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Brief paper

Delayed finite-dimensional observer-based control of 1-D parabolic PDEs☆

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A B S T R A C T

In our recent paper a constructive method for finite-dimensional observer-based control of 1-D linear heat equation was suggested. In the present paper we aim to extend this method to the case of input/output general time-varying delays or sawtooth delays (that correspond to network-based control). We assume known measurement delays and, for the first time under observer-based control of PDEs, unknown input delays. We use a modal decomposition approach, and consider boundary or non-local sensing together with non-local actuation, or Dirichlet actuation with non-local sensing. The dimension of the controller is equal to the number of unstable modes, whereas the observer may have a larger dimension N. Under the Dirichlet actuation we present two methods: a direct one that manages with time-varying input and output delays, and a dynamic-extension-based one that treats constant input and time-varying output delays. To compensate the fast-varying output delay (without any constraints on the delay derivative) that appears in the infinite-dimensional part of the closed-loop system, we combine Lyapunov functionals with Halanay's inequality. For the slowly-varying output delay (with the delay derivative smaller than $d < 1$), we suggest a direct Lyapunov method. We provide LMIs for finding N and upper bounds on the delays that preserve the exponential stability.

1. Introduction

Sampled-data and delayed control of PDEs is becoming an active research area. For sampled-data control of parabolic systems, a modal decomposition approach was suggested in Ghantasala and El-Farra (2012), where a finite-dimensional controller was designed on the basis of a slow system following the approach of Christofides (2001). Rigorous conditions via modal decomposition for 1-D heat equation were recently suggested in Karafyllis and Krstic (2018) for the sampled-data state-feedback boundary control, and in Karafyllis, Ahmed-Ali, and Giri (2019) and Selivanov and Fridman (2019a) for the sampled-data observers under the boundary and non-local measurements respectively. Large constant input delays can be compensated by predictors (Krstic, 2009; Prieur & Trélat, 2018).

Sampled and delayed observers or distributed static output-feedback controllers were suggested for heat equation in Fridman and Bar Am (2013), Fridman and Blighovsky (2012), Selivanov and Fridman (2016) and Selivanov and Fridman (2019b) where, in the case of controllers, the uncertain sampling and delays were considered. Design of an observer-based controller in the presence of unknown input delays is essentially more challenging. See e.g. for ODEs (Kruszewski, Jiang, Fridman, Richard, & Toguyeni, 2012), where unknown delays do not allow for decoupling of the estimation error equation from the state equation. For known input and output delays, boundary controller based on a boundary PDE observer was proposed via modal decomposition in Katz and Fridman (2020). Whereas the knowledge of measurement delay may be justified e.g. by time-stamps in network-based control (Fridman, 2014; Kruszewski et al., 2012), the assumption on the known input delay may be restrictive in applications.

Finite-dimensional observer-based controllers that are attractive in applications do not allow separation of observer and controller designs, and their construction is a challenging control problem (Balas, 1988; Christofides, 2001; Curtain, 1982). Recently an LMI-based method for design of such controllers was introduced (Katz & Fridman, 2020).

The objective of the present paper is finite-dimensional observer-based control of 1-D heat equation in the presence of unknown input and known output delays. We consider either sawtooth delays (for the case of sampled-data or network-based control) or general differentiable time-varying delays. We propose a method which is applicable to the boundary or non-local

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sensing with non-local actuation, or to the Dirichlet actuation with non-local sensing. We use a modal decomposition approach. The dimension of the controller is equal to the number of unstable modes, whereas the observer may have a larger dimension N. For the boundary actuation we present two methods: a direct one that manages with fast-varying (without any constraints on the delay derivative) input delay and slowly-varying (with the delay derivative smaller than $d < 1$) output delay, and a dynamic-extension-based one that treats constant input and fast-varying output delays.

In the stability analysis, the main challenge is due to output delay that appears, for the first time, in the infinite-dimensional tail of the closed-loop system. This is different from the studied till now cases of delayed state-feedback or PDE observer-based controller, where the delay appears in the finite-dimensional states only, and may be treated by known methods for ODEs with delays. For fast-varying output delay, we suggest to combine Lyapunov functionals with Halanay's inequality (as introduced in Fridman & Blighovsky, 2012). For the slowly-varying output delay, we present a direct Lyapunov–Krasovskii method. We provide LMIs for finding as small as possible N, and as large as possible delays as well as the resulting exponential decay rate. We prove that the LMIs are always feasible for large enough N and small enough delays.

Let $L^2(0,1)$ be the Hilbert space of square integrable functions $f : [0,1] \to \mathbb{R}$ with the inner product $\langle f, g\rangle := \int_0^1 f(x)g(x)dx$ and induced norm $\|f\|^2 := \langle f, f\rangle$. $H^1(0,1)$ and $H^2(0,1)$ denote the corresponding Sobolev spaces. The norm on $\mathbb{R}^n$ is denoted by $|\cdot|$. whereas for $A \in \mathbb{R}^{n\times n}$ the induced norm is denoted by $|\cdot|_2$. For $P \in \mathbb{R}^{n\times n}$, the notation $P > 0$ ($P < 0$) means that P is symmetric and positive definite (negative definite). The sub-diagonal elements of a symmetric matrix are denoted by $*$. For $U \in \mathbb{R}^{n\times n}$, $U > 0$ and $x \in \mathbb{R}^n$ we denote $|x|_U^2 := x^T Ux$.

2. Mathematical preliminaries

Lemma 2.1 (*Halanay's Inequality, p.138 of Fridman, 2014*)**.** *Let $0 < \delta_1 < \delta_0$ and let $V : [-\tau_M, \infty) \longrightarrow [0, \infty)$ be an absolutely continuous function that satisfies*

$$\mathcal{H}_\tau := \dot{V}(t) + 2\delta_0 V(t) - 2\delta_1 \sup_{-\tau_M \le \theta \le 0} V(t+\theta) \le 0, \quad t \ge 0. \tag{2.1}$$

Then

$$V(t) \le e^{-2\delta_\tau t} \sup_{-\tau_M \le \theta \le 0} V(\theta), \quad t \ge 0, \tag{2.2}$$

where $\delta_\tau > 0$ is a unique positive solution of

$$\delta_\tau = \delta_0 - \delta_1 \exp(2\delta_\tau \tau_M). \tag{2.3}$$

Recall that the regular Sturm–Liouville eigenvalue problem

$$\phi'' + \lambda\phi = 0, \quad x \in [0,1], \ \phi'(0) = \phi(1) = 0, \tag{2.4}$$

induces a sequence of eigenvalues $\lambda_n = (n - \frac{1}{2})^2\pi^2, n \ge 1$ with corresponding eigenfunctions $\phi_n(x) = \sqrt{2}\cos\left(\sqrt{\lambda_n}x\right)$, $n \ge 1$. Moreover, the $\{\phi_n\}_{n=1}^\infty$ are a complete orthonormal system in $L^2(0,1)$.

Lemma 2.2 (*Katz & Fridman, 2020*)**.** *Let $h \in L^2(0,1)$ be a function such that $h \overset{L^2}{=} \sum_{n=1}^\infty h_n\phi_n$. Then, $h \in H^1(0,1)$, $h(1) = 0$ iff $\sum_{n=1}^\infty \lambda_n h_n^2 < \infty$. Moreover, $\left\|h'\right\|^2 = \sum_{n=1}^\infty \lambda_n h_n^2$.*

Given $N \in \mathbb{N}$ and $h \in L^2(0,1)$ with $h \overset{L^2}{=} \sum_{n=1}^\infty h_n\phi_n$ we will use the following notation:

$$\|h\|_N^2 := \|h\|^2 - \sum_{n=1}^N h_n^2 = \sum_{n=N+1}^\infty h_n^2. \tag{2.5}$$

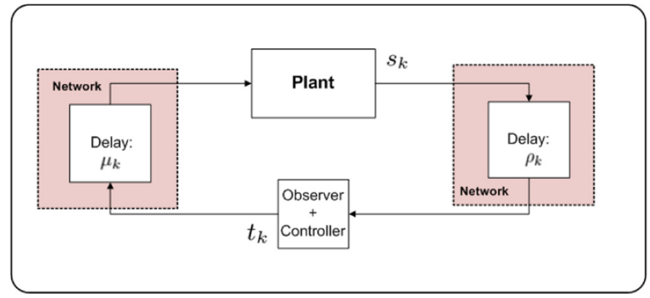

Fig. 1. Network-based control.

3. Delayed non-local measurement and actuation

Consider the reaction–diffusion system

$$\begin{aligned} &z_t(x,t) = z_{xx}(x,t) + qz(x,t) + b(x)u(t-\tau_u(t)), \quad t \ge 0, \\ &z_x(0,t) = 0, \quad z(1,t) = 0, \end{aligned} \tag{3.1}$$

with $b \in L^2(0,1)$. $z(x,t) \in \mathbb{R}$ is the state, $u(t) \in \mathbb{R}$ is the control input, $q \in \mathbb{R}$ is the reaction coefficient and $\tau_u(t)$ is an unknown input delay. We assume delayed non-local measurement

$$\begin{aligned} &y(t) = \int_0^1 c(x)z(x, t-\tau_y(t))dx, \quad t - \tau_y(t) \ge 0, \\ &y(t) = 0, \quad t - \tau_y(t) < 0, \end{aligned} \tag{3.2}$$

where $z_0(x)$ is the initial condition, $\tau_y(t)$ is a known measurement delay and $c \in L^2(0,1)$. We treat two classes of input and output delays: continuously differentiable delays and sawtooth delays, which correspond to network-based control. We assume that τ_y is known, while τ_u is not known and both delays are upper-bounded: $\tau_u(t) \le \tau_M$, $\tau_y(t) \le \tau_M$ with a common $\tau_M > 0$ (for simplicity only). As in Katz and Fridman (2020), our results can be easily extended to a more general Sturm–Liouville operator $\frac{d}{dx}(p(x)z_x(x,t)) + q(x)$ on the right-hand side of (3.1).

