Sub-predictors and classical predictors for finite-dimensional observer-based control of parabolic PDEs

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Abstract—We study constant input delay compensation by using finite-dimensional observer-based controllers in the case of the 1D heat equation. We consider Neumann actuation with nonlocal measurement and employ modal decomposition with $N+1$ modes in the observer. We introduce a chain of $M$ sub-predictors that leads to a closed-loop ODE system coupled with infinite-dimensional tail. Given an input delay $r$, we present LMI stability conditions for finding $M$ and $N$ and the resulting exponential decay rate and prove that the LMIs are always feasible for any $r$. We also consider a classical observer-based predictor and show that the corresponding LMI stability conditions are feasible for any $r$ provided $N$ is large enough. A numerical example demonstrates that the classical predictor leads to a lower-dimensional observer. However, it is known to be hard for implementation due to the distributed input signal.

Index Terms—Distributed parameter systems, observer-based control, time-delay

I. INTRODUCTION

Finite-dimensional observer-based controllers for parabolic systems were designed by the modal decomposition approach in [1], [2], [3], [4], [5]. Recently, the first constructive LMI-based method for finite-dimensional observer-based controller was suggested in [6] for the 1D heat equation under nonlocal or Dirichlet actuation and nonlocal measurement. The observer dimension $N$ and the resulting exponential decay rate were found from simple LMI conditions. Finite-dimensional observer-based control of the Kuramoto-Sivashinsky equation with boundary actuation and point measurement was studied in [7].

Robustness with respect to small delays and/or sampling intervals for the heat equation was studied in [8], [9] for distributed static output-feedback control, in [10] for boundary state-feedback and in [11], [12] for boundary controller based on PDE observer. Delayed implementation of finite-dimensional observer-based controllers for the 1D heat equation was introduced in [13], where in case of time-varying output delay, a combination of Lyapunov functionals with Halanay’s inequality appeared to be an efficient tool.

To compensate large input/output delay, there are two main predictor methods: the classical predictor, which is based on a reduction approach [14] or the backstepping approach [15] and sub-predictors or chain of observers [16], [17], [18], [19]. The classical predictors for state-feedback control of PDEs were suggested in [15], [20], [21]. For the heat equation, a PDE sub-predictor (an observer of the future state) was presented in [11]. A chain of observers for the estimation of heat equation with a large output delay was designed in [22].

In the recent paper [23], reduced-order LMI stability conditions were introduced for finite-dimensional observer-based control. This was presented for the heat equation with Neumann actuation and non-local measurement. The dimension of the LMIs does not grow with the dimension of the observer $N$. Moreover, feasibility of the LMIs for $N$ implies their feasibility for $N+1$. In [23], the classical predictor was extended to finite-dimensional observer-based control. This predictor compensated delay in the finite-dimensional controller, whereas the infinite-dimensional part still depended on the large input delay. It was shown in a numerical example that the predictor allows for larger delays. However, the feasibility of LMIs for arbitrary delays was not proved due to complexity of the analysis in the presence of time-varying output delay.

The present paper is dedicated to predictor methods for finite-dimensional observer-based control of parabolic PDEs with constant input delay $r$. As in [23], we consider the 1D heat equation under Neumann actuation and non-local measurement. The main novelty is in use of sub-predictors for such a system. We show that for any $r$ there exists a chain of $M$ sub-predictors and a large enough number of modes $N+1$ employed in observer that guarantee the stability of the closed-loop system. We present LMI stability conditions for finding $M$, $N$ and the resulting exponential decay rate. We prove that these LMIs are always feasible for all $r$ and large enough $M$ and $N$. We also consider the classical predictor which compensates the delay in the finite-dimensional part, as introduced in [23] (if the time-varying input/output delays are omitted). This is the first time that feasibility guarantees for the resulting LMIs with arbitrary delays are proved for both sub-predictors and predictors. This proof is challenging, due to coupling in the closed-loop system. A numerical example demonstrates that for the same $N$, the classical predictor allows larger delays found from the LMIs, whereas...
for the same delay they employ lower-dimensional observers than the sub-predictors. However, as is well-known [24], [25], they are harder to implement, due to the distributed input term which should be carefully discretized. This paper is an essential step towards the use of sub-predictors and classical predictors for delay compensation in PDEs, via finite-dimensional observers.

