{n \times n}$ implies that $\bar{A}$ is TP and thus, in particular, oscillatory with exponent one.

From here on we consider the following general problem.

**Problem 1.** Consider a measurable and essentially bounded matrix function $A : [0, 1] \rightarrow \mathbb{R}^{n \times n}$. When is

$$\bar{A} := \int_0^1 A(t) \, dt$$

an oscillatory matrix?

Some of the conditions given below actually guarantee that $\bar{A}$ is TP (and thus, in particular, oscillatory with exponent one).

4.3. **Sufficient condition based on the checkerboard partial order**

For $A, B \in \mathbb{R}^{n \times n}$ we write $A \preceq B$ if $a_{ij} \leq b_{ij}$ for all $i, j$.

**Definition 2.** The checkerboard partial order on $\mathbb{R}^{n \times n}$ is defined by

$$A \preceq B \iff D_\pm AD_\pm \preceq D_\pm BD_\pm.$$

In other words, $A \preceq B$ iff

$$(-1)^i \delta_{ij} a_{ij} \leq (-1)^i \delta_{ij} b_{ij}$$

for all $i, j \in \{1, \ldots, n\}$.

Note that (33) implies that the matrix interval

$$\{ C \in \mathbb{R}^{n \times n} : A \preceq C \preceq B \}$$

is compact. For more on such matrix intervals, see Carlin (2003) and the references therein. It is well-known (Fallat & Johnson, 2011) that if $A, B$ are TP and $A \preceq C \preceq B$ then $C$ is TP.

**Theorem 6.** Let $A : [0, 1] \rightarrow \mathbb{R}^{n \times n}$ be a Riemann integrable matrix function. If there exist $\delta > 0$ and TN matrices $G$ and $H$ such that

$$\delta + (-1)^i \varepsilon a_{ij}(t) \leq \delta + (-1)^i \varepsilon b_{ij}$$

for all $i, j$ and all $t \in [0, 1]$ then $\bar{A}$ is TP.

**Proof.** Recall that the set of $n \times n$ TP matrices is dense in the set of $n \times n$ TN matrices (Whitney, 1952). Combining this with (34) implies that there exist TP matrices $P$ and $Q$ such that $P \preceq A(t) \preceq Q$ for all $t \in [0, 1]$.

We claim that this implies that every minor of $\bar{A}$ is positive. We will show that det $\bar{A} > 0$. The proof for any other minor is very similar. Fix $k \in \{1, 2, \ldots\}$ and consider the partition of $[0, 1]$ defined by

$$t_0 := 0, \quad t_1 := 1/k, \quad t_2 := 2/k, \ldots, t_k := 1.$$

Consider the Riemann sum $B := \sum_{\ell=0}^{k-1} (t_{\ell+1} - t_\ell) A(t_\ell)$.

Then (35) holds, so the required condition is that $P \preceq B \preceq Q$. Since $\sum_{\ell=0}^{k-1} (t_{\ell+1} - t_\ell) = t_k - t_0 = 1$, we conclude that $P \preceq B \preceq Q$.

By compactness of the set $\{ C \in \mathbb{R}^{n \times n} : P \preceq C \preceq Q \}$ and the fact that any $C$ in this set is TP, there exists $\alpha > 0$ such that det $B \geq \alpha$. Taking $k \rightarrow \infty$ and using the continuity of the determinant, we conclude that det $\bar{A} \geq \alpha > 0$. □

Suppose that every entry $a_{ij}(t)$ of $A(t)$ attains a maximum value $\bar{a}_{ij}$ and a minimum $\underline{a}_{ij}$ over $[0, 1]$. Define $P, Q$ by

$$p_{ij} := \begin{cases} \bar{a}_{ij}, & \text{if } i + j \text{ is even}, \\ \underline{a}_{ij}, & \text{if } i + j \text{ is odd} \end{cases}$$

and

$$q_{ij} := \begin{cases} \bar{a}_{ij}, & \text{if } i + j \text{ is even}, \\ \underline{a}_{ij}, & \text{if } i + j \text{ is odd} \end{cases}$$

Then (35) holds, so the required condition is that $P$ and $Q$ are TP.