For the case of continuously differentiable delays, we assume that τ_u and τ_y are lower bounded by $\tau_m > 0$. This assumption is employed for well-posedness only. Following Liu and Fridman (2014), we assume there exists a unique $t_* \in [\tau_m, \tau_M]$ such that $t - \tau(t) < 0$ if $t < t_*$ and $t - \tau(t) \ge 0$ if $t \ge t_*$ for $\tau(t) \in \left\{\tau_u(t), \tau_y(t)\right\}$. For the case of sawtooth delays, τ_y and τ_u are induced by two networks: from sensor to controller and from controller to actuator, respectively (see Fig. 1). For the first network, denote the sampling instances on the sensor side by s_k, where $0 = s_0 < s_1 < \dots$, $\lim_{k\to\infty} s_k = \infty$. Let $\rho_k, k \ge 0$ be the transmission delays between the sensor and controller. For simplicity of notation, we assume $\rho_0 = 0$. By using the time-delay approach to networked-control systems (see Section 7.5 of Fridman, 2014; Katz, Fridman, & Selivanov, 2020), the measurement delay is presented as

$$\tau_y(t) = t - s_k, \quad t \in [s_k + \rho_k, s_{k+1} + \rho_{k+1}).$$

Furthermore, we assume that $s_{k+1} - s_k \le \text{MATI}$, $k = 0, 1, \dots$, where MATI is the maximum allowable transmission interval. Similarly, $\rho_k \le \text{MAD}$, $k = 0, 1, \dots$, where MAD is the maximum allowable delay. We assume that the sampling instances and sampling delays $\{\rho_k\}_{k=1}^\infty$ are known. This assumption is valid e.g. when the measurement is sent together with a time-stamp. For the second network, denote the sampling instances on the controller side by t_k, $k = 0, 1, \dots$, where $0 = t_0 < t_1 < \dots$, $\lim_{k\to\infty} t_k = \infty$. Let μ_k, $k = 0, 1, \dots$ be the transmission delays between controller and actuator. We assume that $\mu_0 = 0$. The input delay is modeled as

$$\tau_u(t) = t - t_k, \quad t \in [t_k + \mu_k, t_{k+1} + \mu_{k+1}).$$

We assume $t_{k+1} - t_k \leq$ MATI, $k = 0, 1, \ldots$ and $\mu_k \leq$ MAD, $k = 1, 2, \ldots$. Therefore, $\tau_u(t)$ and $\tau_y(t)$ are upper-bounded by τ_M = MATI + MAD. The input delays μ_k, $k = 1, 2, \ldots$ are assumed to be unknown, differently from Katz et al. (2020). We allow the transmission delays to be larger than the corresponding sampling intervals provided that the updating sequences remain increasing.

We present the solution to (3.1) as

$$z(x,t) = \sum_{n=1}^{\infty} z_n(t)\phi_n(x),\ \ z_n(t) = \langle z(\cdot,t), \phi_n \rangle . \tag{3.3}$$

By differentiating under the integral sign, integrating by parts and using (2.4) we have

$$\begin{aligned} &\dot z_n(t) = (-\lambda_n + q) z_n(t) + b_n u(t - \tau_u(t)),\ \ t \geq 0 \\ &z_n(0) = \langle z_0, \phi_n\rangle =: z_{0,n}, \quad b_n = \langle b, \phi_n\rangle . \end{aligned} \tag{3.4}$$

Let $0 < \delta_1 < \delta_0$, and let $0 < \delta_\tau < \delta$, where $\delta := \delta_0 - \delta_1$, be a desired decay rate satisfying (2.3). Since $\lim_{n\to\infty}\lambda_n = \infty$, there exists some $N_0 \in \mathbb{N}$ such that

$$-\lambda_n + q < -\delta, \quad n > N_0. \tag{3.5}$$

N_0 will define the dimension of the controller, whereas $N \geq N_0$ will be the dimension of the observer. We construct a finite-dimensional observer of the form

$$\hat z(x,t) := \sum_{n=1}^{N} \hat z_n(t)\phi_n(x), \tag{3.6}$$

where $\hat z_n(t)$ satisfy the ODEs

$$\begin{aligned} &\dot{\hat z}_n(t) = (-\lambda_n + q)\hat z_n(t) + b_n u(t) \\ &\qquad - l_n\left[\int_0^1 c(x)\hat z(x, t-\tau_y(t))dx - y(t)\right],\ \ t \geq 0, \\ &\hat z_n(t) = 0, \quad t \leq 0, \qquad 1 \leq n \leq N. \end{aligned} \tag{3.7}$$

Note that (3.7) includes $u(t)$ instead of $u(t - \tau_u(t))$ since τ_u is assumed to be unknown. Denote

$$\begin{aligned} &A_0 = \mathrm{diag}\left\{-\lambda_1 + q, \ldots, -\lambda_{N_0} + q\right\},\ L_0 = \left[l_1, \ldots, l_{N_0}\right]^T, \\ &C_0 = \left[c_1, \ldots, c_{N_0}\right],\ c_n = \langle c, \phi_n\rangle,\quad n \geq 1. \end{aligned} \tag{3.8}$$

We assume that

$$c_n \neq 0, \quad 1 \leq n \leq N_0. \tag{3.9}$$

Then, the pair (A_0, C_0) is observable by the Hautus lemma. We choose $l_1, \ldots, l_{N_0}$ such that L_0 satisfies the following Lyapunov inequality:

$$P_o(A_0 - L_0C_0) + (A_0 - L_0C_0)^T P_o < -2\delta P_o, \tag{3.10}$$

where $0 < P_o \in \mathbb{R}^{N_0\times N_0}$ and $l_n = 0,\ n > N_0$. Similarly, we assume

$$b_n \neq 0, \quad 1 \leq n \leq N, \tag{3.11}$$

where $b_n = \langle b, \phi_n\rangle$, and denote

$$B_0 := \begin{bmatrix} b_1 & \ldots & b_{N_0}\end{bmatrix}^T. \tag{3.12}$$

The pair (A_0, B_0) is controllable. Let $K_0 \in \mathbb{R}^{1\times N_0}$ satisfy

$$P_c(A_0 + B_0K_0) + (A_0 + B_0K_0)^T P_c < -2\delta P_c, \tag{3.13}$$

where $0 < P_c \in \mathbb{R}^{N_0\times N_0}$. We propose the control law

$$\begin{aligned} &u(t) = K_0 \hat z^{N_0}(t),\ \ t \in \mathbb{R}, \\ &\hat z^{N_0}(t) = \mathrm{col}\left\{\hat z_1(t), \ldots, \hat z_{N_0}(t)\right\} \end{aligned} \tag{3.14}$$

which is based on the N-dimensional observer (3.7). Let

$$\begin{aligned} &A_1 = \mathrm{diag}\left\{-\lambda_{N_0+1} + q, \ldots, -\lambda_N + q\right\}, \\ &C_1 = \left[c_{N_0+1}, \ldots, c_N\right],\ B_1 = \left[b_{N_0+1}, \ldots, b_N\right]^T. \end{aligned} \tag{3.15}$$

For well-posedness of the closed-loop system (3.1) and (3.7), with control input (3.14), we define an operator

$$\begin{aligned} &\mathcal{A}_1 : \mathcal{D}(\mathcal{A}_1) \subseteq L^2(0,1) \to L^2(0,1),\ \ \mathcal{A}_1 w = -w'', \\ &\mathcal{D}(\mathcal{A}_1) = \left\{w \in H^2(0,1) : w'(0) = w(1) = 0\right\}. \end{aligned} \tag{3.16}$$

Let $\mathcal{H} := L^2(0,1) \times \mathbb{R}^N$ be a Hilbert space with the norm $\|\cdot\|^2_{\mathcal{H}} := \|\cdot\|^2 + |\cdot|^2$. Let $z_0 \in \mathcal{D}(\mathcal{A}_1)$. We begin with continuously differentiable delays, and use the step method, i.e. prove the well-posedness iteratively on the intervals $[0, t_*]$, $[t_*, (s+1)\tau_m]$, $[(s+1)\tau_m, (s+2)\tau_m], \ldots$, where $s \in \mathbb{N}$ satisfies $s\tau_m \leq t_* < (s+1)\tau_m$ (see Section 1.2 of Fridman, 2014). For $t \in [0, t_*]$, defining the state $\xi(t)$ as

$$\xi(t) = \begin{bmatrix} z(\cdot,t) & \hat z^{N,T}(t)\end{bmatrix}^T,\ \hat z^N(t) = \left[\hat z_1(t), \ldots, \hat z_N(t)\right]^T,$$

the closed-loop system can be presented as

$$\begin{aligned} &\tfrac{d}{dt}\xi(t) + \tilde{\mathcal{A}}\xi(t) = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)}\end{bmatrix},\ \ \tilde{\mathcal{A}} = \begin{bmatrix}\mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2\end{bmatrix}, \\ &\mathcal{A}_2 : \mathbb{R}^N \to \mathbb{R}^N,\ \ \mathcal{A}_2 y = \begin{bmatrix} -(A_0 + B_0K_0) & 0 \\ -B_1K_0 & -A_1\end{bmatrix} y, \\ &f_1^{(1)} = qz(\cdot,t),\ f_2^{(1)} = \mathrm{col}\left\{L_0\langle c, z_0\rangle, 0\right\}. \end{aligned} \tag{3.17}$$

Since $-\tilde{\mathcal{A}}$ is an infinitesimal generator of an analytic semigroup on $\mathcal{H}$ and $f_1^{(1)}, f_2^{(1)}$ are continuously differentiable, by Theorems 6.1.2 and 6.1.5 in Pazy (1983) there exists a unique classical solution

$$\xi \in C\left([0,t_*];\mathcal{H}\right),\ \ \xi \in C^1\left((0,t_*];\mathcal{H}\right) \tag{3.18}$$

such that

$$\xi(t) \in \mathcal{D}\left(\tilde{\mathcal{A}}\right) = \mathcal{D}\left(\mathcal{A}_1\right) \times \mathbb{R}^N \quad \forall t \in [0, t_*]. \tag{3.19}$$

The latter follows from the definition of a classical solution in Pazy (1983) (see Section 4.1 therein). Next, let $t \in [t_*, (s+1)\tau_m]$. We present the closed loop system as (3.17), with $f_1^{(1)}$ and $f_2^{(1)}$ replaced by

$$\begin{aligned} &f_1^{(2)} = qz(\cdot,t) + b(\cdot)K_0\hat z^{N_0}(t - \tau_u(t)), \\ &f_2^{(2)} = \begin{bmatrix} L_0 \\ 0\end{bmatrix}\left[\left\langle c, z(\cdot, t-\tau_y(t))\right\rangle - \left[C_0\ C_1\right]\hat z^N(t-\tau_y(t))\right]. \end{aligned} \tag{3.20}$$

For $t \in [t_*, (s+1)\tau_m]$ we have $t - \tau_y(t) \leq t_*$ and $t - \tau_u(t) \leq t_*$. Thus, the delayed terms in (3.1), (3.7) may be treated as non-homogeneous terms in (3.17). Continuous differentiability of τ_u and τ_y together with (3.18) imply that $f_1^{(2)}, f_2^{(2)}$ satisfy the conditions of Theorems 6.1.2 and 6.1.5 in Pazy (1983). Since $\xi(t_*) \in \mathcal{D}(\tilde{\mathcal{A}})$, there exists a unique classical solution ξ satisfying (3.18) and (3.19) on $[t_*, (s+1)\tau_m]$. Using these arguments step by step on $[(s+k)\tau_m, (s+k+1)\tau_m]$ $(k = 1, 2, \ldots)$ with initial conditions $\xi^{(k)}((s+k)\tau_m) \in \mathcal{D}(\tilde{\mathcal{A}})$, we obtain, for $z_0 \in \mathcal{D}(\mathcal{A}_1)$, existence of a unique solution $\xi \in C([0,\infty),\mathcal{H}) \cap C^1((0,\infty)\setminus J, \mathcal{H})$, where $J = \{0, t_*, (s+j)\tau_m\}_{j=1}^{\infty}$, such that $\xi(t) \in \mathcal{D}(\mathcal{A}_1)\times\mathbb{R}^N$ for all $t \geq 0$.