Notations and preliminaries: $L^2(0, 1)$ is the Hilbert space of Lebesgue measurable and square integrable functions $f : [0, 1] \to \mathbb{R}$ with the inner product $(f, g) := \int_0^1 f(x)g(x)dx$ and induced norm $\|f\|_2 := (f, f)^{1/2}$. $H^1(0, 1)$ is the space of functions $f : [0, 1] \to \mathbb{R}$ with square integrable weak derivative, with the norm $\|f\|_{H^1}^2 := \sum_{j=0}^1 \|f_j\|^2$. The Euclidean norm on $\mathbb{R}^n$ is denoted by $|\cdot|$. For $P \in \mathbb{R}^{n \times n}$, $P > 0$ means that $P$ is symmetric and positive definite. The sub-diagonal elements of a symmetric matrix will be denoted by $\ast \otimes \ast$ is the standard Kronecker product. For $U \in \mathbb{R}^{m \times n}$, $U > 0$ and $x \in \mathbb{R}^n$ let $|x|^2 = x^T U x$. $\mathbb{Z}_+$ is the set of nonnegative integers.

Recall that the Sturm-Liouville eigenvalue problem
\[
\phi'' + \lambda \phi = 0, \quad x \in [0, 1], \quad \phi'(0) = \phi'(1) = 0,
\]
induces a sequence of eigenvalues $\lambda_n = n^2 \pi^2, n \in \mathbb{Z}_+$ with corresponding eigenfunctions
\[
\phi_0(x) = 1, \quad \phi_n(x) = \sqrt{2} \cos (\sqrt{n} \pi x), n \geq 1.
\]
The eigenfunctions form a complete orthonormal system in $L^2(0, 1)$. Given $N \in \mathbb{Z}_+$ and $h \in L^2(0, 1)$ satisfying $h^2 \sum_{n=0}^N h_n \phi_n$ we denote $\|h\|^2_N = \sum_{n=N+1}^\infty h_n^2$.

II. SUB-PREDICTORS VS CLASSICAL PREDICTORS

We consider the PDE
\[
z(x, t) = z_x(x, t) + qz(x, t), \quad x \in [0, 1], \quad t \geq 0,
\]
\[z_x(0, t) = 0, \quad z_x(1, t) = u(t - r)
\]
under delayed Neumann actuation with known delay $r$ and non-local measurement
\[
y(t) = \langle c, z(\cdot, t) \rangle, \quad t \geq 0
\]
with $c \in L^2(0, 1)$. To compensate the delay, we will present in this section both sub-predictors and classical predictors.

Using modal decomposition, we present the solution to (3) as
\[
z(x, t) = \sum_{n=0}^\infty z_n(t) \phi_n(x), \quad z_n(t) = \langle z(\cdot, t), \phi_n \rangle
\]
with $\phi_n$, $n \in \mathbb{Z}_+$ given in (2). Differentiating under the integral, integrating by parts and using (1) and (2) we obtain (similar to [10] and the references therein)
\[
\dot{z}_n(t) = (-\lambda_n + q) z_n(t) + b_n u(t - r), \quad t \geq 0
\]
\[b_0 = 1, \quad b_n = (-1)^n \sqrt{n}, \quad n \geq 1.
\]
Let $\delta > 0$ be a desired decay rate. Since $\lim_{n \to \infty} \lambda_n = \infty$, there exists some $N_0 \in \mathbb{Z}_+$ such that
\[
-\lambda_n + q < -\delta, \quad n > N_0.
\]
Let
\[
A_0 = \text{diag} \{ -\lambda_0 + q, \ldots, -\lambda_{N_0} + q \},
\]
\[
L_0 = [l_0, \ldots, l_{N_0}]^T, \quad B_0 = [b_0, \ldots, b_{N_0}]^T
\]
\[C_0 = [c_0, \ldots, c_{N_0}], \quad c_n = \langle c, \phi_n \rangle, \quad n \in \mathbb{Z}_+.
\]
Assume that $c_n \neq 0$, $0 \leq n \leq N_0$.

Then $(A_0, C_0)$ is observable, by the Hautus lemma. We choose $L_0 = [0, \ldots, l_{N_0}]^T$ which satisfies the following Lyapunov inequality:
\[
P_0 (A_0 - L_0 C_0) + (A_0 - L_0 C_0)^T P_0 < -2\delta P_0,
\]
where $0 < P_0 \in \mathbb{R}^{(N_0+1) \times (N_0+1)}$.

Similarly, by the Hautus lemma, $b_n \neq 0$, $n \in \mathbb{Z}_+$ implies that $(A_0, B_0)$ is controllable. Let $K_0 \in \mathbb{R}^{1 \times (N_0+1)}$ satisfy
\[
P_0 (A_0 - B_0 K_0) + (A_0 - B_0 K_0)^T P_0 < -2\delta P_0,
\]
where $0 < P_0 \in \mathbb{R}^{(N_0+1) \times (N_0+1)}$.

