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Stability of singularly perturbed functional-differential systems: spectrum analysis and LMI approaches

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A singularly perturbed linear functional-differential system is considered. The delay is assumed to be small of the order of a small parameter multiplying a part of derivatives in the system. It is 'not assumed that the fast subsystem is asymptotically stable'. Two approaches to the study of the exponential stability of the singularly perturbed system are suggested. The first one treats systems with constant delays via the analysis of asymptotic behaviour of the roots of their characteristic equation. The second approach develops a direct Lyapunov–Krasovskii method for systems with time-varying delays leading to stability conditions in terms of linear matrix inequalities. Numerical examples illustrate the efficiency of both approaches.

Keywords: functional-differential system; singular perturbation; exponential stability; spectrum analysis; linear matrix inequality.

1. Introduction

Singularly perturbed differential equations, being an adequate mathematical model of real-life multi-time-scale systems, were studied extensively in the literature (see, e.g. Halanay, 1966; Khalil, 2001; Kokotovic *et al.*, 1986; O'Malley, 1991; Vasil'eva *et al.*, 1995; Wasov, 1965 and references therein). One of the important classes of such equations is the class of equations with small time delays of order of a small positive parameter ε multiplying a part of the derivatives in the system. Brief surveys of results in this topic can be found in Glizer (2004a) and Glizer (2009).

One of the important issues, studied in the theory of differential equations, is the stability (see, e.g. Bellman, 1953; Lyapunov, 1966; Halanay, 1966; Rasvan, 1983; Halanay & Rasvan, 1997). Two approaches to the study of stability of the trivial solution to linear constant-coefficients differential systems (without and with time delays) are most spread in the literature. The first (classical one) is based on the spectrum analysis of the system. The second (more recent one) is a Lyapunov-method-based one leading to sufficient conditions in terms of linear matrix inequalities (LMIs).

Spectrum analysis of a linear time-invariant differential system allows to derive many quantitative and qualitative properties of its solutions (see, e.g. Bellman & Cooke, 1963; Halanay, 1966; Hale & Verduyn Lunel, 1993; Hartman, 2002). In this paper, we consider a singularly perturbed linear time-invariant differential system with the general type of small time delay in the state variables. Since the system depends on ε, its characteristic equation also depends on this parameter. The structure of the set of roots of this equation, valid for all sufficiently small ε (robust with respect to ε), is studied.

The structure of the set of roots of the characteristic equation, associated with an undelayed singularly perturbed system, was analysed in a number of works (see, e.g. Kokotovic *et al.*, 1986; Luse & Khalil, 1985; Luse, 1986). The dependence on a parameter of roots of the characteristic equation, associated with a time-delay system, also was studied in the literature. In Hale & Verduyn Lunel (1993) and Halanay (1966), different aspects of behaviour of the spectrum for regularly perturbed time-delay systems were studied. Asymptotic behaviour with respect to a small delay perturbation of critical (pure imaginary) roots was analysed in Chen *et al.* (2006), Chen *et al.* (2008) and Fu *et al.* (2007) for the case of commensurate delays. Conditions, under which such roots become asymptotically stable due to a small perturbation of the delays, were established. The limit behaviour (as $\varepsilon \to +0$) of spectrum of a singularly perturbed time-delay system was studied in Glizer (1999) and Glizer (2003). The separation of this spectrum into two sets, not intersecting each other, also was done.

In the present paper, we continue the study of the asymptotic behaviour (for $\varepsilon \to +0$) of the spectrum of a singularly perturbed time-delay differential system, started in Glizer (1999) and Glizer (2003). The results of this study are applied to analysis of the exponential stability of the original singularly perturbed time-delay system.

The exponential stability and the equivalent to it L^2-stability of linear singularly perturbed systems with small time delays were studied in a number of works in the literature. Thus, in Fridman (1996) and Glizer & Fridman (2000), such a study is based on the exact slow–fast decomposition of the system. In Dragan & Ionita (1999), the exponential stability of a singularly perturbed system with two kinds of state delay (non-small for the slow and small for the fast state variables) was investigated by using the transformations of the slow and fast parts of the original differential system to equivalent integral equations. In Glizer (2004a), the analysis of the exponential stability is based on the block-wise estimate of the fundamental matrix solution of singularly perturbed systems with small time delays established in Glizer (2003). In Glizer (2007), the L^2-stability was studied for a closed-loop system arising in an infinite horizon linear-quadratic optimal control problem for singularly perturbed systems with small state delays. In all these works, the essential condition is the exponential (or L^2) stability of both, slow and fast, subsystems associated with the singularly perturbed system. In the present paper, in contrast with the above mentioned works, the exponential stability of the singularly perturbed time-delay system is analysed also in the case where the fast subsystem is not required to be exponentially stable.

During two recent decades, the LMI method (see, e.g. Boyd *et al.*, 1994) became one of basic approaches to analysis and control of time-delay systems. This approach was extended to singularly perturbed systems with delay (see, e.g. Chen *et al.*, 2010; Fridman, 2002a,b, 2006 and references therein). It is interesting to note that the LMI approach to singularly perturbed systems (Fridman, 2002a) gave an idea of descriptor approach (Fridman, 2002c) to time-delay systems, which allowed for the first time to treat fast-varying delays (i.e. delays without any constraints on the delay derivative) via Krasovskii method (see Fridman & Shaked, 2002). In all the existing LMI-based papers on singularly perturbed systems, the case of exponentially stable fast subsystem was considered. In the present paper, an LMI approach is extended to exponential stability analysis of singularly perturbed time-delay systems with constant coefficients and variable time delays in the case, where there is no assumption on the exponential stability of the fast subsystem.

It should be noted that the method, based on the asymptotic analysis of the spectrum of a singularly perturbed system with delays (the asymptotic method), is infinite dimensional, while the LMI method is finite dimensional. Therefore, they do not replace each other. These two methods are complimentary. Namely, the asymptotic one gives some accurate enough, but mostly qualitative analysis for linear systems with constant delays. Based on the asymptotic analysis, an appropriate LMI approach gives more restrictive sufficient conditions. However, LMI-based conditions are robust and they give an interval for the small parameter ε on which the system has the same decay rate. An LMI method can be also applied to analysis and design of uncertain systems with uncertainties in the coefficients and delays.

The paper is organized as follows. In the next section, the problem is formulated. The objectives of the paper are stated. The separation of roots of the characteristic equation, associated with the original singularly perturbed functional-differential system, is studied in Section 3. In Sections 4 and 5, the sets of slow and fast roots of this characteristic equation are analysed. In Section 6, based on this analysis, the exponential stability of the original singularly perturbed functional-differential system is investigated. In Section 7, the LMI method is developed for study of the stability of a singularly perturbed linear system with point-wise and distributed variable delays. In Section 8, a numerical evaluation of both methods of the stability analysis of the singularly perturbed systems with delays is carried out.

The following main notations are applied in the paper:

(1) E^n denotes the real n-dimensional Euclidean space;
(2) I_n denotes the n-dimensional identity matrix;
(3) $\mathscr{C}$ denotes the set of all complex numbers;
(4) $\mathrm{Re}\lambda$ and $\mathrm{Im}\lambda$ denote the real and imaginary parts, respectively, of a complex number λ;
(5) $\mathrm{col}(x, y)$, where $x \in E^n$ and $y \in E^m$, denotes a column block-vector with the upper block x and the lower block y;
(6) $\|\cdot\|$ denotes the Euclidean norm of a vector and of a matrix;
(7) the superscript $\top$ denotes the transposition of either a matrix or a vector;
(8) the inequality $A > (\geqslant) 0$, where A is a symmetric matrix, means that this matrix is positive definite (semi-definite);
(9) $C[a, b; E^n]$ is the space of continuous functions $f(t)\colon [a, b] \to E^n$;
(10) $\|\cdot\|_C$ denotes the uniform norm in $C[a, b; E^n]$;
(11) $W[a, b; E^n]$ is the Sobolev space of absolutely continuous functions $f(t)\colon [a, b] \to E^n$ with the derivatives, square integrable on the interval $[a, b]$.

2. Problem statement

2.1 *Singularly perturbed system with time-independent delay*

Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \int_{-h}^{0} [\mathrm{d}A_1(\eta)]x(t + \varepsilon\eta) + \int_{-h}^{0} [\mathrm{d}A_2(\eta)]y(t + \varepsilon\eta), \quad t \geqslant 0, \tag{2.1}$$

$$\varepsilon\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \int_{-h}^{0} [\mathrm{d}A_3(\eta)]x(t + \varepsilon\eta) + \int_{-h}^{0} [\mathrm{d}A_4(\eta)]y(t + \varepsilon\eta), \quad t \geqslant 0, \tag{2.2}$$

where $x(t) \in E^n$, $y(t) \in E^m$; $\varepsilon > 0$ is a small parameter ($\varepsilon << 1$); $h > 0$ is a given constant independent of ε and $A_i(\eta)$, $(i = 1, \ldots, 4)$ are given matrices of respective dimensions.

In what follows, we assume:

A1. The matrix-valued functions $A_i(\eta)$, $(i = 1, \ldots, 4)$ are defined for $\eta \in (-\infty, +\infty)$ and satisfy the conditions:

(a_1) $A_i(\eta) = 0, \quad \forall\, \eta \geqslant 0;$

(b$_1$) $A_i(\eta) = A_i(-h), \quad \forall \eta \leqslant -h$;

(c$_1$) $A_i(\eta)$ is continuous from the left for $\eta \in (-h, 0)$;

(d$_1$) $A_i(\eta)$ has bounded variation on the interval $\eta \in [-h, 0]$.

System (2.1)–(2.2) is called 'singularly perturbed by the small parameter ε' or simply 'singularly perturbed'. The state variables $x(\cdot)$ and $y(\cdot)$ are called the 'slow' and 'fast' ones, respectively. Equation (2.1) describes the 'slow mode (motion)' of system (2.1)–(2.2), while (2.2) describes its 'fast mode (motion)'.

Let us write down the characteristic equation for the original singularly perturbed system (2.1)–(2.2). For this purpose, we rewrite (2.1)–(2.2) as follows:

$$E_\varepsilon \frac{\mathrm{d}z(t)}{\mathrm{d}t} = \int_{-h}^{0} [\mathrm{d}A(\eta)] z(t + \varepsilon\eta), \quad t \geqslant 0, \tag{2.3}$$

where $z(\cdot) = \mathrm{col}(x(\cdot), y(\cdot))$, and

$$E_\varepsilon = \begin{bmatrix} I_n & 0 \\ 0 & \varepsilon I_m \end{bmatrix}, \quad A(\eta) = \begin{bmatrix} A_1(\eta) & A_2(\eta) \\ A_3(\eta) & A_4(\eta) \end{bmatrix}. \tag{2.4}$$

Using equivalent form (2.3) of system (2.1)–(2.2), we obtain the characteristic equation (with respect to λ) for this system in the form

$$\det \Delta(\lambda, \varepsilon) = 0, \quad \Delta(\lambda, \varepsilon) \stackrel{\triangle}{=} \int_{-h}^{0} \exp(\varepsilon\lambda\eta) \mathrm{d}A(\eta) - \lambda E_\varepsilon. \tag{2.5}$$

In what follows, we call (2.5) the 'original characteristic equation'.

The spectrum analysis of (2.1)–(2.2), i.e. the analysis of roots of the original characteristic equation, is based on the asymptotic decomposition of this system into two much simpler ε-free subsystems, the fast and slow ones.

2.1.1 *Fast subsystem.* The fast subsystem is derived from the equation for the fast mode (2.2) in two steps. In the first step, the slow state variable $x(\cdot)$ is removed from (2.2). Thus, we obtain the equation

$$\varepsilon \frac{\mathrm{d}y(t)}{\mathrm{d}t} = \int_{-h}^{0} [\mathrm{d}A_4(\eta)] y(t + \varepsilon\eta), \quad t \geqslant 0. \tag{2.6}$$

On the second step, the following transformations of the independent variable and the state are made in this equation:

$$t = \varepsilon\xi, \quad y(\varepsilon\xi) = y_{\mathrm{f}}(\xi), \tag{2.7}$$

where ξ and $y_{\mathrm{f}}(\xi)$ are a new independent variable (the stretched time) and a new state, respectively. By this transformations, (2.6) becomes

$$\frac{\mathrm{d}y_{\mathrm{f}}(\xi)}{\mathrm{d}\xi} = \int_{-h}^{0} [\mathrm{d}A_4(\eta)] y_{\mathrm{f}}(\xi + \eta). \tag{2.8}$$

The fast subsystem (2.8) is ε-free, and it is of a less dimension than the original one (2.1)–(2.2). The characteristic equation (with respect to μ) for the fast subsystem (2.8) is

$$\det \Delta_{\mathrm{f}}(\mu) = 0, \quad \Delta_{\mathrm{f}}(\mu) \stackrel{\triangle}{=} \int_{-h}^{0} \exp(\mu\eta)\mathrm{d}A_4(\eta) - \mu I_m. \tag{2.9}$$

We call the characteristic equation (2.9) for the fast subsystem (2.8) the 'fast characteristic equation'.

In what follows, we assume:

A2. The fast characteristic equation (2.9) has no zero root, i.e. $\det \Delta_{\mathrm{f}}(0) \neq 0$.

2.1.2 *Slow subsystem.* The slow subsystem is obtained from (2.1) to (2.2) by setting there formally $\varepsilon = 0$ and re-denoting the states $x(\cdot)$ and $y(\cdot)$ by $x_{\mathrm{s}}(\cdot)$ and $y_{\mathrm{s}}(\cdot)$, respectively. Thus, we obtain the system

$$\frac{\mathrm{d}x_{\mathrm{s}}(t)}{\mathrm{d}t} = \bar{A}_1 x_{\mathrm{s}}(t) + \bar{A}_2 y_{\mathrm{s}}(t), \tag{2.10}$$

$$0 = \bar{A}_3 x_{\mathrm{s}}(t) + \bar{A}_4 y_{\mathrm{s}}(t), \tag{2.11}$$

where

$$\bar{A}_i \stackrel{\triangle}{=} \int_{-h}^{0} \mathrm{d}A_i(\eta), \quad i = 1, \ldots, 4. \tag{2.12}$$

It is seen that the slow subsystem (2.10)–(2.11) is differential-algebraic, it is independent of ε and has no delays. Under the Assumption A2, the slow subsystem can be converted to a differential equation with respect to $x_{\mathrm{s}}(\cdot)$. Indeed, due to this assumption, $\det \Delta_{\mathrm{f}}(0) \neq 0$. Direct calculation yields $\Delta_{\mathrm{f}}(0) = \bar{A}_4$. Hence, as a consequence of the assumption A2, we have

$$\det \bar{A}_4 \neq 0. \tag{2.13}$$

Thus, under the assumption A2, the original singularly perturbed system (2.1)–(2.2) is standard (see Kokotovic *et al.*, 1986, Chapter 1, Section 2).

Resolving (2.11) with respect to $y_{\mathrm{s}}(t)$ and substituting the obtained result into (2.10), one transforms the slow subsystem as follows:

$$\frac{\mathrm{d}x_{\mathrm{s}}(t)}{\mathrm{d}t} = \mathscr{A}_0 x_{\mathrm{s}}(t), \quad \mathscr{A}_0 \stackrel{\triangle}{=} \bar{A}_1 - \bar{A}_2(\bar{A}_4)^{-1}\bar{A}_3. \tag{2.14}$$

The characteristic equation (with respect to λ) for (2.14) is

$$\det \Delta_{\mathrm{s}}(\lambda) = 0, \quad \Delta_{\mathrm{s}}(\lambda) \stackrel{\triangle}{=} \mathscr{A}_0 - \lambda I_n. \tag{2.15}$$

We call the characteristic equation (2.15) for the slow subsystem (2.14) the 'slow characteristic equation'.

2.1.3 *Asymptotic decomposition of the original characteristic equation.* In this subsection, we show that a proper asymptotic ($\varepsilon \to +0$) decomposition of the original characteristic equation (2.5) yields the slow and fast characteristic equations.

Let us begin with the fast characteristic equation. First, we rewrite the original characteristic equation (2.5) in the equivalent form

$$\det \hat{\Delta}_1(\lambda,\varepsilon)=0, \quad \hat{\Delta}_1(\lambda,\varepsilon) \triangleq \mathscr{E}_\varepsilon \int_{-h}^{0} \exp(\varepsilon\lambda\eta)\mathrm{d}A(\eta) - \varepsilon\lambda I_{n+m}, \tag{2.16}$$

where

$$\mathscr{E}_\varepsilon \triangleq \varepsilon(E_\varepsilon)^{-1} = \begin{bmatrix} \varepsilon I_n & 0 \\ 0 & I_m \end{bmatrix}. \tag{2.17}$$

By the transformation of variables $\lambda = \mu/\varepsilon$, (2.16) becomes

$$\det \hat{\Delta}_2(\mu,\varepsilon)=0, \quad \hat{\Delta}_2(\mu,\varepsilon) \triangleq \mathscr{E}_\varepsilon \int_{-h}^{0} \exp(\mu\eta)\mathrm{d}A(\eta) - \mu I_{n+m}. \tag{2.18}$$

It should be noted that the transformation of variables $\lambda = \mu/\varepsilon$ in (2.16) corresponds to the transformation of the independent variable $t = \varepsilon\xi$ in (2.1)–(2.2).