For sawtooth delays, let $z_0 \in H^1(0,1)$, $z_0(1) = 0$ (3.1) can be presented as:

$$\frac{dz}{dt}(t) + \mathcal{A}_1 z(t) = qz(t) + b(\cdot)u(t_k),\ \ t \in [t_k + \mu_k, t_{k+1} + \mu_{k+1}),$$

where $z(t) = z(\cdot,t)$. Since $b(\cdot)u(t_k)$ is piecewise constant, the step method and Theorems 6.3.1, 6.3.3 (with $\alpha = \frac{1}{2}$) in Pazy (1983) imply the existence of a unique solution $z \in C([0,\infty),\mathcal{H}) \cap C^1((0,\infty)\setminus J, \mathcal{H})$, where $J = \left\{0, t_j + \mu_j\right\}_{j=1}^{\infty}$. Moreover, $z(t) \in \mathcal{D}(\mathcal{A}_1)$ for all $t \geq 0$.

Let

$$e_n(t) = z_n(t) - \hat z_n(t),\ \ 1 \leq n \leq N \tag{3.21}$$

be the estimation error. By using (3.3) and (3.6), the last term on the right-hand side of (3.7) can be written as

$$\begin{aligned}&\int_0^1 c(x)\left[\sum_{n=1}^{N}\hat{z}_n(t-\tau_y(t))\phi_n(x)\right.\\&\qquad\qquad\left.-\sum_{n=1}^{\infty}z_n(t-\tau_y(t))\phi_n(x)\right]dx\\&=-\sum_{n=1}^{N}c_ne_n(t-\tau_y(t))-\zeta(t-\tau_y(t)),\\&\zeta(t)=\sum_{n=N+1}^{\infty}c_nz_n(t).\end{aligned} \tag{3.22}$$

Then the error equations have the form

$$\begin{aligned}\dot{e}_n(t)&=(-\lambda_n+q)e_n(t)+b_nK_0\left(\hat{z}^{N_0}(t-\tau_u(t))-\hat{z}^{N_0}(t)\right)\\&\quad-l_n\left(\sum_{n=1}^{N}c_ne_n(t-\tau_y(t))+\zeta(t-\tau_y(t))\right),\ t\geq 0\end{aligned} \tag{3.23}$$

where $n\leq N$.

We define $e_n(t)=\langle z_0,\phi_n\rangle$ for $t<0$. Denote

$$\begin{aligned}&e^{N_0}(t)=\left[e_1(t),\ldots,e_{N_0}(t)\right]^T,\\&e^{N-N_0}(t)=\left[e_{N_0+1}(t),\ldots,e_N(t)\right]^T,\\&\hat{z}^{N-N_0}(t)=\left[\hat{z}_{N_0+1}(t),\ldots,\hat{z}_N(t)\right]^T,\\&X(t)=\mathrm{col}\left\{\hat{z}^{N_0}(t),e^{N_0}(t),\hat{z}^{N-N_0}(t),e^{N-N_0}(t)\right\},\\&\mathcal{L}=\mathrm{col}\left\{L_0,-L_0,0_{2(N-N_0)\times 1}\right\},\ \tilde{K}=\left[K_0\quad 0_{1\times(2N-N_0)}\right].\end{aligned} \tag{3.24}$$

From (3.4), (3.7), (3.8), (3.12), (3.14), (3.23) and (3.24) we obtain the delayed closed-loop system

$$\begin{aligned}\dot{X}(t)&=FX(t)+F_1X(t-\tau_y(t))\\&\quad+F_2\tilde{K}X(t-\tau_u(t))+\mathcal{L}\zeta(t-\tau_y(t)),\quad t\geq 0,\\\dot{z}_n(t)&=(-\lambda_n+q)z_n(t)+b_n\tilde{K}X(t-\tau_u(t)),\ n>N,\end{aligned} \tag{3.25}$$

where

$$F_1=\mathrm{col}\{L_0,-L_0,0,0\}\cdot\left[0\quad C_0\quad 0\quad C_1\right],$$
$$F=\begin{bmatrix}A_0+B_0K_0&0&0&0\\-B_0K_0&A_0&0&0\\B_1K_0&0&A_1&0\\-B_1K_0&0&0&A_1\end{bmatrix},\ F_2=\begin{bmatrix}0\\B_0\\0\\B_1\end{bmatrix}. \tag{3.26}$$

Note that the Cauchy–Schwarz inequality implies

$$\zeta^2(t)\quad\leq\|c\|_N^2\sum_{n=N+1}^{\infty}z_n^2(t), \tag{3.27}$$

where $\|c\|_N^2$ is defined by (2.5). Define the Lyapunov functional

$$V(t)=V_{nom}(t)+\sum_{i=1}^{2}V_{S_i}(t)+\sum_{i=1}^{2}V_{R_i}(t), \tag{3.28}$$

where

$$V_{nom}(t)=|X(t)|_P^2+\sum_{n=N+1}^{\infty}z_n^2(t), \tag{3.29}$$

and

$$\begin{aligned}&V_{S_1}(t)=\int_{t-\tau_M}^{t}e^{-2\delta_0(t-\tau)}\left|X(\tau)\right|_{S_1}^2d\tau,\\&V_{S_2}(t)=\int_{t-\tau_M}^{t}e^{-2\delta_0(t-\tau)}\left|\tilde{K}X(\tau)\right|_{S_2}^2d\tau,\\&V_{R_1}(t)=\tau_M\int_{-\tau_M}^{0}\int_{t+\theta}^{t}e^{-2\delta_0(t-\tau)}\left|\dot{X}(\tau)\right|_{R_1}^2d\tau d\theta,\\&V_{R_2}(t)=\tau_M\int_{-\tau_M}^{0}\int_{t+\theta}^{t}e^{-2\delta_0(t-\tau)}\left|\tilde{K}\dot{X}(\tau)\right|_{R_2}^2d\tau d\theta.\end{aligned} \tag{3.30}$$

Here $0<P,S_1,R_1\in\mathbb{R}^{2N\times 2N}$ and $0<S_2,R_2$ are scalars. $V_{R_1}(t)$ and $V_{S_1}(t)$ compensate for $\tau_y(t)$ in $X(t-\tau_y(t))$, whereas $V_{R_2}(t)$ and $V_{S_2}(t)$ compensate for $\tau_u(t)$ in $\tilde{K}X(t-\tau_u(t))$. Let

$$\begin{aligned}&\nu_\tau(t)=X(t)-X(t-\tau(t)),\ \tau\in\left\{\tau_y,\tau_u\right\},\\&\theta_\tau(t)=X(t-\tau(t))-X(t-\tau_M),\ \tau\in\left\{\tau_y,\tau_u\right\},\\&F^*=F+F_1+F_2\tilde{K},\quad \varepsilon_M=e^{-2\delta_0\tau_M}.\end{aligned} \tag{3.31}$$

Differentiation of $V_{nom}(t)$ along (3.25) gives

$$\begin{aligned}&\dot{V}_{nom}+2\delta_0V_{nom}=X^T(t)\left[PF^*+(F^*)^TP+2\delta_0P\right]X(t)\\&-2X^T(t)PF_1\nu_{\tau_y}(t)+2X^T(t)P\mathcal{L}\zeta(t-\tau_y(t))\\&-2X^T(t)PF_2\tilde{K}\nu_{\tau_u}(t)\ +2\sum_{n=N+1}^{\infty}(-\lambda_n+q+\delta_0)\,z_n^2(t)\\&+2\sum_{n=N+1}^{\infty}z_n(t)b_n\tilde{K}\left[X(t)-\nu_{\tau_u}(t)\right].\end{aligned} \tag{3.32}$$

Let $\alpha>0$. The last term in (3.32) is bounded using

$$\begin{aligned}&\sum_{n=N+1}^{\infty}2z_n(t)b_n\tilde{K}\left[X(t)-\nu_{\tau_u}(t)\right]\leq\frac{2}{\alpha}\sum_{n=N+1}^{\infty}z_n^2(t)\\&+\alpha\left\|b\right\|_N^2\left|\tilde{K}X(t)\right|^2+\alpha\left\|b\right\|_N^2\left|\tilde{K}\nu_{\tau_u}(t)\right|^2.\end{aligned} \tag{3.33}$$

Differentiation of $V_{S_1}(t)$, $V_{R_1}(t)$ along (3.25) leads to

$$\dot{V}_{S_1}+2\delta_0V_{S_1}\leq|X(t)|_{S_1}^2-\varepsilon_M\left|X(t)-\nu_{\tau_y}(t)-\theta_{\tau_y}(t)\right|_{S_1}^2,$$

$$\dot{V}_{R_1}+2\delta_0V_{R_1}\leq\tau_M^2\left|\dot{X}(t)\right|_{R_1}^2-\tau_M\varepsilon_M\int_{t-\tau_M}^{t}\left|\dot{X}(\tau)\right|_{R_1}^2d\tau. \tag{3.34}$$

Similarly, differentiation of $V_{S_2}(t)$, $V_{R_2}(t)$ along (3.25) gives (3.34) with $X(t)$, $\nu_{\tau_y}(t)$ and $\theta_{\tau_y}(t)$ replaced by $\tilde{K}X(t)$, $\tilde{K}\nu_{\tau_u}(t)$ and $\tilde{K}\theta_{\tau_u}(t)$, respectively. Let $G_1\in\mathbb{R}^{2N}$ and $G_2\in\mathbb{R}$ satisfy

$$\begin{bmatrix}R_1&G_1\\ *&R_1\end{bmatrix}\geq 0,\ \begin{bmatrix}R_2&G_2\\ *&R_2\end{bmatrix}\geq 0. \tag{3.35}$$

