Brief paper

Stabilization by switching of parabolic PDEs with spatially scheduled actuators and sensors

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\textbf{A B S T R A C T}

This paper addresses a switched sampled-data control design for stabilization of Kuramoto-Sivashinsky equation under the Dirichlet/periodic boundary conditions with spatially scheduled actuators. It is supposed that discrete-time point-like or averaged measurements are available. The system is known to be stabilizable by static output-feedback employing several distributed in space actuators and sensors, but is not stabilizable by only one of the actuator–sensor pairs. Does there exist a switching stabilizing static output-feedback such that at all times, only one actuator–sensor pair is active? We give a positive answer and find the appropriate switching sampled-data control law. The proposed switching controller can be implemented either by \( N \) actuators and sensors placed in each subdomain (here switching control may reduce the energy that the system spends) or by using one actuator–sensor pair that can move to the active subdomain. For implementation of the control law by moving actuators and sensors, we take into account a moving time by treating it as an additional switching between the open-loop system and the closed-loop switched system. The guidance of active (or mobile) actuators and sensors is provided by using output-dependent switching. Constructive conditions are derived to ensure that the resulting closed-loop system is regionally stable by means of the Lyapunov approach. Numerical example illustrates the efficiency of the method.

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1. Introduction

In recent years, substantial efforts have been taken to develop switched control of partial differential equations (PDEs) (see e.g. Iftite and Demetriou (2009) and Zuazua (2010)). In Iftite and Demetriou (2009), optimal and switching policies of spatially scheduled actuators were suggested that were based on finite horizon Linear Quadratic Regulator problem. In Zuazua (2010) the following problem was formulated: assuming that one can control a system using two or more actuators, does there exist a control strategy such that at all times, only one actuator is active? The positive answer for the controllability of some classes of PDEs along with the corresponding switching laws was given in Zuazua (2010).

The switching control laws can be implemented either by stationary actuators or by one moving actuator that can move to the active subdomain in the negligible time (Iftime & Demetriou, 2009). In Butkovskiy and Pustyl’nikov (1987) and Demetriou (2010, 2012), moving actuators and sensors guiding laws were suggested for PDEs. Intermittent control of reaction–diffusion equation by time-dependent switching between all working pairs of collocated mobile actuators and sensors and the rest (all not working) has been studied in Wu and Zhang (2019). Furthermore, Wu and Zhang (2020) has dealt with the stabilization problem of linear reaction–diffusion equation with time-varying delay via the projection modification algorithm using collocated mobile actuators and sensors. In Zhao, Lin, and Xue (2012), switching control of closed quantum systems governed by Schrödinger equation has been proposed for the degenerate case via multiple Lyapunov function technique. Note that the mentioned above methods for switching control or for control by mobile actuators and sensors may be inefficient for the unstable open-loop systems, which motivates our study.
Control of Kuramoto-Sivashinsky equation (KSE) has attracted extensive attention owing to its wide range of applications (Armaou & Christofides, 2000; Christofides & Armaou, 2000; El-Farra, Lou, & Christofides, 2003). KSE models a variety of physical-chemical systems including falling liquid films, chemical reactors and interfacial instabilities in viscous flows (El-Farra et al., 2003). Active control of fluid flow which is modeled by KSE can be achieved by injection of polymers, mass transport through porous walls (e.g. blowing/suction) and application of electro-magnetic forcing. In these cases actuators and sensors are in-domain.

In-domain control of KSE with a large number of in-domain actuators and sensors has received significant research attention (Armaou & Christofides, 2000; Christofides & Armaou, 2000; El-Farra et al., 2003). Particularly, El-Farra et al. (2003) studied the motion of a liquid film falling down on a vertical wall, where in-domain control actuators were introduced to suppress the unstable behavior of the falling liquid films. There are also many fruitful works on the boundary control of KSE (see e.g. Cerpa (2010) and Cerpa and Mercado (2011)). In El-Farra et al. (2003) and Ghantasala and El-Farra (2012), for KSE, a switching law reconfigured the control actuators following fault detection. It should be noticed that stabilization by switching of open-loop unstable PDE with several actuators, where the system is not stabilizable by using only one actuator, has not been achieved yet. Thus, the design of a switching controller for open-loop unstable parabolic PDEs is a challenging topic.

Stabilization of unstable systems by switching is a classical, challenging and well-studied problem for ordinary differential equations (ODEs) (Liberzon, 2003). Thus, for linear switched systems with a stable convex combination, stabilization can be achieved by state or output-dependent switching (see e.g. Deaecto (2016) and Hetel and Fridman (2013) and the references therein). The key idea of switching control design for PDEs is to schedule the position of the actuator and sensor in order to achieve the control goal.

This paper addresses a switched sampled-data control design for stabilization of KSE under the Dirichlet/periodic boundary conditions with spatially scheduled actuators. It is supposed that discrete-time point-like or averaged measurements are available. The system is known to be stabilizable by static output-feedback employing several distributed in space actuators and sensors, but is not stabilizable by only one of the actuator–sensor pairs. Does there exist a stabilizing switching static output-feedback such that at all times, only one actuator–sensor pair is active? We give a positive answer and find the appropriate switching sampled-data control law. The proposed switching static output-feedback can be implemented either by N actuators and sensors placed in each subdomain (here the switching control may reduce the energy that the system spends) or by using one actuator–sensor pair that can move to the active subdomain. The guidance of active (or mobile) actuators and sensors is provided by using output-dependent switching. Constructive conditions are derived to ensure that the resulting closed-loop system is regionally stable by means of the Lyapunov approach. For implementation of the control law by moving actuators and sensors, we take into account a moving time by treating it as an additional switching between the open-loop system and the closed-loop switched system. Consistent simulation results that support the proposed theoretical statements are provided. Preliminary results on global stabilization by switching of 1-D semilinear heat equation under averaged measurements were presented in Kang, Fridman, and Liu (2021).

This work is organized as follows. Section 2 presents some preliminaries. In Sections 3 and 4, the switching control strategy for KSE is proposed under the point-like measurements and main theoretical results are presented, whereas extensions to the case of periodic boundary conditions and the case of averaged state measurements are presented in Section 5. Section 6 provides simulation results and Section 7 gives conclusions.

