Regional Stabilization of the 1-D Kuramoto–Sivashinsky Equation via Modal Decomposition

Rami Katz, *Graduate Student Member, IEEE*, and Emilia Fridman, *Fellow, IEEE*

***Abstract*—In this letter, we suggest regional stabilization of the semilinear 1D KSE under nonlocal or boundary actuation. We employ modal decomposition and derive regional H^1 stability conditions for the closed-loop system. Given a decay rate that defines the number of state modes in the controller, we provide LMIs for finding the the controller gain as well as a bound on the domain of attraction. In the case of boundary control, we suggest a dynamic extension with a novel internally stable dynamics. The latter allows to enlarge a bound on the domain of attraction. Numerical examples illustrate the efficiency of the method.**

***Index Terms*—Distributed parameter systems, nonlinear parabolic PDEs, boundary control.**

I. Introduction

The Kuramoto-Sivashinsky equation (KSE) describes a variety of physical phenomena such as magnetized plasmas, flame front propagation, viscous flow problems and chemical reaction-diffusion processes (see, e.g., [1], [2], [3], [4]). Distributed control of the KSE was considered in [5], whereas its boundary control in case of small anti-diffusion parameter was studied in [6]. Most of the results on stabilization for arbitrary anti-diffusion parameter were confined to the linear case. Thus, stabilization of linear KSE was presented in [7], [8], [9] via modal decomposition, whereas its null controllability was considered in [10]. Stability and distributed stabilization of the linear KSE were studied in [11].

Stabilization of the semilinear 1D and 2D KSE with arbitrary anti-diffusion parameter was suggested in [12], [13] via the spatial decomposition method. The latter method may require many sensing and actuation devices, whereas the actuators are supposed to cover almost all the spatial domain. Modal decomposition method may avoid the latter restriction being efficient for boundary control or non-local control, where the shape functions need not cover the whole domain. However, the existing modal decomposition results that employ direct Lyapunov method (providing domain of attractions in the semilinear case) are limited to the linear PDEs [8], [9], [14], [15], [16].

The difficulty of modal decomposition method for nonlinear PDEs lies in coupling of the solution modes, which is introduced by the nonlinearity and may cause a spillover behavior [17]. In our recent paper [18], a direct Lyapunov approach via modal decomposition was introduced for global L^2 stabilization of 1D heat equation with globally Lipschitz nonlinearity. The nonlinear terms in [18] were compensated by using Parseval's equality, leading to efficient and constructive LMIs for finding the controller gain.

In this letter, we suggest regional stabilization of the semilinear 1D KSE under nonlocal or boundary actuation. We derive H^1 stability conditions for the closed-loop system. Given a decay rate that defines the number of state modes in the controller, we provide LMIs for finding the the controller gain as well as a bound on the domain of attraction. In the case of boundary control, we suggest a dynamic extension, which is based on the polynomial change of variables as in [9], but with a novel internally stable dynamics. The latter allows to enlarge a bound on the domain of attraction. Such dynamic extension is inspired by [19], where a trigonometric change of variables was suggested for semilinear heat equations. Numerical examples illustrate the efficiency of the method.

Notations and preliminaries: $L^2(0,1)$ is the Hilbert space of Lebesgue measurable and square integrable functions $f:[0,1]\to\mathbb{R}$ with the inner product $<f,g> := \int_0^1 f(x)g(x)dx$ and induced norm $\|f\|^2 := <f,f>$. $H^1(0,1)$ is the space of functions $f:[0,1]\to\mathbb{R}$ with square integrable weak derivative, with the norm $\|f\|^2_{H^1} := \sum_{j=0}^1 \|f^{(j)}\|^2$. The Euclidean norm on $\mathbb{R}^n$ is denoted by $|\cdot|$. For $P\in\mathbb{R}^{n\times n}$, $P>0$ means that P is symmetric and positive definite. Sub-diagonal elements of a symmetric matrix will be denoted by $*$. For $U\in\mathbb{R}^{n\times n}$, $U>0$ and $x\in\mathbb{R}^n$ let $|x|^2_U = x^TUx$. $\mathbb{N}$ are the natural numbers.

Recall that the Sturm-Liouville eigenvalue problem

$$\phi''+\lambda\phi=0,\quad x\in(0,1),\quad \phi(0)=\phi(1)=0, \tag{1}$$

induces a sequence of eigenvalues $\lambda_n = n^2\pi^2, n\in\mathbb{N}$ with corresponding eigenfunctions

$$\phi_n(x)=\sqrt{2}\sin\left(\sqrt{\lambda_n}x\right),\ n\in\mathbb{N}. \tag{2}$$

Manuscript received September 13, 2021; revised November 6, 2021; accepted November 26, 2021. Date of publication December 7, 2021; date of current version December 22, 2021. This work was supported in part by Israel Science Foundation under Grant 673/19; in part by the Chana and Heinrich Manderman Chair at Tel Aviv University; and in part by the Y. and C. Weinstein Research Institute for Signal Processing. Recommended by Senior Editor C. Prieur. *(Corresponding author: Rami Katz.)*

The authors are with the School of Electrical Engineering, Tel Aviv University, Ashkelon 7843932, Israel (e-mail: rami@benis.co.il; emilia@eng.tau.ac.il).

Digital Object Identifier 10.1109/LCSYS.2021.3133492

The eigenfunctions form a complete and orthonormal system in $L^2(0,1)$.