Setting formally $\varepsilon = 0$ in (2.18) yields

$$\det \tilde{\Delta}(\mu)=0, \quad \tilde{\Delta}(\mu) \triangleq \hat{\Delta}_2(\mu,0) = \mathscr{E}_0 \int_{-h}^{0} \exp(\mu\eta)\mathrm{d}A(\eta) - \mu I_{n+m}, \tag{2.19}$$

where $\mathscr{E}_0 \triangleq \mathscr{E}_\varepsilon|_{\varepsilon=0}$.

By using the block form of the matrix $A(\eta)$ (see (2.4)) and the block form of the matrix $\mathscr{E}_0$, we can rewrite the matrix $\tilde{\Delta}(\mu)$ in the explicit block form

$$\tilde{\Delta}(\mu) = \begin{bmatrix} -\mu I_n & 0 \\ H_3(\mu) & H_4(\mu) - \mu I_m \end{bmatrix}, \quad H_k(\mu) \triangleq \int_{-h}^{0} \exp(\mu\eta)\mathrm{d}A_k(\eta), \quad k=3,4. \tag{2.20}$$

Due to (2.20), (2.19) becomes

$$(-1)^n \mu^n \det(H_4(\mu) - \mu I_n) = 0. \tag{2.21}$$

Comparing (2.21) to (2.9), and using the assumption A2 yield that the set of all roots of the fast characteristic equation coincides with the set of all non-zero roots of (2.21). Moreover, the fast characteristic equation (2.9) can be obtained from the original characteristic equation (2.5) in the following way: (i) equivalent transformation of (2.5) to (2.16); (ii) transformation of variables $\lambda = \mu/\varepsilon$ in (2.16) yielding (2.18); (iii) setting formally $\varepsilon = 0$ in (2.18) yielding (2.21) and (iv) dividing (2.21) by $(-1)^n\mu^n$.

Now, let us proceed to obtaining the slow characteristic equation from the original one.

Setting formally $\varepsilon = 0$ in (2.5), we obtain

$$\det \bar{\Delta}(\lambda)=0, \quad \bar{\Delta}(\lambda) \triangleq \Delta(\lambda,0) = \bar{A} - \lambda E_0, \tag{2.22}$$

where $E_0 \triangleq E_\varepsilon|_{\varepsilon=0}$, and

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ \bar{A}_3 & \bar{A}_4 \end{bmatrix}. \tag{2.23}$$

By using (2.23) and the block form of E_0, the matrix $\bar{\Delta}(\lambda)$ can be rewritten in the explicit block form

$$\bar{\Delta}(\lambda) = \begin{bmatrix} \bar{A}_1 - \lambda I_n & \bar{A}_2 \\ \bar{A}_3 & \bar{A}_4 \end{bmatrix}. \tag{2.24}$$

Applying the formula for the determinant of a block matrix (see Gantmacher, 1974) to (2.24), and taking into account (2.13), we obtain directly that for any complex λ,

$$\det \bar{\Delta}(\lambda) = \det[\bar{A}_1 - \lambda I_n - \bar{A}_2(\bar{A}_4)^{-1}\bar{A}_3] \det \bar{A}_4. \tag{2.25}$$

Comparing (2.25) to (2.14) and (2.15), and using (2.13), we can conclude that the slow characteristic equation and (2.25) have the same roots. Moreover, the slow characteristic equation (2.15) can be obtained from the original characteristic equation (2.5) by setting there formally $\varepsilon = 0$ and dividing the resulting equation by $\det \bar{A}_4$.

2.2 *Singularly perturbed system with time-dependent delay*

A Lyapunov-method-based stability analysis will be developed for linear systems with time-varying delays

$$E_\varepsilon \frac{\mathrm{d}z(t)}{\mathrm{d}t} = Bz(t) + B_h z(t - \varepsilon h(t)) + B_r \int_{-\varepsilon r(t)}^{0} z(t+\theta)\mathrm{d}\theta, \quad t \geqslant 0, \tag{2.26}$$

where B, B_h and B_r are constant matrices.

The functions $h(t)$ and $r(t)$ are piecewise continuous for $t \geqslant 0$, satisfying the inequalities

$$0 \leqslant h(t) \leqslant h_0, \quad 0 \leqslant r(t) \leqslant r_0, \tag{2.27}$$

where $h_0 > 0$ and $r_0 > 0$ are some constants.

Note that for $h(t) \equiv \text{const}$, $r(t) \equiv \text{const}$, the system (2.26) is a particular version of the system (2.3). However, it is not the case when either $h(t)$ or $r(t)$ does not equal identically to a constant. In this case, both systems (2.3) and (2.26) are particular versions of the system

$$E_\varepsilon \frac{\mathrm{d}z(t)}{\mathrm{d}t} = \int_{-r(t)}^{0} [d_\eta \mathscr{A}(t, \eta, \varepsilon)] z(t + \varepsilon\eta), \quad t \geqslant 0, \tag{2.28}$$

with properly chosen matrix-valued function $\mathscr{A}(t, \eta, \varepsilon)$ and function $r(t)$.

2.3 *Objectives of the paper*

The objectives of the paper are:

(I) to study a structure of the set $\mathscr{R}(\varepsilon)$ of roots of the original characteristic equation (2.5), robust with respect to ε;

(II) to obtain asymptotic expansions (with respect to ε) for roots of (2.5);

(III) to apply the results on structure of $\mathscr{R}(\varepsilon)$ and asymptotic expansions of the roots of (2.5) to analysis of stability of the original singularly perturbed system (2.1)–(2.2);

(IV) with the system (2.26), to develop an alternative approach (an LMI approach) to stability analysis of system (2.26) with time-varying delays and

(V) to illustrate the efficiency of the two approaches in numerical examples.

3. Separation of roots of (2.5)

Let $\bar{\lambda}_p$, $(p = 1, \ldots, q \leqslant n)$ be all distinct eigenvalues of the matrix $\mathscr{A}_0$, i.e. all distinct roots of the slow characteristic equation (2.15).

3.1 *Auxiliary lemmas*

LEMMA 3.1 Let the Assumptions A1 and A2 be satisfied. Let $\{\varepsilon_k\}$ and $\{\lambda_k\}$, $(k = 1, 2, \ldots)$ be any sequences such that

(i) $\varepsilon_k > 0, \quad (k = 1, 2, \ldots)$;
(ii) $\lim_{k\to+\infty} \varepsilon_k = 0$;
(iii) $\lim_{k\to+\infty} \varepsilon_k \lambda_k = 0$;
(iv) $\det \Delta(\lambda_k, \varepsilon_k) = 0, \quad (k = 1, 2, \ldots)$, where $\Delta(\lambda, \varepsilon)$ is defined in (2.5).

Then, there exists a subsequence of the sequence $\{\lambda_k\}$, which converges to one of the numbers $\bar{\lambda}_p$, $(p = 1, \ldots, q)$.

Proof. The lemma is proved very similar to Lemma 1 of Glizer (2009). □

Let $\mathscr{M}$ be the set of all distinct roots of the fast characteristic equation (2.9). Let $\mathscr{M}_+ = \{\mu \in \mathscr{M} : \mathrm{Re}\mu > 0\}$, $\mathscr{M}_- = \{\mu \in \mathscr{M} : \mathrm{Re}\mu < 0\}$ and $\mathscr{M}_0 = \{\mu \in \mathscr{M} : \mathrm{Re}\mu = 0\}$. Due to the assumption A2, the set $\mathscr{M}_0$ does not contain $\mu = 0$. Note also that, due to Bellman & Cooke (1963) and Hale & Verduyn Lunel (1993), $\mathscr{M}_+$ and $\mathscr{M}_0$ are finite sets. Moreover, there does not exist a sequence $\{\mu_k\}$, $\mu_k \in \mathscr{M}_-$, $(k = 1, 2, \ldots)$, such that $\mathrm{Re}\mu_k \to 0$ for $k \to +\infty$. Hence, one can find numbers $\chi > 0$, $\gamma > 0$ and $\kappa_2 > \kappa_1 > 0$ such that

$$\mathrm{Re}\mu > \chi, \quad \forall\, \mu \in \mathscr{M}_+, \tag{3.1}$$

$$\mathrm{Re}\mu < -\gamma, \quad \forall\, \mu \in \mathscr{M}_-, \tag{3.2}$$

$$\kappa_1 < |\mathrm{Im}\mu| < \kappa_2, \quad \forall\, \mu \in \mathscr{M}_0. \tag{3.3}$$

Consider the domain

$$\mathscr{D}_{\mathrm{f}} = \mathscr{D}_{\mathrm{f},\chi} \bigcup \mathscr{D}_{\mathrm{f},\gamma} \bigcup \mathscr{D}_{\mathrm{f},\kappa}, \tag{3.4}$$

where

$$\mathscr{D}_{\mathrm{f},\chi} = \{\mu : \mathrm{Re}\mu > \chi\}, \tag{3.5}$$

$$\mathscr{D}_{\mathrm{f},\gamma} = \{\mu : \mathrm{Re}\mu < -\gamma\}, \tag{3.6}$$

$$\mathscr{D}_{\mathrm{f},\kappa} = \{\mu : -\gamma < \mathrm{Re}\mu < \chi,\ \kappa_1 < |\mathrm{Im}\mu| < \kappa_2\}. \tag{3.7}$$

LEMMA 3.2 Let the Assumptions A1 and A2 be satisfied. Let $\{\varepsilon_k\}$ and $\{\mu_k\}$ be any sequences such that

(i) $\varepsilon_k > 0, \quad (k = 1, 2, \ldots)$;
(ii) $\lim_{k\to+\infty} \varepsilon_k = 0$;
(iii) μ_k does not belong to $\mathscr{D}_{\mathrm{f}}$ for all sufficiently large $k \in \{1, 2, \ldots\}$;
(iv) $\hat{\Delta}_2(\mu_k, \varepsilon_k) = 0, \quad (k = 1, 2, \ldots)$, where $\hat{\Delta}_2(\mu, \varepsilon)$ is defined in (2.18).

Then, the sequence $\{\mu_k\}$ converges to zero.

Proof. The lemma is proved very similar to Lemma 2.1 of Glizer (2003). □

3.2 *Main theorem on the roots separation*

Let $\sigma_1 < \sigma_2$ and $\rho_1 < \rho_2$ be numbers, such that

$$\sigma_1 < \mathrm{Re}\bar{\lambda}_p < \sigma_2, \quad \rho_1 < \mathrm{Im}\bar{\lambda}_p < \rho_2, \quad p = 1, \ldots, q. \tag{3.8}$$

Consider the domain

$$\mathscr{D}_{\mathrm{s}} = \{\lambda\colon \sigma_1 < \mathrm{Re}\lambda < \sigma_2, \quad \rho_1 < \mathrm{Im}\lambda < \rho_2\}, \tag{3.9}$$

and, for any $\varepsilon > 0$, the domain

$$\tilde{\mathscr{D}}_{\mathrm{f}}(\varepsilon) = \tilde{\mathscr{D}}_{\mathrm{f},\chi}(\varepsilon) \bigcup \tilde{\mathscr{D}}_{\mathrm{f},\gamma}(\varepsilon) \bigcup \tilde{\mathscr{D}}_{\mathrm{f},\kappa}(\varepsilon), \tag{3.10}$$

where

$$\tilde{\mathscr{D}}_{\mathrm{f},\chi}(\varepsilon) = \{\lambda\colon \mathrm{Re}\lambda > \chi/\varepsilon\}, \tag{3.11}$$

$$\tilde{\mathscr{D}}_{\mathrm{f},\gamma}(\varepsilon) = \{\lambda\colon \mathrm{Re}\lambda < -\gamma/\varepsilon\}, \tag{3.12}$$

$$\tilde{\mathscr{D}}_{\mathrm{f},\kappa}(\varepsilon) = \{\lambda\colon -\gamma/\varepsilon < \mathrm{Re}\lambda < \chi/\varepsilon,\ \kappa_1/\varepsilon < |\mathrm{Im}\lambda| < \kappa_2/\varepsilon\}, \tag{3.13}$$

the positive numbers χ, γ, κ_1 and κ_2 are the same as in (3.1)–(3.3).

THEOREM 3.1 Let the assumptions A1 and A2 be satisfied. Then, there exists a number $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$:

(I) $\mathscr{D}_{\mathrm{s}} \bigcap \tilde{\mathscr{D}}_{\mathrm{f}}(\varepsilon) = \emptyset$;
(II) any root of the characteristic equation (2.5) belongs either to the domain $\mathscr{D}_{\mathrm{s}}$ or to the domain $\tilde{\mathscr{D}}_{\mathrm{f}}(\varepsilon)$.

Proof. The statement (I) of the theorem is directly follows from the structure of the domains $\mathscr{D}_{\mathrm{s}}$ and $\tilde{\mathscr{D}}_{\mathrm{f}}(\varepsilon)$, (see (3.9) and (3.10)–(3.13)).

Proceed to the proof of the statement (II). We prove this statement by contradiction. Namely, assume that the statement (II) is wrong. Then, there exist sequences $\{\varepsilon_k\}$ and $\{\lambda_k\}$ such that

(a) $\varepsilon_k > 0, \quad (k = 1, 2, \ldots)$;
(b) $\lim_{k\to+\infty} \varepsilon_k = 0$;
(c) λ_k does not belong to $\tilde{\mathscr{D}}_{\mathrm{f}}(\varepsilon_k)$ for all $k \in \{1, 2, \ldots\}$;

(d) λ_k does not belong to $\mathscr{D}_{\mathrm{s}}$ for all $k \in \{1, 2, \ldots\}$;

(e) $\det \Delta(\lambda_k, \varepsilon_k) = 0$ for all $k \in \{1, 2, \ldots\}$, where $\Delta(\lambda, \varepsilon)$ is defined in (2.5).

Now, let us consider the sequence $\{\mu_k\}$, where $\mu_k = \varepsilon_k \lambda_k$, $(k = 1, 2, \ldots)$. It is verified directly that the sequences $\{\varepsilon_k\}$ and $\{\mu_k\}$ satisfy all the conditions of Lemma 3.2. Hence, $\lim_{k \to +\infty} \mu_k = 0$, implying that the sequences $\{\varepsilon_k\}$ and $\{\lambda_k\}$ satisfy all the conditions of Lemma 3.1. Due to this lemma, there exist a subsequence $\{\lambda_{k_j}\}$ of $\{\lambda_k\}$ and a number $p \in \{1, \ldots, q\}$ such that $\lim_{j \to +\infty} \lambda_{k_j} = \bar{\lambda}_p$. The latter means that $\lambda_{k_j} \in \mathscr{D}_{\mathrm{s}}$ for all sufficiently large j, which contradicts the property (d) of the sequence $\{\lambda_k\}$. This contradiction proves the theorem. □

Due to Theorem 3.1, for all sufficiently small $\varepsilon > 0$, the set of all roots $\mathscr{R}(\varepsilon)$ of the original characteristic equation (2.5) can be separated into two subsets not intersecting each other. The roots of (2.5), belonging to $\mathscr{D}_{\mathrm{s}}$, are called the 'slow roots', while the ones, belonging to $\tilde{\mathscr{D}}_{\mathrm{f}}(\varepsilon)$, are called the 'fast roots'. We denote the sets of slow and fast roots of the original characteristic equation (2.5) by $\mathscr{R}_{\mathrm{s}}(\varepsilon)$ and $\mathscr{R}_{\mathrm{f}}(\varepsilon)$, respectively.

Since for any $\varepsilon > 0$, (2.5) has roots, at least one of the sets $\mathscr{R}_{\mathrm{s}}(\varepsilon)$ and $\mathscr{R}_{\mathrm{f}}(\varepsilon)$ is not empty. In what follows, it is shown that both sets are not empty, and the structure of each set is studied.

REMARK 3.1 Note that for a singularly perturbed undelayed linear differential equation with constant coefficients, the asymptotic decomposition of the characteristic equation, as well as the separation of its roots, were proposed in Vishik & Lyusternik (1957, 1960).

4. Analysis of the set of slow roots

First of all note that in this section, we assume $n \geqslant 1$. Otherwise, the system (2.1)–(2.2) has no the slow mode, and, consequently, $\mathscr{R}_{\mathrm{s}}(\varepsilon)$ is empty.