2. Mathematical preliminaries

Notation. Throughout the paper the support of a function $g$ is denoted by supp$g$, and conv(supp$g$) represents the convex hull of supp$g$. $L^2(0,L)$ stands for the Hilbert space of square integrable scalar functions $f(x)$ on $(0,L)$ with the corresponding norm $||f||_{L^2(0,L)} = \left\{\int_0^L f^2(x)dx\right\}^{\frac{1}{2}}$. $L^{\infty}(0,L)$ denotes the space of essentially bounded function $f(x)$ on $(0,L)$ with the corresponding norm $||f||_{L^{\infty}(0,L)} = \sup_{x \in [0,L]} |f(x)|$. The Sobolev space $H^1(0,L)$ with $k \in \mathbb{Z}$ is defined as $H^k(0,L) = \{f : f^{(a)} \in L^2(0,L), \forall 0 \leq |a| \leq k \}$ with norm $||f||_{H^k(0,L)} \equiv \left\{(\sum_{|j| \leq k} ||f^{(j)}||_{L^2(0,L)}^2)^{\frac{1}{2}}\right\}$. Moreover, $H^0(0,L) = \{f \in H^1(0,L) | f(0) = f'(0) = \ldots = f^{(k-1)}(0) = 0, f(L) = f'(L) = \ldots = f^{(k-1)}(L) = 0\}$. $\mathbb{Z}^+$ denotes the set of nonnegative numbers.

Lemma 2.1 (Poincaré’s Inequality Fridman & Bar Am, 2013; Hardy, Littlewood, & Polya, 1988). For $a < b$, let $g \in H^1(a,b)$ be a scalar function with $\int_a^b g(x)dx=0$. Then $\|g\|^2_{L^2[a,b]} \leq \frac{b-a}{\pi} \int_a^b |g''(x)|^2dx$.

Cauchy-Schwarz’s inequality leads to the next lemma:

Lemma 2.2 (Jensen’s Inequality Fridman, 2014). For $a < b$, let $c : [a,b] \rightarrow [0,\infty)$ and $g : [a,b] \rightarrow \mathbb{R}$ be such that the integration concerned is well defined. Then $\int_a^b c(x)g(x)dx \leq \left[\int_a^b c(x)dx\right]^{\frac{1}{2}} \left[\int_a^b c(x)g(x)^2dx\right]^{\frac{1}{2}}$.

Lemma 2.3 (Sobolev’s Embedding and Inequality). The embedding $H^1(0,L) \subset C([0,L])$ is compact and for any $z \in H^1_0(0,L)$, it holds $\|z\|_{L^\infty(0,L)} \leq \sqrt{T}\|z\|_{L^2(0,L)}$.

3. Problem formulation

Let $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots$, $\lim_{k \rightarrow \infty} t_k = \infty$ be a sequence of sampling instants. Consider the following plant governed by KSE:

$$\begin{align*}
z_t(x,t) + z_{xx}(x,t) + v z_{x\sigma}(x,t) + z(x,t) z_{x\sigma}(x,t) &= b_{x\sigma}(x)u_{x\sigma}(t), \quad x \in [0,L], \quad t \in [t_k,t_{k+1}], \\
z(x,0) &= z_0(x),
\end{align*}$$

where $k \in \mathbb{Z}^+$, under the Dirichlet boundary conditions:

$$z(0,t) = z(L,t) = 0, \quad z_0(0) = z_0(L) = 0, \quad t > 0.$$  

(3.2)

Here $z(x,t)$ is the state of KSE, $z_0(x)$ is the initial state, and $u_{x\sigma}(t)$ is the control input. The switching function $\sigma_k : k \in \mathbb{Z}^+ \rightarrow \{1, \ldots, N\}$ selects at each sampling time $t_k$ one of the $N$ available actuators corresponding to the shape function $b_{x\sigma}(x)$ that will be shortly defined. We make the following assumptions:

- Spatial sampling: Similar to Azouani and Titi (2014), Fridman and Bar Am (2013), Fridman and Bligheovsky (2012) and Lunasin and Titi (2017), we assume that the points $0 = x_0 < x_1 < \ldots < x_N = L$ divide $[0,L]$ into $N$ equal-length subintervals $\Omega_j = [x_{j-1}, x_j)$ such that $\cup_j \Omega_j = [0,L]$, and $|\Omega_j| = \frac{1}{N}$ meaning that all the subintervals have the same length. The shape functions $b_{x\sigma}(x)$ are chosen to be characteristic functions $b_{x\sigma}(x)$ of $\Omega_j$, as follows:

$$\begin{align*}
b(x) &= \begin{cases}
0, & x \notin \Omega_j, \\
1, & \text{otherwise},
\end{cases} \quad j = 1, \ldots, N.
\end{align*}$$

(3.3)

- Time sampling: The length of sampling subintervals in time is supposed to be uniformly bounded:

$$0 < h_0 \leq t_{k+1} - t_k \leq h, \quad \forall k \in \mathbb{Z}^+.$$  

(3.4)

- Moving time: The moving time $\delta \in (0,h_0)$ for sensors and actuators to the appropriate domain $\Omega_{x\sigma}$ is taken into account.
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We first consider sensors that provide the discrete-time point-like measurements:

\[ y_j(t_k) = \int_{\Omega_j} c_j(x) z(x, t_k) dx, \quad k \in \mathbb{Z}_+ \]  

(3.5)

with

\[ 0 \leq c_j \in L^2(\Omega_j), \quad \int_{\Omega_j} c_j(x) dx = 1, \]

(3.6)

\[ c_j(x) = \begin{cases} 
1, & \text{if } x \in \Omega_j, \\
0, & \text{otherwise},
\end{cases} \quad j = 1, \ldots, N, \]

(3.7)

where \( \Omega_j \) is a subinterval of \( \Omega \) with the length \( \epsilon \) independent of \( j \) (see Fig. 1, where \( x_j \) is the midpoint of \( \Omega_j \)).

We will also consider the averaged state measurements

\[ y_j(t_k) = \frac{\int_{\Omega_j} z(x, t_k) dx}{|\Omega_j|} = \frac{N}{\epsilon} \int_{\Omega_j} z(x, t_k) dx, \quad j = 1, \ldots, N, \quad k \in \mathbb{Z}_+. \]

(3.8)

Differently from point-like measurements, the sensors in the case of averaged measurements cover the whole subdomain. However, our method under the averaged measurements leads to a smaller number of actuators and sensors or allow larger sampling in time (see comparison in Table 1 in Section 6)).

Note that the presented method under the averaged measurements can be extended to \( N = D \) PDEs for any \( N \) on the basis of static output-feedback without switching presented in Bar and Fridman (2014), whereas such extension under the point-like measurements is questionable (see Selivanov and Fridman (2019), where non-switched static output-feedback for heat equation under point-like measurements is confined to \( N \leq 2 \)).