In our finite-dimensional observer-based predictor design, the closed-loop system will be presented as a coupled system of ODEs and the infinite-dimensional tail. This complicates the proof of stabilization for all $r > 0$ under higher-dimensional observers.

Given $N \geq N_0$ denote
\[
z^{N_0}(t) = \text{col} \{ z_0(t), \ldots, z_{N_0}(t) \},
\]
\[A_1 = \text{diag} \{ -\lambda_{N_0+1}, \ldots, -\lambda_N + q \},
\]
\[B_1 = \begin{bmatrix} b_{N_0+1}, \ldots, b_N \end{bmatrix}^T, \quad C_1 = [c_{N_0+1}, \ldots, c_N].
\]

A. Sub-predictors

In order to deal with a large delay $r$, we subdivide $r$ into $M$ parts of equal size $\frac{r}{M}$, where $M \in \mathbb{Z}_+$, $M \geq 1$. We first consider $M \geq 2$ and employ a chain of sub-predictors (observers of the future state)
\[
z_i^{N_0}(t - \frac{M-1}{M}r) \mapsto \cdots \mapsto z_i^{N_0}(t - \frac{M-i+1}{M}r) \mapsto \cdots \mapsto z_i^{N_0}(t).
\]
Here $z_i^{N_0}(t - \frac{M-i+1}{M}r) \mapsto z_i^{N_0}(t - \frac{M-i}{M}r)$ means that $z_i^{N_0}(t)$ predicts the value of $z_{i+1}^{N_0}(t + \frac{r}{M})$. Similarly, $z_i^{M}(t)$ predicts the value of $z_i^{N_0}(t + \frac{r}{M})$. The sub-predictors satisfy the following ODEs for $t \geq 0$
\[
z_i^{N_0}(t) = A_0 z_i^{N_0}(t) + B_0 u(t - \frac{M-1}{M}r) + L_0 [C_1 z_i^{N_0}(t - \frac{r}{M}) - C_1 z_i^{N_0}(t) - y(t)],
\]
whereas $z_i^{N_0}(t)$ satisfies the following ODE
\[
z_i^{N_0}(t) = A_1 z_i^{N_0}(t) + B_1 u(t - r), \quad z_i^{N_0}(t) = 0, \quad t \leq 0.
\]
The finite-dimensional observer \( \hat{z}(x,t) \) of the state \( z(x,t) \), based on \( (M-1)(N_0+1)+N+1 \)-dimensional system of ODEs (14)-(15), is given by

\[
\hat{z}(x,t) = \zeta^{N_0}(t-r) \cdot \text{col} \{ \phi_j(x) \}_{j=0}^{N_0} + \zeta^{N-N_0}(t) \cdot \text{col} \{ \phi_j(x) \}_{j=N_0+1}^{N}.
\]

The controller is further chosen as

\[
u(t) = -K_0 \hat{z}_{1}^{N_0}(t).
\]

In particular, (14) implies \( u(t) = 0 \) for \( t \leq 0 \).

For well-posedness we introduce the change of variables \( w(x,t) = z(x,t) - \frac{1}{2}x^2 u(t-r) \). Then, the closed-loop system is presented as

\[
w_i(x,t) = w_{ix}(x,t) + qw(x,t) + f(x,t), \; x \in [0,1], \; t \geq 0,
\]

\[
w_i(0,t) = f_i(0), \; w_i(x,0) = 0, \;
\]

\[
f(x,t) = -\frac{1}{2}x^2 \hat{u}(t-r) + \left( \frac{x^2}{2} + 1 \right) u(t-r),
\]

the ODEs (14) and (17). Let \( z(\cdot,0) = w(\cdot,0) \in H^2(0,1) \).

We apply the step method on \( \{ [r,(j+1)r] \}_{j=0}^{r} \). For \( t \in [0,r] \) we have that \( f(x,t) = 0 \). By Theorems 6.3.1 and 6.3.3 in [26], (18) has a unique classical solution \( z = w \in C([0,r],L^2(0,1)) \cap L^1((0,1),L^2(0,1)) \) such that \( w_\cdot(\cdot,t) \in H^2(0,1) \) with \( w_\cdot(0,t) = w_\cdot(1,t) = 0 \) for \( t \in [0,r] \). Furthermore, since \( u(t-r) \equiv 0 \) for \( t \in [0,r] \), (15) implies that \( \zeta^{N-N_0}(t) \in C([0,r],L^2(0,1)) \). Since \( y \in C([0,\infty),L^2(0,1)) \), considering (14) on the subintervals \( \{ [r,(j+1)r] \}_{j=0}^{r} \), it can be seen that \( \zeta_{1}^{N_0} \in C([0,r],L^2(0,1)) \).