The next result describes an application of Theorem 6 to a dynamical system.

**Corollary 2.** Consider the nonlinear system:

$$x(k+1) = Ax(k) + eg(x(k)).$$

where $g$ is $C^1$ and $\varepsilon > 0$ is small. Suppose that $A$ is TP, and that the trajectories of (36) evolve on a compact and convex set $\Omega \subset \mathbb{R}^n$. Define $B \in \mathbb{R}^{n \times n}$ by

$$b_{ij} := \max_{x \in \Omega} \left| \frac{\partial g(x)}{\partial x_j} \right|$$

and define matrix functions $P, Q : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ by

$$P(v) := A - vBD_\pm, \quad Q(v) := A + vBD_\pm.$$
Then there exists \( w > 0 \) such that for all \( v \in [0, w) \) and all \( x \in \Omega \),
\[
P(v) \leq 1 \quad \text{and} \quad \forall \epsilon \in [0, w] \quad \text{every solution of (36) emanating from} \ \Omega \ \text{converges to an equilibrium point.}
\]

**Proof.** It follows from (37) that \( P(0) = Q(0) = A \), and \( P(v) \leq 1 \) for all \( v \geq 0 \). By continuity of the minors, there exists \( w > 0 \) such that
\[
P(v), Q(v) \text{ are TP for all } v \in [0, w).
\]
The Jacobian of (36) is \( f(x) = A + \epsilon \frac{\partial^2}{\partial x^2} g(x) \), so for any \( s, r, \in [1, \ldots, n] \) and any \( x \in \Omega \) we have
\[
\begin{align*}
[j_{sr}(x)] &= \left[ a_{sr} + \epsilon \frac{\partial}{\partial x} g(x) \right] \\
&\leq a_{sr} + \epsilon b_{sr}.
\end{align*}
\]
It is straightforward to verify that this implies that for any \( v \geq 0 \) and any \( \epsilon \in [0, v] \) we have
\[
(-1)^{j+r} p_{sr}(v) \leq (-1)^{j+r} j_{sr}(v) \leq (-1)^{j+r} q_{sr}(v),
\]
that is,
\[
P(v) \leq 1 \quad \text{f(x)} \leq Q(v).
\]
Now fix \( \epsilon \in [0, w] \). Pick \( v \in [\epsilon, w] \). Then for these values all the conditions in Theorem 6 hold, so the matrix \( F(a, b) \) in (13) is TP for all \( a, b \in \Omega \), and this completes the proof. \( \square \)

**Example 7.** Consider (36) with \( n = 3 \),
\[
A = 0.65 \begin{bmatrix} 1 & \exp(-1) & \exp(-4) \\ \exp(-1) & 1 & \exp(-1) \\ \exp(-4) & \exp(-1) & 1 \end{bmatrix},
\]
and \( g(k, x(k)) = \begin{bmatrix} \tanh((50 + 50 \sin(k \pi/5))x_3(k)) \\ 0 \\ 0 \end{bmatrix} \). This model represent a cooperative linear chain in which the effect of \( x_3(k) \) on \( x_3(k+1) \) decays exponentially with the “distance” \((i-j)^2\) between \( x_i \) and \( x_j \). It is well-known that \( A \) in (38) is TP (see Cantacher & Krein, 2002, Ch. II). The nonlinear term represents a time-varying and \( T \)-periodic, with \( T = 10 \), positive feedback from \( x_3 \) to \( x_1 \).

It is clear that we can take the “bounding matrix” \( B \) as \( \in \mathbb{R}^{3 \times 3} \) as the matrix with \( b_{13} = 1 \), and zero in all other entries. It is not difficult to verify that for this \( B \) we have that \( P(v), Q(v) \) defined in (37) are TP for all \( v \in [0, w), \) with \( w := 0.65 \exp(-4) \). Fig. 3 depicts the solution of the system with \( \epsilon = 0.0118 < w \) and initial condition \( x(0) = (2/50) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \). It may be seen that every \( x(k) \) converges to a periodic solution with period \( T = 10 \).