Applying Jensen's and Park's inequalities (see, e.g, Section 3.6.3 of Fridman, 2014), we obtain for $\xi=\mathrm{col}\left\{\nu_{\tau_y}(t),\theta_{\tau_y}(t)\right\}$

$$-\tau_M\int_{t-\tau_M}^{t}\left|\dot{X}(\tau)\right|_{R_1}^2d\tau\leq-\xi^T\begin{bmatrix}R_1&G_1\\ *&R_1\end{bmatrix}\xi. \tag{3.36}$$

Similar arguments are applied to $V_{R_2}(t)$. To compensate for $\tau_y(t)$ in $\zeta(t-\tau_y(t))$, we use Halanay's inequality. Using (3.27) we obtain

$$\begin{aligned}&-2\delta_1\sup_{-\tau_M\leq\theta\leq 0}V(t+\theta)\leq-2\delta_1V_{nom}(t-\tau_y(t))\\&\leq-2\delta_1\left|X(t)-\nu_{\tau_y}(t)\right|_P^2-2\delta_1\left\|c\right\|_N^{-2}\zeta^2(t-\tau_y(t)),\end{aligned} \tag{3.37}$$

where $0<\delta_1<\delta_0$. By (3.32), (3.33), (3.34) and (3.36)

$$\mathcal{H}_{\tau_M}\leq\eta(t)^T\Phi^1\eta(t)+\sum_{n=N+1}^{\infty}2W_n^{(1)}z_n^2(t)\leq 0,\ t\geq 0, \tag{3.38}$$

where $\mathcal{H}_{\tau_M}$ is defined in (2.1) and $\eta(t)=\mathrm{col}\left\{X(t),\zeta(t-\tau_y),\nu_{\tau_y}(t),\theta_{\tau_y}(t),\tilde{K}\nu_{\tau_u}(t),\tilde{K}\theta_{\tau_u}(t)\right\}$, if $W_n^{(1)}=-\lambda_n+q+\delta_0+\frac{1}{\alpha}<0$ for $n>N$ and $\Phi^1<0$. Here,

$$\Phi^1=\left[\begin{array}{c|cc}\Omega_1&\Theta_1&\Theta_2\\ *&\multicolumn{2}{c}{\mathrm{diag}(\Omega_2,\Omega_3)}\end{array}\right]+\tau_M^2\left[\Lambda_y^TR_1\Lambda_y+\Lambda_u^T\tilde{K}^TR_2\tilde{K}\Lambda_u\right] \tag{3.39}$$

and

$$\begin{aligned}&\Omega_1=\Omega_0+(1-\varepsilon_M)\,\mathrm{diag}(S_1+\tilde{S}_2\tilde{K},0),\\&\tilde{S}_2=\tilde{K}^TS_2,\ \delta=\delta_0-\delta_1,\\&\Theta_1=\begin{bmatrix}P(2\delta_1I-F_1)+\varepsilon_MS_1&\varepsilon_MS_1\\0&0\end{bmatrix},\ \Lambda_u=[F^*,\mathcal{L},-F_1,0,0,0],\\&\Theta_2=\begin{bmatrix}-PF_2+\varepsilon_M\tilde{S}_2&\varepsilon_M\tilde{S}_2\\0&0\end{bmatrix},\ \Lambda_y=[F^*,\mathcal{L},-F_1,0,-F_2,0].\\&\Omega_0=\begin{bmatrix}PF^*+(F^*)^TP+2\delta P+\alpha\left\|b\right\|_N^2\tilde{K}^T\tilde{K}&P\mathcal{L}\\ *&-2\delta_1\left\|c\right\|_N^{-2}\end{bmatrix},\\&\Omega_2=\begin{bmatrix}-2\delta_1P-\varepsilon_M(R_1+S_1)&-\varepsilon_M(S_1+G_1)\\ *&-\varepsilon_M(R_1+S_1)\end{bmatrix},\\&\Omega_3=\begin{bmatrix}\alpha\left\|b\right\|_N^2-\varepsilon_M\left[S_2+R_2\right]&-\varepsilon_M\left[S_2+G_2\right]\\ *&-\varepsilon_M\left[R_2+S_2\right]\end{bmatrix}.\end{aligned} \tag{3.40}$$

Furthermore, monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$ and Schur complement imply that $W_n^{(1)} < 0,\ n > N$ iff

$$\begin{bmatrix} -\lambda_{N+1} + q + \delta_0 & 1 \\ * & -\alpha \end{bmatrix} < 0. \tag{3.41}$$

From (3.38), the LMIs $\Phi^1 < 0$, (3.35) and (3.41) imply $\mathcal{H}_{\tau_M} \le 0$ for $t \ge 0$. Thus, Halanay's inequality (2.2) holds.

We have for some $M > 0$

$$\sup_{-\tau_M \le \theta \le 0} V(\theta) \le M \|z_0\|^2 \tag{3.42}$$

Note that $z_n^2 + e_n^2 = (z_n - e_n)^2 + e_n^2 \ge 0.5 z_n^2$. Then by Parseval's equality, for $t \ge 0$ we have for some $D > 0$

$$V(t) \ge D \cdot \max(\|z(\cdot,t)\|^2, \|z(\cdot,t) - \hat{z}(\cdot,t)\|^2). \tag{3.43}$$

Finally, (2.2), (3.42) and (3.43) imply

$$\max\left(\|z(\cdot,t)\|^2, \|z(\cdot,t) - \hat{z}(\cdot,t)\|^2\right) \le M e^{-2\delta_\tau t} \|z_0\|^2 \tag{3.44}$$

for some $M > 0$, where $\delta_\tau > 0$ satisfies (2.3).

For asymptotic feasibility of LMIs with large N, $\delta_1 = \delta_0 - \delta$ and small τ_M, let $S_i = 0$, $G_i = 0$ for $i = 1, 2$. Taking $\tau_M \to 0^+$, it is sufficient to show (3.41) and

$$\begin{bmatrix} \Omega_0 & M \\ * & D \end{bmatrix} < 0,\ M = \begin{bmatrix} P(2\delta_1 I - F_1) & 0 & -PF_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ D = \mathrm{diag}\left(-R_1 - 2\delta_1 P\ , -R_1\ , -R_2 + \alpha\|b\|_N^2\ , -R_2\right) \tag{3.45}$$

Let $\alpha = N^{-1}$ and $\delta_1 = N$. Then, Theorem 3.1 in Katz and Fridman (2020) implies that (3.41) and $\Omega_0 < 0$ hold for large enough N. Applying Schur complement and taking $R_1 = R_2 = N^{2.5} I$, we obtain that (3.41) and (3.45) hold for large enough N. By continuity, (3.39) and (3.41) hold for $\tau_M = N^{-2}$ and large enough N. Summarizing, we arrive at:

Theorem 3.1. *Consider (3.1) with $b \in L^2(0,1)$ satisfying (3.11), measurement (3.2) with $c \in L^2(0,1)$ satisfying (3.9), control law (3.14) and $z(\cdot,0) = z_0 \in \mathcal{D}(\mathcal{A}_1)$ (continuously differentiable delays) or $z(\cdot,0) = z_0 \in H^1(0,1),\ z_0(1) = 0$ (sawtooth delays). Given $\delta > 0$ and $N_0 \in \mathbb{N}$ subject to (3.5), let L_0 and K_0 satisfy (3.10) and (3.13). Given $\tau_M > 0$, $N \ge N_0$ and $\delta_0 > 0$, let there exist $0 < P, S_1, R_1 \in \mathbb{R}^{2N\times 2N}$, scalars $0 < R_2, S_2, \alpha$ and $G_1 \in \mathbb{R}^{2N\times 2N}$, $G_2 \in \mathbb{R}$ such that the following LMIs hold with $\delta_1 = \delta_0 - \delta$: LMI $\Phi^1 < 0$ with Φ^1 given in (3.39)–(3.40), LMI (3.35) and LMI (3.41). Then the solution $z(x,t)$ to (3.1) with $z(\cdot,0) = z_0$ under control law (3.14), (3.7) and the corresponding observer $\hat{z}(x,t)$ defined by (3.6) satisfy (3.44) for some $M > 0$, where $\delta_\tau > 0$ satisfies (2.3). Moreover, the above LMIs always hold for large enough δ_0 and N and small enough $\tau_M > 0$.*

4. Delayed boundary measurement and non-local actuation

Consider the system (3.1) with $b \in H^1(0,1),\ \ b(1) = 0$ satisfying (3.11), $z_0 \in \mathcal{D}(\mathcal{A}_1)$ (continuously differentiable delays) or $z_0 \in H^1(0,1),\ z_0(1) = 0$ (sawtooth delays),

$$\begin{aligned} &y(t) = z(0, t - \tau_y(t)),\quad t - \tau_y(t) \ge 0, \\ &y(t) = 0,\quad t - \tau_y(t) < 0. \end{aligned} \tag{4.1}$$

By Lemma 2.2, we have $\sum_{n=1}^{\infty} \lambda_n b_n^2 < \infty$. Recall that the unknown $\tau_u(t)$ and known $\tau_y(t)$ are upper-bounded by τ_M.

We present the solution to (3.1) as (3.3) with $z_n(t)$ satisfying (3.4). Let $N_0 \in \mathbb{N}$ satisfy (3.5) and $N \ge N_0$. We construct a N-dimensional observer of the form (3.6), where $\hat{z}_n(t)$ satisfy

$$\begin{aligned} &\dot{\hat{z}}_n(t) = (-\lambda_n + q)\hat{z}_n(t) + b_n u(t) \\ &- l_n\left[\textstyle\sum_{n=1}^N c_n \hat{z}_n(t - \tau_y(t)) - y(t)\right],\quad t \ge 0, \\ &\hat{z}_n(t) = 0,\quad t \le 0,\quad c_n = \phi_n(0) = \sqrt{2},\quad 1 \le n \le N. \end{aligned} \tag{4.2}$$

Let L_0 defined in (3.8) satisfy (3.10) and $l_n = 0,\ n > N_0$. Define $u(t)$ in (3.14) with $K_0 \in \mathbb{R}^{1\times N_0}$ satisfying (3.13).