Using (6), (14) and (21) we obtain the following dynamics of the estimation errors for \( t \geq 0 \)

\[
c_{N_0}^{N_0}(t) = A_0 c_{N_0}^{N_0}(t) - L_0 c_{N_0}^{N_0}(t - \frac{r}{M}) - L_0 c_{N_0}^{N_0}(t - \frac{r}{M}) - L_0 c_{N_0}^{N_0}(t - \frac{r}{M}) + L_0 c_{N_0}^{N_0}(t - \frac{r}{M}) + L_0 c_{N_0}^{N_0}(t - \frac{r}{M}),
\]

\[
c_{M-1}^{N-1}(t) = A_0 c_{M-1}^{N-1}(t) - L_0 c_{M-1}^{N-1}(t - \frac{r}{M}) + L_0 c_{M-1}^{N-1}(t - \frac{r}{M}) + L_0 c_{M-1}^{N-1}(t - \frac{r}{M}),
\]

\[
c_{i}^{N-i}(t) = A_0 c_{i}^{N-i}(t) - L_0 c_{i}^{N-i}(t - \frac{r}{M}) + L_0 c_{i}^{N-i}(t - \frac{r}{M}), \; 1 \leq i \leq M-2
\]

and

\[
c_{N-N_0}^{N-N_0}(t) = A_1 c_{N-N_0}^{N-N_0}(t).
\]

From (6), (17) and (20), \( \zeta_{N_0}^{N}(t) \) satisfies

\[
z_{N_0}^{N}(t) = (A_0 - B_0 K_0) z_{N_0}^{N}(t) + B_0 K_0 \sum_{i=1}^{M} e_{i}^{N}(t).
\]

We introduce the notations

\[
\mathcal{X}_e(t) = \text{col} \{ e_{i}^{N}(t) \}_{i=1}^{M}, \; \mathcal{V}_e(t) = \mathcal{X}_e(t) - \mathcal{X}_e(t),
\]

\[
\mathcal{F}_e = \text{diag} \{ I_{M-1} \otimes (A_0 - L_0 C_0) + J_{M-1} (0) \otimes L_0 C_0, J_{M-1} \otimes L_0 C_0 \},
\]

\[
\mathcal{G}_e = \text{diag} \{ (M-1) \otimes (C_0 - L_0 C_0) + J_{M-1} (0) \otimes L_0 C_0, -L_0 C_0 \},
\]

\[
\mathcal{L}_e = \text{col} \{ (M-2)(N_0+1) \times 1, L_0, -L_0 \},
\]

\[
\mathcal{K}_e = [K_0, \cdots, K_0] \in \mathbb{R}^{1 \times M(N_0+1)}
\]

where \( J_{M-1}(0) \) is a Jordan block of order \( M-1 \) with zero diagonal. Note that (24) implies \( e_{N-N_0}^{N-N_0}(t - \frac{r}{M}) = e^{-A_1} \sigma \cdot e_{N-N_0}^{N-N_0}(t) \). Then, using (6), (17), (23), (25) and the reduced-order (i.e., decoupled from \( \zeta_{N-N_0}^{N}(t) \)) closed-loop system can be presented as

\[
z_{N_0}^{N}(t) = (A_0 - B_0 K_0) z_{N_0}^{N}(t) + B_0 \mathcal{X}_e(t),
\]

\[
\mathcal{X}_e(t) = \mathcal{F}_e \mathcal{X}_e(t) + \mathcal{G}_e \mathcal{V}_e(t) + \mathcal{L}_e \left( \mathcal{K}_e - \frac{1}{2} \right) \sigma \cdot \zeta_{N-N_0}^{N}(t),
\]

\[
\mathcal{X}_e(t) = (-A_0 + \mathcal{K}_e - \frac{1}{2} \sigma) z_{N_0}^{N}(t) - b_0 \mathcal{X}_e(t), \; n > N.
\]

In the case \( M = 1 \), \( \zeta^{N_0}(t) \) satisfies the first ODE in (14) and predicts \( \zeta_{N_0}^{N}(t+r) \). Here \( \mathcal{X}_e(t) = e_{N_0}^{N}(t) \) and the closed-loop system has the form (24) and (26), where now

\[
F_e = A_0 - L_0 C_0, \; G_e = -L_0 C_0, \; \mathcal{K}_e = K_0, \; \mathcal{L}_e = -L_0.
\]