4.4. Integrating TP Hankel matrices

Recall that \( A \in \mathbb{R}^{n \times n} \) is called a Hankel matrix if for any \( i, j, p, q \) with \( i + j = p + q \) we have \( a_{ij} = a_{pq} \). For example, for \( n = 3 \) a Hankel matrix has the form
\[
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{13} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}
\]
Note that a Hankel matrix is in particular symmetric. Our main result in this subsection is that the integral of a time-varying TP Hankel matrix is TP.

**Theorem 7.** Let \( A : [0, 1] \to \mathbb{R}^{n \times n} \) be a measurable matrix function such that \( A(t) \in L^\infty([0, 1]) \). Suppose that \( A(t) \) is a TP Hankel matrix for almost every \( t \in [0, 1] \). Then \( A \) is TP.

**Remark 4.** Note that for \( n = 2 \) this implies that if \( A : [0, 1] \to \mathbb{R}^{2 \times 2} \) is a continuous matrix function with \( A(t) \) symmetric and TP for all \( t \in [0, 1] \) then \( A \) is TP (compare with Example 2).

To prove Theorem 7 we recall several definitions and results. A set of indices \( I \subseteq \{1, \ldots, n\} \) is called an interval if it has the form \( I = \{p, p+1, p+2, \ldots, q\} \). A square sub-matrix of a matrix \( B \in \mathbb{R}^{n \times n} \) with row indices \( I \subseteq \{1, \ldots, n\} \) and column indices \( J \subseteq \{1, \ldots, n\} \) is called a contiguous sub-matrix if both \( I \) and \( J \) are intervals.

It is well-known and straightforward to show that the following three conditions are equivalent: (1) \( B \in \mathbb{R}^{n \times n} \) is a Hankel matrix; (2) every contiguous sub-matrix of \( B \) is a Hankel matrix; (3) every contiguous sub-matrix of \( B \) is symmetric.

We can now prove Theorem 7.

**Proof.** We start by showing that \( \det A \) is measurable (as it is a polynomial in the entries \( a_{ij}(t), i, j \in \{1, \ldots, n\} \) and essentially bounded. Therefore, it is Lebesgue integrable. For this \( N \in \{1, 2, \ldots, \} \), let
\[
B_N := \{ t \in [0, 1] : \det(A(t)) \geq N^{-1} \}
\]
Since \( A(t) \) is TP for almost every \( t \in [0, 1] \) and \( B_1 \subseteq B_2 \subseteq \ldots \),
the monotone convergence theorem (see e.g. Bogachev, 2007) yields
\[
\lim_{N \to \infty} \mu(B_N) = 1,
\]
where \( \mu \) is the Lebesgue measure on \([0, 1]\). Therefore, there exists \( N_0 \in \mathbb{N} \) such that \( \mu(B_{N_0}) > 1/2 \). Markov’s inequality (see e.g. Bogachev, 2007) yields
\[
\int_0^1 (\det A(t))^{\frac{1}{2}} \, dt \mu(t) \geq N_0^{-\frac{1}{2}} \mu(B_{N_0}) > N_0^{-\frac{1}{2}}/2.
\]
Since \( A(t) \) is Hankel and TP for almost all \( t \in [0, 1] \), it is symmetric with positive principal minors, so \( A(t) \) is positive-definite for almost all \( t \in [0, 1] \). Minkowski’s determinant inequality (see e.g. Marcus & Minc, 1992, p. 115) states that \( B \mapsto (\det B)^{\frac{1}{2}} \) is a concave function over the space of semi-positive definite
and study the effect of the sampling time. It may be of interest to classify nonlinear systems that yield ODTS system follows from discretization of a continuous-time system, control inputs, as was done for continuous-time monotones systems by Angeli and Sontag (2003). Finally, when the discrete-time system follows from discretization of a continuous-time system, it may be of interest to classify nonlinear systems that yield ODTS and study the effect of the sampling time.