The following lemma will be used (see, e.g., [20], [21]):

Lemma 1: Let $h \in L^2(0,1)$ satisfy $h \overset{L^2}{=} \sum_{n=1}^{\infty} h_n\phi_n$. Then $h \in H^1(0,1)$ with $h(0) = h(1) = 0$ iff $\sum_{n=1}^{\infty} \lambda_n h_n^2 < \infty$. Moreover,

$$\|h'\|^2 = \sum_{n=1}^{\infty} \lambda_n h_n^2. \tag{3}$$

Finally, given $N \in \mathbb{N}$ and $h \in L^2(0,1)$ satisfying $h \overset{L^2}{=} \sum_{n=1}^{\infty} h_n\phi_n$ we denote $\|h\|_N^2 = \sum_{n=N+1}^{\infty} h_n^2$.

II. Stabilization of the Semilinear 1D KSE

A. Non-Local Actuation

In this section we consider disctributed stabilization of the semilinear 1D Kuramoto-Sivashinsky equation

$$z_t(x,t) = -z_{xxxx}(x,t) - \nu z_{xx}(x,t) - \frac{1}{2}\left(z^2(x,t)\right)_x + b(x)u(t),$$
$$z(0,t) = 0, \quad z(1,t) = 0, \quad z_{xx}(0,t) = 0, \quad z_{xx}(1,t) = 0 \tag{4}$$

where $t \geq 0$, $x \in (0,1)$, $z(x,t) \in \mathbb{R}$, $\nu > 0$ is the "anti-diffusion" coefficient, $b \in L^2(0,1)$ and $u(t) \in \mathbb{R}$ is the control input.

We show below the existence of a unique classical solution to (4), which extends to $t \in [0,\infty)$ (see (12) - (15)). Using modal decomposition, we present the solution to (4) as

$$z(x,t) = \sum_{n=1}^{\infty} z_n(t)\phi_n(x), \;\; z_n(t) = \langle z(\cdot,t), \phi_n\rangle \tag{5}$$

with $\{\phi_n\}_{n\in\mathbb{N}}$ defined in (2). Differentiating under the integral sign, integrating by parts and using (1) we have

$$\begin{aligned} &\dot{z}_n(t) = (-\lambda_n^2 + \nu\lambda_n)z_n(t) + b_n u(t) - z_n^{(1)}(t),\\ &z_n(0) = \langle z(\cdot,0), \phi_n\rangle,\\ &z_n^{(1)}(t) = \langle z(\cdot,t)z_x(\cdot,t), \phi_n\rangle, \quad b_n = \langle b, \phi_n\rangle. \end{aligned} \tag{6}$$

Let $\delta > 0$ be a desired decay rate. Since $\lim_{n\to\infty} \lambda_n = \infty$, there exists some $N \in \mathbb{N}$ such that

$$-\lambda_n^2 + \nu\lambda_n < -\delta, \quad n > N. \tag{7}$$

Denote

$$A_0 = \mathrm{diag}\left\{-\lambda_n^2 + \nu\lambda_n\right\}_{n=1}^{N}, \;\; B_0 = \mathrm{col}\{b_n\}_{n=1}^{N}. \tag{8}$$

Assumption 1: We assume

$$b_n \neq 0, \quad 1 \leq n \leq N. \tag{9}$$

Under Assumption 1, it can be verified that the pair (A_0, B_0) is controllable. Let $K_0 \in \mathbb{R}^{1\times N}$ be a controller gain, to be found later from LMIs. We propose a controller of the form

$$\begin{aligned} &u(t) = -K_0 z^N(t), \;\; z^N(t) = \mathrm{col}\{z_n(t)\}_{n=1}^{N},\\ &\dot{z}^N(t) = (A_0 - B_0K_0)z^N(t) - z^{N,(1)}(t),\\ &z^{N,(1)}(t) = \mathrm{col}\left\{z_n^{(1)}(t)\right\}_{n=1}^{N}. \end{aligned} \tag{10}$$

Using (6) and (10), we arrive at the following closed-loop system for $t \geq 0$:

$$\begin{aligned} &\dot{z}^N(t) = (A_0 - B_0K_0)z^N(t) - z^{N,(1)}(t),\\ &\dot{z}_n(t) = (-\lambda_n^2 + \nu\lambda_n)z_n(t) - b_nK_0z^N(t) - z_n^{(1)}(t), \quad n > N. \end{aligned} \tag{11}$$

For well-posedness of the closed-loop system (4) and (10), subject to the assumptions of Theorem 1. Consider the operator

$$\mathscr{A} : \mathscr{D}(\mathscr{A}) \to L^2(0,1), \;\; \mathscr{A} = \partial_{xxxx} + \nu\partial_{xx},$$
$$\mathscr{D}(\mathscr{A}) = \left\{h \in H^4(0,1) \mid h(0) = h(1) = h''(0) = h''(1) = 0\right\}.$$

Let $\theta > 0$ and $\mathscr{A}_\theta = \mathscr{A} + \theta I$. Given $h \in \mathscr{D}(\mathscr{A})$, integration by parts leads to $< \mathscr{A}_\theta h, h >= \|h''\|^2 - \nu\|h'\|^2 + \theta\|h\|^2$. Using the Gagliardo–Nirenberg inequality (see, e.g., [22]), $< \mathscr{A}_\theta h, h >> 0$ for large enough θ. By [21, Sec. 2.6], $-\mathscr{A}_\theta$ is diagonalizable and the spectrum of $-\mathscr{A}_\theta$ is given by $\sigma(-\mathscr{A}_\theta) = \{-\lambda_n^2 + \nu\lambda_n - \theta\}_{n=1}^{\infty} \subset (-\infty, 0)$. Thus, $\{\mu \in \mathbb{C} \mid \mathrm{Re}(\mu) > 0\} \subseteq \rho(-\mathscr{A}_\theta)$, where $\rho(-\mathscr{A}_\theta)$ is the resolvent set of $-\mathscr{A}_\theta$. By [20, Th. 12.31], $-\mathscr{A}_\theta$ is a sectorial operator generating an analytic semigroup on $L^2(0,1)$. Let $\mathscr{H} = L^2(0,1) \times \mathbb{R}^N$ be a Hilbert space with the norm $\|\cdot\|_{\mathscr{H}}^2 := \|\cdot\|^2 + |\cdot|^2$. Introducing the state