For both measurements, our aim is to find a sampled-data switching law and a sampled-data regionally exponentially stabilizing controller for KSE (3.1) implemented by zero-order hold device. As already mentioned, in this paper we will take into account the moving time \( \delta \) for sensors and actuators. For the actuators moving time, we consider additional switching between the open-loop system (when the actuator is moving) during the part of the sampling interval and the closed-loop switched system during the remaining part of the interval, where

\[ u_{\alpha_j}(t) = \begin{cases} 
0, & \text{if } t \in [t_k, t_k + \delta), \\
-K\alpha_j(t_k), & \text{if } t \in [t_k + \delta, t_{k+1})
\end{cases} \]

(3.9)

with some \( K > 0 \). The switching signal \( \alpha_j \) is calculated at time \( t_k \), whereas it takes \( \delta \) seconds for actuators and sensors to move to the domain \( \Omega_j \).

Our main objective is to find an appropriate output-depending switching law. Denote the characteristic function of the time interval \([t_k, t_k + \delta] \) by \( \chi_{[t_k, t_k + \delta]}(t) \). Consider first the case of the averaged state measurements (3.8), where the closed-loop system (3.1), (3.9) has the form

\[ z_i(x, t) + z_{\alpha}(x, t) + v z_{\alpha max}(x, t) + z(x, t) z_{\alpha}(x, t) = 0, \quad x \in (0, L), \quad t \in [t_k, t_{k+1}) \]

subject to (3.2). Note that if \( b_{\alpha_k}(x) u_{\alpha_j}(t) \) in (3.1) is changed by \( \sum_{j=1}^N b_j(x) u_j(t) \), then there exists a \( K > 0 \) that regionally exponentially stabilizes the system by \( u_j(t) = -K\eta_j(t) \) (Kang & Fridman, 2018). The latter means that the average of systems (3.1) with \( b_{\alpha_k}(x) u_{\alpha_j}(t) \) changed by \( b_j(x) u_j(t) \) is stabilizable by the static output-feedback (3.9). Similar to state-dependent switching for ODEs in the case of stable convex combination of systems (Hetel & Fridman, 2013), we will define a min-type switching function by using the corresponding Lyapunov function \( V(t) \) according to

\[ \alpha_j = \arg \min_{\alpha_k} V(t) \]

for \( t \in [t_k + \delta, t_{k+1}) \) along the closed-loop system. Thus, for

\[ V(t) = \int_0^L z^2(x, t) dx \]

we have

\[ V(t) = -\frac{KN}{\epsilon} \int_{\Omega_j} z(x, t_k) dx \int_{\Omega_j} z(x, t_k) dx \]

that leads (for small enough \( h \)) to

\[ \arg \min_{\alpha_j} V(t) = \arg \min_j \left[ -\int_{\Omega_j} z(x, t_k) dx \int_{\Omega_j} z(x, t_k) dx \right] \]

i.e. to the following discrete-time switching law:

\[ \alpha_j = \arg \max_j \left[ \int_{\Omega_j} z(x, t_k) dx \right]^2. \]

(3.11)

Similarly to (3.11) for the point-like measurements we choose

\[ \alpha_j = \arg \max_j \left[ \int_{\Omega_j} c_j(x) z(x, t_k) dx \right]^2. \]

(3.12)

Our sampled-data switching law (3.12) with (3.4) and \( \lim_{t \to \infty} t_k = \infty \) rules out the possibility of Zeno behavior. Note that (3.12) is calculated at time \( t_k \). The law (3.12) means that the \( \alpha_j \)-th mode is active if

\[ \int_{\Omega_j} c_j(x) z(x, t_k) dx \leq \left[ \int_{\Omega_{\alpha_j}} c_{\alpha}(x) z(x, t_k) dx \right]^2. \]

(3.13)

4. Main results

In this section, we will analyze the well-posedness and regional exponential stability of the system (3.1) under the static output-feedback (3.9) and the switching law (3.9) (where \( c_j = 1 \) in the case of averaged measurements).

4.1. Well-posedness of the controlled system

We establish the existence, uniqueness and regularity of the system (3.1) under the switching control law (3.9), (3.12) and Dirichlet boundary conditions (3.2) by using the step method (see e.g. Section 1.2 in Fridman (2014)). For the switching law (3.9), (3.12), we assume that the \( \alpha_j \)-th mode is active. We first consider \( t \in [0, \delta] \). Then (3.1), (3.2) becomes

\[ \begin{align*}
z_i(x, t) + z_{\alpha}(x, t) + v z_{\alpha max}(x, t) + z(x, t) z_{\alpha}(x, t) &= 0, \\
x &\in (0, L), \quad t \in [0, \delta], \\
z(0, t) &= z(L, t) = 0, \quad z_i(0, t) = z_i(L, t) = 0, \\
z(x, 0) &= z_0(x).
\end{align*} \]

(4.1)

Define the system operator \( A : D(A) \subset L^2(0, L) \to L^2(0, L) \) as follows:

\[ A f = -v \frac{\partial f}{\partial x}. \]

\[ D(A) = H^1(0, L) \cap H_0^2(0, L). \]

It is well-known that \( A \) is a dissipative operator, and \( A \) generates an analytic semigroup. Operator \( -A \) is positive implying that its
square root $(\cdot)^{\frac{1}{2}}$ is also positive. Moreover, $D((-A)^{\frac{1}{2}}) = H_0^2(0, L)$ with the norm $\|f\|_{D((-A)^{\frac{1}{2}})} = v^2 \|f''(\cdot)\|_{L^2(0, L)}$. Then the system (4.1) can be represented as an evolution equation:

$$
\begin{align*}
\frac{d}{dt}z(\cdot, t) &= Az(\cdot, t) + F(z(\cdot, t)), \\
z(\cdot, 0) &= z_0(\cdot),
\end{align*}
$$

(4.2)

where the nonlinear term $F$ is defined on function $z(\cdot, t)$ according to

$$
F(z(\cdot, t)) = -z(x, \cdot)\bar{z}(x, t) - z\alpha(x, t), \quad t \in [0, \delta].
$$

It should be noticed that the nonlinear term $F$ is locally Lipschitz continuous, that is, there exists a positive constant $l(M)$ such that

$$
\|F(z_1) - F(z_2)\|_{L^2(0, L)} \leq l(M)\|z_1 - z_2\|_{H_0^2(0, L)}
$$

holds for $z_1, z_2 \in H_0^2(0, L)$ with $\|z_1\|_{H_0^2(0, L)} \leq M$, $\|z_2\|_{H_0^2(0, L)} \leq M$.

Therefore, Theorem 3.3.3 of Henry (1981) is applicable to (4.2).

For any initial condition $z_0 \in H_0^2(0, L)$, there exists a unique local strong solution of (4.2) on some interval $[0, T] \subset [0, \delta]$, where $T = T(z_0) > 0$.

$$
z \in C([0, T]; H_0^2(0, L)) \cap L^2([0, T]; D(A)),
$$

$z \in L^2([0, T]; L^2(0, L)).$

From Theorem 6.23.5 of Krasnoselskii, Zabreiko, Pustyl’nyi, and Sobolevskii (1976), we obtain that if the solution admits a priori estimate (i.e. bounded), then it exists on the entire interval $[0, \delta]$.