Example 8. Consider the nonlinear system:

\[
x_1(k + 1) = h_1(x_1(k)) + g(x_1(k), x_2(k)),
\]

\[
x_2(k + 1) = h_2(x_2(k)) + g(x_1(k), x_2(k)),
\]

with \( h_1, h_2, g \in C^1 \), whose trajectories evolve on a compact and convex state-space \( \Omega \subset \mathbb{R}^2 \). Suppose that \( \frac{\partial}{\partial x_i} g(x_1, x_2) = \frac{\partial}{\partial x_2} g(x_1, x_2) \) for all \( x_1, x_2 \in \Omega \) (e.g. \( g(x_1, x_2) = \tanh(x_1 + x_2) \)). Note that this implies that the Jacobian

\[
J(x) = \begin{bmatrix}
\frac{\partial}{\partial x_1} g(x_1, x_2) & \frac{\partial}{\partial x_2} g(x_1, x_2) \\
\frac{\partial}{\partial x_2} g(x_1, x_2) & \frac{\partial}{\partial x_1} g(x_1, x_2)
\end{bmatrix}
\]

is symmetric. If \( J(x_1, x_2) \) is TP for all \( (x_1, x_2) \in \Omega \) then combining Corollary 1 and Remark 4 implies that any solution of (40) emanating from \( \Omega \) converges to an equilibrium point.

5. Conclusion

We introduced a new class of positive discrete-time LTV systems called ODTSs of order \( p \). Discrete-time nonlinear systems, whose variational system is an ODTS of order \( p \), have a well-ordered behavior. More precisely, if the map defining the dynamical system is \( T \)-periodic then every solution either leaves any compact set or converges to a \( (pT) \)-periodic solution, i.e. a sub-harmonic solution. This is important because, as noted by Smith (1998), “...in the class of all discrete dynamical systems, we do not know so many special classes which have relatively simple dynamics”.

The ODTS framework requires establishing that certain line integrals of the Jacobian of the time-varying nonlinear system are oscillatory matrices. This is non-trivial, as the sum of two oscillatory matrices is not necessarily oscillatory, and this naturally extends to integrals. We derived several sufficient conditions guaranteeing that the line integral of a matrix is oscillatory (or TP).

Topics for further research include the following. First, extending the oscillatory framework to other dynamical models e.g. systems with time-delays or discretized PDEs. Second, cooperative discrete-time systems frequently arise as the Poincaré maps of continuous-time systems (see, e.g. Golubiatnikov & Minushkina, 2019). It may be of interest to explore the implications of oscillatory Poincaré maps. Third, it may be of interest to generalize the ODTS framework to discrete-time systems with control inputs, as was done for continuous-time monotone systems by Angeli and Sontag (2003). Finally, when the discrete-time system follows from discretization of a continuous-time system, it may be of interest to classify nonlinear systems that yield ODTS and study the effect of the sampling time.

Appendix

Proof of Lemma 1. Let \( z := D_p(x - y) \). Then (14) implies that \( v := D_p z \ll 0 \). Thus, the vector \( z = D_p v \) is alternating, with

\[
\langle P z \rangle_1 = v_1 < 0.
\]

Applying the VDP (6) yields

\[
n - 1 = s - \langle P z \rangle \leq s -\langle z \rangle.
\]

Thus, \( s -\langle z \rangle = n - 1 \), i.e. \( z \) is alternating. Recall that if a matrix \( H \) is TN, non-singular, and \( s\langle H q \rangle = s\langle q \rangle \) for some \( q \in \mathbb{R}^n \setminus \{0\} \) then the first non-zero entry in \( H q \) and the first non-zero entry in \( q \) have the same sign (Gantmacher & Krein, 2002, p. 254). Since \( s\langle P z \rangle = s\langle z \rangle = n - 1 \), and \( \langle P z \rangle_1 < 0 \), the first non-zero entry of \( z \) is negative. Since \( z \) is alternating this implies that \( D_p z \ll 0 \), and this completes the proof.

References


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Linear Algebra and its Applications, 424(2), 466–479.