$$\xi(t) = \mathrm{col}\{\xi_1(t), \xi_2(t)\}, \;\; \xi_1(t) = z(\cdot,t), \;\; \xi_2(t) = z^N(t), \tag{12}$$

the closed-loop system can be presented as

$$\begin{aligned} &\frac{d\xi}{dt}(t) + \mathscr{B}\xi(t) = \begin{bmatrix} f_1(\xi)\\ f_2(\xi)\end{bmatrix}, \;\; \mathscr{B} = \mathrm{diag}\{\mathscr{A}_\theta, -A_0 + B_0K_0\},\\ &\mathscr{D}(\mathscr{B}) = \mathscr{D}(\mathscr{A}_\theta) \times \mathbb{R}^N, \;\; f_1(\xi) = \theta z - zz_x - bK_0z^N,\\ &f_2(\xi) = -z^{N,(1)} \end{aligned} \tag{13}$$

where $-\mathscr{B}$ generates an analytic semigroup on $\mathscr{H}$. Since $\mathscr{A}_\theta$ is positive and diagonalizable, $\mathscr{A}_\theta^{\frac{1}{4}}$ is well-defined [21] with domain $\{h \in L^2(0,1) = \sum_{n=1}^{\infty} \lambda_n h_n^2 < \infty\} \overset{\text{Lemma 1}}{=} H_0^1(0,1)$. Fix $\varphi \in H_0^1(0,1) \times \mathbb{R}^N$ and let η, ζ belong to the open ball $B(\varphi, R)$, $R > 0$. By the Sobolev inequality

$$\begin{aligned} &\|\eta_1\partial_x\eta_1 - \zeta_1\partial_x\zeta_1\| \leq \|\eta_1(\partial_x\eta_1 - \partial_x\zeta_1)\| + \|\partial_x\zeta_1(\eta_1 - \zeta_1)\|\\ &\overset{\text{Sobolev}}{\leq} 2\max(\|\partial_x\eta_1\|, \|\partial_x\zeta_1\|)\|\partial_x\eta_1 - \partial_x\zeta_1\|\\ &\leq 2(\|\varphi\|_{\mathscr{H}} + R)\|\eta - \zeta\|_{\mathscr{H}} = C(\varphi)\|\eta - \zeta\|_{\mathscr{H}}, \end{aligned}$$

where the constant $C(\varphi)$ depends on φ. By [23, Lemma 3.3.2 and Th. 3.3.3], we have that given any $z(\cdot,0) \in H_0^1(0,1)$, (13) has a unique local strong solution

$$\begin{aligned} &\xi \in C([0,T]; H_0^1(0,1) \times \mathbb{R}^N) \cap L^2((0,T); \mathscr{D}(\mathscr{A}) \times \mathbb{R}^N),\\ &\frac{d}{dt}\xi \in L^2((0,T); L^2(0,1)). \end{aligned} \tag{14}$$

Moreover, since a classical solution is also a strong solution [24, Ths. 6.3.1 and 6.3.3] imply that ξ is also a unique local classical solution satisfying, in addition,

$$\xi \in C^1((0,T); \mathscr{H}), \quad \xi(t) \in \mathscr{D}(\mathscr{A}) \times \mathbb{R}^N, \quad t \in (0,T). \tag{15}$$

For H^1-stability analysis of the closed-loop system (11), we consider the Lyapunov function

$$V(t) = \left|z^N(t)\right|_P^2 + \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t), \;\; t \in [0,T) \tag{16}$$

where $0 < P \in \mathbb{R}^{N\times N}$. Note that $V(t)$ is equivalent to the H^1-norm of the solution $z(x,t)$, by (3) and Wirtinger's inequality [25] (see (32) below).

The following Lyapunov analysis is carried out for $t \in (0,T)$. Differentiation of $V(t)$ along the solution of (11) gives

$$\begin{aligned} \dot{V} + 2\delta V = &\left(z^N(t)\right)^T\big[P(A_0 - B_0K_0) + (A_0 - B_0K_0)^TP\\ &+ 2\delta P\big]z^N(t) - 2\left(z^N(t)\right)^TPz^{N,(1)}(t)\\ &+ 2\sum_{n=N+1}^{\infty}(-\lambda_n^3 + \nu\lambda_n^2 + \delta\lambda_n)z_n^2(t) \end{aligned}$$

$$-2\sum_{n=N+1}^{\infty}\lambda_n z_n(t)b_nK_0z^N(t)-2\sum_{n=N+1}^{\infty}\lambda_n z_n(t)z_n^{(1)}(t). \tag{17}$$

By using the Young inequality, we have

$$\begin{aligned}&2\sum_{n=N+1}^{\infty}\lambda_n z_n(t)(-b_n)K_0z^N(t)\\ &\leq \alpha\sum_{n=N+1}^{\infty}\lambda_n^2z_n^2(t)+\frac{1}{\alpha}\|b\|_N^2\left|K_0z^N(t)\right|^2\end{aligned} \tag{18}$$

where $\alpha > 0$. Let $0 < \sigma \in \mathbb{R}$ and assume that

$$\|z_x(\cdot,t)\|^2 < \sigma^2, \quad t \in [0, T). \tag{19}$$

Then, for some $\alpha_1 > 0$, we obtain by Young's inequality

$$\begin{aligned}-2\sum_{n=N+1}^{\infty}\lambda_n z_n(t)z_n^{(1)}(t) &\leq \alpha_1\sum_{n=N+1}^{\infty}\lambda_n^2z_n^2(t)\\ &-\frac{1}{\alpha_1}\left|z^{N,(1)}(t)\right|^2+\frac{1}{\alpha_1}\sum_{n=1}^{\infty}\left[z_n^{(1)}(t)\right]^2.\end{aligned} \tag{20}$$