Let $\bar{\lambda}_p$, $p \in \{1, \ldots, q\}$ be a chosen root of the slow characteristic equation (2.15). Let n_p, $(1 \leqslant n_p \leqslant n)$ be the algebraic multiplicity of $\bar{\lambda}_p$. Hence, the left-hand side of (2.15) can be represented as

$$\det \Delta_{\mathrm{s}}(\lambda) = (\lambda - \bar{\lambda}_p)^{n_p} \mathscr{F}_{\mathrm{s},p}(\lambda), \quad \forall \lambda \in \mathscr{C}, \tag{4.1}$$

where $\mathscr{F}_{\mathrm{s},p}(\lambda)$ is a known polynomial of order $n - n_p$, and

$$\mathscr{F}_{\mathrm{s},p}(\bar{\lambda}_p) \neq 0. \tag{4.2}$$

Let $\delta_p > 0$ be such that

$$\mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \delta_p) \stackrel{\triangle}{=} \{\lambda \colon |\lambda - \bar{\lambda}_p| \leqslant \delta_p\} \subset \mathscr{D}_{\mathrm{s}}, \tag{4.3}$$

and all the roots of (2.15), excepting $\bar{\lambda}_p$, lie outside the circle $\mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \delta_p)$. The latter leads to the inequality

$$\det \Delta_{\mathrm{s}}(\lambda) \neq 0, \quad \forall \lambda \in \mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \delta_p)/\{\bar{\lambda}_p\}. \tag{4.4}$$

This inequality, along with (4.1) and (4.2), implies that

$$\mathscr{F}_{\mathrm{s},p}(\lambda) \neq 0, \quad \forall \lambda \in \mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \delta_p). \tag{4.5}$$

We begin the analysis of the set $\mathscr{R}_{\mathrm{s}}(\varepsilon)$ with an analysis of the set of all roots of (2.5), belonging to $\mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \delta_p)$ for sufficiently small ε and δ_p. For the sake of saving the space and non-overloading the paper, we restrict our analysis to the case $n_p = 1$.

4.1 *Asymptotic behaviour of a slow root*

Let us consider the following function of two variables λ and ε in the domain $\Omega_{\mathrm{s}} \stackrel{\triangle}{=} \{(\lambda, \varepsilon)\colon \lambda \in \mathscr{D}_{\mathrm{s}},\ \varepsilon \in [0, \varepsilon^*]\}$

$$g_{\mathrm{s}}(\lambda, \varepsilon) \stackrel{\triangle}{=} \det \Delta(\lambda, \varepsilon). \tag{4.6}$$

This function is continuous and it has continuous partial derivatives of any order with respect to both arguments.

LEMMA 4.1 Let the Assumptions A1 and A2 be satisfied. Let $n_p = 1$. Then, there exist a positive number $\delta_p = \bar{\delta}_p^*$, satisfying (4.3) and (4.5), and a positive number $\bar{\varepsilon}_p^*$, such that for all $\varepsilon \in (0, \bar{\varepsilon}_p^*]$, the original characteristic equation (2.5) has the unique root $\lambda_p(\varepsilon)$, belonging to the circle $\mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \bar{\delta}_p^*)$. This root is a continuous function of ε on the interval $(0, \bar{\varepsilon}_p^*]$, and

$$\lim_{\varepsilon \to +0} \lambda_p(\varepsilon) = \bar{\lambda}_p. \tag{4.7}$$

Proof. Let $\bar{\delta}_p > 0$ be a number such that (4.3) and (4.5) are satisfied for $\delta_p = \bar{\delta}_p$. Consider the equation (2.5) in the domain $\Omega_{\mathrm{s},p} \stackrel{\triangle}{=} \{(\lambda, \varepsilon)\colon \lambda \in \mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \bar{\delta}_p),\ \varepsilon \in [0, \varepsilon^*]\}$.

Using (2.14) and (2.15), as well as (2.22), (2.25), (4.1) and (4.6), one can rewrite (2.5) in the equivalent form

$$(\lambda - \bar{\lambda}_p)\mathscr{F}_{\mathrm{s},p}(\lambda) \det \bar{A}_4 + g(\lambda, \varepsilon) - g(\lambda, 0) = 0. \tag{4.8}$$

By virtue of (2.13) and (4.5), (4.8) is transformed to the equivalent equation

$$\mathscr{H}(\lambda, \varepsilon) \stackrel{\triangle}{=} \lambda - \bar{\lambda}_p + \varepsilon \mathscr{G}_{\mathrm{s}}(\lambda, \varepsilon) = 0, \tag{4.9}$$

where $\mathscr{G}_{\mathrm{s}}(\lambda, \varepsilon)$ is given by

$$\mathscr{G}_{\mathrm{s}}(\lambda, \varepsilon) = \frac{g_{\mathrm{s}}(\lambda, \varepsilon) - g_{\mathrm{s}}(\lambda, 0)}{\varepsilon} (\mathscr{F}_{\mathrm{s},p}(\lambda) \det \bar{A}_4)^{-1}, \quad \lambda \in \mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \bar{\delta}_p), \quad \varepsilon \in (0, \varepsilon^*], \tag{4.10}$$

$$\mathscr{G}_{\mathrm{s}}(\lambda, 0) = \frac{\partial g_{\mathrm{s}}(\lambda, 0)}{\partial \varepsilon} (\mathscr{F}_{\mathrm{s},p}(\lambda) \det \bar{A}_4)^{-1}, \quad \lambda \in \mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \bar{\delta}_p). \tag{4.11}$$

Due to the above mentioned smoothness of $g_{\mathrm{s}}(\lambda, \varepsilon)$, the function $\mathscr{G}(\lambda, \varepsilon)$ is continuous and it has continuous partial derivatives of any order with respect to both arguments in the domain $\Omega_{\mathrm{s},p}$.

By direct calculations, one obtains that $\mathscr{H}(\bar{\lambda}_p, 0) = 0$ and $\partial \mathscr{H}(\bar{\lambda}_p, 0)/\partial \lambda = 1 \neq 0$.

Now, the statements of the lemma directly follow from the Implicit Function Theorem (see, e.g. Schwartz, 1967) applied to (4.9). □

Lemma 4.1 implies that the unique root $\lambda_p(\varepsilon)$ of the original characteristic equation (2.5) in the circle $\mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \bar{\delta}_p^*)$ can be approximate by $\bar{\lambda}_p$ with an error, tending to zero for $\varepsilon \to +0$. The following corollary gives an estimate of this error and proposes a more accurate approximation for $\lambda_p(\varepsilon)$.

COROLLARY 4.1 Let the assumptions A1 and A2 be satisfied. Let $n_p = 1$. Then, for all $\varepsilon \in (0, \bar{\varepsilon}_p^*]$, the root $\lambda_p(\varepsilon)$ of (2.5) can be represented as

$$\lambda_p(\varepsilon) = \bar{\lambda}_p + \varepsilon \bar{\lambda}_p^1 + \varepsilon f_\lambda(\varepsilon), \tag{4.12}$$

where

$$\bar{\lambda}_p^1 = -\mathscr{G}_{\mathrm{s}}(\bar{\lambda}_p, 0), \tag{4.13}$$

and $f_\lambda(\varepsilon)$ is a known function of ε satisfying the condition

$$\lim_{\varepsilon \to +0} f_\lambda(\varepsilon) = 0. \tag{4.14}$$

Proof. Substituting (4.12) into (4.9) and dropping the notation of the dependence of f_λ on ε yield after some rearrangement

$$\mathscr{H}_1(f_\lambda, \varepsilon) \stackrel{\triangle}{=} \bar{\lambda}_p^1 + f_\lambda + \mathscr{G}_{\mathrm{s}}(\bar{\lambda}_p + \varepsilon\bar{\lambda}_p^1 + \varepsilon f_\lambda, 0) + \varepsilon\mathscr{G}_{\mathrm{s},1}(\bar{\lambda}_p + \varepsilon\bar{\lambda}_p^1 + \varepsilon f_\lambda, \varepsilon) = 0, \tag{4.15}$$

where

$$\mathscr{G}_{\mathrm{s},1}(\lambda, \varepsilon) = \frac{\mathscr{G}_{\mathrm{s}}(\lambda, \varepsilon) - \mathscr{G}_{\mathrm{s}}(\lambda, 0)}{\varepsilon}, \quad \lambda \in \mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \bar{\delta}_p^*), \ \ \varepsilon \in (0, \bar{\varepsilon}_p^*], \tag{4.16}$$

$$\mathscr{G}_{\mathrm{s},1}(\lambda, 0) = \frac{\partial \mathscr{G}_{\mathrm{s}}(\lambda, 0)}{\partial \varepsilon}, \quad \lambda \in \mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \bar{\delta}_p^*). \tag{4.17}$$

Due to the smoothness of the function $\mathscr{G}_{\mathrm{s}}(\lambda, \varepsilon)$, mentioned in the proof of Lemma 4.1, the function $\mathscr{G}_{\mathrm{s},1}(\lambda, \varepsilon)$ is continuous and it has continuous partial derivatives of any order with respect to both arguments in the domain $\bar{\Omega}_{\mathrm{s},p}^* = \{(\lambda, \varepsilon)\colon \lambda \in \mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \bar{\delta}_p^*),\ \varepsilon \in [0, \bar{\varepsilon}_p^*]\}$.

Using (4.13) and (4.15)–(4.17), we obtain that $\mathscr{H}_1(0, 0) = 0$ and $\partial\mathscr{H}_1(0, 0)/\partial f_\lambda = 1 \neq 0$. By virtue of the Implicit Function Theorem and Lemma 4.1, one directly has the existence of the unique root $f_\lambda(\varepsilon)$ of (4.15) for all $\varepsilon \in (0, \bar{\varepsilon}_p^*]$, and this root satisfies (4.14). Thus, the corollary is proved. □

4.2 *Structure of the set of slow roots*

The following theorem gives the structure of the set $\mathscr{R}_{\mathrm{s}}(\varepsilon)$ for all sufficiently small $\varepsilon > 0$ in the case where for each $p \in \{1, \ldots, q\}$ the assumptions of Lemma 4.1 are valid.

Let, for each $p \in \{1, \ldots, q\}$ and each $\varepsilon \in (0, \bar{\varepsilon}_p^*]$, $\mathscr{R}_{\mathrm{s},p}(\varepsilon)$ be the set of all roots of the original characteristic equation (2.5) belonging to the circle $\mathscr{O}_{\mathrm{s}}(\bar{\lambda}_p, \bar{\delta}_p^*)$ according to Lemma 4.1. Let $\bar{\varepsilon}_{\mathrm{s}} = \min_{p\in\{1,\ldots,q\}} \bar{\varepsilon}_p^*$. Due to Lemma 4.1, for all $\varepsilon \in (0, \bar{\varepsilon}_{\mathrm{s}}]$, we have the following:

$$\mathscr{R}_{\mathrm{s},p_1}(\varepsilon) \bigcap \mathscr{R}_{\mathrm{s},p_2}(\varepsilon) = \emptyset, \quad \forall\, p_1, p_2 \in \{1, \ldots, q\}, \quad p_1 \neq p_2. \tag{4.18}$$

THEOREM 4.1 Let for each $p \in \{1, \ldots, q\}$ the assumptions of Lemma 4.1 be valid. Then, there exists a positive number $\varepsilon_{\mathrm{s}}^*$, $(\varepsilon_{\mathrm{s}}^* \leqslant \bar{\varepsilon}_{\mathrm{s}})$, such that for all $\varepsilon \in (0, \varepsilon_{\mathrm{s}}^*]$, the set $\mathscr{R}_{\mathrm{s}}(\varepsilon)$ of the slow roots of the original characteristic equation (2.5) has the form

$$\mathscr{R}_{\mathrm{s}}(\varepsilon) = \bigcup_{p=1}^{q} \mathscr{R}_{\mathrm{s},p}(\varepsilon). \tag{4.19}$$

Moreover, the slow roots of (2.5) are simple, and each of them has the respective asymptotic form given by Corollary 4.1.

Proof. The statements of the theorem directly follow from Lemma 4.1, Corollary 4.1, Lemma 3.1 and Theorem 3.1. □

5. Analysis of the set of fast roots

In order to study the set $\mathscr{R}_{\rm f}(\varepsilon)$ of the fast roots of the original characteristic equation (2.5), the transformation of variables $\lambda = \mu/\varepsilon$ is made in (2.5) yielding (2.18) with respect to μ. Thus, the analysis of the set $\mathscr{R}_{\rm f}(\varepsilon)$ is reduced to analysis of the set $\mathscr{R}^{\mu}_{\rm f}(\varepsilon)$ of those roots of (2.18), which satisfy the inclusion

$$\mu(\varepsilon) \in \mathscr{D}_{\rm f}, \quad \forall\, \varepsilon \in (0, \varepsilon^*]. \tag{5.1}$$

This analysis is based on the following properties of the set of roots of the fast characteristic equation (2.9). Namely, due to Bellman & Cooke (1963) and Hale & Verduyn Lunel (1993), if (2.9) does not degenerate to a polynomial one, the set of its roots is an infinite countable set with a single limit point at infinity. The multiplicity of each root is finite. Moreover, for any real constant $\tilde{\gamma}$, there exists no more than a finite number of roots of (2.9) satisfying the inequality ${\rm Re}\mu > \tilde{\gamma}$. Using these properties of the set of roots of (2.9), the set $\mathscr{R}^{\mu}_{\rm f}(\varepsilon)$ is analysed in the way similar to that for the analysis of $\mathscr{R}_{\rm s}(\varepsilon)$. This analysis yields, for all sufficiently small $\varepsilon > 0$, the structure of

$$\mathscr{R}_{{\rm f},\chi}(\varepsilon) \stackrel{\triangle}{=} \{\mu(\varepsilon)\colon \det \hat{\Delta}_2(\mu(\varepsilon), \varepsilon) = 0,\ \mu(\varepsilon) \in \mathscr{D}_{{\rm f},\chi}\}, \tag{5.2}$$

$$\mathscr{R}_{{\rm f},\kappa}(\varepsilon) \stackrel{\triangle}{=} \{\mu(\varepsilon) : \det \hat{\Delta}_2(\mu(\varepsilon), \varepsilon) = 0,\ \mu(\varepsilon) \in \mathscr{D}_{{\rm f},\kappa}\}, \tag{5.3}$$

$$\mathscr{R}_{{\rm f},\gamma}(\tilde{\gamma}, \varepsilon) \stackrel{\triangle}{=} \{\mu(\varepsilon)\colon \det \hat{\Delta}_2(\mu(\varepsilon), \varepsilon) = 0,\ \mu(\varepsilon) \in \mathscr{S}_{{\rm f},\gamma}(\tilde{\gamma})\}, \tag{5.4}$$

where $\tilde{\gamma} < -\gamma$ is a given number, such that for any root μ of the fast characteristic equation (2.9), ${\rm Re}\mu \neq \tilde{\gamma}$, and

$$\mathscr{S}_{{\rm f},\gamma}(\tilde{\gamma}) \stackrel{\triangle}{=} \{\mu : \mu \in \mathscr{D}_{{\rm f},\gamma},\ {\rm Re}\mu > \tilde{\gamma}\}. \tag{5.5}$$

Allowing to $\tilde{\gamma}$ to be infinity, we can represent the set of fast roots $\mathscr{R}_{\rm f}(\varepsilon)$ of the original characteristic equation (2.5) in the form

$$\mathscr{R}_{\rm f}(\varepsilon) = \{\lambda(\varepsilon) = \mu(\varepsilon)/\varepsilon\colon \mu(\varepsilon) \in \mathscr{R}_{{\rm f},\chi}(\varepsilon) \bigcup \mathscr{R}_{{\rm f},\kappa}(\varepsilon) \bigcup \mathscr{R}_{{\rm f},\gamma}(-\infty, \varepsilon)\}. \tag{5.6}$$

Let denote by $\mathscr{P}_{\mathscr{R},{\rm f}}(\varepsilon)$ any of the sets $\mathscr{R}_{{\rm f},\chi}(\varepsilon)$, $\mathscr{R}_{{\rm f},\kappa}(\varepsilon)$, $\mathscr{R}_{{\rm f},\gamma}(\tilde{\gamma}, \varepsilon)$. Let $\mathscr{Q}_{\mathscr{D},{\rm f}}$ be one of the sets $\mathscr{D}_{{\rm f},\chi}$, $\mathscr{D}_{{\rm f},\kappa}$, $\mathscr{S}_{{\rm f},\gamma}(\tilde{\gamma})$, corresponding to $\mathscr{P}_{\mathscr{R},{\rm f}}(\varepsilon)$ according to the definitions (5.2), (5.3), (5.4)–(5.5). It is clear that there exists a finite number of distinct roots of the fast characteristic equation (2.9), belonging to $\mathscr{Q}_{\mathscr{D},{\rm f}}$, and each such root has a finite multiplicity. Let β be the number of such roots, and $\tilde{\mu}_\alpha, \alpha \in \{1, \ldots, \beta\}$ be one of such roots arbitrary chosen. Let m_α, $(m_\alpha \geqslant 1)$ be the multiplicity of $\tilde{\mu}_\alpha$. Hence, the left-hand side of (2.9) can be represented as

$$\det \Delta_{\rm f}(\mu) = (\mu - \tilde{\mu}_\alpha)^{m_\alpha} \mathscr{F}_{{\rm f},\alpha}(\mu), \quad \forall\, \mu \in \mathscr{C}, \tag{5.7}$$

where $\mathscr{F}_{{\rm f},\alpha}(\mu)$ is a known infinitely differentiable function, and

$$\mathscr{F}_{{\rm f},\alpha}(\tilde{\mu}_\alpha) \neq 0. \tag{5.8}$$

Let $\tilde{\delta}_\alpha > 0$ be such that

$$\mathscr{O}_{\rm f}(\tilde{\mu}_\alpha, \tilde{\delta}_\alpha) \stackrel{\triangle}{=} \{\mu\colon |\mu - \tilde{\mu}_\alpha| \leqslant \tilde{\delta}_\alpha\} \subset \mathscr{Q}_{\mathscr{D},{\rm f}}, \tag{5.9}$$

and all the roots of (2.9), belonging to $\mathcal{Q}_{\mathcal{D},\mathrm{f}}$ (excepting $\tilde{\mu}_\alpha$), lie outside the circle $\mathcal{O}_\mathrm{f}(\tilde{\mu}_\alpha, \tilde{\delta}_\alpha)$. This leads to the inequality

$$\det \Delta_\mathrm{f}(\mu) \neq 0, \quad \forall\, \mu \in \mathcal{O}_\mathrm{f}(\tilde{\mu}_\alpha, \tilde{\delta}_\alpha)/\{\tilde{\mu}_\alpha\}. \tag{5.10}$$

The latter, along with (5.7) and (5.8), implies that

$$\mathcal{F}_{\mathrm{f},\alpha}(\mu) \neq 0, \quad \forall\, \mu \in \mathcal{O}_\mathrm{f}(\tilde{\mu}_\alpha, \tilde{\delta}_\alpha). \tag{5.11}$$

For the same reasons, as in Section 4, we restrict our analysis to the case $m_\alpha = 1$.