Differently from the existing finite-dimensional controllers [6], [13], where the closed-loop systems is written in terms of the observer and the tail \( z_{N_0}^{N}(t) \) (\( n > N \)), here (26) is presented in terms of the state \( z_{N_0}^{N}(t) \), the estimation errors \( \mathcal{X}_e(t) \) and the tail. This allows to eliminate the delay \( r \) from the ODEs of \( \zeta_{N_0}^{N}(t) \) and \( \mathcal{X}_e(t) \). However, as \( n \) increases decreasing it to \( \frac{r}{M} \) (which is small for large \( M \)) in the ODE of \( \mathcal{X}_e(t) \).

**Remark 1:** In the case of sub-predictors for linear ODEs, the closed-loop system is given by (23) and (25), where \( \zeta = 0 \) and \( e_{N-N_0}^{N}(t) = 0 \). Thus, exponential stability of

\[
e_{N_0}^{N}(t) = (A_0 - L_0 C_0) e_{N_0}^{N}(t) - L_0 C_0 \mathcal{V}_e(t),
\]

where \( \mathcal{V}_e(t) = e_{M}(t - \frac{r}{M}) - e_{N}(t) \), guarantees the stability of the closed-loop system due to ISS of the \( e_{i}^{N}(1 \leq i \leq M-1) \) systems with respect to \( e_{i}^{N}(t) \). This is different from
the infinite-dimensional closed-loop system (26), where the
finite-dimensional part of the system is coupled via \( \zeta(t) \) with
the infinite-dimensional tail \( \zeta_n \) \((n > N)\). Here the proof of
stabilization for any delay \( r > 0 \) provided \( M \) and \( N \) are large
enough becomes challenging.

For \( L^2 \)-stability analysis of (24) and (26) we define the
Lyapunov functional

\[
V(t) := V_0(t) + V_e(t) + V_Q(t) + \rho e^{\alpha N - N_0}(t)^2,
\]

where \( 0 < P_0 \in \mathbb{R}^{(N_0 + 1) \times (N_0 + 1)} \), \( Q = Q^T \geq 0 \), \( \rho > 0 \in \mathbb{R}^m \)
and \( V_e(t) \) is given by (22), whereas \( V_0(t) \) compensates \( g(t) \)
in the ODEs of the estimation errors. Differentiation of \( V_Q(t) \) yields

\[
\dot{V}_e(t) + 2\delta V_e(t) = Q \xi^2(t) - \varepsilon_M Q \xi^2 \left( t - \frac{r}{M} \right),
\]

Differentiating \( V_0(t) \) along (26) we obtain

\[
\dot{V}_0(t) + 2\delta V_0(t) = \left( z_0(t) + 2\delta P_0 + 2\delta A_0 B_0 K_0 \right)^T (t) 2\delta P_0 + 2\delta A_0 B_0 K_0 + 2\delta (z_0(t) + 2\delta P_0)^T (t) P_0 B_0 \zeta(t)
\]

\[
+ 2\delta N_0 \left( -\lambda_0 + \gamma + \delta + \xi(t) \right) z_0(t)
\]

where \( 0 < \delta N_0 \) and \( \lambda_0 \) are given in (6), and the estimate

\[
\sum_{n = N_0 + 1}^{\infty} b_n^2 \leq 2\alpha^2 \frac{N}{N \pi^2}.
\]

To compensate \( \xi^2(t) \) we use monotonicity of \( \lambda_n \), \( n \in \mathbb{Z}_+ \),
and (31) and (32) to obtain

\[
2 \sum_{n = N_0 + 1}^{\infty} \left( -\lambda_0 + \gamma + \delta + \frac{1}{2} \alpha + \frac{1}{2} \beta \right) \xi_n^2(t)
\]

\[
\leq 2 \left( -\lambda_0 + \gamma + \delta + \frac{1}{2} \alpha + \frac{1}{2} \beta \right) \xi_{N_0}^2(t) \| e \|_N^2 \leq \xi(t)
\]

From (31)-(35) we have

\[
\dot{V}(t) + 2\delta V(t) \leq \eta(t) + \frac{\xi(t)}{N} \xi^2(t) \leq 0, \quad t \geq 0,
\]

provided

\[
\Psi_1 = \Psi_{full} + \left( \frac{1}{M} \right)^2 \Lambda \epsilon R \Lambda < 0,
\]

where \( \xi(t) \) is defined in (31). Finally, (7) yields \( \gamma \leq 0 \).