Using the boundary conditions in (4),

$$z^2(x,t) \leq \left[\int_0^x |z_x(s,t)|ds\right]^2 \leq \|z_x(\cdot,t)\|^2 < \sigma^2, \tag{21}$$

and Parseval's equality, we obtain

$$\begin{aligned}\frac{1}{\alpha_1}\sum_{n=1}^{\infty}\left[z_n^{(1)}(t)\right]^2 &= \frac{1}{\alpha_1}\int_0^1 z_x^2(s,t)z^2(s,t)ds \leq \frac{\sigma^2}{\alpha_1}\|z_x(\cdot,t)\|^2\\ &\overset{(3)}{=} \frac{\sigma^2}{\alpha_1}\left|z^N(t)\right|_\Lambda^2+\frac{\sigma^2}{\alpha_1}\sum_{n=N+1}^{\infty}\lambda_n z_n^2(t),\ \ \Lambda=\text{diag}\{\lambda_n\}_{n=1}^N.\end{aligned} \tag{22}$$

Let $\eta(t) = \text{col}\{z^N(t), z^{N,(1)}(t)\}$. From (17)-(22), we have

$$\dot{V}+2\delta V \leq \eta^T(t)\Psi\eta(t)+2\sum_{n=N+1}^{N}\mu_n\lambda_n^2z_n^2(t) \leq 0, \tag{23}$$

if $\mu_n = -\lambda_n+\nu+\frac{\delta}{\lambda_n}+\frac{\alpha}{2}+\frac{\alpha_1}{2}+\frac{\sigma^2}{2\alpha_1\lambda_n}<0,\ n>N$ and

$$\begin{aligned}\Psi &= \begin{bmatrix}P(A_0-B_0K_0)+(A_0-B_0K_0)^TP+2\delta P+\psi & -P\\ * & -\frac{1}{\alpha_1}\end{bmatrix} < 0,\\ \psi &= \frac{\sigma^2}{\alpha_1}\Lambda+\frac{\|b\|_N^2}{\alpha}K_0^TK_0.\end{aligned} \tag{24}$$

From monotonicity of $\lambda_n,\ n \in \mathbb{N}$ we have that $\mu_n<0,\ n>N$ iff $\mu_{N+1} < 0$. By Schur complement, the latter holds iff

$$\begin{bmatrix}-\lambda_{N+1}+\nu+\frac{\delta}{\lambda_{N+1}}+\frac{\alpha}{2}+\frac{\alpha_1}{2} & 1\\ * & -\frac{2\alpha_1\lambda_{N+1}}{\sigma^2}\end{bmatrix} < 0. \tag{25}$$

To obtain equivalent LMIs for the design of controller gains we multiply Ψ on the left and right by $\text{diag}\{P^{-1}, I\}$. Then, introducing the notations

$$Q = P^{-1}, \quad Y_0 = QK_0^T \tag{26}$$

and applying Schur complement twice, it can be verified that (24) is equivalent to

$$\begin{bmatrix}A_0Q+QA_0^T-Y_0B_0^T-B_0Y_0^T+2\delta Q+\alpha_1 I & Y_0 & Q\Lambda^{\frac{1}{2}}\\ * & -\alpha\|b\|_N^{-2} & 0\\ * & * & -\frac{\alpha_1}{\sigma^2}\end{bmatrix} < 0. \tag{27}$$

Note that (25) and (27) are LMIs in Q, Y_0, α and α_1. Furthermore, feasibility of LMIs allows to recover the controller gains as $K_0 = Y_0^TQ^{-1}$. Finally, by arguments similar to [16, Th. 3.1], it can be shown that (24) and (25) (and hence (25) and (27)) are always feasible for small enough $\sigma > 0$ and large enough N. Summarizing, we arrive at:

Proposition 1: Consider (4) subject to non-local actuation with $b \in L^2(0,1)$ and control law (10). Let $z(\cdot,0) \in H_0^1(0,1)$. Let $\delta > 0$ and $N \in \mathbb{N}$ satisfy (7). Given $\sigma > 0$, assume that (19) holds. If there exist $0 < Q \in \mathbb{R}^{N\times N}$, $Y_0 \in \mathbb{R}^N$ and scalars $0 < \alpha, \alpha_1$ such that the LMIs (25) and (27) hold, then the solution $z(x,t)$ to (4) subject to the control law (10) with $K_0 = Y_0^TQ^{-1}$ satisfies

$$V(t) \leq e^{-2\delta t}V(0),\ \ t \in [0, T). \tag{28}$$

Furthermore, the LMIs (25) and (27) are always feasible for small enough σ and large enough N.

We provide next an estimate on the radius of the ball of attraction, starting from which the solution of the closed-loop system exists for all $t \geq 0$ and is exponentially decaying.