5.1 *Asymptotic behaviour of a fast root*

Consider the following function of two variables μ and ε in the domain $\Omega_\mathrm{f} \stackrel{\triangle}{=} \{(\mu, \varepsilon)\colon \mu \in \mathcal{Q}_{\mathcal{D},\mathrm{f}},\ \varepsilon \in [0, \varepsilon^*]\}$

$$g_\mathrm{f}(\mu.\varepsilon) \stackrel{\triangle}{=} \det \hat{\Delta}_2(\mu, \varepsilon). \tag{5.12}$$

This function is continuous and it has continuous partial derivatives of any order with respect to both arguments.

Along with (5.12), let us consider the function $\mathcal{G}_\mathrm{f}(\mu, \varepsilon)$ given as follows:

$$\mathcal{G}_\mathrm{f}(\mu, \varepsilon) = (-1)^n \frac{g_\mathrm{f}(\mu, \varepsilon) - g_\mathrm{f}(\mu, 0)}{\varepsilon} (\mu^n \mathcal{F}_{\mathrm{f},\alpha}(\mu))^{-1},$$

$$\mu \in \mathcal{O}_\mathrm{f}(\tilde{\mu}_\alpha, \tilde{\delta}_\alpha), \quad \varepsilon \in (0, \varepsilon^*], \tag{5.13}$$

$$\mathcal{G}_\mathrm{f}(\mu, 0) = (-1)^n \frac{\partial g_\mathrm{f}(\mu, 0)}{\partial \varepsilon} \left(\mu^n \mathcal{F}_{\mathrm{f},\alpha}(\mu)\right)^{-1}, \quad \mu \in \mathcal{O}_\mathrm{f}(\tilde{\mu}_\alpha, \tilde{\delta}_\alpha), \tag{5.14}$$

where $\tilde{\delta}_\alpha > 0$ is any given number satisfying (5.9) and (5.11). Due to the above mentioned smoothness of $g_\mathrm{f}(\mu, \varepsilon)$, the function $\mathcal{G}_\mathrm{f}(\mu, \varepsilon)$ is continuous and it has continuous partial derivatives of any order with respect to both arguments in the domain $\Omega_{\mathrm{f},\alpha} \stackrel{\triangle}{=} \{(\mu, \varepsilon)\colon \mu \in \mathcal{O}_\mathrm{f}(\tilde{\mu}_\alpha, \tilde{\delta}_\alpha),\ \varepsilon \in [0, \varepsilon^*]\}$.

LEMMA 5.1 Let the Assumptions A1 and A2 be satisfied. Let $m_\alpha = 1$. Then, there exist a positive number $\delta_\alpha = \tilde{\delta}^*_\alpha$, satisfying (5.9) and (5.11), and a positive number $\tilde{\varepsilon}^*_\alpha$, such that for all $\varepsilon \in (0, \tilde{\varepsilon}^*_\alpha]$, (2.18) has the unique root $\mu_\alpha(\varepsilon)$ belonging to the circle $\mathcal{O}_\mathrm{f}(\tilde{\mu}_\alpha, \tilde{\delta}^*_\alpha)$. This root is a continuous function of ε on the interval $(0, \tilde{\varepsilon}^*_\alpha]$, and it can be represented as

$$\mu_\alpha(\varepsilon) = \tilde{\mu}_\alpha + \varepsilon \tilde{\mu}^1_\alpha + \varepsilon f_\mu(\varepsilon), \tag{5.15}$$

where

$$\tilde{\mu}^1_\alpha = -\mathcal{G}_\mathrm{f}(\tilde{\mu}_\alpha, 0), \tag{5.16}$$

and $f_\mu(\varepsilon)$ is a known function of ε satisfying the condition

$$\lim_{\varepsilon \to +0} f_\mu(\varepsilon) = 0. \tag{5.17}$$

Proof. The lemma is proved similar to Lemma 4.1 and Corollary 4.1, using (2.9), (2.19)–(2.21), (5.7), (5.11) and (5.12)–(5.14). □

5.2 *Structure of the set of fast roots*

Let, for each $\alpha \in \{1, \ldots, \beta\}$ and each $\varepsilon \in (0, \tilde{\varepsilon}_\alpha^*]$, $\mathscr{P}_{\mathrm{f},\alpha}(\varepsilon)$ be the set of all roots of (2.18), belonging to the circle $\mathscr{O}_\mathrm{f}(\tilde{\mu}_\alpha, \tilde{\delta}_\alpha^*)$ according to Lemma 5.1. Let $\tilde{\varepsilon}_\mathrm{f} = \min_{\alpha\in\{1,\ldots,\beta\}} \tilde{\varepsilon}_\alpha^*$. Due to Lemma 5.1, for all $\varepsilon \in (0, \tilde{\varepsilon}_\mathrm{f}]$, we have that

$$\mathscr{P}_{\mathrm{f},\alpha_1}(\varepsilon) \bigcap \mathscr{P}_{\mathrm{f},\alpha_2}(\varepsilon) = \emptyset, \quad \forall\, \alpha_1, \alpha_2 \in \{1, \ldots, \beta\}, \quad \alpha_1 \neq \alpha_2. \tag{5.18}$$

The following theorem is obtained similar to Theorem 4.1.

THEOREM 5.1 Let for each $\alpha \in \{1, \ldots, \beta\}$, the assumptions of Lemma 5.1 be valid. Then, there exists a positive number ε_f^*, $(\varepsilon_\mathrm{f}^* \leqslant \tilde{\varepsilon}_\mathrm{f})$, such that for all $\varepsilon \in (0, \varepsilon_\mathrm{f}^*]$, the set $\mathscr{P}_{\mathscr{R},\mathrm{f}}(\varepsilon)$ of roots of (2.18) has the form

$$\mathscr{P}_{\mathscr{R},\mathrm{f}}(\varepsilon) = \bigcup_{\alpha=1}^{\beta} \mathscr{P}_{\mathrm{f},\alpha}(\varepsilon). \tag{5.19}$$

Moreover, the roots of (2.18), belonging to $\mathscr{P}_{\mathscr{R},\mathrm{f}}(\varepsilon)$, are simple and each of them has the respective asymptotic form given by (5.15).

6. Stability analysis of (2.3): spectrum structure approach

In this section, we consider some applications of the above obtained results on the structure of the spectrum of the system (2.1)–(2.2) to its stability analysis. This analysis is carried out for its equivalent form (2.3).

6.1 *Case of exponentially stable fast subsystem*

In this subsection, the following case is treated:

$$\bar{a}_\lambda^{\max} \triangleq \max_{p\in\{1,\ldots,q\}} \mathrm{Re}\bar{\lambda}_p < 0 \tag{6.1}$$

and

$$\mathscr{M}_+ = \emptyset, \quad \mathscr{M}_0 = \emptyset. \tag{6.2}$$

Remember that $\bar{\lambda}_p$, $(p = 1, \ldots, q)$ are all distinct roots of the slow characteristic equation (2.15), $\mathscr{M}_+$, $\mathscr{M}_0$ and $\mathscr{M}_-$ are the sets of all distinct roots of the fast characteristic equation (2.9) with positive, zero and negative real part, respectively.

Consider the following initial condition for the system (2.3):

$$z(\tau) = \varphi(\tau), \quad \tau \in [-\varepsilon h, 0], \tag{6.3}$$

where $\varphi(\cdot) \in C[-\varepsilon_0 h, 0; E^{n+m}]$ is any given; ε_0 is some positive constant.

Represent the vector-valued function $\varphi(\tau)$ in the block form

$$\varphi(\tau) = \mathrm{col}(\varphi_x(\tau), \varphi_y(\tau)), \quad \varphi_x(\cdot) \in C[-\varepsilon_0 h, 0; E^n], \quad \varphi_y(\cdot) \in C[-\varepsilon_0 h, 0; E^m]. \tag{6.4}$$

THEOREM 6.1 Let the Assumptions A1 and A2 be satisfied. Let the conditions (6.1) and (6.2) hold. Let $\gamma > 0$ and ν be any given constants, satisfying the inequalities (3.2) and

$$0 < \nu < |\bar{a}_{\lambda}^{\max}|, \tag{6.5}$$

respectively. Then, there exist positive numbers $\bar{\varepsilon}(\nu, \gamma) \leqslant \varepsilon_0$ and $c(\nu, \gamma)$, such that the solution $z(t, \varepsilon) = \mathrm{col}(x(t, \varepsilon), y(t, \varepsilon))$ of the initial-value problem (IVP) (2.3), (6.3) satisfies the following inequalities for any $\varepsilon \in (0, \bar{\varepsilon}(\nu, \gamma)]$:

$$\|x(t, \varepsilon)\| \leqslant c(\nu, \gamma) \exp(-\nu t)[\|\varphi_x(0)\| + \varepsilon(\|\varphi_x(\cdot)\|_C + \|\varphi_y(\cdot)\|_C)], \quad t \geqslant 0, \tag{6.6}$$

$$\|y(t, \varepsilon)\| \leqslant c(\nu, \gamma) \exp(-\nu t)[\|\varphi_x(0)\| + \varepsilon(\|\varphi_x(\cdot)\|_C + \|\varphi_y(\cdot)\|_C)] + c(\nu, \gamma) \exp\left(-\frac{\gamma t}{\varepsilon}\right)(\|\varphi_x(\cdot)\|_C + \|\varphi_y(\cdot)\|_C), \quad t \geqslant 0, \tag{6.7}$$

where $\|\varphi_x(\cdot)\|_C$ and $\|\varphi_y(\cdot)\|_C$ are the uniform norm of $\varphi_x(\cdot)$ and $\varphi_y(\cdot)$, respectively, on the interval $[-\varepsilon_0 h, 0]$.

Proof. Let us prove the inequality (6.7). The inequality (6.6) is proved similarly.

Let, for any $\varepsilon > 0$, $\Psi(t, \varepsilon)$, $t \geqslant 0$ be the fundamental matrix of the system (2.3). By using the variation of constant formula (see, e.g. Hale & Verduyn Lunel, 1993), we obtain the solution of the IVP (2.3), (6.3) in the form

$$z(t, \varepsilon) = \Lambda_1(t, \varepsilon) + \Lambda_2(t, \varepsilon), \quad t \geqslant 0, \tag{6.8}$$

where

$$\Lambda_1(t, \varepsilon) = \Psi(t, \varepsilon)\varphi(0), \quad \Lambda_2(t, \varepsilon) = \int_0^h \Psi(t - \varepsilon s, \varepsilon)\mathscr{E}_\varepsilon \left\{\int_{-h}^{-s} [\mathrm{d}A(\eta)]\varphi(\varepsilon(s + \eta))\right\} \mathrm{d}s, \tag{6.9}$$

$\mathscr{E}_\varepsilon$ is given by (2.17).

For the sake of the further consideration, let us partition the matrix $\Psi(t, \varepsilon)$ and the vectors $\Lambda_i(t, \varepsilon)$, $(i = 1, 2)$ into blocks as follows:

$$\Psi(t, \varepsilon) = \begin{pmatrix} \Psi_1(t, \varepsilon) & \Psi_2(t, \varepsilon) \\ \Psi_3(t, \varepsilon) & \Psi_4(t, \varepsilon) \end{pmatrix}, \quad \Lambda_i(t, \varepsilon) = \begin{pmatrix} \Lambda_{i1}(t, \varepsilon) \\ \Lambda_{i2}(t, \varepsilon) \end{pmatrix}, \quad i = 1, 2, \tag{6.10}$$

where the blocks $\Psi_1(t, \varepsilon)$ and $\Psi_4(t, \varepsilon)$ are of the dimensions $n \times n$ and $m \times m$, respectively; the blocks $\Lambda_{i1}(t, \varepsilon)$ and $\Lambda_{i2}(t, \varepsilon)$, $(i = 1, 2)$ are of the dimensions n and m, respectively.

Thus,

$$y(t, \varepsilon) = \Lambda_{12}(t, \varepsilon) + \Lambda_{22}(t, \varepsilon), \tag{6.11}$$

and in order to prove the inequality (6.7), one has to estimates the vector-valued functions $\Lambda_{i2}(t, \varepsilon)$, $(i = 1, 2)$. Let us start with $\Lambda_{12}(t, \varepsilon)$. Due to (6.4), (6.9) and (6.10),

$$\Lambda_{12}(t, \varepsilon) = \Psi_3(t, \varepsilon)\varphi_x(0) + \Psi_4(t, \varepsilon)\varphi_y(0). \tag{6.12}$$

By virtue of the condition (6.1)–(6.2), the inequalities (3.2) and (6.5), and the results of Glizer (2003, Theorem 2.3), there exist positive numbers $\bar{\varepsilon}_1(\nu, \gamma)$ and $c_1(\nu, \gamma)$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_1(\nu, \gamma)]$ the following inequalities are satisfied:

$$\|\Psi_3(t,\varepsilon)\| \leqslant c_1(\nu,\gamma)\exp(-\nu t), \quad \|\Psi_4(t,\varepsilon)\| \leqslant c_1(\nu,\gamma)\left[\varepsilon\exp(-\nu t)+\exp\left(-\frac{\gamma t}{\varepsilon}\right)\right], \quad t \geqslant 0. \tag{6.13}$$

By using (6.12) and (6.13), we obtain the following estimate of $\Lambda_{12}(t,\varepsilon)$ for any $\varepsilon \in (0, \bar{\varepsilon}_1(\nu,\gamma)]$:

$$\|\Lambda_{12}(t,\varepsilon)\| \leqslant c_1(\nu,\gamma)\exp(-\nu t)(\|\varphi_x(0)\|+\varepsilon\|\varphi_y(0)\|)+c_1(\nu,\gamma)\exp\left(-\frac{\gamma t}{\varepsilon}\right)\|\varphi_y(0)\|, \quad t \geqslant 0. \tag{6.14}$$

Now, proceed to the vector-valued function $\Lambda_{22}(t,\varepsilon)$. Due to (2.17), (6.4), (6.9) and (6.10),

$$\begin{aligned}\Lambda_{22}(t,\varepsilon) = \varepsilon\int_0^h \Psi_3(t-\varepsilon s,\varepsilon)\left\{\int_{-h}^{-s}[\mathrm{d}A_1(\eta)]\varphi_x(\varepsilon(s+\eta))+\int_{-h}^{-s}[\mathrm{d}A_2(\eta)]\varphi_y(\varepsilon(s+\eta))\right\}\mathrm{d}s \\ +\int_0^h \Psi_4(t-\varepsilon s,\varepsilon)\left\{\int_{-h}^{-s}[\mathrm{d}A_3(\eta)]\varphi_x(\varepsilon(s+\eta))+\int_{-h}^{-s}[\mathrm{d}A_4(\eta)]\varphi_y(\varepsilon(s+\eta))\right\}\mathrm{d}s.\end{aligned} \tag{6.15}$$

By using (6.13), (6.15) and results of Kolmogorov & Fomin (1975, Chapter VI, Section 6), we obtain the existence of positive numbers $\bar{\varepsilon}_2(\nu,\gamma) \leqslant \min\{\varepsilon_0, \bar{\varepsilon}_1(\nu,\gamma)\}$ and $c_2(\nu,\gamma)$ such that the following estimate of $\Lambda_{22}(t,\varepsilon)$ holds for any $\varepsilon \in (0, \bar{\varepsilon}_2(\nu,\gamma)]$:

$$\|\Lambda_{22}(t,\varepsilon)\| \leqslant c_2(\nu,\gamma)\left[\varepsilon\exp(-\nu t)+\exp\left(-\frac{\gamma t}{\varepsilon}\right)\right]\left(\|\varphi_x(\cdot)\|_C+\|\varphi_y(\cdot)\|_C\right), \quad t \geqslant 0. \tag{6.16}$$

Now, the inequality (6.7) is a direct consequence of (6.11) and the inequalities (6.14) and (6.16). □

The fulfilment of the inequalities (6.6)–(6.7) means the exponential stability of the system (2.3) uniformly with respect to ε for all sufficiently small $\varepsilon > 0$. In Theorem 6.1, such a stability was obtained under the condition that all roots of the slow and fast characteristic equations have negative real parts. It is clear that the negativeness of real parts of the roots of the slow characteristic equation is necessary for the uniform exponential stability of the system (2.3). However, such a statement is not correct with respect to the roots of the fast characteristic equation. Below, the uniform exponential stability of the system (2.3) is established under a weaker assumption on the set of these roots than the assumption (6.2) of Theorem 6.1.