For well-posedness of (3.1) under (3.13) in the case of continuously differentiable delays, let $\mathcal{A}_1$, defined in (3.16). Since $\mathcal{A}_1 > 0$ is self-adjoint, it has a unique square root

$$\mathcal{A}_1^{\frac{1}{2}} : \mathcal{D}\left(\mathcal{A}_1^{\frac{1}{2}}\right) \to L^2(0,1),$$
$$\mathcal{D}\left(\mathcal{A}_1^{\frac{1}{2}}\right) = \left\{w \in H^1(0,1) | w(1) = 0\right\} \supseteq \mathcal{D}(\mathcal{A}_1).$$

Let $\mathcal{G} := \mathcal{D}\left(\mathcal{A}_1^{\frac{1}{2}}\right) \times \mathbb{R}^N \subseteq \mathcal{H}$ be a Hilbert space with norm $\|\cdot\|_{\mathcal{G}}^2 = \|\cdot\|_{H^1}^2 + |\cdot|^2$. Define the state as

$$\xi(t) = \left[z(\cdot,t)\quad \hat{z}^{N,T}(t)\right]^T,\quad \hat{z}^N(t) = \left[\hat{z}_1(t), \ldots, \hat{z}_N(t)\right]^T.$$

We apply the step method: for $t \in [0, t_*]$, the closed-loop system (3.1) and (4.2), with control input (3.14) can be presented as (3.17) with $f_2^{(1)} = \mathrm{col}\left\{L_0^T z_0(0), 0\right\}$. Since $z_0 \in \mathcal{D}(\mathcal{A}_1) \subseteq \mathcal{D}\left(\mathcal{A}_1^{\frac{1}{2}}\right)$, Lipschitz continuity of $f_1^{(1)}, f_2^{(1)}$ and Theorems 6.3.1 and 6.3.3 in Pazy (1983) with $\alpha = \frac{1}{2}$ imply the existence of a unique classical solution ξ, satisfying (3.18) and (3.19). Furthermore, ξ is Lipschitz continuous on $[0, t_*]$. Next, consider the interval $t \in [t_*, (s+1)\tau_m]$, where $s \in \mathbb{N}$ satisfies $s\tau_m \le t_* < (s+1)\tau_m$. We present the closed-loop system as (3.17) and (3.20) with

$$f_2^{(2)}(t) = \begin{bmatrix} L_0 \\ 0 \end{bmatrix}\left[z(0, t - \tau_y(t)) - [C_0\ C_1]\hat{z}^N(t - \tau_y(t))\right].$$

Since $t - \tau_y(t) \le t_*$ for $t \in [t_*, (s+1)\tau_m]$, Lipschitz continuity of ξ on $[0, t_*]$, the identity

$$z(0, t - \tau_y(t)) = \int_0^1 z_x(x, t - \tau_y(t))dx$$

and continuous differentiability of τ_u and τ_y imply

$$\left|f_2^{(2)}(t_1) - f_2^{(2)}(t_2)\right| \le M^{(1)}|t_1 - t_2|,$$

for $t_1, t_2 \in [t_*, (s+1)\tau_m]$ and some $M^{(1)} > 0$. By Theorems 6.3.1 and 6.3.3 in Pazy (1983) with $\alpha = \frac{1}{2}$, the system (3.17) and (3.20) on $[t_*, (s+1)\tau_m]$, with initial condition $\xi(t_*) \in \mathcal{D}(\tilde{\mathcal{A}})$ has a unique classical solution $\xi(t)$ satisfying (3.18) and (3.19) on $[t_*, (s+1)\tau_m]$. Furthermore, ξ is Lipschitz continuous on $[t_*, (s+1)\tau_m]$. Continuing as in Section 3, we obtain for $z_0 \in \mathcal{D}(\mathcal{A}_1)$ the existence of a unique solution $\xi \in C([0,\infty), \mathcal{H}) \cap C^1((0,\infty) \setminus J, \mathcal{H})$, where $J = \{0, t_*, (s+j)\tau_m\}_{j=1}^{\infty}$. Moreover $\xi(t) \in \mathcal{D}(\tilde{\mathcal{A}}) = \mathcal{D}(\mathcal{A}_1) \times \mathbb{R}^N$ for all $t > 0$. For sawtooth delays, the proof of well-posedness is identical to Section 3.

By using (3.4) and the estimation error (3.21), the last term on the right-hand side of (4.2) can be presented as

$$\begin{aligned} &\textstyle\sum_{n=1}^N \phi_n(0)\hat{z}_n(t - \tau_y(t)) - y(t) \\ &= -\textstyle\sum_{n=1}^N c_n e_n(t - \tau_y(t)) - \zeta(t), \\ &\zeta(t) = z(0,t) - \textstyle\sum_{n=1}^N c_n z_n(t). \end{aligned} \tag{4.3}$$

Furthermore,

$$\zeta^2(t) \le \left\|z_x(\cdot,t) - \textstyle\sum_{n=1}^N \phi_n'(\cdot) z_n(t)\right\|^2 = \textstyle\sum_{n=N+1}^{\infty} \lambda_n z_n^2(t). \tag{4.4}$$

By (4.3), (4.4) and $X(t)$ defined in (3.24), we obtain the closed-loop system (3.25)–(3.26). Taking into account (4.4), for exponential H^1-stability we consider the Lyapunov functional (3.28) with $V_{nom}(t)$ given by

$$V_{nom}(t) := |X(t)|_P^2 + \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t) \tag{4.5}$$

and $V_{S_i}, V_{R_i},\ i=1,2$ given in (3.29), (3.30). Differentiating (3.28) along (3.25) and using (3.32), (3.34), (4.4) and arguments similar to (3.37) with $\|c\|_N^2$ replaced by 1 we obtain

$$\mathcal{H}_{\tau_M} \le \eta(t)^T \Phi^2 \eta(t) + \sum_{n=N+1}^{\infty} 2W_n^{(1)} z_n^2(t) \le 0, \quad t \ge 0. \tag{4.6}$$

Here $\eta(t) = \text{col}\left\{X(t), \zeta(t-\tau_y), \nu_{\tau_y}(t), \theta_{\tau_y}(t), \tilde{K}\nu_{\tau_u}(t), \tilde{K}\theta_{\tau_u}(t)\right\}$, $W_n^{(1)}$ is given in (3.38) and Φ^2 is a symmetric block matrix, which differs from Φ^1, given in (3.39) and (3.40) by replacing $\|c\|_N$ with 1 in Ω_0 and $\|b\|_N^2$ by $\left\|b'\right\|_N^2$ in Ω_0 and Ω_3. Furthermore, $W_n^{(1)} < 0$ iff (3.41) holds. By arguments similar to Theorem 3.1 we arrive at:

Theorem 4.1. *Consider (3.1) with $b \in H^1(0,1)$, $b(1)=0$ satisfying (3.11), measurement (4.1), control law (3.14) and $z(\cdot,0) = z_0 \in \mathcal{D}(\mathcal{A}_1)$ (continuously differentiable delays) or $z(\cdot,0)=z_0 \in H^1(0,1)$, $z_0(1)=0$ (sawtooth delays). Given $\delta>0$ and $N_0 \in \mathbb{N}$ subject to (3.5), let L_0 and K_0 satisfy (3.10) and (3.13). Given $\tau_M>0$, $N \ge N_0$ and $\delta_0>0$, let there exist $0 < P, S_1, R_1 \in \mathbb{R}^{2N\times 2N}$, scalars $0 < R_2, S_2, \alpha$ and $G_1 \in \mathbb{R}^{2N\times 2N}$, $G_2 \in \mathbb{R}$ such that the following LMIs hold with $\delta_1 = \delta_0 - \delta$: LMI $\Phi^2 < 0$ with Φ^2 given in (4.6), LMI (3.35) and LMI (3.41). Then the solution $z(x,t)$ to (3.1) with $z(\cdot,0)=z_0$ under control law (3.14), (4.2) and observer $\hat{z}(x,t)$ defined by (3.6) satisfy (3.44) with the L^2-norm replaced by the H^1-norm. Moreover, the above LMIs always hold for large enough N and δ_0 and small enough $\tau_M>0$.*

5. Dirichlet actuation/non-local measurement

We consider Dirichlet actuation and non-local measurement. We present two cases. The first case corresponds to time-varying $\tau_y(t) \le \tau_M$ and $\tau_u(t) \le \tau_M$, where the former satisfies $\dot{\tau}_y \le d < 1$ for some constant d. The second case corresponds to a constant $\tau_u(t) \equiv r$ and time-varying $\tau_y(t) \le \tau_M$. For the first case we present a direct approach, whereas for the second we use a method that employs dynamic extension.

5.1. Time-varying input and output delays

Consider the system

$$\begin{aligned} &z_t(x,t) = z_{xx}(x,t) + qz(x,t), \quad t\ge 0,\\ &z_x(0,t)=0, \quad z(1,t) = u(t-\tau_u(t)), \end{aligned} \tag{5.1}$$

and measurement (3.2) with continuously differentiable and slowly-varying $\tau_y(t) \ge \tau_m > 0$ such that $\dot{\tau}_y \le d < 1$ for some d. Here $\tau_y(t) \le \tau_M$ is known, $\tau_u(t) \le \tau_M$ is an unknown delay and $c \in H^1(0,1)$, $c(1)=0$ satisfies (3.9).

By presenting the solution to (5.1) as (3.3), we find that $z_n(t)$, $n\ge 1$ satisfy

$$\begin{aligned} &\dot{z}_n(t) = (-\lambda_n + q) z_n(t) + b_n u(t-\tau_u(t)), \quad t \ge 0,\\ &b_n = \sqrt{2}(-1)^{n+1}\left(n-\frac{1}{2}\right)\pi = (-1)^{n+1}\sqrt{2\lambda_n}. \end{aligned} \tag{5.2}$$

In particular, $b_n \to \infty$ as $n\to\infty$ and assumption (3.11) is satisfied for all $N \in \mathbb{N}$. Moreover, we have

$$\sum_{n=N+1}^{\infty} \frac{b_n^2}{\lambda_n^2} \le \frac{8}{\pi^2} \sum_{n=N+1}^{\infty} \frac{1}{(2n-1)^2} \le \frac{4}{\pi^2(2N-1)}. \tag{5.3}$$

Let $N_0 \in \mathbb{N}$ satisfy (3.5) with $\delta = \delta_0 > 0$. Let $N \in \mathbb{N}$ satisfy $N_0 \le N$. We construct a N-dimensional observer of the form (3.6), where $\hat{z}_n(t)$ satisfy (3.7). Let L_0 defined in (3.8) satisfy (3.10) and $l_n = 0$, $n > N_0$. Define the controller (3.14) with $K_0 \in \mathbb{R}^{1\times N_0}$ subject to (3.13).