Theorem 1: Let the conditions of Proposition 1 hold for some $0 < P \in \mathbb{R}^{N\times N}$ and$\sigma, \alpha, \alpha_1 > 0$. Recall Λ, given in (22), and let

$$\rho = \sigma\sqrt{\frac{\min(\sigma_{\min}(P^*),1)}{\max(\sigma_{\max}(P^*),1)}},\ \ P^* = \Lambda^{-\frac{1}{2}}P\Lambda^{-\frac{1}{2}}. \tag{29}$$

If $\|z_x(\cdot,0)\|^2 < \rho^2$ then solution of the closed-loop system (4) and (10) can be extended to $[0,\infty)$, (19) holds for $t \in [0,\infty)$ and $z(x,t)$ satisfies

$$\|z(\cdot,t)\|_{H^1}^2 \leq Me^{-2\delta t}\|z(\cdot,0)\|_{H^1}^2,\ \ t \in [0,\infty) \tag{30}$$

for some $M \geq 1$.

Proof: We begin by showing that (19) holds. Since $\rho \leq \sigma$, $\|z_x(\cdot,0)\|^2 < \rho^2$ implies that $\|z_x(\cdot,0)\|^2 \leq \sigma^2$ for $t \in [0,\tau)$ and some $\tau > 0$. Let

$$\tau_* = \sup\left\{\tau \in [0,T)\ |\ \|z_x(\cdot,t)\|^2 \leq \sigma^2,\ \ t \in [0,\tau)\right\}. \tag{31}$$

We claim that $\tau_* = T$. Assume by contradiction that $\tau_* < T$. By (23) and the comparison principle, we have (28) with T replaced by τ_*. By (3) and (16) we obtain

$$\begin{aligned}V(t) &\overset{(3)}{\geq} \min\left(\sigma_{\min}(P^*),1\right)\|z_x(\cdot,t)\|^2,\\ V(t) &\overset{(3)}{\leq} \max\left(\sigma_{\max}(P^*),1\right)\|z_x(\cdot,t)\|^2\end{aligned} \tag{32}$$

which yield

$$\|z_x(\cdot,t)\|^2 \leq e^{-2\delta t}\sigma^2, \quad t \in [0,\tau_*). \tag{33}$$

The latter implies $\|z_x(\cdot,\tau_*)\|^2 < \sigma^2$ and, by continuity, there exists some $\tau_* < \tau_1 < T$ with $\|z_x(\cdot,t)\|^2 < \sigma^2$ for $t \in [0,\tau_1)$, contradicting (31). Recall the notations (12) and (13). Since (19) and (21) hold, we have for $t \in [0,T)$

$$\|\xi_1\partial_x\xi_1\| \leq \|\partial_x\xi_1\|^2 \leq \sigma^2. \tag{34}$$

The latter estimate, Proposition 1 and proof of [24, Th. 6.3.3] (which shows that extending the solution to $T = \infty$ is possible provided the apriori bound (34) holds) imply that the solution of (13) can be extended to $T = \infty$. Finally, (30) follows from (28) and (32) with $\tau_* = T = \infty$. ■

B. Dirichlet Actuation

Here we consider boundary stabilization of the following semilinear 1D KSE

$$\begin{aligned}&z_t(x,t) = -z_{xxxx}(x,t)-\nu z_{xx}(x,t)-\frac{1}{2}\left(z^2(x,t)\right)_x,\\ &z(0,t)=0,\quad z(1,t)=u(t),\quad z_{xx}(0,t)=0,\quad z_{xx}(1,t)=0\end{aligned} \tag{35}$$

where $u(t)$ is the control input to be designed.

Following [14] we introduce the change of variables

$$w(x,t) = z(x,t) - r(x)u(t), \quad r(x) = x \tag{36}$$

to obtain the following PDE:

$$\begin{aligned} w_t(x,t) &= -w_{xxxx}(x,t) - \nu w_{xx}(x,t) - r(x)\dot{u}(t) \\ &\quad - [w(x,t) + xu(t)][w_x(x,t) + u(t)]. \end{aligned} \tag{37}$$

Let $\kappa \geq 0$. We will henceforth treat $u(t)$ as an additional state variable subject to the dynamics

$$\dot{u}(t) = -\kappa u(t) + v(t), \quad u(0) = 0 \tag{38}$$

where $v(t)$ is the new control input. Given $v(t)$, $u(t)$ can be computed by integrating (38), thereby resulting in a PI controller for the original PDE (35). From (37) and (38) we obtain the following equivalent ODE-PDE system

$$\begin{aligned} &\dot{u}(t) = -\kappa u(t) + v(t), \ u(0) = 0, \\ &w_t(x,t) = -w_{xxxx}(x,t) - \nu w_{xx}(x,t) + \kappa r(x)u(t) \\ &\qquad - r(x)v(t) - [w(x,t) + xu(t)][w_x(x,t) + u(t)], \\ &w(0,t) = w(1,t) = w_{xx}(0,t) = w_{xx}(1,t) = 0. \end{aligned} \tag{39}$$

Remark 1: A dynamic extension with internally stable dynamics for the heat equation was introduced in [19], where $r(x)$ in (36) was chosen as a solution of a Sturm-Liouville problem with a modified boundary condition, rsulting in trigonometric $r(x)$. Extension of such $r(x)$ construction to the KSE with the fourth order spatial operator leads to a complicated expression of $r(x)$ in terms of hyperbolic functions. Thus, we consider a polynomial $r(x)$ as in [9]. However, differently from existing works that employed dynamic extension based on polynomial functions, where the dynamics of $u(t)$ was given by (38) with $\kappa = 0$ (see, e.g., [14], [26] for the heat equation and [9] for the KSE), we consider $\kappa \geq 0$ to be a tuning parameter. Numerical simulations of the resulting LMIs (see (55) and (56)) show that taking $\kappa > 0$ (i.e., choosing internally stable (38)) reduces the magnitude of the controller gain K and leads to an essentially larger bound on the ball of attraction ρ (see (29)).