6.2 *Case of no exponential stability for the fast subsystem*

In what follows, we assume

$$\mathscr{M}_+ = \emptyset, \quad \mathscr{M}_0 \neq \emptyset. \tag{6.17}$$

Let $\tilde{\mu}_\alpha$, $(\alpha = 1,\ldots,\beta)$, be all distinct pure imaginary roots of the fast characteristic equation (2.9). Denote

$$\tilde{a}_\mu^{\min} \triangleq \min_{\alpha\in\{1,\ldots,\beta\}} \mathrm{Re}(\mathscr{G}_{\mathrm{f}}(\tilde{\mu}_\alpha, 0)). \tag{6.18}$$

LEMMA 6.1 Let the Assumptions A1 and A2 be satisfied. Let the condition (6.17) hold, and all pure imaginary roots $\tilde{\mu}_\alpha$, $(\alpha = 1, \ldots, \beta)$ of the fast characteristic equation (2.9) be simple. Let

$$\mathrm{Re}(\mathscr{G}_{\mathrm{f}}(\tilde{\mu}_\alpha, 0)) > 0, \quad \alpha = 1,\ldots,\beta, \tag{6.19}$$

where $\mathscr{G}_{\mathrm{f}}(\tilde{\mu}_{\alpha}, 0)$ is given in Lemma 5.1.

Let the condition (6.1) hold. Let $\gamma > 0$ and ν be any given constants, satisfying the inequalities (3.2) and

$$0 < \nu < \min\left\{|\bar{a}_{\lambda}^{\max}|, \tilde{a}_{\mu}^{\min}\right\}, \tag{6.20}$$

respectively. Then, there exists a positive number $\bar{\varepsilon}(\nu, \gamma)$, such that the following inequality is valid:

$$\sup_{\varepsilon\in(0,\bar{\varepsilon}(\nu,\gamma)]} \mathrm{Re}\lambda(\varepsilon) < -\nu, \tag{6.21}$$

where $\lambda(\varepsilon)$ is any root of (2.5).

Proof. First, note that, due to the definitions of $\tilde{a}_{\mu}^{\min}$ and the inequality (6.19), the value $\tilde{a}_{\mu}^{\min}$ is positive, meaning the correctness of the inequality (6.20).

Consider the set $\mathscr{R}_{\mathrm{s}}(\varepsilon)$ of slow roots of the characteristic equation (2.5). By setting $\sigma_2 = -\nu$ in (3.8)–(3.9) and using Theorem 3.1, we obtain the existence of $\bar{\varepsilon}_1(\nu) > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}_1(\nu)]$, the following inequality is valid:

$$\sup_{\varepsilon\in(0,\bar{\varepsilon}_1(\nu)]} \mathrm{Re}\lambda(\varepsilon) < -\nu, \quad \lambda(\varepsilon) \in \mathscr{R}_{\mathrm{s}}(\varepsilon). \tag{6.22}$$

Proceed to the sets $\mathscr{R}_{\mathrm{f},\chi}(\varepsilon)$, $\mathscr{R}_{\mathrm{f},\kappa}(\varepsilon)$ and $\mathscr{R}_{\mathrm{f},\gamma}(\tilde{\gamma}, \varepsilon)$ defined by (5.2), (5.3) and (5.4), respectively. Since $\mathscr{M}_{+} = \emptyset$, then for a given $\chi > 0$, there exists a number $\bar{\varepsilon}_2 > 0$, such that for all $\varepsilon \in (0, \bar{\varepsilon}_2]$,

$$\mathscr{R}_{\mathrm{f},\chi}(\varepsilon) = \emptyset. \tag{6.23}$$

By virtue of Theorem 3.1, the elements of the set $\mathscr{R}_{\mathrm{f},\gamma}(\tilde{\gamma}, \varepsilon)$ satisfy the following inequality for any $\tilde{\gamma} < -\gamma$ and all $\varepsilon \in (0, \bar{\varepsilon}_3(\gamma)]$ with some $0 < \bar{\varepsilon}_3(\gamma) \leqslant \bar{\varepsilon}_1(\nu)$:

$$\mathrm{Re}\mu(\varepsilon) < -\gamma, \quad \mu(\varepsilon) \in \mathscr{R}_{\mathrm{f},\gamma}(\tilde{\gamma}, \varepsilon). \tag{6.24}$$

By using Lemma 5.1, Theorem 5.1, (6.18) and the inequalities (6.19), (6.20), one obtains the existence of a number $\bar{\varepsilon}_4(\nu)$, $(0 < \bar{\varepsilon}_4(\nu) \leqslant \varepsilon_{\mathrm{f}}^{*})$, such that all elements of the set $\mathscr{R}_{\mathrm{f},\kappa}(\varepsilon)$ satisfy the following inequality:

$$\sup_{\varepsilon\in(0,\bar{\varepsilon}_4(\nu)]} \mathrm{Re}\left(\frac{1}{\varepsilon}\mu(\varepsilon)\right) < -\nu, \quad \mu(\varepsilon) \in \mathscr{R}_{\mathrm{f},\kappa}(\varepsilon). \tag{6.25}$$

By using (5.6) and (6.23), and the inequalities (6.22), (6.24) and (6.25), one directly obtains the statement of the lemma with $\bar{\varepsilon}(\nu, \gamma) = \min(\bar{\varepsilon}_1(\nu), \bar{\varepsilon}_2, \bar{\varepsilon}_3(\gamma), \bar{\varepsilon}_4(\nu))$. □

Thus, under the conditions of Lemma 6.1, for all $\varepsilon \in (0, \bar{\varepsilon}(\nu, \gamma)]$, any root of the original characteristic equation (2.5) belongs either to the domain $\mathscr{D}_{\mathrm{s}}$ with $\sigma_2 = -\nu$ or to the domain $\tilde{\mathscr{D}}_{\mathrm{f},\kappa}(\varepsilon)$ with $\chi = -\varepsilon\nu$ or to the domain $\tilde{\mathscr{D}}_{\mathrm{f},\gamma}(\varepsilon)$.

LEMMA 6.2 Let the conditions of Lemma 6.1 be satisfied. Then, the fundamental matrix $\Psi(t, \varepsilon)$ of the system (2.3) can be represented in the form

$$\Psi(t,\varepsilon) = \frac{1}{2\pi\mathrm{i}}\left\{\int_{\partial\mathscr{D}_{\mathrm{s}}} \Theta(\lambda, t, \varepsilon)\mathrm{d}\lambda + \lim_{\psi\to+\infty}\int_{-\gamma/\varepsilon-\mathrm{i}\psi}^{-\gamma/\varepsilon+\mathrm{i}\psi} \Theta(\lambda, t, \varepsilon)\mathrm{d}\lambda \right.$$
$$\left. + \int_{\partial\tilde{\mathscr{D}}_{\mathrm{f},\kappa}^{+}(\varepsilon)} \Theta(\lambda, t, \varepsilon)\mathrm{d}\lambda + \int_{\partial\tilde{\mathscr{D}}_{\mathrm{f},\kappa}^{-}(\varepsilon)} \Theta(\lambda, t, \varepsilon)\mathrm{d}\lambda \right\}, \quad t > 0, \quad \varepsilon \in (0, \bar{\varepsilon}(\nu, \gamma)], \tag{6.26}$$

where

$$\Theta(\lambda, t, \varepsilon) = \exp(\lambda t)\Delta^{-1}(\lambda, \varepsilon)E_\varepsilon, \quad t > 0, \tag{6.27}$$

$$\tilde{\mathscr{D}}^{+}_{\mathrm{f},\kappa}(\varepsilon) = \{\lambda\colon -\gamma/\varepsilon < \mathrm{Re}\lambda < -\nu, \ \ \kappa_1/\varepsilon < \mathrm{Im}\lambda < \kappa_2/\varepsilon\}, \tag{6.28}$$

$$\tilde{\mathscr{D}}^{-}_{\mathrm{f},\kappa}(\varepsilon) = \{\lambda\colon -\gamma/\varepsilon < \mathrm{Re}\lambda < -\nu, \ \ -\kappa_2/\varepsilon < \mathrm{Im}\lambda < -\kappa_1/\varepsilon\}, \tag{6.29}$$

$\partial\mathscr{D}$ is the boundary of a set $\mathscr{D}$ in a complex plane, the direction of motion along each of the curves $\partial\mathscr{D}_\mathrm{s}$, $\partial\mathscr{D}^{+}_{\mathrm{f},\kappa}(\varepsilon)$ and $\partial\mathscr{D}^{-}_{\mathrm{f},\kappa}(\varepsilon)$ is opposite to the clockwise one, the curve of the integration in the second integral in the right-hand side of (6.26) is the straight-line segment connecting the initial and terminal points.

Proof. By using Lemma 6.1 and the result of Hale & Verduyn Lunel (1993) on the representation of the fundamental matrix of a linear autonomous time-delay system, one obtains for all $\varepsilon \in (0, \bar{\varepsilon}(\nu, \gamma)]$

$$\Psi(t, \varepsilon) = \frac{1}{2\pi\mathrm{i}} \lim_{\psi\to+\infty} \int_{-\nu+\mathrm{i}\psi}^{-\nu+\mathrm{i}\psi} \Theta(\lambda, t, \varepsilon)\mathrm{d}\lambda, \quad t > 0, \tag{6.30}$$

where the curve of the integration is the straight-line segment connecting the initial and terminal points.

Let $\varepsilon \in (0, \bar{\varepsilon}(\nu, \gamma)]$ be any but fixed. For any $\psi > \kappa_2/\varepsilon$, consider the domains

$$\mathscr{D}_1(\varepsilon) = \{\lambda\colon -\gamma/\varepsilon < \mathrm{Re}\lambda < -\nu, \ \ \kappa_2/\varepsilon < \mathrm{Im}\lambda < \psi\}, \tag{6.31}$$

$$\mathscr{D}_2(\varepsilon) = \{\lambda\colon -\gamma/\varepsilon < \mathrm{Re}\lambda < \sigma_1, \ \ -\kappa_1/\varepsilon < \mathrm{Im}\lambda < \kappa_1/\varepsilon\}, \tag{6.32}$$

$$\mathscr{D}_3(\varepsilon) = \{\lambda\colon \sigma_1 < \mathrm{Re}\lambda < -\nu, \ \ \rho_2 < \mathrm{Im}\lambda < \kappa_1/\varepsilon\}, \tag{6.33}$$

$$\mathscr{D}_4(\varepsilon) = \{\lambda\colon \sigma_1 < \mathrm{Re}\lambda < -\nu, \ \ -\kappa_1/\varepsilon < \mathrm{Im}\lambda < \rho_1\}, \tag{6.34}$$

$$\mathscr{D}_5(\varepsilon) = \{\lambda\colon -\gamma/\varepsilon < \mathrm{Re}\lambda < -\nu, \ \ -\psi < \mathrm{Im}\lambda < -\kappa_2/\varepsilon\}. \tag{6.35}$$

By virtue of the Cauchy theorem

$$\int_{\partial\mathscr{D}_l(\varepsilon)} \Theta(\lambda, t, \varepsilon)\mathrm{d}\lambda = 0, \quad l = 1, \ldots, 5, \tag{6.36}$$

where the direction along the boundary $\partial\mathscr{D}_j(\varepsilon)$ of the domain $\mathscr{D}_j(\varepsilon)$ is clockwise.

Using (6.36) yields the following chain of equalities:

$$
\begin{aligned}
\int_{-\nu+\mathrm{i}\psi}^{-\nu+\mathrm{i}\psi} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda &= \int_{-\nu+\mathrm{i}\psi}^{-\nu+\mathrm{i}\psi} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda + \sum_{l=1}^{5} \int_{\partial\mathscr{D}_l(\varepsilon)} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda \\
&= \int_{\partial\mathscr{D}_{\mathrm{s}}} \Theta(\lambda,t,\varepsilon) + \int_{\partial\tilde{\mathscr{D}}^{+}_{\mathrm{f},\kappa}(\varepsilon)} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda + \int_{\partial\tilde{\mathscr{D}}^{-}_{\mathrm{f},\kappa}(\varepsilon)} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda \\
&\quad + \int_{-\gamma/\varepsilon-\mathrm{i}\psi}^{-\gamma/\varepsilon+\mathrm{i}\psi} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda + \int_{-\gamma/\varepsilon+\mathrm{i}\psi}^{-\nu+\mathrm{i}\psi} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda \\
&\quad + \int_{-\nu-\mathrm{i}\psi}^{-\gamma/\varepsilon-\mathrm{i}\psi} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda.
\end{aligned}
\tag{6.37}
$$

By using Glizer (2003, Lemma 2.5), one has

$$
\lim_{\psi\to+\infty} \int_{-\gamma/\varepsilon+\mathrm{i}\psi}^{-\nu+\mathrm{i}\psi} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda = 0, \qquad \lim_{\psi\to+\infty} \int_{-\nu-\mathrm{i}\psi}^{-\gamma/\varepsilon-\mathrm{i}\psi} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda = 0. \tag{6.38}
$$

Now, the statement of the lemma is a direct consequence of (6.30), (6.37) and (6.38). □

LEMMA 6.3 Let the conditions of Lemma 6.1 be satisfied. Then, there exist a positive number ε_ν, such that for all $\varepsilon \in (0, \varepsilon_\nu]$, the following inequality is satisfied:

$$
\left\| \int_{\partial\mathscr{D}_{\mathrm{s}}} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda \right\| \leqslant c\exp(-\nu t), \quad t > 0, \tag{6.39}
$$

where $c > 0$ is some positive constant independent of ε.

Proof. The lemma is proved very similar to Glizer (2003, Lemma 2.3). □

LEMMA 6.4 Let the conditions of Lemma 6.1 be satisfied. Then, there exist a positive number ε_γ, such that for all $\varepsilon \in (0, \varepsilon_\gamma]$, the following inequality is satisfied:

$$
\left\| \lim_{\psi\to+\infty} \int_{-\gamma/\varepsilon-\mathrm{i}\psi}^{-\gamma/\varepsilon+\mathrm{i}\psi} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda \right\| \leqslant c\exp\left(-\frac{\gamma t}{\varepsilon}\right), \quad t > 0, \tag{6.40}
$$

where $c > 0$ is some positive constant independent of ε.

Proof. The lemma is proved very similar to Glizer (2003, Lemma 2.4). □

LEMMA 6.5 Let the conditions of Lemma 6.1 be satisfied. Then, there exist a positive number $\tilde{\varepsilon}(\nu,\gamma)$, such that for all $\varepsilon \in (0, \tilde{\varepsilon}(\nu,\gamma)]$, the following inequalities are satisfied:

$$
\left\| \int_{\partial\tilde{\mathscr{D}}^{+}_{\mathrm{f},\kappa}(\varepsilon)} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda \right\| \leqslant c\exp(-\nu t), \quad t > 0, \tag{6.41}
$$

$$
\left\| \int_{\partial\tilde{\mathscr{D}}^{-}_{\mathrm{f},\kappa}(\varepsilon)} \Theta(\lambda,t,\varepsilon)\mathrm{d}\lambda \right\| \leqslant c\exp(-\nu t), \quad t > 0, \tag{6.42}
$$

where $c > 0$ is some positive constant independent of ε.

