PDE-based deployment of multi-agents measuring relative position to one neighbour

Anton Selivanov and Emilia Fridman

Abstract— **We develop a PDE-based approach to multi-agent deployment where each agent measures its relative position to only one neighbour. First, we show that such systems can be modelled by a first-order hyperbolic partial differential equation (PDE) whose L^2 stability implies the stability of the multi-agent system for a large enough number of agents. Then, we show that PDE modelling helps to construct a Lyapunov function for the multi-agent system using spatial discretisation. Then, we use the PDE model to estimate the leader input delay preserving the stability.**

Index Terms— **Multi-agent systems, Partial Differential Equations, Time-delay systems, Linear Matrix Inequalities**

I. Introduction

Multi-agent systems are traditionally modelled by ordinary differential equations (ODEs) [1]. Such models become complicated for a large number of agents (see Section III-B). This scalability problem does not arise if a large-scale multi-agent system is modelled by a partial differential equation (PDE), whose complexity does not change when the number of agents grows. In particular, PDEs provide convenient models for highway traffic [2], [3], animal swarms [4], [5], and self-driven particles [6], [7]. They are used to design robot swarm [8]–[10], control vehicle formations [11]–[13], and deploy multi-agent systems [14]–[20]. Note that PDE-based traffic models are not used for deployment since they describe traffic density and ignore relative vehicle positions.

Most existing papers on deployment assume that each agent knows its distance to at least two neighbours. In this paper, we assume that each agent knows its relative position with respect to only one neighbour. This leads to a first-order hyperbolic PDE that has not been considered in the context of multi-agent deployment. We show that its L^2 stability implies the stability of the multi-agent system (Section II). The traditional, pointwise relation between the multi-agent and PDE states require a stronger H^1 stability (Remark 2). In Section III, we demonstrate how PDE-based analysis can provide insights into the stability of a multi-agent system. In Section IV, we use the PDE model to study the stability of a multi-agent system where the leader has a time-varying input delay. Section V provides a numerical demonstration of the main results.

Notations: $|\cdot|$ is the Euclidean norm, $\|\cdot\|$ is the L^2 norm. The minimum eigenvalue of $P \in \mathbb{R}^{n\times n}$ is denoted by $\lambda_{\min}P$. The symmetric elements of a matrix are denoted by "$*$". Partial derivatives are denoted by indices, e.g., $y_x := \partial y/\partial x$. Other notations are standard.

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II. From connected ODEs to a hyperbolic PDE

Consider $N+1$ agents governed by

$$\dot z_i(t) = f(t, z_i(t)) + u_i(t), \quad i = 0, \dots, N \tag{1}$$

with states $z_i\colon [0,\infty) \to \mathbb{R}^n$, control inputs $u_i\colon [0,\infty) \to \mathbb{R}^n$, and $f\colon [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ describing local dynamics. The objective is to deploy the agents onto a curve given by $\gamma \in C^1([0,1],\mathbb{R}^n)$. Namely, we are looking for u_i such that

$$\lim_{t\to\infty} z_i(t) = \gamma_i := \gamma\left(\tfrac{i}{N}\right), \quad i = 0, \dots, N. \tag{2}$$

We assume that

1) f is continuous and satisfies a Lipschitz condition in the second argument, i.e., there exists $L>0$ such that $|f(t,x)-f(t,y)| \le L|x-y|,\ \forall t\ge 0,\ \forall x,y\in\mathbb{R}^n$.
2) Agent 0 (the leader) measures $z_0(t)-\gamma_0$ and knows $f(t,\gamma_0)$;
3) Each agent $i \in \{1,\dots,N\}$ (a follower) measures $z_i(t)-z_{i-1}(t)$, knows $\gamma_i-\gamma_{i-1}$, and knows $f(t,\gamma_i)$.

The leader is the only agent that knows its position relative to the target curve. In case of formation control, where the absolute positions do not matter, this assumption is not needed. Each follower knows the current and desired differences between its state and the state of one neighbour. This way, if $z_{i-1}(t) = \gamma_{i-1}$, then agent i can get to γ_i without knowing where it is. Finally, each agent needs $f(t,\gamma_i)$ to maintain its position when on the target curve.

These assumptions allow for the following controllers

$$\begin{aligned} u_0(t) &= -k(z_0(t)-\gamma_0) - f(t,\gamma_0),\\ u_i(t) &= -\sigma\left[(z_i(t)-z_{i-1}(t)) - (\gamma_i-\gamma_{i-1})\right] - f(t,\gamma_i),\\ &\qquad\qquad i = 1,\dots,N, \end{aligned} \tag{3}$$

where $k>0$ and $\sigma>0$ are design parameters. The leader's controller tries to steer its state $z_0(t)$ to γ_0, while the other controllers try to achieve the desired relative positions. The $f(t,\gamma_i)$ terms guarantee that $\dot z_i(t) = 0$ on the target curve.