Let $z_0 \in L^2(0,1)$. For well-posedness of (5.1) with sawtooth τ_u, by arguments similar to Theorem 2.1 and Corollary 2.2 in Karafyllis and Krstic (2018), (5.1) has a unique solution $z \in C([0,\infty); L^2(0,1))$. Moreover, $z \in C^1(I\times[0,1])$, $z(\cdot,t) \in C^2(0,1)$ for all $t>0$ and $z(\cdot,0)=z_0$. Here, $I := [0,\infty) \setminus \{t_k+\mu_k\}_{k=1}^{\infty}$. For continuously differentiable τ_u, arguments of well-posedness in Section 3, together with Theorem 6.1.2 in Pazy (1983) imply the existence of a unique mild solution $\xi \in C([0,\infty], \mathcal{H})$, with $\mathcal{H} = L^2(0,1)\times\mathbb{R}^N$.

By using the estimation error (3.21) and

$$\begin{aligned} &\rho_n(t) = \lambda_n^{-\frac{1}{2}} \hat{z}_n(t), \quad \upsilon_n(t) = \lambda_n^{-\frac{1}{2}} e_n(t), \quad N_0+1 \le n \le N,\\ &\rho^{N-N_0}(t) = \left[\rho_{N_0+1}(t), \dots, \rho_N(t)\right]^T,\\ &\upsilon^{N-N_0}(t) = \left[\upsilon_{N_0+1}(t), \dots, \upsilon_N(t)\right]^T,\\ &X(t) = \text{col}\left\{\hat{z}^{N_0}(t), e^{N_0}(t), \rho^{N-N_0}(t), \upsilon^{N-N_0}(t)\right\}. \end{aligned} \tag{5.4}$$

we obtain the closed-loop system (3.24)–(3.26), with L_0C_1 and B_1K_0 in (3.26) replaced by $L_0\tilde{C}_1$ and $\tilde{B}_1K_0$, respectively, where

$$\begin{aligned} &\tilde{C}_1 = \left[\lambda_{N_0+1}^{\frac{1}{2}} c_{N_0+1}, \dots, \lambda_N^{\frac{1}{2}} c_N\right],\\ &\tilde{B}_1 = \left[\lambda_{N_0+1}^{-\frac{1}{2}} b_{N_0+1}, \dots, \lambda_N^{-\frac{1}{2}} b_N\right]^T. \end{aligned} \tag{5.5}$$

Note that by the Cauchy–Schwarz inequality

$$\zeta^2(t) \le \left\|c'\right\|_N^2 \sum_{n=N+1}^{\infty} \tfrac{1}{\lambda_n} z_n^2(t). \tag{5.6}$$

For convergence analysis of the closed-loop system, we introduce the Lyapunov functional $V_1(t) = V(t) + V_Q(t)$, where $V(t)$ is given by (3.28) with $V_{S_1}(t), V_{S_2}(t), V_{R_1}(t), V_{R_2}(t)$ appearing in (3.30),

$$\begin{aligned} V_{nom}(t) = &|X(t)|_P^2 + \sum_{n=N+1}^{\infty} \tfrac{1}{\lambda_n} z_n^2(t)\\ &+ q_1 \int_{t-\tau_y(t)}^{t} e^{-2\delta_0(t-\tau)} \zeta^2(\tau) d\tau, \end{aligned} \tag{5.7}$$

and

$$V_Q(t) = \int_{t-\tau_y(t)}^{t} e^{-2\delta_0(t-\tau)} |X(\tau)|_Q^2 \, d\tau \tag{5.8}$$

with $Q>0$. Here $P, S_1, R_1, Q > 0$ are matrices and $q_1, S_2, R_2 > 0$ are scalars. Using (3.31), differentiation of $V_{nom}(t)$ along (3.25), the Cauchy–Schwarz inequality, (5.3) and (5.6) give

$$\begin{aligned} &\dot{V}_{nom} + 2\delta V_{nom} \le X^T(t)\left[PF^* + (F^*)^T P + 2\delta_0 P\right] X(t)\\ &- 2X^T(t) PF_1 \nu_{\tau_y}(t) - 2X^T(t) PF_2 \tilde{K} \nu_{\tau_u}(t)\\ &+ 2X^T(t) P\mathcal{L}\zeta(t-\tau_y(t)) - q_1(1-d)\varepsilon_M \zeta^2(t-\tau_y(t))\\ &+ \tfrac{4\alpha}{\pi^2(2N-1)} X^T(t)\tilde{K}^T\tilde{K}X(t) + \tfrac{4\alpha}{\pi^2(2N-1)} \nu_{\tau_u}^T(t)\tilde{K}^T\tilde{K}\nu_{\tau_u}(t)\\ &+ 2\sum_{n=N+1}^{\infty}\left(-1 + \tfrac{q+\delta_0}{\lambda_n} + \tfrac{1}{\alpha} + \tfrac{q_1\left\|c'\right\|_N^2}{2\lambda_n}\right) z_n^2(t) \end{aligned} \tag{5.9}$$

where $\alpha > 0$. Differentiation of V_Q in (5.8) gives

$$\dot{V}_Q + 2\delta_0 V_Q \le |X(t)|_Q^2 + (1-d)\varepsilon_M \left|X(t-\tau_y(t))\right|_Q^2.$$

Let $G_1 \in \mathbb{R}^{2N}$ and $G_2 \in \mathbb{R}$ satisfy (3.35). By differentiating V_{S_i}, V_{R_i}, $1 \le i \le 2$ along the closed-loop system and applying Jensen's and Park's inequalities, we obtain for $t \ge 0$

$$\dot{V}_1 + 2\delta_0 V_1 \le \eta(t)^T \Phi^3 \eta(t) + 2\sum_{n=N+1}^{\infty} W_n^{(3)} z_n^2(t) \le 0 \tag{5.10}$$

if $W_n^{(3)} = -1 + \frac{q+\delta_0}{\lambda_n} + \frac{1}{\alpha} + \frac{q_1\left\|c'\right\|_N^2}{2\lambda_n} < 0$, $n > N$ and $\Phi^3 < 0$. Here, $\eta(t) = \text{col}\left\{X(t), \zeta(t-\tau_y), \nu_{\tau_y}(t), \theta_{\tau_y}(t), \tilde{K}\nu_{\tau_u}(t), \tilde{K}\theta_{\tau_u}(t)\right\}$ and $\Phi^3 = \left\{\Phi_{ij}^3\right\}$ is a symmetric block matrix obtained from Φ^1 in (3.39)–(3.40) by substituting $\delta_1 = 0$, $\delta = \delta_0$, $G_1 = 0$, and replacing

Ω_0, Θ_1, Ω_2 and Ω_3 by

$$\begin{aligned}
&\bar{\Omega}_0 = \begin{bmatrix} PF^* + (F^*)^T P + 2\delta_0 P + \frac{4p\alpha}{\pi^2(2N-1)}\tilde{K}^T\tilde{K} & P\mathcal{L} \\ * & -q_1(1-d)\varepsilon_M \end{bmatrix} \\
&\quad +(1-(1-d)\varepsilon_M)\,\mathrm{diag}\,(Q,0)\,, \\
&\bar{\Theta}_1 = \Theta_1 + (1-d)\varepsilon_M\,\mathrm{diag}\,(Q,0)\,, \\
&\bar{\Omega}_2 = \Omega_2 - (1-d)\varepsilon_M\,\mathrm{diag}\,(Q,0)\,, \\
&\bar{\Omega}_3 = \begin{bmatrix} \frac{4p\alpha}{\pi^2(2N-1)} - \varepsilon_M[S_2+R_2] & -\varepsilon_M[S_2+G_2] \\ * & -\varepsilon_M[R_2+S_2] \end{bmatrix},
\end{aligned} \tag{5.11}$$

respectively. By Schur complement, $W_n^{(3)} < 0$ for $t \geq 0$ iff

$$\begin{bmatrix} -1 + \frac{q+\delta_0}{\lambda_n} + \frac{q_1\|c'\|_N^2}{2\lambda_n} & 1 \\ * & -\alpha \end{bmatrix} < 0. \tag{5.12}$$

By using further arguments of Theorem 3.1 we arrive at:

Theorem 5.1. *Consider (5.1), measurement (3.2) with $c \in H^1(0,1)$, $c(1) = 0$ satisfying (3.9), control law (3.14) and $z(\cdot,0) = z_0 \in L^2(0,1)$. Let $\delta_0 > 0$ be a desired decay rate and let $N_0 \in \mathbb{N}$ satisfy (3.5) with $\delta = \delta_0$. Assume that L_0 and K_0 are obtained using (3.10) and (3.13), respectively. Given $\tau_M > 0$ and $N \geq N_0$, let there exist positive definite matrices $P, S_1, R_1 \in \mathbb{R}^{2N\times 2N}$, scalars $R_2, S_2, q_1, \alpha, p > 0$, $G_1 \in \mathbb{R}^{2N\times 2N}$ and $G_2 \in \mathbb{R}$ such that the following LMIs hold: LMI $\Phi^3 < 0$ with Φ^3 given in (5.11), LMI (3.35) and LMI (5.12). Then $V(t) \leq e^{-2\delta_0 t}V(0)$, $t \geq 0$, with V given by (3.29), (5.7). Moreover, the above LMIs always hold for large enough N and small enough $\tau_M > 0$.*

5.2. Time-varying output delay and constant input delay

Consider the system

$$\begin{aligned} &z_t(x,t) = z_{xx}(x,t) + qz(x,t), \quad t \geq 0, \\ &z_x(0,t) = 0, \quad z(1,t) = u(t-r) \end{aligned} \tag{5.13}$$

with unknown constant input delay $r > 0$, measurement (3.2) with known time-varying delay and $c \in L^2(0,1)$ satisfying (3.9). Following Prieur and Trélat (2018), we introduce

$$w(x,t) = z(x,t) - u(t-r), \tag{5.14}$$

to obtain the following ODE–PDE system

$$\begin{aligned} &\dot{u}(t) = v(t), \quad u(0) = 0, \quad t \geq 0, \\ &w_t(x,t) = w_{xx}(x,t) + qw(x,t) + qu(t-r) - v(t-r), \\ &w_x(0,t) = 0, \quad w(1,t) = 0, \end{aligned} \tag{5.15}$$

where we treat $u(t)$ as an additional state variable and $v(t)$ as the control input with non-local actuation with $b \equiv 1 \in L^2(0,1)$. Note that once the control input $v(t)$ is specified, the value of $u(t)$ can be computed online. Using (5.14), the measurement (3.2) can be presented as

$$\begin{aligned} y(t) = &\int_0^1 c(x)w(x,t-\tau_y(t))dx \\ &+ gu(t-\tau_y(t)-r), \quad t-\tau_y(t) > 0 \end{aligned}$$

where $g := \langle c, 1\rangle$. By presenting the solution to (5.15) as in (3.3) and using arguments similar to (3.4) for $b \equiv 1$, we find that $w_n(t)$, $n \geq 1$ satisfy

$$\begin{aligned} &\dot{w}_n(t) = (-\lambda_n + q)w_n(t) + b_n[qu(t-r) - v(t-r)]\,, \ t \geq 0, \\ &b_n = \sqrt{2}(-1)^{n+1}\left[\left(n-\tfrac{1}{2}\right)\pi\right]^{-1}. \end{aligned} \tag{5.16}$$

Note that (3.11) is satisfied for all $N \in \mathbb{N}$. Furthermore,

$$\sum_{n=N+1}^{\infty} b_n^2 \leq 4\pi^{-2}(2N-1)^{-1}. \tag{5.17}$$

Well-posedness of (5.15) follows from arguments similar to proof of well-posedness in Section 3.