Proof. Let us start with the proof of the inequality (6.41). First of all note that, due to Lemma 6.1, there exists a positive number $\tilde{\varepsilon}_1(\nu, \gamma)$, $(\tilde{\varepsilon}_1(\nu, \gamma) \leqslant \bar{\varepsilon}(\nu, \gamma))$, such that the matrix $\Delta^{-1}(t, \varepsilon)$ exists for all $\varepsilon \in (0, \tilde{\varepsilon}_1(\nu, \gamma)]$ and all $\lambda \in \partial\tilde{\mathscr{D}}^+_{\mathrm{f},\kappa}(\varepsilon)$. Rewrite the matrix $\Delta^{-1}(\lambda, \varepsilon)E_\varepsilon$ in the form

$$\Delta^{-1}(\lambda, \varepsilon)E_\varepsilon = N(\lambda, \varepsilon) - \frac{1}{\lambda} I_{n+m}, \quad \varepsilon \in (0, \tilde{\varepsilon}_1(\nu, \gamma)], \ \ \lambda \in \partial\tilde{\mathscr{D}}^+_{\mathrm{f},\kappa}(\varepsilon), \tag{6.43}$$

where

$$N(\lambda, \varepsilon) = \frac{1}{\lambda}\Delta^{-1}(\lambda, \varepsilon)\int_{-h}^{0} \exp(\varepsilon\lambda\eta)\mathrm{d}A(\eta). \tag{6.44}$$

Hence,

$$\int_{\partial\tilde{\mathscr{D}}^+_{\mathrm{f},\kappa}(\varepsilon)} \Theta(\lambda, t, \varepsilon)\mathrm{d}\lambda = \int_{\partial\tilde{\mathscr{D}}^+_{\mathrm{f},\kappa}(\varepsilon)} \exp(\lambda t)N(\lambda, \varepsilon)\mathrm{d}\lambda - \int_{\partial\tilde{\mathscr{D}}^+_{\mathrm{f},\kappa}(\varepsilon)} \frac{\exp(\lambda t)}{\lambda} I_{n+m}\,\mathrm{d}\lambda. \tag{6.45}$$

Let us estimate the integrals in the right-hand side of (6.45). We start with the second integral. Due to (6.28), one has for all $\varepsilon \in (0, \tilde{\varepsilon}_1(\nu, \gamma)]$,

$$|\exp(\lambda t)| \leqslant \exp(-\nu t), \quad \lambda \in \partial\tilde{\mathscr{D}}^+_{\mathrm{f},\kappa}(\varepsilon), \quad t > 0, \tag{6.46}$$

$$\frac{1}{|\lambda|} \leqslant c_1\varepsilon, \quad \lambda \in \partial\tilde{\mathscr{D}}^+_{\mathrm{f},\kappa}(\varepsilon), \tag{6.47}$$

where $c_1 > 0$ is some constant independent of ε. Hence,

$$\left\| \int_{\partial\tilde{\mathscr{D}}^+_{\mathrm{f},\kappa}(\varepsilon)} \frac{\exp(\lambda t)}{\lambda} I_{n+m}\,\mathrm{d}\lambda \right\| \leqslant c_2 \exp(-\nu t), \quad \varepsilon \in (0, \tilde{\varepsilon}_1(\nu, \gamma)], \quad t > 0, \tag{6.48}$$

where $c_2 > 0$ is some constant independent of ε.

Proceed to the first integral in the right-hand side of (6.45). By using (6.28), one can show very similar to Glizer (2003, Proof of Lemma 2.4) that the matrix $N(\lambda, \varepsilon)$ is bounded, i.e. the following inequality is satisfied:

$$\|N(\lambda, \varepsilon)\| \leqslant c_3, \quad 0 < \varepsilon \leqslant \tilde{\varepsilon}_2(\nu, \gamma) \leqslant \tilde{\varepsilon}_1(\nu, \gamma), \quad \lambda \in \partial\mathscr{D}^+_{\mathrm{f},\kappa}(\varepsilon), \tag{6.49}$$

with some positive constants $\tilde{\varepsilon}_2(\nu, \gamma)$ and c_3 independent of ε. By virtue of the inequalities (6.46) and (6.49), we obtain

$$\left\| \int_{\partial\tilde{\mathscr{D}}^+_{\mathrm{f},\kappa}(\varepsilon)} \exp(\lambda t)N(\lambda, \varepsilon)\mathrm{d}\lambda \right\| \leqslant c_4 \exp(-\nu t), \quad \varepsilon \in (0, \tilde{\varepsilon}_2(\nu, \gamma)], \quad t > 0, \tag{6.50}$$

where $c_4 > 0$ is some constant.

Now, (6.45) and the inequalities (6.48) and (6.50) prove the inequality (6.41). The inequality (6.42) is proved similarly. Thus, the proof of the lemma is completed. □

Based on Lemmas 6.2–6.5, one directly obtain the following lemma.

LEMMA 6.6 Let the conditions of Lemma 6.1 be satisfied. Then, there exists a positive number $\hat{\varepsilon}(\nu, \gamma)$, such that for all $\varepsilon \in (0, \hat{\varepsilon}(\nu, \gamma)]$, the fundamental matrix $\Psi(t, \varepsilon)$ of the system (2.3) satisfies the inequality $\|\Psi(t, \varepsilon)\| \leqslant c \exp(-\nu t), \quad t > 0$, where $c > 0$ is some constant independent of ε.

THEOREM 6.2 Let the conditions of Lemma 6.1 be satisfied. Then, there exist positive numbers $\hat{\varepsilon}(\nu, \gamma) \leqslant \varepsilon_0$ and $\hat{c}(\nu, \gamma)$, such that the solution $z(t, \varepsilon)$ of the IVP (2.3), (6.3) satisfies the following inequality for any $\varepsilon \in (0, \hat{\varepsilon}(\nu, \gamma)]$:

$$\|z(t, \varepsilon)\| \leqslant \hat{c}(\nu, \gamma) \exp(-\nu t)\|\varphi(\cdot)\|_C \quad \forall t \geqslant 0, \tag{6.51}$$

where $\|\varphi(\cdot)\|_C$ is the uniform norm of $\varphi(\cdot)$ on the interval $[-\varepsilon_0 h, 0]$.

Proof. The statement of the theorem directly follows from Lemma 6.6 and the equations (6.8)–(6.9). □

7. An LMI approach to exponential stability

In this section, we analyse the exponential stability of the system (2.26) with time-varying delays. Our objective is to derive LMI conditions that guarantee such a kind of stability of this system for all sufficiently small values of ε. Note that the results of this section can be easily extended to the case of a finite number of discrete and distributed delays.

Let us partition the matrices B, B_h and B_r into blocks as follows:

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}, \quad B_h = \begin{bmatrix} B_{h,1} & B_{h,2} \\ B_{h,3} & B_{h,4}, \end{bmatrix}, \quad B_r = \begin{bmatrix} B_{r,1} & B_{r,2} \\ B_{r,3} & B_{r,4} \end{bmatrix}, \tag{7.1}$$

where the blocks B_1, $B_{h,1}$ and $B_{r,1}$ are of the dimension $n \times n$, while the blocks B_4, $B_{h,4}$ and $B_{r,4}$ are of the dimension $m \times m$.

It was shown in Fridman (2002a) that ε-independent LMI conditions for the asymptotic stability of (2.26) imply the exponential stability of the fast subsystem, associated with this system. However, in this paper, we do not assume that the fast subsystem is exponentially stable. In this situation, we cannot apply the Lyapunov theorem for the asymptotic stability. Instead, we will look for the exponential stability conditions of (2.26) with a given decay rate.

We represent (2.26) in the form

$$E_\varepsilon \dot{z}(t) = (B + B_h)z(t) - B_h \int_{t-\varepsilon h(t)}^{t} \dot{z}(s)\mathrm{d}s + B_r \int_{t-\varepsilon r(t)}^{t} z(s)\mathrm{d}s \tag{7.2}$$

and consider the following Lyapunov–Krasovskii functional (see Fridman & Shaked, 2002 for regular systems with time-varying delays)

$$\begin{aligned} V(z_t, \dot{z}_t, \varepsilon) &= z^\top(t) E_\varepsilon P_\varepsilon z(t) + \varepsilon h_0 \int_{-\varepsilon h_0}^{0} \int_{t+\theta}^{t} \exp(2\nu(s-t))\dot{z}^\top(s) R_h \dot{z}(s)\mathrm{d}s\, \mathrm{d}\theta \\ &\quad + \int_{-\varepsilon r_0}^{0} \int_{t+\theta}^{t} \exp(2\nu(s-t)) z^\top(s) R_r z(s)\mathrm{d}s\, \mathrm{d}\theta, \end{aligned} \tag{7.3}$$

where $z_t = \{z(s),\ s \in [t - \varepsilon \max\{h_0, r_0\}, t]\}$; $\dot{z}_t = \{\dot{z}(s),\ s \in [t - \varepsilon \max\{h_0, r_0\}, t]\}$; R_h and R_r are some positive definite matrices of corresponding dimensions; $\nu > 0$ is some scalar value.

The matrix P_ε has the block form

$$P_\varepsilon = \begin{bmatrix} P_1 & \varepsilon P_2^\top \\ P_2 & P_3 \end{bmatrix}, \quad P_1 > 0, \quad P_3 > 0, \tag{7.4}$$

where the blocks P_1, P_2 and P_3 have the same dimensions as the respective blocks of the matrix E_ε, and

$$E_e P_\varepsilon > 0, \quad \varepsilon > 0. \tag{7.5}$$

For the system (2.26), we consider the initial condition

$$z(t_0 + \theta) = \phi(\theta), \quad t_0 \geqslant 0, \quad \theta \in [-\varepsilon \max\{h_0 \cdot r_0\}, 0], \tag{7.6}$$

where $\phi(\cdot) \in W[-\varepsilon_0 \max\{h_0 \cdot r_0\}, 0; E^{n+m}]$; $\varepsilon_0 > 0$ is some constant.
If for all $\varepsilon \in [0, \varepsilon_0]$,

$$\frac{\mathrm{d}}{\mathrm{d}t} V + 2\nu V \leqslant 0, \tag{7.7}$$

then by comparison principle

$$z^\top(t) E_\varepsilon P_\varepsilon z(t) \leqslant V(z_t, \dot{z}_t, \varepsilon) \leqslant \exp(-2\nu(t - t_0)) V(\phi, \dot{\phi}, \varepsilon), \quad \varepsilon \in [0, \varepsilon_0]. \tag{7.8}$$

Therefore, if (7.7) holds, then for all initial functions $\phi(\cdot) \in W[-\varepsilon_0 \max\{h_0 \cdot r_0\}, 0; E^{n+m}]$, there exists a constant $C(\varepsilon) > 0$ such that the solution of the problem (2.26), (7.6) satisfies the inequality

$$\|z(t)\| \leqslant \exp(-\nu(t - t_0)) C(\varepsilon) \left[\varepsilon r_0 \max_{\theta \in [-\varepsilon r_0, 0]} \|\phi(\theta)\|^2 + \int_{-\varepsilon h_0}^{0} \|\dot{\phi}(\theta)\|^2 \,\mathrm{d}\theta \right]^{0.5} \tag{7.9}$$

for all $\varepsilon \in [0, \varepsilon_0]$, i.e. (2.26) is exponentially stable with the ε-independent decay rate $\nu > 0$.
We obtain for $\varepsilon \in (0, \varepsilon_0]$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V + 2\nu V \leqslant\ & 2z^\top(t) P_\varepsilon^\top \left[(B + B_h) z(t) - B_h \int_{t-\varepsilon h(t)}^{t} \dot{z}(s) \mathrm{d}s + B_r \int_{t-\varepsilon r(t)}^{t} z(t+\theta) \mathrm{d}\theta \right] \\ & + 2\nu z^\top(t) E_\varepsilon P_\varepsilon z(t) + \varepsilon^2 h_0^2 \dot{z}^\top(t) R_h \dot{z}(t) - \varepsilon h_0 \exp(-2\varepsilon\nu h_0) \int_{t-h(t)}^{t} \dot{z}^\top(s) R_h \dot{z}(s) \mathrm{d}s \\ & - \exp(-2\varepsilon\nu r_0) \int_{t-\varepsilon r(t)}^{t} z^\top(s) R_r z(s) \mathrm{d}s + \varepsilon r_0 z^\top(t) R_r z(t). \end{aligned} \tag{7.10}$$

We apply further the Jensen's inequality (see, e.g. Gu *et al.*, 2003)

$$\varepsilon h_0 \int_{t-\varepsilon h(t)}^{t} \dot{z}^\top(s) R_h \dot{z}(s) \mathrm{d}s \geqslant \int_{t-\varepsilon h(t)}^{t} \dot{z}^\top(s) \mathrm{d}s\, R_h \int_{t-\varepsilon h(t)}^{t} \dot{z}(s) \mathrm{d}s, \tag{7.11}$$

$$\int_{t-\varepsilon r(t)}^{t} z^\top(s) R_r z(s) \mathrm{d}s \geqslant \frac{1}{\varepsilon r_0} \int_{t-\varepsilon r(t)}^{t} z^\top(s) \mathrm{d}s\, R_r \int_{t-\varepsilon r(t)}^{t} z(s) \mathrm{d}s. \tag{7.12}$$

Using (7.11)–(7.12) and setting
$\eta(t) = \mathrm{col}\Big(z(t), \int_{t-\varepsilon h(t)}^{t} \dot{z}(s)\mathrm{d}s, \frac{1}{\varepsilon r_0}\int_{t-\varepsilon r(t)}^{t} z(s)\mathrm{d}s\Big)$, one can rewrite (7.10) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}V + 2\nu V \leqslant \eta^{\top}(t)\Phi\eta(t) + \varepsilon^2 h_0^2 \dot{z}^{\top}(t) R_h \dot{z}(t), \tag{7.13}$$

where

$$\Phi = \begin{bmatrix} \Gamma_\varepsilon & -P_\varepsilon^{\top} B_h & \varepsilon r_0 P_\varepsilon^{\top} B_r \\ * & -R_h \exp(-2\varepsilon\nu h_0) & 0 \\ * & * & -\varepsilon r_0 R_r \exp(-2\varepsilon\nu r_0) \end{bmatrix}, \tag{7.14}$$

$$\Gamma_\varepsilon = P_\varepsilon^{\top}(B + B_h) + (B + B_h)^{\top} P_\varepsilon + \varepsilon r_0 R_r + 2\nu E_\varepsilon P_\varepsilon. \tag{7.15}$$

Substituting the right-hand side of (7.2) into $\varepsilon^2 h_0^2 \dot{z}^{\top}(t) R_h \dot{z}(t)$ and applying further the Schur complements, we find that the inequality (7.7) is satisfied if

$$\Psi_\varepsilon = \begin{bmatrix} \Gamma_\varepsilon & -P_\varepsilon^{\top} B_h & \varepsilon r_0 P_\varepsilon^{\top} B_r & h_0 (B + B_h)^{\top} J_\varepsilon R_h \\ * & -R_h \exp(-2\varepsilon\nu h_0) & 0 & -h_0 B_h^{\top} J_\varepsilon R_h \\ * & * & -\varepsilon r_0 R_r \exp(-2\varepsilon\nu r_0) & \varepsilon r_0 h_0 B_r^{\top} J_\varepsilon R_h \\ * & * & * & -R_h \end{bmatrix} \leqslant 0. \tag{7.16}$$

where

$$J_\varepsilon = \begin{bmatrix} \varepsilon I_n & 0 \\ 0 & I_m \end{bmatrix}. \tag{7.17}$$

If (7.16) is feasible for $\varepsilon = 0$, then the following slow LMI

$$P_0^{\top}(B + B_h) + (B + B_h)^{\top} P_0 + 2\nu E_0 P_0 \leqslant 0 \tag{7.18}$$

and the following fast LMI

$$\Psi_{\mathrm{f}} = \begin{bmatrix} P_3(B_4 + B_{h,4}) + (B_4 + B_{h,4})^{\top} P_3 & -P_3 B_{h,4} & h_0 (B_4 + B_{h,4})^{\top} R_{h,3} \\ * & -R_{h,3} & h_0 B_{h,4}^{\top} R_{h,3} \\ * & * & -R_{h,3} \end{bmatrix} \leqslant 0, \tag{7.19}$$

where $R_{h,3} = [0 \quad I_m] R_h [0 \quad I_m]^{\top}$, are feasible.