Remark 1 (The target curve): By imposing additional restrictions on the target positions, we can relax the above assumptions and simplify the controllers. For example, if $f(t,\gamma_i) = f(\gamma_i) = 0$, then the controllers do not need the $f(t,\gamma_i)$ terms. Moreover, we may consider time-varying target positions $\gamma_i \in C^1[0,\infty)$ that satisfy $\dot\gamma_i = f(t,\gamma_i)$. Then the agents do not need the $f(t,\gamma_i)$ terms, but they do need to know the time-varying differences $\gamma_i(t)-\gamma_{i-1}(t)$,

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which seems impractical. If $\dot{\gamma}_i = -\sigma(\gamma_i - \gamma_{i-1}) + f(t, \gamma_i)$ for $i = 1, \dots, N$, then the followers should use $u_i(t) = -\sigma[z_i(t) - z_{i-1}(t)]$. Such simplifying conditions are common for the PDE-based multi-agent deployment (see, e.g., [15]).

For the deviations $y_i(t) := z_i(t) - \gamma_i$, we obtain

$$\dot{y}_0 = -ky_0 + \Delta f_0(t, y_0), \tag{4a}$$
$$\dot{y}_i = -\sigma(y_i - y_{i-1}) + \Delta f_i(t, y_i), \quad i = 1, \dots, N, \tag{4b}$$

where $\Delta f_i(t, y_i) := f(t, \gamma_i + y_i(t)) - f(t, \gamma_i)$ satisfies $|\Delta f_i(t, y)| \le L|y|$. We will look for a PDE with state $y\colon [0, \infty) \times [0, 1] \to \mathbb{R}^n$ such that $y \in C^1$, $y_t(t, \cdot) \in C[0, 1]$, and, for $i = 0, \dots, N$,

$$y_i(t) = \frac{1}{h} \int_{ih}^{(i+1)h} y(t, x)\, dx, \quad h := \frac{1}{N+1}. \tag{5}$$

Then (4a) becomes

$$\frac{1}{h} \int_0^h y_t(t, x)\, dx = \\ = -\frac{k}{h} \int_0^h y(t, x)\, dx + \Delta f_0 \left(t, \frac{1}{h} \int_0^h y(t, x)\, dx \right).$$

By continuity, for $N \to \infty$ we obtain

$$y_t(t, 0) = -ky(t, 0) + F(t, 0, y(t, 0)),$$

where $F(t, x, y) := f(t, \gamma(x) + y) - f(t, \gamma(x))$ satisfies

$$|F(t, x, y)| \le L|y|, \quad \forall t \in [0, \infty), \ x \in [0, 1], \ y \in \mathbb{R}^n. \tag{6}$$

For the followers ($i = 1, \dots, N$), we have

$$y_i(t) - y_{i-1}(t) = \frac{1}{h} \int_{ih}^{(i+1)h} y(t, x) dx - \frac{1}{h} \int_{(i-1)h}^{ih} y(t, x) dx \\ = \int_{ih}^{(i+1)h} \frac{y(t, x) - y(t, x-h)}{h}\, dx.$$

Then (4b) becomes

$$\frac{1}{h} \int_{ih}^{(i+1)h} y_t(t, x)\, dx = \\ -\sigma \int_{ih}^{(i+1)h} \frac{y(t, x) - y(t, x-h)}{h}\, dx \\ + \Delta f_i \left(t, \frac{1}{h} \int_{ih}^{(i+1)h} y(t, \xi)\, d\xi \right). \tag{7}$$

Since $y(t, \cdot)$ and γ are uniformly continuous (see Section II-A), for any $\delta > 0$ there is $\bar{h}$ such that for any $h < \bar{h}$ and any $x \in [ih, (i+1)h]$,

$$\left| y(t, x) - \frac{1}{h} \int_{ih}^{(i+1)h} y(t, \xi)\, d\xi \right| < \delta, \quad |\gamma(x) - \gamma_i| < \delta.$$

Since $f(t, \cdot)$ is also uniformly continuous, for any $\varepsilon > 0$ there is $\delta > 0$ such that the above relations imply

$$\left| F(t, x, y(t, x)) - \Delta f_i \left(t, \frac{1}{h} \int_{ih}^{(i+1)h} y(t, \xi)\, d\xi \right) \right| < \varepsilon.$$