The a priori estimate on the solutions will be guaranteed by the conditions that we will provide (see Theorem 1).

For $t \in [\delta, t_1]$, the system (3.1) under the switching control law (3.9), (3.12) can be also written in the form of (4.2) with the following nonlinearity

$$
F(z(\cdot, t)) = -z(x, t)\bar{z}(x, t) - z\alpha(x, t) - Kb_{\alpha}(x)\int_{\Omega}c_\omega(x)z_\omega(x, t)dx, \quad t \in [\delta, t_1].
$$

Since $F$ is locally Lipschitz continuous, the same line of reasoning is applied to the time interval $[\delta, t_1]$. The strong solution exists on $[\delta, t_1]$ due to a priori estimate on the solutions starting from the domain of attraction, which is guaranteed by the stability conditions of Theorem 1.

### 4.2. Stability analysis of the switched system

By the mean-value theorem, from (3.6) it follows that there exists $x_0^* \in \text{conv supp}(\nu^\prime)$ (see Fig. 1) such that

$$
\int_{\Omega_\delta} c_\omega(x)z_\omega(x, t)dx = z(x_0^*, t), \quad t \in [t_\delta, t_{k+1}].
$$

Denote

$$
f_1(x, t) = z(x, t) - z(x_0^*, t), \quad t \in [t_\delta, t_{k+1}].
$$

(4.3)

$$
\rho_1(t) = \int_{\Omega_\delta} c_\omega(x)z_\omega(x, t)ds, \quad t \in [t_{k+1}].
$$

(4.4)

Then the switching controller (3.9) can be rewritten as

$$
u_\delta(t) = \begin{cases}
0, & t \in [t_\delta, t_\delta + \delta], \\
-K[z(x_\delta, t) - f_\delta(x_\delta, t) - \rho_\delta(t)], & t \in [t_\delta + \delta, t_{k+1}].
\end{cases}
$$

(4.5)

whereas the switching law chooses $\sigma_k$ that satisfies

$$
\int_{\Omega_\delta} [z(x, t) - f_\delta(x, t) - \rho_\delta(t)]^2 dx
\leq \int_{\Omega_\delta} [z(x, t) - f_\delta(x, t) - \rho_\delta(t)]^2 dx, \quad j = 1, 2, \ldots, N.
$$

(4.6)

Thus, under the controller (4.5), the closed-loop system becomes

$$
z_{t_\delta}(x, t) + z\alpha(x, t) + v_{\text{ssxx}}(x, t) + z(x, t)\bar{z}(x, t) = -Kb_{\alpha}(x) \left[1 - \chi_{[t_\delta, t_{k+1}]}(t)\right] z(x, t) - f_\delta(x, t) - \rho_\delta(t),
$$

(4.7)

$x \in (0, L), \quad t \in [t_\delta, t_{k+1}],$

subject to (3.2), (3.12).

Note that (3.1) may be not stabilizable with a desired decay rate by the non-switched control. The challenge in the stability analysis is to take efficiently into account the switching condition (3.12) in order to derive feasible stability conditions (see (4.24) below and the resulting expression in (4.25)).

Now we focus on the stability of the closed-loop system that switches at times $t_k$ and $t_k + \delta$. Consider the following Lyapunov-Krasovskii functional:

$$
V(t) = V_1(t) + V_2(t) + V_3(t), \quad t \in [t_k, t_{k+1}]
$$

(4.8)

where

$$
\begin{align*}
V_1(t) &= \int_0^t \nu_1^2(z(x, t))dt, \\
V_2(t) &= \int_0^t \nu_2^2(z_\alpha(x, t))dt, \\
V_3(t) &= \int_0^t \nu_3^2[z_{\text{ssxx}}(x, t)]dt.
\end{align*}
$$

(4.9)

with $P_1 > 0, P_2 > 0, R > 0$. Here $\rho_\delta(s)$ is the derivative of $\rho(s)$ with respect to $s$. By Wirtinger’s inequality, $V_3(t)$ is non-negative (see Lemma 1 in Selivanov and Fridman (2013)), it does not grow in the switching times $t_k$ and it is continuous in the switching times $t_k + \delta$. Moreover, $V_k$ extends the corresponding terms in Selivanov and Fridman (2019) to Wirtinger-based Lyapunov functional.

For $z(\cdot, t) \in H_0^2(0, L)$ we define

$$
\|z(\cdot, t)\|_{L^2} = P_1\|z(\cdot, t)\|_{L^2(\Omega_\delta)} + P_2\|\nu_\delta(\cdot, t)\|^2_{L^2(\Omega_\delta)}
$$

(4.10)

with $P_1 > 0, P_2 > 0$.

Remark 4.1. In order to find a bound on the domain of attraction for closed-loop system (4.7) subject to (3.2), we use positive invariance principle in Theorem 1: we show that if $\rho_\delta < 0$, $\Psi_1 < 0$ and $\Psi_2 < 0$, where $\Psi_0, \Psi_1, \Psi_2$ are given by (4.12)–(4.14), then $V(t) \leq V(0)$ for all $t \geq 0$. Matrices $\Psi_0, \Psi_1, \Psi_2$ are affine in $x$.

Let $C > 0$ be the upper bound of $z_\omega, \text{i.e.}, \max_{x \in [0, L]} |z_\omega(x, t)| \leq C$ for all $t \geq 0$. Then it is sufficient to verify the matrix inequalities $\Psi_0 < 0, \Psi_1 < 0$ and $\Psi_2 < 0$ in the vertices $z_\omega = \pm C$ (see (4.9)–(4.11)).

The following result provides sufficient stability conditions in the form of linear matrix inequalities (LMIs) for the closed-loop system (4.7), (3.2), (3.12):

**Theorem 1.** Consider the closed-loop system (4.7) subject to (3.2) and the switching law (3.12). Given positive scalars $h, K$ and tuning parameter $C > 0$, $c_0 > 0$ such that $\alpha h_0 > (c_0 + \alpha)h$, let there exist scalars $R > 0, P_1 > 0, \lambda_\gamma > 0 (n = 1, 2, 3)$ that satisfy the inequalities:

$$
\begin{align*}
\Psi_1 &< 0, \\
\Psi_2 &< 0, \\
\Psi_3 &< 0,
\end{align*}
$$

(4.11)

where

$$
\Psi_1 = \begin{bmatrix}
\psi_{11} & \psi_{12} & \psi_{13} & \lambda_1 & 0 & 0 \\
\psi_{12} & -P_2 & 0 & 0 & 0 & 0 \\
\psi_{13} & -P_1 & 0 & 0 & 0 & 0 \\
-\lambda_2 & -\lambda_2 & -\lambda_2 & -\lambda_2 & 0 & 0 \\
\lambda_1 & 0 & 0 & 0 & -\lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & -\lambda_1
\end{bmatrix}
$$