By Schur’s complement we have that \( \Psi_2 < 0 \) if

\[
\left[ \frac{-\lambda_0 + \gamma + \delta + \alpha \eta(t)}{N} \right] \Lambda \xi(t) \xi(t) \leq 0.
\]

Finally, that (7) yields \( \gamma \leq 0 \). Therefore, applying Schur
complement and taking \( p \rightarrow \infty \) we find that \( \Psi_1 < 0 \)

with \( \Lambda_0 \) given in (36). Note that (37) and (38) are reduced-order LMI’s whose dimension is independent of \( N \). Summarizing, we arrive at:

**Theorem 1:** Consider (3), measurement (4) with \( c \in L^2(0,1) \) satisfying (9) and control law (17). Let \( \delta > 0 \) be a desired decay rate. Let \( N_0 \in \mathbb{Z}_+ \) satisfy (7) and \( N \geq N_0 + 1 \). Assume that \( L_0 \) and \( K_0 \) are obtained using (10) and (11), respectively. Given \( M \in \mathbb{Z}_+ \), \( M \geq 1 \) and \( r > 0 \), let there exist positive definite matrices \( P_0, P_e, S_e, R_e \) and scalars \( Q, \alpha, \alpha_0 > 0 \) such that (37) and (38) hold. Then the solution \( z(x,t) \) to (3) under the control law (17) and the corresponding subpredictor-based observer \( \hat{z}(x,t) \) defined by (15), (14) and (16) satisfy

\[
\| z(t) - \hat{z}(t) \| + \| z(t) - \hat{z}(t) \| \leq De^{-\delta t} \| z(t) \|, \quad \text{for some constant } D > 0.
\]

We show next that (37) and (38) are feasible for any delay \( r > 0 \) provided \( M \) and \( N \) are large enough. For this purpose consider (27) and

\[
V_M(t) = e_{\xi_0}(t)^T P_{s, e} \xi_0(t) + V_{R_M}(t)
\]

with \( V_{s,e}(t), V_{R_M}(t) \) as in (29), where \( 0 < P, S, R \in \mathbb{R}^{N_0 + 1}, \)
and (40)
with $h = \frac{\epsilon}{M}$ guarantees $V_M(t) + 2\delta V_M(t) \leq 0$ along (27). Given $\delta > 0$, (10) implies that (40) is feasible for small enough $h > 0$ (see e.g. [27]).

**Proposition 1:** Given $h > 0$, let $0 < P, S, R \in \mathbb{R}^{N+1}$ such that (40) holds. Then, given $r > 0$ and $M > \frac{\epsilon}{M}$, there exists some $N_r$ such that for all $N > N_r$, (37) and (38) are feasible.

**Proof:** We first show that there exist $0 < P_c, S_c, R_c \in \mathbb{R}^{M(N+1) \times M(N+1)}$ such that

$$\Phi(P_c, S_c, R_c) + \left( \frac{r}{M} \right)^2 \begin{bmatrix} F_T^T & G_T^T \end{bmatrix} \begin{bmatrix} R_c & F_c \end{bmatrix} < 0$$

with $\Phi(P_c, S_c, R_c)$ given in (36). Consider the ODE

$$\dot{X}_c = F_c X_c(t) + G_c V_c(t)$$

obtained from (26) by setting $\zeta(t) \equiv 0$ and $e^{N-N(t)} \equiv 0$. For $V_c(t)$, given in (28), by standard arguments it can be easily verified that (41) guarantees $V_c(t) + 2\delta V_c(t) \leq 0$. We will construct $V_c(t)$ recursively, by using $P, S$ and $R$, thereby obtaining $P_c, S_c$ and $R_c$. First, consider the ODE (27). Since $\frac{\epsilon}{M} < h$, (40) holds with $h$ replaced by $\frac{\epsilon}{M}$. Next, consider (27) and the ODE of $e^{M-1}(t)$ in (42):

$$e^{M-1}(t) = (A_0 - L_0 C_0) e^{M-1}(t) - L_0 C_0 \left( V_c, M-1(t) - e^{M}(t) \right)$$