Since (7) should hold for large enough N (i.e., small enough $h > 0$), we obtain

$$y_t(t, x) = -\sigma h y_x(t, x) + F(t, x, y(t, x)), \ t > 0, \ x \in (0, 1).$$

Therefore, (4) is approximated by the hyperbolic PDE

$$\begin{aligned} & y_t = -\nu y_x + F(t, x, y), \quad x \in (0, 1), \quad t > 0, \\ & y_t(t, 0) = -ky(t, 0) + F(t, 0, y(t, 0)), \quad t > 0, \end{aligned} \tag{8}$$

where

$$\nu = \sigma h = \frac{\sigma}{N+1}.$$

The value of ν is the propagation speed. In particular, it takes $1/\nu$ time units for a change in the leader dynamics to affect the last agent. Note that $\nu \to 0$ as $N \to \infty$. This reflects the fact that the information from the leader needs more time to reach the last follower when the number of agents is higher. Since only the leader knows its relative position with respect to the target curve, it seems reasonable to have ν as large as possible, which requires large σ in the control (3). However, if $z_i - z_{i-1}$ are measured with noise, this noise is multiplied by σ, which, therefore, should not be too large. We show below that (8) is exponentially stable in the L^2 norm for any $\nu > 0$, though smaller ν leads to string instability (see Section V).

Remark 2: In fact, it is quite obvious that (4) is the finite-difference approximation of (8) when one assumes that $y_i(t) = y(t, ih)$. However, when we use the finite-volume method (5), the Jensen inequality implies

$$\frac{1}{N+1} \sum_{i=0}^{N} y_i^2(t) = h \sum_{i=0}^{N} \frac{1}{h^2} \left(\int_{ih}^{(i+1)h} y(t, x)\, dx \right)^2 \\ \le \sum_{i=0}^{N} \int_{ih}^{(i+1)h} y^2(t, x)\, dx = \|y(t, \cdot)\|^2.$$

Therefore, if $\|y(t, \cdot)\| \xrightarrow[t \to \infty]{} 0$, then $\frac{1}{N+1} \sum_{i=0}^{N} y_i^2(t) \xrightarrow[t \to \infty]{} 0$. That is, the L^2-convergence of the PDE state implies the convergence of the ODE state. This does not hold for $y_i(t) = y(t, ih)$, where a stronger H^1-convergence is required.

A. Well-posedness of the hyperbolic PDE

The boundary condition of (8) is a well-posed ODE with a unique solution $y(\cdot, 0) \in C^1[0, \infty)$. For a given $y(\cdot, 0)$, $w(t, x) = y(t, x) - y(t, 0)$ satisfies

$$\begin{aligned} & w_t = -\nu w_x + \Delta F + ky(t, 0), \\ & w(t, 0) = 0, \end{aligned}$$

where $\Delta F(t, x, w) = F(t, x, y(t, 0) + w) - F(t, x, y(t, 0))$. The operator $Af = -\nu f'$ defined on $D(A) \subset X$, where

$$D(A) := \{\psi \in H^2[0, 1] \mid \psi(0) = 0\}, \qquad X := L^2[0, 1],$$

is the infinitesimal generator of a C_0 semigroup

$$(T(t)\psi)(x) = \begin{cases} \psi(x - \nu t), & x > \nu t, \\ 0, & x \le \nu t. \end{cases}$$

Since the map $(t, w) \mapsto \Delta F(t, \cdot, w) + ky(t, 0) \in X$ is continuous in t and Lipschitz in w, there is a unique mild

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solution for $w(0,\cdot)\in X$ [21, Theorem 6.1.2]. Therefore, (8) has a unique mild solution $y\in C([0,\infty),X)$ for $y(0,\cdot)\in L^2[0,1]$. Moreover, if $y(0,\cdot)$ and f from (1) are smooth and

$$y(0,\cdot)\in D(A), \tag{9}$$

this mild solution is a smooth classical solution [21, Theorem 6.1.5]. Note that we need $y(t,\cdot)\in H^2$ to ensure that $y_t(t,\cdot)\in C[0,1]$, which is used in the derivation of (8).

III. Stability analysis

In this section, we demonstrate how a PDE model can simplify stability analysis even for a relatively simple multi-agent system. Namely, we construct a Lyapunov functional for the PDE and show that its discretisation is better than the intuitive Lyapunov function for the multi-agent system.