We present the solution to (39) as

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\phi_n(x), \ w_n(t) = \langle w(\cdot,t), \phi_n \rangle \tag{40}$$

with $\{\phi_n\}_{n\in\mathbb{N}}$ defined in (2). Differentiating under the integral sign, integrating by parts and using (1) we have

$$\begin{aligned} \dot{w}_n(t) &= \left(-\lambda_n^2 + \nu\lambda_n\right)w_n(t) + \kappa b_n u(t) - b_n v(t) \\ &\quad - w_n^{(1)}(t) - w_n^{(2)}(t), \ t \geq 0, \\ w_n(0) &= \langle w(\cdot,0), \phi_n \rangle, \ n \in \mathbb{N}. \end{aligned} \tag{41}$$

where

$$\begin{aligned} &w_n^{(1)}(t) = \langle [w(\cdot,t) + \cdot u(t)]w_x(\cdot,t), \phi_n \rangle, \\ &w_n^{(2)}(t) = \langle w(\cdot,t) + \cdot u(t), \phi_n \rangle u(t), \\ &b_n = \langle r, \phi_n \rangle = \frac{(-1)^{n+1}\sqrt{2}}{\sqrt{\lambda_n}}, \quad n \geq 1. \end{aligned} \tag{42}$$

Let $\delta > 0$ be a desired decay rate and $N \in \mathbb{N}$ satisfy (7). We introduce

$$\begin{aligned} A &= \operatorname{diag}\left\{0_{1\times1}, -\lambda_n^2 + \nu\lambda_n\right\}_{n=1}^N - \kappa B \cdot \mathbb{1}, \\ B &= \operatorname{col}\{1, -b_n\}_{n=1}^N, \ \mathbb{1} = [1, 0_{1\times N}]. \end{aligned} \tag{43}$$

It can be verified, using the Hautus lemma, that the pair (A, B) is controllable. Let $K \in \mathbb{R}^{1\times(N+1)}$ be a controller gain, to be found later from LMIs. We propose the controller

$$\begin{aligned} v(t) &= -Kw^N(t), \ w^N(t) = \operatorname{col}\{u(t), w_n(t)\}_{n=1}^N, \\ \dot{w}^N(t) &= (A - BK)w^N(t) - w^{N,(1)}(t) - w^{N,(2)}(t), \\ w^{N,(j)}(t) &= \operatorname{col}\left\{0_{1\times1}, w_n^{(j)}(t)\right\}_{n=1}^N, \ j \in \{1,2\}. \end{aligned} \tag{44}$$

Using (41), (42), (43) and (44), we arrive at the following closed-loop system for $t \geq 0$:

$$\begin{aligned} \dot{w}^N(t) &= (A - BK)w^N(t) - w^{N,(1)}(t) - w^{N,(2)}(t), \\ \dot{w}_n(t) &= \left(-\lambda_n^2 + \nu\lambda_n\right)w_n(t) + \kappa b_n u(t) - b_n v(t) \\ &\quad - w_n^{(1)}(t) - w_n^{(2)}(t), \ t \geq 0. \end{aligned} \tag{45}$$

Well-posedness of the closed-loop system (39) and (44) follows by arguments similar to the well-posedness in the previous section. In particular, given $z(\cdot,0) = w(\cdot,0) \in H_0^1(0,1)$ satisfying $\|w_x(\cdot,0)\| < \rho$ with $\rho > 0$ in (29), we obtain the existence of a unique classical solution $\xi(t) = \operatorname{col}\{w(\cdot,t), w^N(t)\}$ satisfying (15) with $T = \infty$ and N replaced by $N+1$.

For H^1-stability analysis of the closed-loop system (45), we consider the Lyapunov function

$$V(t) = \left|w^N(t)\right|_P^2 + \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t), \tag{46}$$

where $0 < P \in \mathbb{R}^{(N+1)\times(N+1)}$. We use the Lyapunov function (46) to show that the closed-loop system is locally exponentially stable and derive an estimate on the domain of attraction. Differentiation of $V(t)$ along solution of (45) gives

$$\begin{aligned} \dot{V} + 2\delta V &= \left(w^N(t)\right)^T\left[P(A - BK) + (A - BK)^T P + 2\delta P\right] \\ &\quad \times w^N(t) - 2\left(w^N(t)\right)^T P w^{N,(1)}(t) - 2\left(w^N(t)\right)^T P w^{N,(2)}(t) \\ &\quad + 2\sum_{n=N+1}^{\infty}(-\lambda_n^3 + \nu\lambda_n^2 + \delta\lambda_n)w_n^2(t) \\ &\quad + 2\kappa\sum_{n=N+1}^{\infty}\lambda_n w_n(t)b_n u(t) + 2\sum_{n=N+1}^{\infty}\lambda_n w_n(t)b_n K w^N(t) \\ &\quad - 2\sum_{n=N+1}^{\infty}\lambda_n w_n(t)w_n^{(1)}(t) - 2\sum_{n=N+1}^{\infty}\lambda_n w_n(t)w_n^{(2)}(t). \end{aligned} \tag{47}$$