(4.12)
\[
\Psi_2 = \begin{bmatrix}
\tilde{\psi}_{11} & \tilde{\psi}_{12} & \tilde{\psi}_{13} & P_3K - \lambda_1 & P_3K - \lambda_1 \\
* & \tilde{\psi}_{22} & -P_2K & P_2K & 0 \\
* & * & \tilde{\psi}_{33} & \lambda_1 - \lambda_2 & 0 \\
* & * & * & \lambda_1 & -\lambda_1 - Re^{-2\mu h} \\
-2\alpha_0P_1 & 0 & -P_1 - P_4 & P_4 & -2\alpha_0P_1v - 2P_4v
\end{bmatrix},
\]
(4.13)
\[
\Psi_0 = \begin{bmatrix}
-\lambda_2 & 0 & 0 & 0 & 0 \\
0 & -P_1 - P_2z_k & 0 & P_2 & 0 \\
0 & 0 & -P_3 - P_2z_k & 0 & P_2 \\
0 & 0 & 0 & -P_3 - P_2z_k - 2P_4 & 0 \\
0 & 0 & 0 & 0 & -P_3 - P_2z_k - 2P_4K
\end{bmatrix},
\]
(4.14)

\[
\psi_{11} = 2\alpha_1P_1 - \lambda_2\frac{1}{2}x_1^2 - \frac{\lambda_1}{N^2},
\]
\[
\psi_{12} = P_1 - P_3 - P_2z_k,
\]
\[
\psi_{13} = -P_3 - \frac{\lambda_2}{2}\frac{1}{2}\frac{x_2^2}{2},
\]
\[
\psi_{22} = R\frac{\alpha_0^2}{\pi^2} + 2P_2,
\]
\[
\psi_{33} = 2\alpha_0P_2v - 2P_2v + \lambda_3,
\]
\[
\tilde{\psi}_{11} = 2\alpha_1P_1 - \lambda_2\frac{1}{2}x_1^2 - 2P_3K + \lambda_1,
\]
\[
\tilde{\psi}_{12} = P_1 - P_3 - P_2z_k - P_3K.
\]

Let \( \alpha_1 \) be subject to
\[
0 < \alpha_1h_0 \leq \alpha h_0 - (\alpha_0 + \alpha)\delta.
\]
(4.15)

Then for any initial function \( z_0 \in H^2(0, L) \) that satisfies the bound
\[
\|z_0\|_V < \sqrt{\frac{\rho V}{\epsilon C}}.
\]
the closed-loop system (4.7) subject to (3.2) and (3.12) is exponentially stable with a decay rate \( \alpha_1 \), i.e. the following holds
\[
\|z(\cdot, t)\|_V^2 \leq V(t) \leq \epsilon^{-2\alpha_1(t-h_0)+2\alpha_2\delta}V(0).
\]

**Proof.** **Step 1:** Let us only highlight that there exists a unique local strong solution of (4.1) on some interval \( [0, T] \subset [0, \delta] \), where \( T = T(z_0) \). Due to Theorem 6.23.5 of Krasnoselskii et al. (1976), the solution exists on the entire interval \( [0, \delta] \) provided it is bounded. Therefore, by applying the same arguments at \( \delta, \delta_1, \) and any step \( k \in \mathbb{N} \), one can conclude that the strong solution exists for all \( t \geq 0 \).

**Step 2:** Assume formally that strong solution of (4.7) subject to (3.2) starting from \( \|z_0\|_V < \sqrt{\frac{\rho V}{\epsilon C}} \) exists for all \( t \geq 0 \). We first derive sufficient LMI-based conditions to guarantee that \( \dot{\psi} + 2\alpha_1V \leq 0 \) for \( [\delta, \delta+\epsilon, \delta+\epsilon+1] \). Differentiating \( V(t) \) along the solution of the closed-loop system and integrating by parts, we obtain
\[
\dot{V}(t) + 2\alpha_1V(t) = 2P_1\int_0^t z[x(t), t]z_t[x(t), t]dx
\]
\[
+2P_2v\int_0^t z^2[x(t), t]dx + 2P_3v\int_0^t z^2[x(t), t]dx + 2P_4v\int_0^t z^2[x(t), t]dx
\]
\[
+R\frac{\alpha_0^2}{\pi^2} \sum_{j=1}^N \int_{\Omega_j} [\varphi_j(t)]^2 \|z_t[x, t]\|_V^2 dx - Re^{-2\mu h} \sum_{j=1}^N \int_{\Omega_j} [\varphi_j(t)]^2 \|z_t[x, t]\|_V^2 dx.
\]
(4.16)

Jensen's inequality leads to
\[
\int_{\Omega_j} [\varphi_j(t)]^2 \|z_t[x, t]\|_V^2 dx \leq \left( \frac{\lambda_1}{\pi^2} \int_{\Omega_j} \varphi_j(x)z_t^2(x, t)dx \right) \left( \int_{\Omega_j} \varphi_j(x)z_t^2(x, t)dx \right) \leq \frac{\lambda_1}{\pi^2} \int_{\Omega_j} z_t^2(x, t)dx.
\]
(4.17)

Note that \( \int_{\Omega_j} [\varphi_j(t)]^2 \|z_t[x, t]\|_V^2 dx \geq \frac{\lambda_1}{\pi^2} \int_{\Omega_j} z_t^2(x, t)dx \)

Then, application of Wirtinger's inequality yields
\[
\int_{\Omega_j} [\varphi_j(t)]^2 \|z_t[x, t]\|_V^2 dx \leq \frac{\lambda_1}{\pi^2} \int_{\Omega_j} z_t^2(x, t)dx.
\]
(4.18)

Moreover, we obtain
\[
\|z(\cdot, t)\|_V^2 \leq \left( \frac{\lambda_1}{\pi^2} \right)^2 \|z_{x\alpha}(\cdot, t)\|_V^2.
\]
(4.19)

Therefore, (4.6), (4.18) and (4.19) lead to
\[
-\lambda_1^\mu \sum_{j=1}^N \int_{\Omega_j} [z(x, t) - f_j(x, t) - \varphi_j(t)]^2 dx + \lambda_1 \int_{\Omega} [z(x, t) - f_{\Omega}(x, t) - \varphi_{\Omega}(t)]^2 dx \geq 0,
\]
\[
\lambda_2 \left( \frac{\lambda_1^\mu + \epsilon^2}{\pi^2} \right) \int_{\Omega} [z(x, t) - f_{\Omega}(x, t) - \varphi_{\Omega}(t)]^2 dx \geq 0
\]
(4.21)
\[
\lambda_3 \int_{\Omega} [z(x, t) - f_{\Omega}(x, t) - \varphi_{\Omega}(t)]^2 \|z_t(x, t)\|_V^2 dx \geq 0.
\]
(4.22)