A. Stability of the PDE

First, we construct a Lyapunov functional for (8). To take advantage of the $-ky(t,0)$ and $-\nu y_x$ terms on the right-hand side of (8), it is natural to look for a Lyapunov functional in the form

$$V=W_0+V_0, \tag{10}$$

where

$$W_0=|y(t,0)|^2 \quad\text{and}\quad V_0=\int_0^1 p(x)|y(t,x)|^2\,dx$$

with a design parameter $p\in C^1([0,1],(0,\infty))$. We would like to have $\dot V+2\alpha V\le 0$ so that $V(t)\le e^{-2\alpha t}V(0)$, which implies the exponential stability of (8) with the decay rate α. Using (6), we obtain

$$\begin{aligned}\dot W_0+2\alpha W_0&=2y^T(t,0)y_t(t,0)+2\alpha W_0\\&=2y^T(t,0)\left[-ky(t,0)+F(t,0,y(t,0))\right]+2\alpha W_0\\&\le -2(k-L-\alpha)|y(t,0)|^2.\end{aligned}$$

Furthermore,

$$\dot V_0+2V_0=2\int_0^1 py^Ty_t+2V_0=2\int_0^1 py^T[-\nu y_x+F]+2V_0.$$

Integrating by parts, we obtain

$$\int_0^1 py^Ty_x=p|y|^2|_0^1-\int_0^1 p'|y|^2-\int_0^1 py_x^Ty.$$

Moving the last term to the left and multiplying both sides by $-\nu$, we obtain

$$-2\nu\int_0^1 py^Ty_x=-\nu p|y|^2|_0^1+\nu\int_0^1 p'|y|^2.$$

Since $p(x)>0$, we have

$$2\int_0^1 py^TF\le 2\int_0^1 p|y^TF|\le 2\int_0^1 p|y||F|\overset{(6)}{\le}2L\int_0^1 p|y|^2.$$

Therefore,

$$\dot V_0+2\alpha V_0\le -\nu p|y|^2|_0^1+\int_0^1(\nu p'+2Lp+2\alpha p)|y|^2.$$

Summing up, we obtain

$$\begin{aligned}\dot V+2\alpha V&=V_0+W_0+2\alpha V_0+2\alpha W_0\\&\le -\nu p(1)|y(t,1)|^2+[\nu p(0)-2(k-L-\alpha)]|y(t,0)|^2\\&\qquad+\int_0^1(\nu p'+2Lp+2\alpha p)|y|^2.\end{aligned}$$

The first term is negative provided $p(1)>0$. To zero the other terms, we solve the Cauchy problem

$$\nu p'(x)+2Lp(x)+2\alpha p(x)=0,\quad p(0)=\frac{2}{\nu}(k-L-\alpha),$$

which has a unique solution

$$p(x)=\frac{2}{\nu}(k-L-\alpha)e^{-2(L+\alpha)x/\nu},\quad x\in[0,1]. \tag{11}$$

Note that $p(x)>0$ only if $k>L+\alpha$. That is, if $k>L+\alpha$, then we can design a Lyapunov functional guaranteeing the exponential stability of (8) with the decay rate α.

Proposition 1: If a continuous F satisfies (6), $\nu>0$, and $k>L+\alpha$, then (8), (9) is exponentially stable with the decay rate α, i.e., there is $M\ge 1$ such that

$$|y(t,0)|^2+\|y(t,\cdot)\|^2\le Ce^{-2\alpha t}(|y(0,0)|^2+\|y(0,\cdot)\|^2). \tag{12}$$

Lyapunov functionals similar to (10), (11) are well-known for 1D hyperbolic PDEs [22]. The purpose of this section was to demonstrate how it can be designed by solving an appropriate Cauchy problem.

B. Stability of the ODEs

Now we show how to design a Lyapunov function for (4) without using PDEs. For simplicity, let $n=1$. Consider

$$V=\sum_{i=0}^N q_iy_i^2,\qquad q_i>0. \tag{13}$$

A more general approach is to use $V=\bar y^TP\bar y$, where $\bar y=\operatorname{col}\{y_0,\dots,y_N\}$. However, this leads to high-dimensional conditions when $N\to\infty$.