The slow LMI guarantees that the slow subsystem

$$E_0 \frac{\mathrm{d}\bar{z}}{\mathrm{d}t} = (B + B_h)\bar{z}(t), \quad t \geqslant 0, \tag{7.20}$$

is exponentially stable with the decay rate ν. The slow subsystem is an autonomous descriptor (differential-algebraic) system without delays. The fast LMI guarantees that the fast subsystem

$$\frac{\mathrm{d}\tilde{y}(\xi)}{\mathrm{d}\xi} = B_4\tilde{y}(\xi) + B_{h,4}\tilde{y}(\xi - \tilde{h}(\xi)), \quad \xi \geqslant 0, \tag{7.21}$$

with a piecewise-continuous delay $\tilde{h} \in [0, h_0]$ is stable by Lyapunov (see, e.g. Hale & Verduyn Lunel, 1993). This follows from the fact that the fast LMI guarantees the fulfilment of the inequality $\mathrm{d}V_{\mathrm{f}}/\mathrm{d}\xi \leqslant 0$ for the Lyapunov functional of the form

$$V_{\mathrm{f}}(\dot{\tilde{y}}_\xi) = \tilde{y}^\top(\xi)P_3\tilde{y}(\xi) + h_0\int_{-h_0}^{0}\int_{\xi+\zeta}^{\xi}\dot{\tilde{y}}^\top(\varrho)R_{h,3}\dot{\tilde{y}}(\varrho)\mathrm{d}\varrho\,\mathrm{d}\zeta. \tag{7.22}$$

REMARK 7.1 Note that by using Remark 2.2 of Glizer (2004b), a different fast subsystem of (2.26) can be obtained. Namely,

$$\frac{\mathrm{d}\tilde{y}(\xi)}{\mathrm{d}\xi} = B_4\tilde{y}(\xi) + B_{h,4}\tilde{y}(\xi - h(t)), \quad \xi \geqslant 0, \tag{7.23}$$

where ξ is an independent variable (the stretched time), while $t \geqslant 0$ is a parameter. Thus, for any given $t \geqslant 0$, the fast subsystem (7.23) is an autonomous differential system with a constant point-wise delay.

We consider further (2.26) with $B_r = 0$, i.e. with the discrete delay only. Our next objective is to find LMI conditions that guarantee the exponential decay rate $\nu > 0$ for all $\varepsilon \in [\varepsilon_1, \varepsilon_0]$, where $0 \leqslant \varepsilon_1 < \varepsilon_0$. For this purpose, we first find sufficient LMI conditions that are affine in ε:

$$\Psi_\varepsilon\big|_{r_0=0} \leqslant \tilde{\Psi}_\varepsilon = \begin{bmatrix} \Gamma_\varepsilon\big|_{r_0=0} & -P_\varepsilon^\top B_h & 0 & h_0(B+B_h)^\top J_\varepsilon R_h \\ * & -R\exp(-2\varepsilon_0\nu h_0) & 0 & -h_0 B_h^\top J_\varepsilon R_h \\ * & * & 0 & 0 \\ * & * & * & -R_h \end{bmatrix} \leqslant 0. \tag{7.24}$$

Since $E_\varepsilon P_\varepsilon$ and $\tilde{\Psi}_\varepsilon$ are affine in the constant parameter ε, then LMIs (7.5), (7.24) are feasible for any $\varepsilon \in [\varepsilon_1, \varepsilon_0]$ if these LMIs hold for $\varepsilon = \varepsilon_1$ and for $\varepsilon = \varepsilon_0$ with the same matrices P_2 and P_3 (because these matrices multiply ε in $E_\varepsilon P_\varepsilon$ and in $\tilde{\Psi}_\varepsilon$; Boyd *et al.*, 1994). Therefore, we arrive to the four LMIs

$$E_{\varepsilon_i}P_{\varepsilon_i}^{(i)} > 0, \quad P_{\varepsilon_i}^{(i)} = \begin{bmatrix} P_1^{(i)} & \varepsilon_i P_2^\top \\ P_2 & P_3 \end{bmatrix}, \quad i = 0, 1, \tag{7.25}$$

$$\bar{\Psi}_{\varepsilon_i} = \begin{bmatrix} \Gamma_{\varepsilon_i}^{(i)} & -P_{\varepsilon_i}^{(i)\top}B_h & h_0(B+B_h)^\top J_{\varepsilon_i}R_h^{(i)} \\ * & -R_h^{(i)}\exp(-2\varepsilon_0\nu h_0) & -h_0B_h^\top J_{\varepsilon_i}R_h^{(i)} \\ * & * & -R^{(i)} \end{bmatrix} \leqslant 0, \tag{7.26}$$

where

$$\Gamma_{\varepsilon_i}^{(i)} = P_{\varepsilon_i}^{(i)\top}(B + B_h) + (B + B_h)^\top P_{\varepsilon_i}^{(i)} + 2\nu E_{\varepsilon_i} P_{\varepsilon_i}^{(i)}. \tag{7.27}$$

Note that multiplication of LMIs (7.25)–(7.26) with $i = 0$ by $\frac{\varepsilon-\varepsilon_1}{\varepsilon_0-\varepsilon_1}$ and with $i = 1$ by $\frac{\varepsilon_0-\varepsilon}{\varepsilon_0-\varepsilon_1}$, and then summation of the resulting LMIs, imply the feasibility of $E_\varepsilon P_\varepsilon > 0$ and $\tilde{\Psi}_\varepsilon \leqslant 0$ for $\varepsilon \in [\varepsilon_1, \varepsilon_0]$ with

$$P_1 = \frac{\varepsilon - \varepsilon_1}{\varepsilon_0 - \varepsilon_1} P_1^{(0)} + \frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 - \varepsilon_1} P_1^{(1)}, \quad R_h = \frac{\varepsilon - \varepsilon_1}{\varepsilon_0 - \varepsilon_1} R_h^{(0)} + \frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 - \varepsilon_1} R_h^{(1)}. \tag{7.28}$$

Therefore, the feasibility of (7.25)–(7.26) guarantees the exponential stability of (2.26) with the decay rate ν for $\varepsilon \in [\varepsilon_1, \varepsilon_0]$. Finally, if $P_1 > 0$, $P_3 > 0$ and $\Psi_0 < 0$, then $E_\varepsilon P_\varepsilon > 0$ and $\Psi_\varepsilon < 0$ for all small enough $\varepsilon \geqslant 0$. We note that the strict LMI $\Psi_0 < 0$ can be feasible only if the fast system (7.21) is asymptotically stable.

Summarizing, we have proved the following theorem.

THEOREM 7.1 For a given $\nu > 0$, consider (2.26).
(i) Let there exist an $n \times n$-matrix $P_1 > 0$, an $m \times n$-matrix P_2, an $m \times m$-matrix $P_3 > 0$ and $(n + m) \times (n + m)$-matrices $R_h > 0$ and $R_r > 0$ such that the LMI $\Psi_0 < 0$ is feasible, where Ψ_ε is given by (7.16). Then the fast system (7.21) is asymptotically stable, whereas the full order system (2.26) is exponentially stable with the decay rate ν for all small enough $\varepsilon \geqslant 0$.
(ii) For a given $\varepsilon > 0$, if there exist an $n \times n$-matrix $P_1 > 0$, an $m \times n$-matrix P_2, an $m \times m$-matrix $P_3 > 0$ and $(n + m) \times (n + m)$-matrices $R_h > 0$ and $R_r > 0$ such that LMIs (7.5) and (7.16) are feasible. Then (2.26) is exponentially stable with the decay rate ν.
(iii) For a given $\varepsilon_0 > 0$, let there exist $n \times n$-matrices $P_1^{(0)} > 0$, $P_1^{(1)} > 0$, an $m \times n$-matrix P_2, an $m \times m$-matrix $P_3 > 0$ and $(n + m) \times (n + m)$-matrices $R_h^{(0)} > 0$, $R_h^{(1)} > 0$ such that the LMIs $E_{\varepsilon_0} P_{\varepsilon_0}^{(0)} > 0$, $\bar{\Psi}_{\varepsilon_0} \leqslant 0$ and $\bar{\Psi}_{\varepsilon_1}|_{\varepsilon_1=0} \leqslant 0$ are feasible with the notations given in (7.25)–(7.26). Then (2.26) with $B_r = 0$ is exponentially stable with the decay rate ν for all $\varepsilon \in [0, \varepsilon_0]$.

REMARK 7.2 If the fast system is not asymptotically stable, but we are looking for conditions in the form of the strict LMIs (in order to use LMI Toolbox of Matlab), we suggest the following: if for small enough $\varepsilon_1 \in (0, \varepsilon_0)$, the strict version of LMIs (7.25)–(7.26) is feasible, then the system (2.26) with $B_r = 0$ is exponentially stable with the decay rate ν for all $\varepsilon \in [\varepsilon_1, \varepsilon_0]$.

8. Examples

8.1 *Example 1*

Consider the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} \frac{\mathrm{d}z(t)}{\mathrm{d}t} = \begin{bmatrix} -2 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} z(t - \varepsilon h(t)), \quad t \geqslant 0, \tag{8.1}$$

where $z(t) \in E^3$, $\varepsilon > 0$ is a small parameter.

We note that in this example the fast system (7.23)

$$\frac{\mathrm{d}\tilde{y}(\xi)}{\mathrm{d}\xi} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tilde{y}(\xi - h(t)), \quad \xi \geqslant 0, \tag{8.2}$$

is not asymptotically stable even for $h(t) \equiv 0$.

(a) Consider first (8.1) without delay, i.e. for $h(t) \equiv 0$. In this case, the characteristic equation of (8.1) is

$$\varepsilon^2\lambda^3 + 2\varepsilon^2\lambda^2 + (1 - 3\varepsilon)\lambda + 1 = 0. \tag{8.3}$$

The roots $\lambda_i(\varepsilon)$, $(i = 1, 2, 3)$ of (8.3) have the following asymptotic expansions with respect to ε for all its sufficiently small values:

$$\lambda_1(\varepsilon) = -1 + \mathrm{O}(\varepsilon), \tag{8.4}$$

$$\mathrm{Re}\lambda_2(\varepsilon) = -1/2 + \mathrm{O}(\varepsilon), \quad \mathrm{Im}\lambda_2(\varepsilon) = (\mathrm{i}/\varepsilon)(1 - 3\varepsilon/2 + \mathrm{O}(\varepsilon^2)), \tag{8.5}$$

$$\mathrm{Re}\lambda_3(\varepsilon) = -1/2 + \mathrm{O}(\varepsilon), \quad \mathrm{Im}\lambda_3(\varepsilon) = -(\mathrm{i}/\varepsilon)(1 - 3\varepsilon/2 + \mathrm{O}(\varepsilon^2)), \tag{8.6}$$

where i is the imaginary unit.

It is seen that the root $\lambda_1(\varepsilon)$ is slow, while $\lambda_2(\varepsilon)$ and $\lambda_3(\varepsilon)$ are fast roots.

Let us write down the slow and fast subsystems, associated with the original singularly perturbed system (8.1). The slow subsystem is

$$\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} = -\bar{x}(t). \tag{8.7}$$

The characteristic equation of (8.7) (the slow characteristic equation) has the form

$$\bar{\lambda} + 1 = 0, \tag{8.8}$$

yielding the root $\bar{\lambda} = -1$.

The fast system is given by (8.2), where $h(t) \equiv 0$. The characteristic equation of the fast system (the fast characteristic equation) has the form

$$\tilde{\mu}^2 + 1 = 0. \tag{8.9}$$

The roots of (8.9) are $\tilde{\mu}_1 = i$, $\tilde{\mu}_2 = -i$.

It is directly obtained that

$$\lim_{\varepsilon \to +0} \lambda_1(\varepsilon) = \bar{\lambda}, \quad \lim_{\varepsilon \to +0} \varepsilon\lambda_2(\varepsilon) = \tilde{\mu}_1, \quad \lim_{\varepsilon \to +0} \varepsilon\lambda_3(\varepsilon) = \tilde{\mu}_2. \tag{8.10}$$

Comparing $\lambda_2(\varepsilon)$, $\lambda_3(\varepsilon)$ with $\tilde{\mu}_1$, $\tilde{\mu}_2$, one can conclude that although the roots of the fast characteristic equation are pure imaginary, the fast roots of the original characteristic equation have negative real parts.

We further use LMI Toolbox of MATLAB to verify the feasibility of the strict LMIs of Theorem 7.1 for exponential stability of (8.1) with $h(t) \equiv 0$. Given $\varepsilon = 0.01$, by solving the strict LMIs (7.5) and $\Gamma_\varepsilon < 0$ with Γ_ε defined by (7.15), where $R_r = 0$, $r_0 = h_0 = 0$, we find that the system (8.1) is exponentially stable with the decay rate $\nu = 0.484$. We note that the matrix $E_{0.01}^{-1}B$ in this example has eigenvalues with real parts -0.4845. For $\varepsilon = 0$, the strict LMI $\Gamma_0 < 0$ is not feasible since the fast system is not asymptotically stable. Next, verifying the feasibility of the strict LMIs $\Gamma_{0.01}^{(0)} < 0$ and $\Gamma_{0.011}^{(1)} < 0$ with the same decision variables P_2 and P_3, we find that the system is exponentially stable with the decay rate $\nu = 0.43$ for $\varepsilon \in [0.01, 0.011]$.

(b) Consider the case of time-varying delay $h(t) \leqslant h_0$. By verifying the feasibility of the strict LMIs (7.16) for $\varepsilon = 0.01$, we find that the system (8.1) is exponentially stable with the decay rate $\nu = 0.28$ for $h(t) \leqslant 0.002$. Solving the strict LMIs (7.25)–(7.26) with $\varepsilon_0 = 0.011$, $\varepsilon_1 = 0.01$ and with the same decision variables P_2 and P_3, we find that for $\varepsilon \in [0.01, 0.011]$, the system is exponentially stable with the decay rate $\nu = 0.23$ for $h(t) \leqslant 0.002$.

8.2 *Example 2*

Consider the system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -4x(t) + 2x(t - \varepsilon h(t)) + y(t), \quad t \geqslant 0, \tag{8.11}$$

$$\varepsilon \frac{\mathrm{d}y(t)}{\mathrm{d}t} = x(t) - \omega y(t - \varepsilon h(t)), \quad t \geqslant 0, \tag{8.12}$$

where $x(t)$ and $y(t)$ are scalar, ω is a given positive constant.

(a) Consider the case of constant delay h given by

$$h = \frac{\pi}{2\omega}. \tag{8.13}$$

First of all, note that the system (8.11)–(8.12) is a particular case of the system (2.1)–(2.2) with the following scalar functions $A_i(\eta)$, $(i = 1, \ldots, 4)$:

$$A_1(\eta) = \begin{cases} 2, & -\infty < \eta \leqslant -h, \\ 4, & -h < \eta < 0, \\ 0, & 0 \leqslant \eta < +\infty, \end{cases} \qquad A_2(\eta) = \begin{cases} -1, & -\infty < \eta \leqslant -h, \\ -1, & -h < \eta < 0, \\ 0, & 0 \leqslant \eta < +\infty, \end{cases} \tag{8.14}$$

$$A_3(\eta) = \begin{cases} -1, & -\infty < \eta \leqslant -h, \\ -1, & -h < \eta < 0, \\ 0, & 0 \leqslant \eta < +\infty, \end{cases} \qquad A_4(\eta) = \begin{cases} \omega, & -\infty < \eta \leqslant -h, \\ 0, & -h < \eta < 0, \\ 0, & 0 \leqslant \eta < +\infty. \end{cases} \tag{8.15}$$

Now, let us write down the characteristic equation with respect to λ for the system (8.11)–(8.12)

$$g_{\mathrm{s}}(\lambda, \varepsilon) \stackrel{\triangle}{=} [4 - 2\exp(-\varepsilon\lambda h) + \lambda][\omega \exp(-\varepsilon\lambda h) + \varepsilon\lambda] - 1 = 0. \tag{8.16}$$

Transforming the variable $\varepsilon\lambda = \mu$ in (8.16) and multiplying the resulting equation by ε, one can rewrite this characteristic equation in the form

$$g_{\mathrm{f}}(\mu, \varepsilon) \stackrel{\triangle}{=} [4\varepsilon - 2\varepsilon \exp(-\mu h) + \mu][\omega \exp(-\mu h) + \mu] - \varepsilon = 0. \tag{8.17}$$

The slow subsystem, associated with the original system (8.11)–(8.12), has the form

$$\frac{\mathrm{d}x_{\mathrm{s}}(t)}{\mathrm{d}t} = -\left(2 - \frac{1}{\omega}\right) x_{\mathrm{s}}(t), \quad t \geqslant 0, \tag{8.18}$$

and its characteristic equation (the slow characteristic equation) is

$$\lambda + \left(2 - \frac{1}{\omega}\right) = 0, \tag{8.19}$$

yielding the root

$$\bar{\lambda}_1 = -\left(2 - \frac{1}{\omega}\right). \tag{8.20}$$

It is clear that this root is negative if and only if $\omega > 1/2$.