We further apply the descriptor method (see Section 3.5 in Fridman (2014)), where the left-hand side of the following equation
\[
\int_0^t [P_1z(x, t) + P_2z_k(x, t)][z_t(x, t) - z_{x\alpha}(x, t) - z_k(x, t) - z_{x\alpha}(x, t) - \varphi_{\Omega}(x, t) - \varphi_{\Omega}(x, t)]dx = 0
\]
(4.23)

with some \( P_2 > 0 \) is added to \( \dot{V} \). Then adding the left-hand sides of (4.20)–(4.22) to (4.16) and taking into account (4.17), we obtain
\[
\dot{V}(t) + 2\alpha_1V(t) = 2P_1 \sum_{j=1}^N \int_{\Omega_j} z(x, t)z_t(x, t)dx
\]
\[
+ \left( 2\alpha_0P_1 - \lambda_3 \frac{\pi^2}{L^2} \right) \sum_{j=1}^N \int_{\Omega_j} z_t^2(x, t)dx
\]
\[
- 2P_3 + \lambda_3 \frac{(\lambda_1^\mu + \epsilon^2)}{\pi^2} \sum_{j=1}^N \int_{\Omega_j} z_t^2(x, t)dx
\]
\[
+ (2\alpha_0P_2v - 2P_2v + \lambda_3) \sum_{j=1}^N \int_{\Omega_j} z_t^2(x, t)dx
\]
\[
+ \left( \frac{4h^2L}{\pi^2N^2} - 2P_2 \right) \sum_{j=1}^N \int_{\Omega_j} z_t^2(x, t)dx
\]
\[
- 2P_4 \sum_{j=1}^N \int_{\Omega_j} z_t(x, t)[z(x, t) - z_{x\alpha}(x, t) + z_k(x, t)]dx + 2P_4 \sum_{j=1}^N \int_{\Omega_j} z_t^2(x, t)dx
\]
\[
+ \left( 2\alpha_0P_1 - \lambda_3 \frac{\pi^2}{L^2} \right) \int_{\Omega} z_t^2(x, t)dx
\]
(4.24)
Under the assumption (4.26), we have

\[ V(t) \leq e^{2\alpha(t-t_0)}V(t_0), \quad \forall t \in [t_k + \delta, t_{k+1}). \]

Since \( \alpha < \alpha' \) and \( t_{k+1} - t_k \geq h_0, \) (4.15) implies

\[(\alpha_1 - \alpha)(t_{k+1} - t_k) \leq (\alpha_1 - \alpha)h_0 \leq -(\alpha_0 + \alpha')\delta, \]

which together with (4.28) leads to

\[ V(t_{k+1}) \leq e^{2\alpha(t_{k+1}-t_k-\delta)}V(t_k) \leq e^{-2\alpha(t_{k+1}-t_k)}V(t_k). \]  

From (4.28) it follows

\[ V(t) \leq e^{2\alpha(t-t_0)+2\alpha\delta}V(0), \quad \forall t \geq 0. \]

The latter bound guarantees the existence of these strong solutions for all \( t \in [0, t_*]. \) Then using step method (Fridman, 2014), we conclude that the strong solution exists for all \( t \geq 0. \)

We will prove next that (4.26) holds. On one hand, for \( t = 0, \) inequality (4.26) holds by hypothesis in Theorem 1. On the other hand, let \( (4.26) \) be false for some \( t \geq 0 \) and let \( t^* \) be the smallest instant such that \( V(t^*) \geq \frac{\lambda^2}{\delta}C^2. \) Since \( V \) is continuous in time, we have \( V(t^*) = \frac{\lambda}{\delta}C^2 \) and \( V(t) < \frac{\lambda}{\delta}C^2 \) for \( t \in [0, t^*]. \) Since \( z_0(0, t) = 0, \) the Sobolev inequality (Lemma 2.3) implies:

\[ \|z_0(x, t)|^2 \leq \Omega_1 \|z_0(x, t)|^2 \leq \frac{1}{\lambda^2}V(t) \leq \frac{1}{\lambda^2}V(0) = \frac{1}{\lambda^2}\|z_0(0)|^2 < C^2 \]  

for \( t \in [0, t^*]. \) Thus, the feasibility of LMIs (4.9)-(4.11) guarantees that (4.27) is true for all \( t \in [0, t^*]. \) Hence, by continuity, \( V(t) \leq V(0) < \frac{\lambda}{\delta}C^2 \) for all \( t \in [0, t^*], \) which contradicts the definition of \( t^* \). Therefore, (4.26) holds.

Remark 4.2. The LMIs in Theorem 1 are always feasible for appropriate decision variables and small enough \( \alpha, \alpha_0, h, \delta, C, \) \( \varepsilon, P_2 \) and large enough \( N \) such that \( P_2K \) and \( \Omega_1 \) are small. Indeed, consider \( \Psi_1, \Psi_2 \) and \( \Phi_0 \) given by (4.12), (4.13) and (4.14) respectively. We will show that strict inequalities (4.9)-(4.11) hold with \( \alpha = h = C = 0. \) Then LMIs (4.9)-(4.11) hold with small enough \( \alpha > 0, h > 0 \) and \( C > 0. \) Set \( R = \lambda_2 > 2K, \) \( P_1 = P_2 = P_3 = 1, \) \( \lambda_1 = K \) such that \( \psi_1 = \psi_2 = 0 \) and \( P_2K - \lambda_1 = 0. \) By applying Schur complement, for small enough \( h \) and large enough \( N, \) we obtain

\[ \psi_1|_{x=0} < 0 \iff \begin{bmatrix} \psi_{11} + \frac{2\alpha}{\lambda_2 + 2\pi^2} & \psi_{13} \\ \psi_{13} & \psi_{33} + \frac{P_2}{2} \end{bmatrix} < 0, \]

\[ \psi_{21} < 0 \iff \lambda_2 - 2K > P_2K, \]

\[ \begin{bmatrix} \psi_{11} + \frac{P_2K}{2 - 2\alpha^2} & \psi_{13} + \frac{P_2K}{2 - 2\alpha^2} \\ \psi_{13} + \frac{P_2K}{2 - 2\alpha^2} & \psi_{33} + \frac{P_2}{2} \end{bmatrix} < 0, \]

\[ \psi_{01} < 0 \iff \begin{bmatrix} -\alpha_0 & 0 & -2P_2 & 0 \\ 0 & -1 & 0 & -2P_2 & 0 & 2v \\ 0 & 0 & -2\alpha_0 & 0 & 0 & 2P_2 \end{bmatrix} < 0, \]

where

\[ \psi_{11} = -\lambda_3 \frac{\pi^4}{4} - \frac{K}{N - 1}, \quad \psi_{13} = -\lambda_3 \frac{\pi^2}{4} - K, \quad \psi_{33} = -2v + \lambda_3, \quad \psi_{13} = -\lambda_3 \frac{\pi^4}{4} - K. \]
Then the latter LMIs are feasible with appropriate \( v - \sqrt{v^2 - \frac{1^2}{\pi^2}} < \lambda_3 < v + \sqrt{v^2 - \frac{1^2}{\pi^2}} \), small enough \( \alpha_0, \varepsilon, P_2 \) and large enough \( N \) such that \( P_2 \lambda^2 \epsilon = \frac{1}{\pi^2} \) are small.