Using (**??**), we obtain

$$\begin{aligned}\dot V&=2\sum_{i=0}^N q_iy_i\dot y_i=2q_0y_0[-ky_0+\Delta f_0(t,y_0)]\\&\quad+2\sum_{i=1}^N q_iy_i\left[-\sigma(y_i-y_{i-1})+\Delta f_i(t,y_i)\right]\\&\le -2q_0[k-L]y_0^2-2\sigma\sum_{i=1}^N q_iy_i(y_i-y_{i-1})+2L\sum_{i=1}^N q_iy_i^2\\&=-2q_0[k-L]y_0^2+2\sigma q_1y_1y_0-2\sigma\bar y^TM\bar y+2L\bar y^TQ\bar y,\end{aligned}$$

where $\bar y=\operatorname{col}\{y_1,y_2,\dots,y_N\}$,

$$M=\begin{bmatrix}q_1&0&\dots&\dots&0\\-q_2&q_2&0&\dots&\vdots\\0&-q_3&q_3&\dots&\vdots\\\vdots&\vdots&\ddots&\ddots&0\\0&\dots&\dots&-q_N&q_N\end{bmatrix},$$

$$Q=\operatorname{diag}\{q_1,\dots,q_N\}.$$

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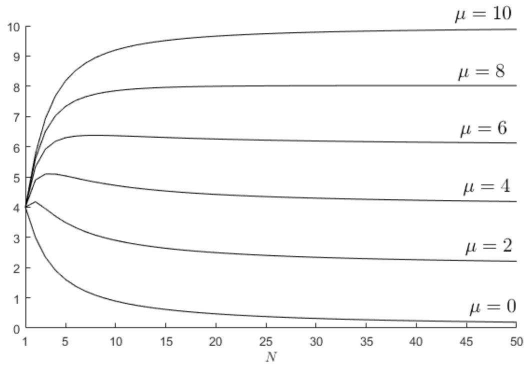

Fig. 1. The value of $(N+1)\lambda_{\min}\left\{Q^{-\frac{1}{2}}(M^T+M)Q^{-\frac{1}{2}}\right\}$ for different N and μ

For stability, we need $k > L$ and

$$-\sigma(M^T+M)+2LQ<0.$$

The latter is equivalent to

$$\sigma\lambda_{\min}\left\{Q^{-\frac{1}{2}}(M^T+M)Q^{-\frac{1}{2}}\right\} > 2L. \tag{14}$$

That is, if $k > L$ and (14) holds, we can design a Lyapunov function guaranteeing the stability of (4). Note that (14) contains the design parameters $q_i > 0$, $i = 1, \dots, N$.

C. Comparison of the PDE and ODE approaches

The naive approach is to take $q_i = 1$, $i = 1, \dots, N$ in (13). In this case, $\lambda_{\min}(M^T+M) = 2\left(1-\cos\frac{\pi}{N+1}\right)$ and (14) turns into

$$\sigma\left(1-\cos\frac{\pi}{N+1}\right) > L.$$

If $\sigma = \nu(N+1)$, as the PDE suggests, this becomes

$$\frac{L}{\nu\pi} < \frac{1-\cos\frac{\pi}{N+1}}{\frac{\pi}{N+1}} \xrightarrow[N\to\infty]{} 0.$$

That is, (14) does not hold for any L and ν if N is large enough. Therefore, the above naive approach is more restrictive than the PDE approach, which only requires $k > L$.

Now let us consider the PDE-inspired weights

$$q_i = \exp\left\{-\frac{i\mu}{N+1}\right\}, \quad i = 1, \dots, N. \tag{15}$$

For these weights, it is more difficult to compute the minimum eigenvalue in (14). Figure 1 shows the values of $(N+1)\lambda_{\min}\left\{Q^{-\frac{1}{2}}(M^T+M)Q^{-\frac{1}{2}}\right\}$ found numerically for different N and μ. The limit of each line is the corresponding μ. Therefore, (14) holds with $\sigma = \nu(N+1)$ for large enough μ and N.

It is difficult to guess appropriate values of q_i by studying (14), while the PDE model (8) leads to a simple ODE whose solution, (11), can be discretised to obtain suitable q_i.

Note that the stability of the PDE (8) does not depend on ν. Therefore, taking $\sigma = \nu(N+1)$, we can be sure that (4) is stable for a large enough N no matter what the nonlinearity is. However, when ν gets smaller, the overshoot becomes larger (see Section V). For the multi-agent system (4), the overshoot gets larger for agents that are further from the leader. This behaviour is reminiscent of the string instability occuring in vehicle platoons [23], [24]. This is why it is reasonable to use the decaying weights (15). In particular, smaller ν leads to a faster decaying function in (11).

There are other ways of studying the stability of (4). For example, since (4) is a positive system, one can consider $V = \sum_{i=0}^{N} q_i y_i$. Similar analysis can be done for (8), which is also a positive system. Here we demonstrated what insights a PDE can bring when comparible methods are used.