The fast subsystem has the form

$$\frac{\mathrm{d}y_{\mathrm{f}}(\xi)}{\mathrm{d}\xi} = -\omega y_{\mathrm{f}}(\xi - h), \quad \xi \geqslant 0, \tag{8.21}$$

and its characteristic equation (the fast characteristic equation) is

$$\det \Delta_{\mathrm{f}}(\mu) \stackrel{\triangle}{=} -\omega \exp(-\mu h) - \mu = 0. \tag{8.22}$$

It can be verified directly that (8.22) has two simple pure imaginary roots

$$\tilde{\mu}_1 = \mathrm{i}\,\omega, \quad \tilde{\mu}_2 = -\mathrm{i}\,\omega, \tag{8.23}$$

where i is the imaginary unit.

Consider any root $\tilde{\mu} \neq \pm\mathrm{i}\omega$ of (8.22). It can be observed immediately that $\mathrm{Re}\tilde{\mu} \neq 0$. Let us show that $\mathrm{Re}\tilde{\mu} < 0$. One can represent this root as $\tilde{\mu} = \tilde{\mu}_{\mathrm{Re}} + \mathrm{i}\tilde{\mu}_{\mathrm{Im}}$, where $\tilde{\mu}_{\mathrm{Re}}$ and $\tilde{\mu}_{\mathrm{Im}}$ are real values. Substituting this representation into (8.22) instead of μ, substituting $\pi/(2\omega)$ instead of h, and equating separately the real and imaginary values on both sides of the resulting equation yields (after some rearrangement) the following system of equations with respect to $\tilde{\mu}_{\mathrm{Re}}$ and $\tilde{\mu}_{\mathrm{Im}}$:

$$\exp\left(-\frac{\pi}{2}\cdot\frac{\tilde{\mu}_{\mathrm{Re}}}{\omega}\right)\cos\left(\frac{\pi}{2}\cdot\frac{\tilde{\mu}_{\mathrm{Im}}}{\omega}\right) = -\frac{\tilde{\mu}_{\mathrm{Re}}}{\omega}, \tag{8.24}$$

$$\exp\left(-\frac{\pi}{2}\cdot\frac{\tilde{\mu}_{\mathrm{Re}}}{\omega}\right)\sin\left(\frac{\pi}{2}\cdot\frac{\tilde{\mu}_{\mathrm{Im}}}{\omega}\right) = \frac{\tilde{\mu}_{\mathrm{Im}}}{\omega}. \tag{8.25}$$

Assume that $\tilde{\mu}_{\mathrm{Re}} > 0$. Then, $0 < \exp\left(-\frac{\pi}{2}\cdot\frac{\tilde{\mu}_{\mathrm{Re}}}{\omega}\right) < 1$, and , due to (8.25), $\left|\frac{\tilde{\mu}_{\mathrm{Im}}}{\omega}\right| < 1$. The latter means that $\cos\left(\frac{\pi}{2}\cdot\frac{\tilde{\mu}_{\mathrm{Im}}}{\omega}\right) > 0$. Hence, the expression in the left-hand side of (8.24) is positive. However, according to the above made assumption that $\tilde{\mu}_{\mathrm{Re}} > 0$, the expression in the right-hand side of (8.24) is negative. This contradiction implies that $\tilde{\mu}_{\mathrm{Re}} < 0$, i.e. any root $\tilde{\mu} \neq \pm\mathrm{i}\omega$ of (8.22) has negative real part.

Using the above presented analysis of roots of the slow and fast characteristic equations and results of Sections 4 and 5, one can conclude that, for all sufficiently small $\varepsilon > 0$, all roots $\lambda(\varepsilon)$ of the original characteristic equation (8.16) (excepting three ones) satisfy the inequality $\mathrm{Re}\lambda(\varepsilon) < -\gamma/\varepsilon$ with some positive constant γ independent of ε. It is clear that all these roots are the fast roots of (8.16). The three other roots of (8.16) are: (a) the slow root $\lambda_{\mathrm{s}1}(\varepsilon)$, corresponding to the root $\bar{\lambda}_1$ of the slow characteristic equation (8.19); (b) the fast roots $\lambda_{\mathrm{f}1}(\varepsilon) = \mu_1(\varepsilon)/\varepsilon$ and $\lambda_{\mathrm{f}2}(\varepsilon) = \mu_2(\varepsilon)/\varepsilon$, where $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$ are the roots of (8.17), corresponding to the roots $\tilde{\mu}_1$ and $\tilde{\mu}_2$, respectively, of the fast characteristic equation (8.22). Below, based on results of Sections 4 and 5, we construct the first-order asymptotic expansions for $\lambda_{s1}(\varepsilon)$ and $\mu_\alpha(\varepsilon)$, $(\alpha = 1, 2)$. We start with $\lambda_{s1}(\varepsilon)$. Using Corollary 4.1, (2.12), (4.11), (8.15), (8.16), (8.20) and the fact that $\bar{\lambda}_1$ is a simple root of the slow characteristic equation (8.19) directly yields that, for all sufficiently small $\varepsilon > 0$, the root $\lambda_1(\varepsilon)$ can be represented in the form (4.12) with $p = 1$, where

$$\bar{\lambda}_1^1 = \frac{1}{\omega}\left(\frac{1}{\omega} - 2\right)\left(2h\omega - h + \frac{1}{\omega}\right). \tag{8.26}$$

Now, proceed to $\mu_\alpha(\varepsilon)$, $(\alpha = 1, 2)$. By using Lemma 5.1, (5.7), (5.14), (8.17), (8.22), (8.23) and the fact that $\mu_\alpha(\varepsilon)$, $(\alpha = 1, 2)$ are simple roots of (8.17), one immediately obtains that, for all sufficiently small $\varepsilon > 0$, the roots $\mu_\alpha(\varepsilon)$, $(\alpha = 1, 2)$ can be represented in the form (5.15), where

$$\tilde{\mu}_1^1 = -\rho - \mathrm{i}\theta, \quad \tilde{\mu}_2^1 = -\rho + \mathrm{i}\theta, \quad \rho = \frac{h\omega^2}{\omega^2(h^2\omega^2 + 1)}, \quad \theta = \frac{1}{\omega(h^2\omega^2 + 1)}. \tag{8.27}$$

The latter means that

$$\lim_{\varepsilon \to +0} \mathrm{Re}(\lambda_{\mathrm{f},\alpha}(\varepsilon)) = -\rho < 0, \quad \alpha = 1, 2. \tag{8.28}$$

Therefore, by virtue of Lemma 6.1, we obtain that for any constant ν, satisfying the inequality

$$0 < \nu < \min\left\{\left(2 - \frac{1}{\omega}\right), \rho\right\}, \tag{8.29}$$

there exists a number $\varepsilon(\nu) > 0$ such that, for all $\varepsilon \in (0, \varepsilon(\nu)]$, any root $\lambda(\varepsilon)$ of the original characteristic equation (8.16) satisfies the inequality $\mathrm{Re}\lambda(\varepsilon) \leqslant -\nu$. Hence, due to Theorem 6.2, the system (8.11)–(8.12) is exponentially stable uniformly with respect to ε for all sufficiently small $\varepsilon > 0$ with the decay rate $\bar{\nu} < \nu$.

(b) Consider the case of time-varying delay $h(t) \leqslant h_0$ and $\omega = 1$. Applying the item (iii) of Theorem 7.1 and verifying the feasibility of the corresponding LMIs with the same decision variables P_2 and P_3, we find that for $\varepsilon \in [0, 0.5]$, the system is exponentially stable with the decay rate $\nu = 0.2$ for $h(t) \leqslant 0.4$.

We note that in this example our LMI approach, which is based on the simple Lyapunov–Krasovskii functional, can treat only comparatively small delays, where the fast subsystem is exponentially stable.

9. Conclusions

In this paper, the singularly perturbed linear differential system with a small delay of order of the singular perturbation parameter $\varepsilon > 0$ was treated. In the case of time-invariant system, the asymptotic behaviour of the set of roots of its characteristic equation has been investigated. For this purpose, the original singularly perturbed system was decomposed asymptotically into two much simpler ε-free subsystems, the slow and fast ones. The characteristic equations of these subsystems (the slow and fast characteristic equations) also are ε-free, and they are much simpler than the one (the original characteristic equation) for the original singularly perturbed system. It was also shown that the original characteristic equation can be decomposed asymptotically into two much simpler ε-free equations of the polynomial and quasipolynomial type. The connection between the asymptotic decomposition of the original singularly perturbed system and the asymptotic decomposition of its characteristic equation was established.

Based on the assumed structure of the sets of roots of the slow and fast characteristic equations, the structure of the set of roots of the original characteristic equation (including qualitative properties and asymptotic expansion of the roots with respect to ε) has been derived. It is important to note that the assumptions, on which the final result is based, are ε-free. The intermediate results also are ε-free. However, the final result, the structure of the set of roots of the original characteristic equation, is valid for all sufficiently small $\varepsilon > 0$, i.e. robustly with respect to this parameter. The results on the structure of roots of the characteristic equation have been applied to the analysis of the exponential stability of the original singularly perturbed system. The exponential stability of this system was established not only in the case where the fast subsystem is exponentially stable but also in the case where the characteristic equation of the fast subsystem has pure imaginary roots.

Along with the method of studying the exponential stability of a singularly perturbed time delay system, based on the asymptotic analysis of its spectrum, an LMI method also was developed for singularly perturbed systems with time-varying (point-wise and distributed) delays. Like the former, the latter also includes the case of no exponential stability of the fast subsystem. In this case, the LMI method can guarantee the exponential stability of the full order system uniformly in $\varepsilon \in [\varepsilon_1, \varepsilon_0]$, where $\varepsilon_1 > 0$ can be chosen arbitrary close to zero.

In the case of a constant delay, the LMI method is more conservative than the method of spectrum analysis. However, in the case of a variable delay, the latter is not applicable to the study of the exponential stability, while the former is. It was shown how these two methods can complement each other in the analysis of the exponential stability of singularly perturbed time delay systems.

REFERENCES

BELLMAN, R. (1953) *Stability Theory of Differential Equations.* New York: McGraw-Hill.

BELLMAN, R. & COOKE, K. L. (1963) *Differential-Difference Equations.* New York: Academic Press.

BOYD, S., EL GHAOUI, L., FERON, E. & BALAKRISHNAN, V. (1994) *Linear Matrix Inequality in Systems and Control Theory.* Philadelphia, PA: SIAM.

CHEN, J., FU, P. & NICULESCU, S.-I. (2006) Asymptotic behavior of imaginary zeros of linear systems with commensurate delays. *Proceedings of the 45th IEEE Conference on Decision and Control*, San-Diego, CA, USA, December 13–15. New York: Institute of Electrical and Electronics Engineers, pp. 1375–1380.

CHEN, J., FU, P. & NICULESCU, S.-I. (2008) An eigenvalue perturbation stability analysis approach with applications to time-delay and polynomially dependent systems. *Proceedings of the 7th World Congress on Intelligent Control and Automation*, Chongqing, China, June 25–27. New York: Institute of Electrical and Electronics Engineers, pp. 307–312.

CHEN, W.-H., YANG, SH.-T., LU, X. & SHEN, YA. (2010) Exponential stability and exponential stabilization of singularly perturbed stochastic systems with time-varying delay. *Int. J. Robust Nonlinear Control*, **20**, 2021–2044.

DRAGAN, V. & IONITA, A. (1999) Exponential stability for singularly perturbed systems with state delays. *Proceedings of the 6th Colloquium on the Qualitative Theory of Differential Equations*, Szeged, Hungary: Szeged University August 10–14, pp. 1–8.

FRIDMAN, E. (1996) Decoupling transformation of singularly perturbed systems with small delays. *Z. Angew. Math. Mech.*, **82**,201–204.

FRIDMAN, E. (2002a) Effects of small delays on stability of singularly perturbed systems. *Automat. J. IFAC*, **38**, 897–902.

FRIDMAN, E. (2002b) Stability of singularly perturbed differential-difference systems: an LMI approach, *Dyn. Continuous Discrete Impulsive Syst. Ser. B Appl. Algorithms*, **9**, 201–212.

FRIDMAN, E. (2002c) Stability of linear descriptor systems with delay: a Lyapunov-based approach. *J. Math. Anal. Appl.*, **273**, 24–44.

FRIDMAN, E. (2006) Robust sampled-data H_∞ control of linear singularly perturbed systems. *IEEE Trans. Automat. Control*, **51**, 470–475.

FRIDMAN, E. & SHAKED, U. (2002) An improved stabilization method for linear time-delay systems. *IEEE Trans. Automat. Control*, **47**, 1931–1937.

FU, P., CHEN, J. & NICULESCU, S.-I. (2007) High-order analysis of critical stability properties of linear time-delay systems. *Proceedings of the 2007 American Control Conference*, New York City, USA, July 11–13. New York: Institute of Electrical and Electronics Engineers , pp. 4921–4926.

GANTMACHER, F. R. (1974) *The Theory of Matrices*, Vol. 2. New York: Chelsea.

GLIZER, V. Y. (1999) Stabilizability and detectability of singularly perturbed linear time-invariant systems with delays in state and control. *J. Dyn. Control Syst.*, **5**, 153–172.

GLIZER, V. Y. (2003) Blockwise estimate of the fundamental matrix of linear singularly perturbed differential systems with small delay and its application to uniform asymptotic solution. *J. Math. Anal. Appl.*, **278**, 409–433.

GLIZER, V. Y. (2004a) On stabilization of nonstandard singularly perturbed systems with small delays in state and control. *IEEE Trans. Automat. Control*, **49**, 1012–1016.

GLIZER, V. Y. (2004b) Observability of singularly perturbed linear time-dependent differential systems with small delay, *J. Dyn. Control Syst.*, **10**, 329–363.

GLIZER, V. Y. (2007) Infinite horizon quadratic control of linear singularly perturbed systems with small state delays: an asymptotic solution of Riccati-type equations. *IMA J. Math. Control Inf.*, **24**, 435–459.

GLIZER, V. Y. (2009) L^2-stabilizability conditions for a class of nonstandard singularly perturbed functional-differential systems. *Dyn. Continuous Discrete Impulsive Syst. Ser. B Appl. Algorithms*, **16**, 181–213.

GLIZER, V. Y. & FRIDMAN, E. (2000) H_∞ control of linear singularly perturbed systems with small state delay. *J. Math. Anal. Appl.*, **250**, 49–85.

GU, K., KHARITONOV, V. & CHEN, J. (2003) *Stability of Time-Delay Systems*. Boston: Birkhauser.

HALANAY, A. (1966) *Differential Equations: Stability, Oscillations, Time Lags*. New York: Academic Press.

HALANAY, A. & RASVAN, V. (1997) Stability radii for some propagation models. *IMA J. Math. Control Inf.*, **14**, 95–107.

HALE, J. K. & VERDUYN LUNEL, S. M. (1993) *Introduction to Functional Differential Equations*. New York: Springer.

HARTMAN, P. (2002) *Ordinary Differential Equations*. Philadelphia, PA: SIAM.

KHALIL, H. K. (2001) *Nonlinear Systems*. Upper Saddle River, NJ: Prentice Hall.

KOKOTOVIC, P. V., KHALIL, H. K. & O'REILLY, J. (1986) *Singular Perturbation Methods in Control: Analysis and Design*. London: Academic Press.

KOLMOGOROV, A. N. & FOMIN, S. V. (1975) *Introductory Real Analysis*. Mineola, NY: Dover Publications Inc.

LUSE, D. W. (1986) Frequency domain results for systems with multiple time scales. *IEEE Trans. Automat. Control*, **31**, 918–924.

LUSE, D. W. & KHALIL, H. K. (1985) Frequency domain results for systems with slow and fast dynamics. *IEEE Trans. Automat. Control*, **30**, 1171–1179.

LYAPUNOV, A. M. (1966) *Stability of Motion*. New York: Academic Press.

O'MALLEY JR, R. E. (1991) *Singular Perturbation Methods for Ordinary Differential Equations*. New York: Springer.

RASVAN, V. (1983) *Absolute Stability of Time Lag Control Systems*. Moscow-Leningrad: Nauka (in Russian).

SCHWARTZ, L. (1967) *Analyse Mathematique*, Vol. I. Paris, France: Hermann.

VASIL'EVA, A. B., BUTUZOV, V. F. & KALACHEV, L. V. (1995) *The Boundary Function Method for Singular Perturbation Problems*. Philadelphia, PA: SIAM.

VISHIK, M. I. & LYUSTERNIK, L. A. (1957) Regular degeneration and boundary layer for linear differential equations with small parameter. *Uspekhi Mat. Nauk*, **12**, 3–122 (in Russian).

VISHIK, M. I. & LYUSTERNIK, L. A. (1960) The solution of some perturbation problems for matrices and self-adjoint or non-self-adjoint differential equations I. *Russ. Math. Surv.*, **15**, 1–73.

WASOV, W. (1965) *Asymptotic Expansions for Ordinary Differential Equations*. New York: John Wiley.