**Remark 4.3.** Technically our Lyapunov-based approach employs the following novel tools compared to Kang and Fridman (2018): (a) The inequality (3.13) (that results from the switching law) and (4.20) that allow to derive feasible LMIs. (b) For the moving time we have derived a new LMI that guarantees a bound on \( V \) in step 3 of Theorem 1 proof (see p.7) and the bound on the dwelling time \( \delta \) in step 4. (c) For point-like measurements that were not considered in Kang and Fridman (2018), we use a mean value theorem ((4.3)) and further Wirtinger’s inequality in (4.18), which is different from the direct application of the Poincaré inequality in Kang and Fridman (2018) for the averaged measurements. Moreover, we employ the Wirtinger-based Lyapunov functional for sampled-data control with \( \rho_j \) which includes \( \zeta_j \), which leads to simpler LMIs comparatively to Lyapunov functional of Kang and Fridman (2018) with \( \zeta_j = 1 \).

Under the averaged state measurements (3.8) we have the following result:

**Theorem 2.** Consider the closed-loop system (3.10) subject to (3.2) and the switching law (3.12) with \( \zeta_j = 1 \). Given positive scalars \( h, \alpha, K \) and tuning parameter \( C > 0, \alpha_0 > 0 \) such that \( \alpha \alpha_0 > (\alpha_0 + \alpha) \delta \), let there exist scalars \( R > 0, P_n > 0, \lambda_n \geq 0 (n = 1, 2, 3) \) that satisfy the LMIs:

\[
\theta_1 |z_n| c < 0,
\]

\[
\theta_2 |z_n| c < 0,
\]

\[
\psi_0 |z_n| c < 0,
\]

where

\[
\theta_1 = \psi_1 + \Pi,
\]

\[
\theta_2 = \psi_2 + \Pi,
\]

where \( \psi_1, \psi_2, \psi_0 \) are given by (4.12), (4.13), (4.14) respectively, and

\[
\Pi = \begin{bmatrix}
0 & 0 & \frac{\lambda^2}{2} \left( \frac{\pi^2}{\alpha_0} + \epsilon^2 \right) & 0

0 & R \frac{\lambda^2}{2} \left( 1 - \frac{1}{\pi^2} \right) & 0 & 0

0 & 0 & -\epsilon \alpha_0 & 0

0 & 0 & 0 & 0
\end{bmatrix}
\]

Then for any initial function \( z_0 \in H^1_0(0, L) \) subject to \( \|z_0\|_V < \sqrt{\frac{L}{\pi^2}} C \), the closed-loop system (3.10) subject to (3.2) is exponentially stable with a decay rate \( \alpha_1 > 0 \) which is subject to (4.15).

**Proof.** See the Appendix.

**Remark 4.4.** For the negligible moving time with \( \delta \rightarrow 0 \), the LMI conditions of Theorem 1 are reduced to (4.9) and (4.10), and the LMI conditions of Theorem 2 are reduced to (4.30) and (4.31).

### 5. Extensions

The proposed method is efficient for various classes of PDEs. In this section we will discuss its extension to KSE under the periodic boundary conditions and to the reaction–diffusion equation.

Consider (3.10) under the periodic boundary conditions:

\[
\frac{\partial^m z}{\partial x^m}(0, t) = \frac{\partial^m z}{\partial x^m}(L, t), \quad t > 0, \quad m = 0, 1, 2, 3
\]

and the switching law (3.12). The well-posedness under the periodic boundary conditions can be established similar to the case of Dirichlet boundary conditions. Denote

\[
H^2_{per}(0, L) \triangleq \{ g \in H^2(0, L) \mid g(0) = g(L), g'(0) = g'(L) \}.
\]

We employ the extension of Sobolev’s inequality (Kang & Fridman, 2018, 2019):

**Lemma 5.1.** Let \( z \in H^2(0, L) \) be a scalar function. Then

\[
\max_{x \in [0, L]} |z(x, t)|^2 \leq (1 + \frac{1}{L}) \|z(\cdot, t)\|_{H^2(0, L)}^2 + \|z_x(\cdot, t)\|_{L^2(0, L)}^2.
\]

Note that Wirtinger’s inequality cannot be applied for the case of the periodic boundary conditions (5.1). Therefore, from Lemma 5.1 we directly obtain the following result:

- If the LMI conditions of Theorem 1/Theorem 2 with \( \lambda_1 = 0 \) hold, then for the initial function \( z_0 \in H^2_{per}(0, L) \) satisfying

\[
\|z_0\|_V < \sqrt{\frac{L}{\pi^2}} C,
\]

there exists a unique strong solution of the corresponding closed-loop system and the closed-loop system is stable, where \( M \triangleq \max \left\{ \frac{\pi^2}{\alpha_0}, \frac{\pi^2}{\alpha_0} \right\} \).

**Remark 5.1.** In the companion conference paper (Hetel & Fridman, 2013), the presented switching control sampled-data control was presented for the semilinear heat equation

\[
z_i(x, t) = \frac{\partial}{\partial x} [a(x)z_i(x, t)] + \phi(z(x, t))z_i(x, t) + b_m(x) u_m(t),
\]

under the averaged measurements and the Dirichlet boundary conditions \( z(0, t) = z(1, t) = 0, \quad t > 0 \). The functions \( a \) and \( \psi \) are of class \( C^1 \) and may be unknown. Global sampled-data stabilization by switching was achieved under assumption that the inequalities \( a(x) \geq a_0 > 0, \quad \psi_m \leq \psi \leq \psi_M \) hold with known bounds \( a_0, \psi_m \) and \( \psi_M \).

### 6. Numerical example

In this section, we present a numerical example which verifies the effectiveness of the proposed method for KSE. For the heat equation see example in Kang et al. (2021).