IV. Time-delayed measurements in the leader

In this section, we assume that the leader has an unknown time-varying measurement delay $\tau(t) \in [0, \tau_M]$ with a known $\tau_M > 0$. In this case, the control takes the form

$$\begin{aligned} u_0(t) &= -k(z_0(t-\tau(t))-\gamma_0) - f(t,\gamma_0),\\ u_i(t) &= -\sigma\left[(z_i(t)-z_{i-1}(t))-(\gamma_i-\gamma_{i-1})\right] - f(t,\gamma_i),\\ &\qquad\qquad i = 1, \dots, N, \end{aligned} \tag{16}$$

where we use $z_0(t) = 0$ for $t < 0$. Note that γ_0 and $f(t,\gamma_0)$ are assumed to be known and, therefore, are not affected by the delay. The error system (8) takes the form

$$\begin{aligned} y_t(t,x) &= -\nu y_x(t,x) + F(t,x,y), \ x \in (0,1), \quad t \geq 0,\\ y_t(t,0) &= -ky(t-\tau(t),0) + F(t,0,y(t,0)), \quad t \geq 0, \end{aligned} \tag{17}$$

where $y(t) = 0$ for $t < 0$ and F is given above (6).

A. Well-posedness of the PDE with time-delay

Following [25], we simplify the well-posedness analysis of (17) by assuming that

$$\exists\, t^* \geq 0\colon \begin{cases} t-\tau(t) < 0, & \forall t \in [0,t^*),\\ t-\tau(t) \geq 0, & \forall t \in [t^*,\infty). \end{cases}$$

Consider the boundary value $v(t) := y(t,0)$. From (17),

$$\dot v(t) = g(t,v(t)), \qquad t \in [0,t^*), \tag{18a}$$

$$\dot v(t) = -kv(t-\tau(t)) + g(t,v(t)), \quad t \in [t^*,\infty), \tag{18b}$$

where $g(t,v) := F(t,0,v)$ is continuous and satisfies a Lipschitz condition in the second argument. Therefore, (18a) has a unique solution $v \in C^1[0,t^*]$ that continuously depends on the initial condition $v(0) \in \mathbb{R}^n$. Since the right-hand side of (18b) does not depend on $v(s)$ with $s < 0$, we can formally set $v(s) = v(0)$ for $s < 0$ to obtain $v|_{[t^*-\tau_M,t^*]} \in C[t^*-\tau_M,t^*]$. Then Theorem 2.1 and comment D. on p. 42 of [26] imply that (18b) has a unique solution $v \in C^1[t^*,\infty)$ that continuously depends on the initial condition $v|_{[t^*-\tau_M,t^*]} \in C[t^*-\tau_M,t^*]$. Therefore, for any $y(0,0) \in \mathbb{R}^n$, the boundary condition of (17) has a unique solution $y(\cdot,0) \in C[0,\infty) \cap C^1(I^*)$, $I^* := [0,\infty)\setminus\{t^*\}$, that continuously depends on $y(0,0) \in \mathbb{R}^n$. Using the reasoning of Section II-A on $[0,t^*)$ and $[t^*,\infty)$, we conclude that (17) has a unique mild solution $y \in C([0,\infty),X)$ for $y(0,\cdot) \in X$, which becomes a classical solution for smooth $y(0,\cdot)$ and f and initial conditions from (9).

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B. Stability of the PDE with time-delay

In Section III, we saw that (17) is stable for $\tau(t) \equiv 0$ if $k > L$. In this section, we show that this remains true if $\tau_M = \sup_t \tau(t)$ is small enough. Moreover, we derive linear matrix inequalities that allow us to find an admissible τ_M.

Theorem 1: Consider (17) with F satisfying (6). Given controller gain k, delay bound $\tau_M > 0$, and decay rate $\alpha > 0$, let there exist positive ρ, r, and η such that

$$\Phi = \begin{bmatrix} \Phi_1 & -k + re^{-2\alpha\tau_M} & 1 \\ * & \tau_M^2 rk^2 - re^{-2\alpha\tau_M} & -\tau_M^2 rk \\ * & * & \tau_M^2 r - \eta \end{bmatrix} < 0,$$

where $\Phi_1 = \nu\rho + \eta L^2 + 2\alpha - re^{-2\alpha\tau_M}$. Then (17) is exponentially stable in the sense of (12). Moreover, if $k > L + \alpha$ and τ_M is small enough, such ρ, r, and η always exist.