Similarly to (18), since $\kappa > 0$, by Young's inequality we have

$$\begin{aligned} &2\kappa\sum_{n=N+1}^{\infty}\lambda_n w_n(t)b_n u(t) \leq \alpha_0\kappa\sum_{n=N+1}^{\infty}\lambda_n^2 w_n^2(t) \\ &\quad + \frac{\kappa}{\alpha_0}\left(\sum_{n=N+1}^{\infty} b_n^2\right)\left|\mathbb{1}w^N(t)\right|^2 \leq \alpha_0\kappa\sum_{n=N+1}^{\infty}\lambda_n^2 w_n^2(t) \\ &\quad + \frac{2\kappa}{\alpha_0\pi^2 N}\left|\mathbb{1}w^N(t)\right|^2, \\ &2\sum_{n=N+1}^{\infty}\lambda_n w_n(t)b_n K w^N(t) \\ &\leq \alpha_1\sum_{n=N+1}^{\infty}\lambda_n^2 z_n^2(t) + \frac{2}{\alpha_1\pi^2 N}\left|Kw^N(t)\right|^2 \end{aligned} \tag{48}$$

for some $\alpha_0, \alpha_1 > 0$. Here we used the estimate

$$\sum_{n=N+1}^{\infty} b_n^2 \overset{(42)}{=} \frac{2}{\pi^2}\sum_{n=N+1}^{\infty}\frac{1}{n^2} \leq \frac{2}{\pi^2 N}.$$

Let $0 < \sigma \in \mathbb{R}$ and assume that

$$\|w_x(\cdot,t)\|^2 + u^2(t) < \sigma^2, \quad t \in [0,\infty). \tag{49}$$

We will derive an estimate on $\|z_x(\cdot,0)\| = \|w_x(\cdot,0)\|$, which guarantees that (49) holds (recall that $u(0)=0$, by (39)). Then, by using Young's inequality we obtain

$$-2\sum_{n=N+1}^{\infty}\lambda_n w_n(t)w_n^{(j)}(t) \le \alpha_2 \sum_{n=1}^{\infty}\lambda_n^2 w_n^2(t) - \frac{1}{\alpha_{j+1}}\left|w^{N,(j)}(t)\right|^2 + \frac{1}{\alpha_{j+1}}\sum_{n=1}^{\infty}\left[w_n^{(j)}(t)\right]^2, \ j \in \{1,2\}. \tag{50}$$

Applying Parseval's equality we have

$$\begin{aligned}\frac{1}{\alpha_2}\sum_{n=1}^{\infty}\left[w_n^{(1)}(t)\right]^2 &\overset{(42)}{=} \frac{1}{\alpha_2}\int_0^1 [w(x,t)+xu(t)]^2 w_x^2(x,t)dx \\ &\le \frac{2\sigma^2}{\alpha_2}\|w_x(\cdot,t)\|^2 \overset{(3)}{=} \frac{2\sigma^2}{\alpha_2}\left|w^N(t)\right|_\Lambda^2 + \frac{2\sigma^2}{\alpha_2}\sum_{n=N+1}^{\infty}\lambda_n w_n^2(t), \\ \frac{1}{\alpha_3}\sum_{n=1}^{\infty}\left[w_n^{(2)}(t)\right]^2 &\overset{(42)}{=} \frac{1}{\alpha_3}|u(t)|^2\int_0^1 [w(x,t)+xu(t)]^2 dx \\ &\le \frac{2\sigma^2}{\alpha_3}\left|\mathbb{1}w^N(t)\right|^2, \quad \Lambda = \mathrm{diag}\{0,\lambda_1,\ldots,\lambda_N\}\end{aligned} \tag{51}$$

where the following estimate was used

$$\begin{aligned}[w(x,t)+xu(t)]^2 &\le 2\left[w^2(x,t)+u^2(t)\right] \\ &\overset{(39),\text{Sobolev}}{\le} 2\left[\|w_x(\cdot,t)\|^2+u^2(t)\right] \overset{(49)}{<} 2\sigma^2, \ 0\le x\le 1, t\ge 0\end{aligned} \tag{52}$$

Let $\eta(t) = \mathrm{col}\{w^N(t), w^{N,(1)}(t), w^{N,(2)}(t)\}$. From (47)-(52), we have

$$\dot V + 2\delta V \le \eta^T(t)\Phi\eta(t) + 2\sum_{n=N+1}^{N}\theta_n\lambda_n^2 z_n^2(t) \le 0, \ t \ge 0 \tag{53}$$

if $\theta_n = -\lambda_n + \nu + \frac{\delta}{\lambda_n} + \frac{\alpha_0\kappa+\alpha_1+\alpha_2+\alpha_3}{2} + \frac{\sigma^2}{\alpha_2\lambda_n} < 0, \ n > N$ and

$$\Phi = \begin{bmatrix} P(A-BK)+(A-BK)^TP+2\delta P+\phi & -P & -P \\ * & -\frac{1}{\alpha_2} & 0 \\ * & * & -\frac{1}{\alpha_3}\end{bmatrix} < 0,$$
$$\phi = \left[\frac{2\kappa}{\alpha_0\pi^2 N} + \frac{2\sigma^2}{\alpha_3}\right]\mathbb{1}^T\mathbb{1} + \frac{2}{\alpha_1\pi^2 N}K^TK + \frac{2\sigma^2}{\alpha_2}\Lambda. \tag{54}$$

From monotonicity of $\lambda_n, \ n \in \mathbb{N}$ we have that $\theta_n < 0, \ n > N$ iff $\theta_{N+1} < 0$. By Schur complement, the latter holds iff

$$\begin{bmatrix} -\lambda_{N+1} + \nu + \frac{\delta}{\lambda_{N+1}} + \frac{\alpha_0\kappa+\alpha_1+\alpha_2+\alpha_3}{2} & 1 \\ * & -\frac{\alpha_2\lambda_{N+1}}{\sigma^2}\end{bmatrix} < 0. \tag{55}$$