Let $\mu > 0$ and define

$$V_{M-1} = \left( e^{M}(t) \right)^2 + V_s, M-1 \left( t \right) + V_r, M-1( t ) + \mu V_M( t )$$

where $V_M(t)$ is rescaled by $\mu$. Using (27) and (43), the following LMI guarantees $V_{M-1}(t) + 2\delta V_{M-1}(t) \leq 0$:

$$\begin{bmatrix} \psi & \left( \frac{\epsilon}{M} \right)^2 \end{bmatrix} \begin{bmatrix} P_{c}, S_{c}, R_{c} \end{bmatrix} + \mu \left( \frac{\epsilon}{M} \right)^2 \begin{bmatrix} P_{c}, S_{c}, R_{c} \end{bmatrix} \begin{bmatrix} 0 \\ \mu \psi \end{bmatrix} 0 \end{bmatrix} < 0 \quad \begin{bmatrix} 0 \end{bmatrix}$$

Since $\psi < 0$, taking $\mu$ large enough and applying Schur complement it can be seen that (44) holds. Repeating these arguments by backward induction, i.e. choosing

$$V_{M-j}(t) = \left( e^{N}(t) \right)^2 + V_s, M-j \left( t \right) + V_r, M-j(t) + \mu V_{M-j+1}( t )$$

and increasing $\mu$ at each step, we obtain that (41) holds with $W_c = \text{diag} \left( \mu W_{j} \right)_{j=0}^{M-1}$, $W \in \{P, S, R\}$. Next, recall (37) and (38), with $\Psi_0$ given in (36). Set $\alpha = \alpha_1 = 2$ and let $\beta > 0$. Rescaling, we replace $\Phi(P_c, S_c, R_c)$ with $\Phi(\beta P_c, \beta S_c, \beta R_c) = \beta \Phi(P_c, S_c, R_c)$. Let $P_0 = P_r$, given in (11), resulting in $\Psi < 0$ in (37). Setting $\beta > 0$ to be large enough, then choosing $Q = N$ large enough and applying Schur complement twice in (38), we find that (37) and (38) hold.

**B. Classical observer-based predictor**

For the case of a classical predictor, we consider a $N+1$ dimensional observer of the form

$$\dot{\bar{z}}(x, t) = \frac{\epsilon}{M} \dot{z}(t) \cdot \text{col} \{\phi_j(x)\}_{j=0}^N$$

$$+ \frac{\epsilon}{M} \dot{z}^{N-N}(t) \cdot \text{col} \{\phi_j(x)\}_{j=N+1}$$

Here $\frac{\epsilon}{M}$ is defined in (12) and satisfies (15), whereas $\frac{\epsilon}{M}$ satisfies the following ODE

$$\begin{align*}
\dot{z}(t) &= A_0 z(t) + B_0 u(t - r) \\
&- L_0 \left[ C_0 z(t) + C_1 z^{N-N}(t) \right], t \geq 0;
\end{align*}$$

$$\dot{z}(t) = 0, t \leq 0.$$  

Recall $e^{N-N(t)}$ given in (19) and satisfying (24). Define $e^{N-N(t)} = z^{N-N}(t)$, where $z^{N-N}(t)$ is given in (12). The innovation term in (47) can be presented as

$$C_0 z^{N-N}(t) + C_1 z^{N-N}(t) - \zeta(t) = -C_0 e^{N-N}(t) - C_1 e^{N-N}(t) - \zeta(t)$$

with $\zeta(t)$, given in (21), subject to (22). Using these notations with (6) and (47), we obtain

$$e^{N-N}(t) = (A_0 - L_0 C_0) e^{N-N}(t) - L_0 C_1 e^{N-N}(t) - L_0 \zeta(t).$$

As in [23], we propose the predictor-based control law

$$\bar{z}(t) = e^{\alpha r} z(t) + \int_0^t e^{\alpha (t-s)} B_0 u(s) ds, u(t) = -K_0 \bar{z}(t)$$

Note that exponential decay of $\bar{z}(t)$ implies exponential decay of $\frac{\epsilon}{M}$ with the same decay rate. Differentiating $\bar{z}(t)$ and using (47) and (49) we obtain

$$\dot{\bar{z}}(t) = (A_0 - B_0 K_0) \bar{z}(t) + e^{\alpha r} L_0 \left[ C_0 e^{N-N}(t) + C_1 e^{N-N}(t) + \zeta(t) \right], t \geq 0.$$  