Proof: First, consider the functions (10) where $p(x) = \rho e^{-\mu x}$ with constant $\rho > 0$ and $\mu > 0$. Repeating the calculations of Section III-A, we obtain

$$\begin{aligned} &\dot{V}_0 + 2\alpha V_0 \\ &\quad \le -\nu\rho e^{-\mu x}|y|^2\big|_0^1 + \int_0^1 (-\nu\mu + 2L + 2\alpha)\rho e^{-\mu x}|y|^2, \end{aligned} \tag{19}$$

$$\begin{aligned} \dot{W}_0 + 2\alpha W_0 &= 2y^T(t,0)y_t(t,0) + 2\alpha W_0 \\ &= -2ky^T(t,0)y(t-\tau(t),0) \\ &\quad + 2y^T(t,0)F(t,0,y(t,0)) + 2\alpha W_0. \end{aligned} \tag{20}$$

Similarly to [27], we compensate the delayed term using

$$V_r = \tau_M r \int_{-\tau_M}^{0} \int_{t+\theta}^{t} e^{-2\alpha(t-s)}|y_s(s,0)|^2\, ds\, d\theta.$$

Using Jensen's inequality [28, Proposition B.8], we obtain

$$\begin{aligned} \dot{V}_r + 2\alpha V_r &= \tau_M^2 r|y_t(t,0)|^2 \\ &\quad - \tau_M r \int_{t-\tau_M}^{t} e^{-2\alpha(t-s)}|y_s(s,0)|^2\, ds \\ &\le \tau_M^2 r|y_t(t,0)|^2 - re^{-2\alpha\tau_M}\left|\int_{t-\tau(t)}^{t} y_t(s,0)\, ds\right|^2 \\ &= \tau_M^2 r|-ky(t-\tau(t),0) + F(t,0,y(t,0))|^2 \\ &\quad - re^{-2\alpha\tau_M}|y(t,0) - y(t-\tau(t),0)|^2. \end{aligned} \tag{21}$$

From (6), we obtain (recall that $\eta > 0$)

$$0 \le \eta\left[L^2|y(t,0)|^2 - |F(t,0,y(t,0))|^2\right]. \tag{22}$$

Consider $V = V_0 + W_0 + V_r$. Summing up the right-hand sides of (19)–(22), we obtain

$$\begin{aligned} \dot{V} + 2\alpha V \le &-(\nu\mu - 2L - 2\alpha)\int_0^1 \rho e^{-\mu x}|y(t,x)|^2\, dx \\ &- \nu\rho e^{-\mu}|y(t,1)|^2 + \varphi^T(t)(\Phi \otimes I_n)\varphi(t), \end{aligned}$$

where $\varphi(t) = \mathrm{col}\{y(t,0), y(t-\tau(t),0), F(t,0,y(t,0))\}$ and $\otimes$ is the Kronecker product. If $\Phi < 0$, then $\dot{V} \le -2\alpha V$ for a large enough μ, which implies (12).

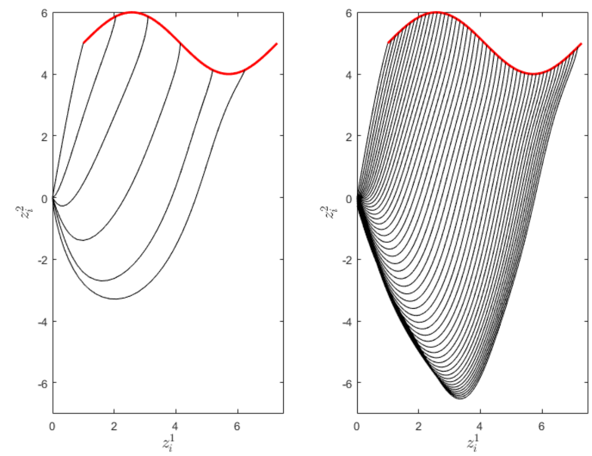

Fig. 2. The phase portraits of (1), (3) with $k = 1$ and $\sigma = 0.1 \cdot (N+1)$ for $N = 5$ (left) and $N = 50$ (right).

If $\tau_M = 0$, then $\Phi < 0$ turns into

$$\begin{bmatrix} -r + \rho\nu + \eta L^2 + 2\alpha & -k + r & 1 \\ * & -r & 0 \\ * & * & -\eta \end{bmatrix} < 0.$$

By the Schur complement lemma, this is equivalent to

$$-r + \nu p + \eta L^2 + 2\alpha + \frac{(r-k)^2}{r} + \frac{1}{\eta} < 0,$$

which for $\eta = 1/L > 0$ boils down to

$$\nu\rho + 2L + 2\alpha - 2k + \frac{k^2}{r} < 0.$$

If $k > L + \alpha$, the above holds for small enough $\rho > 0$ and large enough $r > 0$. By continuity, Φ will stay negative for a small enough $\tau_M > 0$. ∎