Let N_0 satisfy (3.5) with $\delta > 0$. Let $N \in \mathbb{N}$ satisfy $N_0 \leq N$. To approximate $w(x,t)$, we construct a N-dimensional observer of the form

$$\hat{w}(x,t) := \sum_{n=1}^{N} \hat{w}_n(t)\phi_n(x), \tag{5.18}$$

where $\hat{w}_n(t)$ satisfy

$$\begin{aligned} &\dot{\hat{w}}_n(t) = (-\lambda_n + q)\hat{z}_n(t) + b_n[qu(t) - v(t)] - l_n \times \\ &\left[\int_0^1 c(x)\hat{w}(x,t-\tau_y(t))dx + gu(t-\tau_y(t)) - y(t)\right], \ t \geq 0 \\ &\hat{w}_n(t) = 0, \quad t \leq 0, \qquad 1 \leq n \leq N. \end{aligned} \tag{5.19}$$

Let

$$e_n(t) = w_n(t) - \hat{w}_n(t), \ 1 \leq n \leq N \tag{5.20}$$

be the estimation error. By arguments similar to (3.22), the last term on the right-hand side of (5.19) can be written as

$$\begin{aligned} &\textstyle\int_0^1 c(x)\hat{w}(x,t-\tau_y(t))dx + gu(t-\tau_y(t)) - y(t) = \\ &-C_0e^{N_0}(t-\tau_y(t)) - C_1e^{N-N_0}(t-\tau_y(t)) \\ &-\zeta(t-\tau_y(t)) - gu(t-\tau_y(t)-r) + gu(t-\tau_y(t)), \end{aligned} \tag{5.21}$$

with $e^{N_0}(t)$, $e^{N-N_0}(t)$, C_0 and C_1 introduced in (3.8), (3.24), respectively and $\zeta(t) = \sum_{n=N+1}^{\infty} c_n w_n(t)$.

Let L_0 defined in (3.8) satisfy (3.10) and $l_n = 0$, $n > N_0$. We consider the control law

$$\begin{aligned} &v(t) = K_0\hat{w}^{N_0}(t), \ t \in \mathbb{R}, \\ &\hat{w}^{N_0}(t) = \left[u(t), \hat{w}_1(t), \ldots, \hat{w}_{N_0}(t)\right]^T. \end{aligned} \tag{5.22}$$

that is based on the N-dimensional observer (5.18) and $u(t)$, which is computed online. Defining the state as

$$\begin{aligned} &X(t) = \mathrm{col}\left\{\hat{w}^{N_0}(t), e^{N_0}(t), \hat{w}^{N-N_0}(t), e^{N-N_0}(t)\right\}, \\ &\hat{w}^{N-N_0}(t) = \left[w_{N_0+1}(t), \ldots, w_N(t)\right]^T, \end{aligned}$$

and using (5.20) and arguments similar to (3.23), we obtain the following closed-loop system

$$\begin{aligned} \dot{X}(t) &= FX(t) + F_1X(t-\tau_y(t)) + \mathcal{L}\zeta(t-\tau_y(t)) \\ &\quad + F_2\tilde{K}_qX(t-r) + \mathcal{L}F_3X(t-\tau_y(t)-r), \quad t \geq 0, \\ \dot{w}_n(t) &= (-\lambda_n + q)w_n(t) - b_n\tilde{K}_qX(t-r), \ n > N, \end{aligned} \tag{5.23}$$

where

$$\begin{aligned} &K_q = K_0 - [q, 0_{1\times N_0}], \ F_3 = [g, 0_{1\times 2N}], \\ &\tilde{B}_0 = \begin{bmatrix} 1 \\ -B_0 \end{bmatrix}, \ \tilde{L}_0 = \begin{bmatrix} 0 \\ L_0 \end{bmatrix}, \ \tilde{K}_q = [K_q, 0_{1\times(2N-N_0)}], \\ &F = \begin{bmatrix} \tilde{A}_0 + \tilde{B}_0K_0 & 0 & 0 & 0 \\ B_0K_q & A_0 & 0 & 0 \\ -B_1K_q & 0 & A_1 & 0 \\ B_1K_q & 0 & 0 & A_1 \end{bmatrix}, \ \mathcal{L} = \begin{bmatrix} \tilde{L}_0 \\ -L_0 \\ 0 \\ 0 \end{bmatrix}, \ F_2 = \begin{bmatrix} 0 \\ -B_0 \\ 0 \\ -B_1 \end{bmatrix}, \\ &F_1 = \begin{bmatrix} -\tilde{L}_0\cdot[g, 0_{1\times N_0}] & \tilde{L}_0C_0 & 0 & \tilde{L}_0C_1 \\ L_0\cdot[g, 0_{1\times N_0}] & -L_0C_0 & 0 & -L_0C_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \tilde{A}_0 = \begin{bmatrix} 0 & 0 \\ qB_0 & A_0 \end{bmatrix}, \end{aligned} \tag{5.24}$$

A_0, L_0 and A_1, B_1, C_1 are given by (3.8) and (3.15), respectively. By the Hautus lemma the pair $(\tilde{A}_0, \tilde{B}_0)$ is controllable. Let $K_0 \in$

$\mathbb{R}^{1\times(N_0+1)}$ satisfy

$$P_c(\tilde{A}_0+\tilde{B}_0K_0)+(\tilde{A}_0+\tilde{B}_0K_0)^TP_c<-2\delta P_c, \tag{5.25}$$

where $0<P_c\in\mathbb{R}^{(N_0+1)\times(N_0+1)}$. Furthermore, (3.27) holds with $z_n(t)$ replaced by $w_n(t)$.

For stability analysis, introduce the Lyapunov functional

$$V(t):=V_{nom}(t)+\sum_{i=0}^{2}V_{S_i}(t)+\sum_{i=0}^{2}V_{R_i}(t), \tag{5.26}$$

where $V_{nom}(t)$ given by (3.29) with $z_n(t)$ replaced by $w_n(t)$, V_{S_2}, V_{R_2} are given by (3.30) with τ_M and $\tilde{K}X$ replaced by r and X, respectively,

$$\begin{aligned}&V_{S_0}(t):=\int_{t-r-\tau_M}^{t-r}e^{-2\delta_0(t-\tau)}|F_3X(\tau)|^2_{S_0}\,d\tau,\\&V_{R_0}(t):=\tau_M\int_{-r-\tau_M}^{-r}\int_{t+\theta}^{t}e^{-2\delta_0(t-\tau)}\left|F_3\dot{X}(\tau)\right|^2_{R_0}d\tau d\theta,\end{aligned} \tag{5.27}$$

and $V_{S_1}(t)$, $V_{R_1}(t)$ appear in (3.29). Here $P,S_2,R_2>0$ are matrices and $S_0,R_0>0$ are scalars. Let

$$\begin{aligned}&\nu_\tau(t)=X(t)-X(t-\tau(t)),\ \ \tau\in\left\{\tau_y,r\right\},\\&\nu_{r,\tau_y}(t)=X(t-r)-X(t-\tau_y(t)-r),\\&\theta_{\tau_y}(t)=X(t-\tau_y(t))-X(t-\tau_M),\\&\theta_{r,\tau_y}(t)=X(t-r-\tau_y(t))-X(t-r-\tau_M),\\&\varepsilon_M=e^{-2\delta_0\tau_M},\ \ \varepsilon_r=e^{-2\delta_0 r},\\&F^*=F+F_1+F_4,\ \ F_4=F_2\tilde{K}_q+\mathcal{L}F_3.\end{aligned} \tag{5.28}$$

Differentiation of $V_{nom}(t)$ along (5.23), the Cauchy–Schwarz inequality and (5.17) give

$$\begin{aligned}&\dot{V}_{nom}+2\delta_0V_{nom}\le X^T(t)\left[PF^*+(F^*)^TP+2\delta_0P\right]X(t)\\&-2X^T(t)PF_1\nu_{\tau_y}(t)-2X^T(t)PF_4\nu_r(t)\\&-2X^T(t)P\mathcal{L}F_3\nu_{r,\tau_y}(t)+2X^T(t)P\mathcal{L}\zeta(t-\tau_y(t))\\&+\tfrac{4\alpha}{\pi^2(2N-1)}\left(X^T(t)\tilde{K}_q^T\tilde{K}_qX(t)+\nu_r^T(t)\tilde{K}_q^T\tilde{K}_q\nu_r(t)\right)\\&+2\textstyle\sum_{n=N+1}^{\infty}\left(-\lambda_n+q+\delta_0+\frac{1}{\alpha}\right)w_n^2(t).\end{aligned} \tag{5.29}$$