Then, the reduced-order (decoupled from $e^{N-N(t)}$) closed-loop system is given by non-delayed ODEs (24), (48), (50) and the tail

$$z_n(t) = (-\alpha + q) z_n(t) - \beta_n K_{0} \zeta(t), n > N$$

which depends on the delay. Note also that in the case of state-feedback (see e.g. [21]), the predictor is given by (49) with $z^{N-N}$ changed by $\frac{\epsilon}{M}$ leading to decoupled from the tail ODE (50) with $L_0 = 0$. The latter simplifies the stability analysis of the closed-loop system and makes the proof of LMI feasibility trivial. Next, we consider $L^2$-stability analysis of the closed-loop system, which is delay-independent for $\delta = 0$. Define the Lyapunov functional

$$V(t) = V_0(t) + \int_0^t \left[ \Psi_1(t) \right]^2 + \int_0^t e^{-2\delta(t-s)} ||z(s)||^2 ds,$$

$$V_0(t) = ||z(t)||^2 + \sum_{n=N+1}^{\infty} \frac{\zeta_n(t)}{2} + P_r e^{N-N}(t),$$

where $P_r, P_s, S > 0$ are matrices of appropriate dimensions and $P_r > 0$ is a scalar. By arguments similar to (31)-(38) the following LMI guarantees $\dot{V} + 2\delta \dot{V} \leq 0$ for $P_r \rightarrow \infty$:

$$\begin{bmatrix} \Psi_1 & *, \Psi_2 \\ *, \Psi_2 \end{bmatrix} < 0,$$

$$\begin{bmatrix} P_{0} e^{\alpha r} L_0 & P_r e^{\alpha r} L_0 \\ *, *, 0 \end{bmatrix} < 0,$$

$$\Psi_1 = P_{0} (A_0 - L_0 C_0) + (A_0 - L_0 C_0) \Psi_2 + 2\delta \Psi_2,$$

$$\Psi_2 = \frac{1}{2 ||z||^2} \left[ -A_{N} + q + \delta + \frac{\epsilon}{M} A_{N+1} \right],$$

where $\Psi$ is given in (36). Fix any $r > 0$. Let $\alpha = 1, S = \frac{1}{\sqrt{\epsilon}}, P_r$ such that $\psi = -2I$ and $P_r$ such that $\psi = -2I$.  

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Choosing first $\beta > 0$ large enough and then $N$ large enough and applying Schur complement, (53) holds. Summarizing:

**Proposition 2:** Consider (3), measurement (4) with $c \in L^2(0,1)$ satisfying (9) and control law (49). Let $\delta > 0$ be a desired decay rate. Let $N_0 \in \mathbb{Z}_+$ satisfy (7) and $N \geq N_0 + 1$. Let $L_0$ and $K_0$ be obtained using (10) and (11), respectively. Given $r > 0$, let there exist positive definite matrices $P_r, P_r, S$ and scalar $\alpha > 0$ such that LMIs (53) hold. Then the solution $\bar{z}(x,t)$ to (3) under the control law (49) and the observer $\hat{z}(x,t)$ defined by (46) satisfy (39) for some constant $D > 0$.

Furthermore, given any $r > 0$, the LMI (53) is feasible provided $N$ is large enough.

**C. Numerical example**

We consider (3) with $q = 3$, resulting in an unstable open-loop system. We consider (4) with $c(x) = X_{0.3,0.6}$ (an indicator function). We fix $\delta = 0.1$ which results in $N_0 = 0$.

The controller and observer gains are found using (10) and (11) as $K_0 = 8.8, \quad L_0 = 14.66$.

We start with sub-predictors. Given various values of $r > 0$, the LMIs of Theorem 1 were verified for $1 \leq N + M \leq 110$ and $1 \leq M \leq 20$ by using the standard Matlab LMI toolbox.

Table I presents the minimal values of $N$ and $M$ found to guarantee the feasibility (i.e. the exponential stability of the closed-loop system with a decay rate 0.1).

For classical predictors, the LMIs of Proposition 2 were verified for $1 \leq N \leq 100$. Table II presents the minimal values of $N$ which guarantee feasibility of the LMIs. It is seen from the tables that for the same values of $r$, the classical predictor employs a lower-order $N+1$-dimensional observer compared to $(M-1)(N_0+1)+N+1$-dimensional sub-predictors.

Numerical simulations appear in the arXiv version [28].

### TABLE I

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<td>6</td>
<td>6</td>
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<tr>
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**TABLE II**

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</table>

**III. Conclusion**

We studied constant input delay compensation by finite-dimensional observer-based controllers for the 1D heat equation. We proved that both sub-predictors and classical predictors theoretically compensate any delay provided the observer dimension is large. Classical predictors are known to be less friendly in application to uncertain systems (see e.g. Remark 3 in [19]). The suggested predictor methods can be extended in the future to various parabolic PDEs.

**References**


[7] ——, “Finite-dimensional control of the Kuramoto-Sivashinsky equation under point measurement and actuation,” in 59th IEEE Conference on Decision and Control, 2020.