V. Simulations

Consider the multi-agent system (1) with $n = 2$ and $f(t, z_i) = 0.1(z_i + \sin(z_i))$. Let the target curve be

$$\gamma(x) = \begin{bmatrix} 1 + 2\pi x \\ 5 + \sin(2\pi x) \end{bmatrix}, \quad x \in [0,1].$$

The conditions of Proposition 1 hold for $k = 1$, $\alpha = 0.1$, and $\nu = 0.1$. Therefore, the control law (3) with $\sigma = \nu(N+1)$ guarantees (2) if the number of agents, N, is high enough. Figure 2 shows successful deployment onto the red target curve for $N = 5$ and $N = 50$, where

$$z_i(t) = \begin{bmatrix} z_i^1(t) \\ z_i^2(t) \end{bmatrix}, \quad z_i(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad i = 0, \ldots, N.$$

The stability of (8) does not depend on ν, but the overshoot is larger and the convergence is slower for smaller ν. In particular, Figures 3 and 4 show how the differences $|z_i(t) - \gamma_i|$, $i = 0, \ldots, 5$, change in time when $\nu = 0.1$ and $\nu = 0.07$. Note that the maximum of $|z_5(t)|$ is around 9 when $\nu = 0.1$ and above 10 when $\nu = 0.07$. This behaviour is similar to string instability occuring in vehicle platoons [23], [24].

Now consider (1) under the delayed control (16). The system is unstable for $\tau(t) \equiv \tau_M \ge 1.4$, $\nu = 0.1$, $k = 1$.

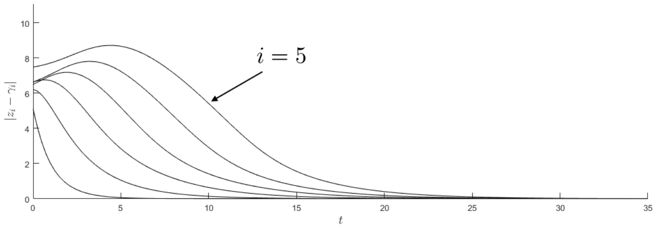

Fig. 3. $|z_i(t) - \gamma_i|$, $i = 0, \ldots, 5$, for $\nu = 0.1$, $k = 1$.

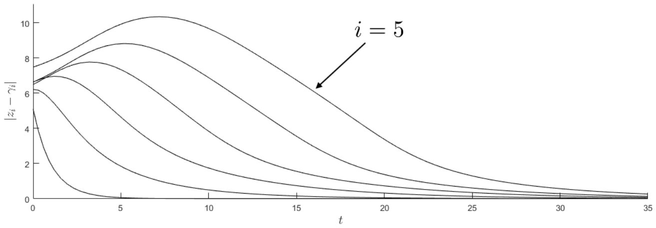

Fig. 4. $|z_i(t) - \gamma_i|$, $i = 0, \ldots, 5$, for $\nu = 0.07$, $k = 1$.

The maximum delay for which the LMIs of Theorem 1 are feasible is $\tau_M \approx 0.97$ ($\alpha = 0$). The LMIs remain feasible for $\alpha = 0.1$ and $\tau_M = 0.89$. In Fig. 5, one can see the norms of errors for different values of N when $\tau(t) \equiv 0.89$. Since the PDE (17) is a continuum limit of (1), (16), the lines get closer when N grows.

VI. Conclusions

We demonstrated that PDEs can be helpful in studying the stability of large-scale multi-agent systems. In particular, we showed that PDE-based analysis can help to design a Lyapunov function for a multi-agent system. Then, we used PDE modelling to derive LMIs characterising the admissible bound on the input delay in the leader.

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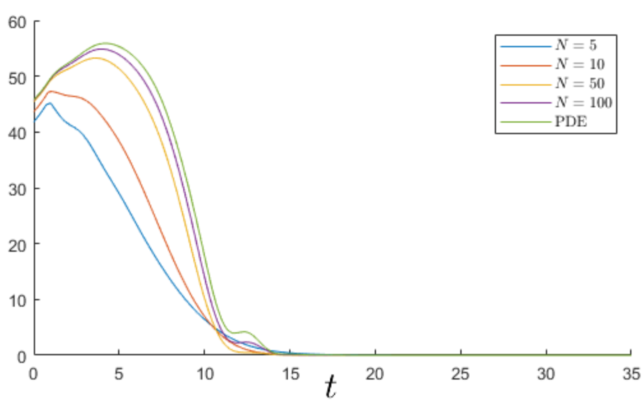

Fig. 5. $\frac{1}{N+1}\sum_0^N |z_i(t) - \gamma_i|^2$ governed by (1), (16) for different N and $\|y(t, \cdot)\|^2$ governed by (17)

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