#### 6.1. Case of the Dirichlet boundary conditions

Consider KSE (3.1) under the Dirichlet boundary conditions (3.2) with \( L = 2\pi \) and instability parameter \( \nu = 0.8 < 1 \). The initial function is chosen as follows:

\[
z(x, 0) = z_0(x) = 0.015(1 - \cos x) \sin x.
\]

Fig. 2 demonstrates the profile of the open-loop system initialized by \( z_0(x) \). It is shown that the unforced system is unstable.

We choose \( K = 25, a_0 = 0.32, \alpha = 0.07, C = 1 \) and verify LMIs of Theorems 1 and 2 under point-like and averaged measurements. The results are given in Table 1, which shows that as \( N \) increases, the maximum value of \( h \) increases. Since the averaged measurements contain more information than the point-like measurements, the corresponding switched closed-loop system under the averaged measurements preserves the stability for larger \( h \). Note that e.g. for \( N = 15 \) and \( \epsilon = \pi / 30 \) the closed-loop system preserves the exponential stability within a given domain of initial conditions \( \|z_0\|_V < 0.743 \) for 0.0059 \( \leq t_{k+1} - t_k \leq h \leq 0.006 \). Note also that for the choice of \( K \) we used trials and errors.
Table 1  Maximal values of $h, \delta$ under the Dirichlet boundary conditions.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$h$</th>
<th>$\delta$</th>
<th>$\varepsilon$</th>
<th>$\sigma$</th>
</tr>
</thead>
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<td>0.003</td>
<td>$\pi/90$</td>
<td>0.0003</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>$\pi/30$</td>
<td>0.0008</td>
</tr>
<tr>
<td>13</td>
<td>0.027</td>
<td>0.004</td>
<td>$\pi/90$</td>
<td>0.0045</td>
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<tr>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
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<td>0.005</td>
<td>$\pi/90$</td>
<td>0.0050</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\pi/30$</td>
<td>0.0060</td>
</tr>
</tbody>
</table>

6.2. Case of the periodic boundary conditions

We proceed further with the case of periodic boundary conditions (5.1) under the same initial conditions. For the switching controller (3.9) via (3.12), by using the Yalmip, we verify LMI conditions of Theorem 1 with $\lambda_2 = 0, N = 15, K = 25, C = 1$, $\alpha = 0.07, \alpha_0 = 0.32, \delta = 0.001$, and $\varepsilon = \pi/30$. We find that the closed-loop system (4.7), (5.1) preserves exponential stability within a given domain of initial conditions $\|z_0\|_{\Omega} < 0.773$ for $0.0059 \leq t_{k+1} - t_k \leq \bar{h} \leq 0.006$. In the numerical simulation, a finite difference method is utilized to compute the numerical solution of the closed-loop system (4.7), (5.1) under the output-feedback (3.9) via the switching law (3.12) with $|\Omega_{\bar{a}}| = \frac{2\pi}{15}$ and $t_{k+1} - t_k = 0.006$. Numerical simulations of the solutions confirm the theoretical results that follow from LMIs.
7. Conclusions

The present paper introduced stabilization by switched sampled-data controller of parabolic PDEs via the employment of either stationary or moving actuating devices that can move to the active subdomain. Extension of the results to other PDEs may be a topic for future research.

Appendix. Proof of Theorem 2

For the case of switched controller under the averaged measurements, by arguments of Theorem 1, the well-posedness of (3.10) subject to (3.2) can be established via the step method.

Denote $\tilde{f}(x, t) = z(x, t) = \frac{\int_{\Omega} \tilde{z}(x, t) \, dx}{|\Omega|}$, 

Then the switching controller (3.9) via the switching law (3.12) with $\zeta_j = 1$ can be rewritten as

$$u_{\theta}(t) = -K z(x, t) - \tilde{f}_q(x, t) - \tilde{p}_r(t). \quad (A.1)$$

We choose the Lyapunov function $V$ with $\tilde{p}_j$ instead of $\rho_j$. Differentiating $V$ along the solution of the closed-loop system (3.10) subject to (3.2), we get (4.16) with $\tilde{p}_j$ instead of $\rho_j$. The substitution $\tilde{f} \rightarrow \tilde{f} + \tilde{p}_j \tilde{p}_j$ in Theorem leads to the following changes:

$$\int_{\Omega} \tilde{p}_j(t)^2 \, dx = \frac{1}{|\Omega|} \left( \int_{\Omega} \tilde{f}_j(x, t) \, dx \right)^2 \leq \int_{\Omega} \tilde{z}_j^2(x, t) \, dx,$$

$$-\frac{\lambda_1}{N^2} \sum_{j=1}^{N} \int_{\Omega} \tilde{p}_j(x, t) \tilde{z}_j(x, t) \tilde{p}_j(t)^2 \, dx$$

$$+ \lambda_1 \int_{\Omega} \tilde{z}_j(x, t) \tilde{p}_j(x, t) \tilde{p}_j(t)^2 \, dx \geq 0, \quad (A.2)$$

For any $\lambda_1 \geq 0, \lambda_2 \geq 0$.

Set $\bar{\eta}_1 = \text{col}(\tilde{z}_1(x, t), z_1(x, t), z_0(x, t), \tilde{f}_j(x, t), \tilde{p}_j(t), \tilde{p}_j(t), \tilde{p}_j(t))$, $\bar{\eta}_2 = \text{col}(\tilde{z}_1(x, t), z_1(x, t), z_0(x, t), \tilde{f}_j(x, t), \tilde{p}_j(x, t), \tilde{p}_j(x, t), \tilde{p}_j(x, t))$, $\eta_0 = \text{col}(\tilde{z}_1(x, t), z_1(x, t), z_0(x, t))$. Applying descriptor method and adding the left-hand sides of (A.2)–(A.2.2) and (A.3) to $V$, we finally obtain

$$V(t) + 2aV(t) \leq \sum_{j \neq k} \int_{\Omega} \bar{\eta}_1^t \Theta_1 \bar{\eta}_1 \, dx + \int_{\Omega} \bar{\eta}_2^t \Theta_2 \bar{\eta}_2 \, dx,$$

$$t \in [t_k + \delta, t_{k+1}^1),$$

where $\Theta_1(l = 1, 2)$ and $\psi_0$ are given by (4.33), (4.34) and (4.14), respectively.

Hence, $V(t) + 2aV(t) \leq \psi_0(t) - 2a\psi_0(t) \leq 0$, if $\theta_l < 0$ (l = 1, 2) and $\psi_0 < 0$ hold for all $-C \leq z \leq C$. Matrices $\Theta_1(l = 1, 2)$ and $\psi_0$ given by (4.33), (4.34) and (4.14) are affine in $z_e$. Hence, $\theta_l < 0$ (l = 1, 2) and $\psi_0 < 0$ for all $-C \leq z \leq C$ if these inequalities hold in the vertices $z = \pm C$, i.e. if LMIs (4.30)–(4.32) are feasible.

References


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