Let (3.35) be satisfied with R_2 and G_2 replaced by R_0 and G_0, respectively. By differentiating V_{S_i},V_{R_i}, $0\le i\le 2$ along the closed-loop system, applying Jensen's and Park's inequalities, (3.27) with $z_n(t)$ replaced by $w_n(t)$ and (3.37) we obtain

$$\mathcal{H}_{\tau_M}\le\eta(t)^T\Phi^4\eta(t)+\textstyle\sum_{n=N+1}^{\infty}2W_n^{(1)}w_n^2(t)\le 0,\ \ t\ge 0, \tag{5.30}$$

where $\mathcal{H}_{\tau_M}$ is defined in (2.1) and $\eta(t)=\mathrm{col}\left\{X(t),\zeta(t-\tau_y),\nu_{\tau_y}(t),\theta_{\tau_y}(t),\nu_r(t),F_3\nu_{r,\tau_y}(t),F_3\theta_{r,\tau_y}(t)\right\}$, provided $W_n^{(1)}=-\lambda_n+q+\delta_0+\frac{1}{\alpha}<0$ for $n>N$ and

$$\begin{aligned}\Phi^4:=&\left[\begin{array}{c|cc}\Omega_1&\Theta_1&\Theta_3\\\hline *&\mathrm{diag}(\Omega_2,\Omega_3)\end{array}\right]+r^2\Lambda_y^TR_2\Lambda_y\\&+\Lambda_y^T\left[\tau_M^2\left(R_1+F_3^TR_0F_3\right)\right]\Lambda_y<0.\end{aligned} \tag{5.31}$$

Here

$$\begin{aligned}&\Omega_1=\Omega_0+(1-\varepsilon_M)\,\mathrm{diag}(S_1,0)+(1-\varepsilon_r)\,\mathrm{diag}(S_2,0)\\&\quad+(\varepsilon_r-\varepsilon_M\varepsilon_r)\,\mathrm{diag}(\tilde{S}_0,0),\ \ \delta=\delta_0-\delta_1,\ \ \tilde{S}_0=F_3^TS_0F_3,\\&\Theta_3=\begin{bmatrix}\theta&-P\mathcal{L}+\varepsilon_M\varepsilon_rF_3^TS_0&\varepsilon_M\varepsilon_rF_3^TS_0\\0&0&0\end{bmatrix},\\&\theta=-PF_4+\varepsilon_rS_2-(\varepsilon_r-\varepsilon_M\varepsilon_r)\tilde{S}_0\\&\Omega_3=\begin{bmatrix}\omega&-\varepsilon_M\varepsilon_rF_3^TS_0&-\varepsilon_M\varepsilon_rF_3^TS_0\\ *&-\varepsilon_M\varepsilon_r\left[R_0+S_0\right]&-\varepsilon_M\varepsilon_r\left(S_0+G_0\right)\\ *&*&-\varepsilon_M\varepsilon_r(S_0+R_0)\end{bmatrix},\\&\omega=\tfrac{4\alpha}{\pi^2(2N-1)}\tilde{K}_q^T\tilde{K}_q-\varepsilon_r(S_2+R_2)+(\varepsilon_r-\varepsilon_M\varepsilon_r)\tilde{S}_0,\\&\Lambda_y=[F^*,\mathcal{L},-F_1,0,-F_4,-\mathcal{L},0],\end{aligned} \tag{5.32}$$

Table 1
Chosen gains L_0 and K_0.

	S3	S4	S5.1	S5.2
b	ϕ_1	ϕ_2	–	–
c	ϕ_1	ϕ_1	ϕ_2	ϕ_1
L_0	11.65	5.67	37.33	8.75
K_0	−23.01	−23.01	−5.86	col (−54.15, −47.82)

Ω_0 is given in (3.40) with $\|b\|_N^2\tilde{K}^T\tilde{K}$ replaced by $\frac{4}{\pi^2(2N-1)}\tilde{K}_q^T\tilde{K}_q$ and Ω_2, Θ_1 are given in (3.40). Furthermore, $W_n^{(1)}<0$, $n>N$ for all $t\ge 0$ iff (3.41) holds. Feasibility of $\Phi^4<0$, (3.35) with R_2 and G_2 replaced by R_0 and G_0, respectively, and (3.41) implies (3.44) for some $M_0>0$ with $z(x,t)$ and $\hat{z}(x,t)$ replaced by $w(x,t)$ and $\hat{w}(x,t)$, respectively. Moreover, $|u(t)|^2\le M_1e^{-2\delta_\tau}\|z_0\|^2$ for some $M_1>0$. By arguments of Theorem 3.1 we arrive at:

Theorem 5.2. *Consider (5.13) with measurement (3.2) and $c\in L^2(0,1)$ satisfying (3.9), control law (5.22) and $z(\cdot,0)=z_0\in\mathcal{D}(\mathcal{A}_1)$ (continuously differentiable delays) or $z(\cdot,0)=z_0\in H^1(0,1)$, $z_0(1)=0$ (sawtooth delays). Given $\delta>0$ and $N_0\in\mathbb{N}$ subject to (3.5), let L_0 and K_0 satisfy (3.10)* and *(5.25). Given $r,\tau_M>0$, $N\ge N_0$ and $\delta_0>0$, let there exist $0<P,S_1,S_2,R_1,R_2\in\mathbb{R}^{(2N+1)\times(2N+1)}$, scalars $0<R_0,S_0,\alpha$ and $G_1\in\mathbb{R}^{(2N+1)\times(2N+1)}$, $G_0\in\mathbb{R}$ such that the following three LMIs hold with $\delta_1=\delta_0-\delta$: LMI $\Phi^4<0$ with Φ^4 given by (5.31)* and *(5.32), LMI (3.35) with R_2 and G_2 replaced by R_0 and G_0, respectively, and LMI (3.41). Then the solution $z(x,t)$ to (5.13) under the control law (5.22), (5.19) and the corresponding observer $\hat{w}(x,t)$ defined by (5.18) with $z(\cdot,0)=z_0$ satisfy (3.44) for some $M>0$, where $\delta_\tau>0$ satisfies (2.3). Moreover, the above LMIs are always feasible for large enough N and δ_0 and small enough r and $\tau_M>0$.*

6. Numerical examples

In all examples, we choose $q=5$ which results in an unstable open-loop system. We consider four cases corresponding to Sections 3, 4, 5.1 and 5.2: S3- non-local actuation and measurement, S4- non-local actuation and boundary measurement, S5.1- boundary actuation with fast-varying τ_u and slow-varying τ_y with either $d=0$ or $d=0.3$ and S5.2- boundary actuation via dynamic extension with constant $\tau_u=r$ and fast-varying τ_y. In each case the kernels b and c are chosen according to Table 1, where

$$\begin{aligned}&\phi_1(x)=\sqrt{2}\chi_{[0.25,0.75]}(x),\\&\phi_2(x)=\sqrt{2}\left[(4x-1)\,\chi_{[0.25,0.5]}+(-4x+3)\,\chi_{(0.5,0.75]}\right].\end{aligned}$$

We choose $N_0=1$ and $\delta=0.5$. The gains L_0 and K_0 (see Table 1) were found from (3.10), (3.13) and (5.25). For $N\in\{4,5,6,7,8\}$ and various values of δ_0, the maximum value of τ_M (as shown in Table 2) was obtained by verifying the LMIs of Theorems 3.1, 4.1, 5.1 and 5.2. The presented values of N start from the smallest that guarantee the feasibility of LMIs. All LMIs are verified by using the standard Matlab LMI toolbox. It is seen from Table 2 that larger values of N lead to larger delays. We believe that the latter can be proved theoretically, but this is not in the scope of the present paper.

For simulations of solutions to the closed-loop systems, we choose the initial condition $z_0(x)=0.5x^2-1$. In S3 and S4 we consider network-based control. Given τ_M, we used $s_{k+1}-s_k\equiv 0.5\tau_M$ and $\rho_k=0.5\tau_M$ for the network between sensor and controller (see Fig. 1). For the network between controller and actuator, we used $t_{k+1}-t_k\equiv 0.5\tau_M$, whereas $\{\mu_k\}_{k=1}^{\infty}$ were randomly chosen in $[0.49\tau_M,0.5\tau_M]$. In S5.1, we consider one network between controller and actuator with the same t_k and μ_k as in S3 and constant $\tau_y\equiv\tau_M$, which corresponds to $d=0$. In S5.2 we consider one network between sensor and

Table 2
Maximum values of τ_M.

N	S3		S4		S5.1, $d=0$		S5.1, $d=0.3$		S5.2	
	δ_0	τ_M	δ_0	τ_M	δ_0	τ_M	δ_0	τ_M	δ_0	τ_M
4	6	0.023	–	–	0.5	0.014	0.5	0.008	4	0.0042
5	6	0.027	–	–	0.5	0.026	0.5	0.021	4	0.005
6	6	0.029	8	0.018	0.5	0.031	0.5	0.028	4	0.0053
7	6	0.031	9	0.021	0.5	0.034	0.5	0.03	4	0.0059
8	6	0.032	9	0.026	0.5	0.035	0.5	0.033	4	0.006

Table 3
Theoretical δ_τ vs. linear fits from simulations of solutions.

	N	δ_0	δ_1	τ_M	δ_τ	a_z	a_e
S3	4	6	5.5	0.023	0.3983	0.4418	0.4363
S4	6	8	7.5	0.018	0.3931	0.4401	0.4482
S5.1a	4	0.5	–	0.014	0.5	0.611	0.607
S5.2	4	4	3.5	0.0042	0.4856	0.5224	0.5301

controller with the same s_k and ρ_k as in S3 and constant input delay $r=\tau_M$. The norms $\|z_x(\cdot,t)\|_{L^2}$ and $\left\|z_x(\cdot,t)-\hat{z}_x(\cdot,t)\right\|_{L^2}$ for $t>0$ were estimated using $\left\|h'\right\|^2=\sum_{n=1}^{\infty}\lambda_n h_n^2$ with $\|z_x\|_{L^2}^2\approx\sum_{n=1}^{40}\lambda_n z_n^2$, whereas z_n were found from simulation of state ODEs. Similar truncation (with 40 coefficients) was done for the L^2-norm. In S5.1 we use the truncation $\sum_{n=1}^{40}\frac{1}{\lambda_n}z_n^2(t)$ to approximate $\sum_{n=1}^{\infty}\frac{1}{\lambda_n}z_n^2(t)$. The closed-loop systems were simulated for final time $t_f=10$. In each case, for both the state and the estimation error norms we compute linear fits versus time on a log-linear scale. The fits are denoted by $p_l(t)=a_l\cdot t+b_l,\quad l\in\{z,e\}$, respectively. The parameters for each case, as well as a_z and a_e are given in Table 3. Note that the decay rates obtained from simulations are close to the theoretical values of δ_τ. Furthermore, simulations of the solutions to the closed-loop system show that the maximum value of τ_M which preserves stability is 2–3 times larger than the delay bound found from the LMIs, meaning that our approach (employing simple Lyapunov functionals) is rather efficient.

7. Conclusion

We presented a design method for finite-dimensional observer-based control of a 1-D linear heat equation with fast-varying unknown input and known output delays. Based on modal decomposition, our approach was applied to the cases where at least one of the control or observation operators was bounded.

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