To obtain equivalent LMIs for the design of controller gains we multiply Ψ on the left and right by $\mathrm{diag}\{P^{-1}, I\}$. Then, introducing the notations (26) with Y_0, K_0 replaced by Y and K, respectively, and applying Schur complement twice, it can be verified that (54) is equivalent to

$$\begin{bmatrix} \chi & Y & Q\Lambda^{\frac12} & Q\mathbb{1}^T & Q\mathbb{1}^T \\ * & -\frac{\alpha_1\pi^2 N}{2} & 0 & 0 & 0 \\ * & * & -\frac{\alpha_2}{2\sigma^2} & 0 & 0 \\ * & * & * & -\frac{\alpha_3}{2\sigma^2} & 0 \\ * & * & * & * & -\frac{\alpha_0\pi^2 N}{2\kappa}\end{bmatrix} < 0,$$

$$\chi = AQ + QA^T - YB^T - BY^T + 2\delta Q + (\alpha_2+\alpha_3)I. \tag{56}$$

Note that (55) and (56) are LMIs in Q, Y and $\alpha_i, \ i \in \{0,1,2,3\}$. Furthermore, feasibility of LMIs allows to recover the controller gains as $K = Y^TQ^{-1}$.

Finally, let (54) and (55) hold with some $0 < P \in \mathbb{R}^{(N+1)\times(N+1)}$ and $0 < \alpha_i, \ i \in \{0,1,2,3\}$. Recall Λ, given in (51), and let $\Lambda_1 = \mathrm{diag}\{1,\Lambda\}$. Let $\rho > 0$ be given in (29) with Λ replaced by Λ_1. Then, using arguments similar to (28) and (32), it can be verified that if $\|z_x(\cdot,0)\|^2 = \|w_x(\cdot,0)\|^2 < \rho^2 \le \sigma^2$ (recall that $u(0)=0$, by (39)), then (49) holds and we obtain H^1-stability of the closed-loop system. Summarizing:

Theorem 2: Consider (39) with some $\kappa \ge 0$ and control law (44). Let $w(\cdot,0) \in H_0^1(0,1)$. Let $\delta > 0$ and $N \in \mathbb{N}$ satisfy (7). Given $\sigma > 0$, assume that $\|w_x(\cdot,0)\| < \rho$ with $\rho > 0$ given in (29), where Λ is replaced by $\Lambda_1 = \mathrm{diag}\{1,\lambda_n\}_{n=1}^N$. If there exist $0 < Q \in \mathbb{R}^{(N+1)\times(N+1)}$, $Y \in \mathbb{R}^{N+1}$ and scalars $0 < \alpha_i, \ i \in \{0,1,2,3\}$ such that the LMIs (55) and (56) hold, then the solution $w(x,t)$, $u(t)$ to (39) subject to the control law (44) with $K = Y^TQ^{-1}$ satisfies

$$\|w(\cdot,t)\|_{H^1}^2 + u^2(t) \le Me^{-2\delta t}\|w(\cdot,0)\|_{H^1}^2, \ t \ge 0, \tag{57}$$

for some $M > 0$. Furthermore, the LMIs (55) and (56) are always feasible for small enough σ and large enough N.

Corollary 1: Under the conditions of Theorem 2, the solution $z(x,t)$ of (35) satisfies

$$\|z(\cdot,t)\|_{H^1}^2 \le M_1 e^{-2\delta t}\|z(\cdot,0)\|_{H^1}^2, \ t \ge 0, \tag{58}$$

for some $M_1 > 0$, provided $\|z_x(\cdot,0)\| < \rho$.

Proof: Since $u(0) = 0$ by (39), we have $\|w_x(\cdot,0)\| = \|z_x(\cdot,0)\| < \rho$, with $w(x,t)$ in (36). Then, by Theorem 2, (57) holds. The desired result then follows by the triangle and Young's inequalities and (58). ■

Remark 2: The results in this manuscript can be applied to Burgers' equation with Dirichlet boundary conditions

$$z_t(x,t) = z_{xx}(x,t) - z(x,t)z_x(x,t), \ x \in (0,1), \tag{59}$$

subject to non-local or boundary control. Note that this PDE is open loop stable. Our approach allows to regionally enlarge the H^1 convergence rate. This is outside the scope of this letter.

Remark 3: Note that to enlarge the estimate of the region of attraction, feasibility of the strict LMIs of Theorems 1 and 2 may be checked for $\delta = 0$. Indeed, the LMIs feasibility for $\delta = 0$ implies their feasibility with some $\delta_1 > 0$, due to continuity of the eigenvalues.

III. Numerical Examples

In all numerical examples we consider the case $\delta = 0.1$ (see (7)). Consider first the case of non-local actuation (4) with $b(x) = \chi_{[0.3,0.9]}(x)$ (i.e., an indicator function). Let $\nu \in \{10, 39.5\}$, which result in $N = 1$ and $N = 2$ unstable modes, respectively. Feasibility of the LMIs of Theorem 1 was verified using MATLAB. The simulation results, including the resulting controller gains (see (26)), the value of σ (see (19)) and bound on the radius of the ball of attraction, ρ (see (29)) are given in Table I.

Next, for the case of boundary actuation (35), we consider $\nu = 10$, leading to $N = 1$ unstable mode. Feasibility of the LMIs of Theorem 2 was verified using MATLAB, where $\kappa > 0$, given in (38), was treated as a tuning parameter. The simulation results are given in Table I. For boundary actuation with $\kappa = 0$, the obtained ρ was 0.0101. Thus, treating κ

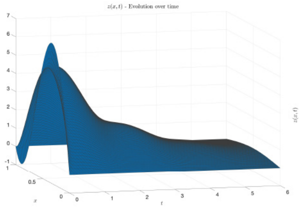

Fig. 1. Surface plot of solution $z(x, t)$.

TABLE I
LMI FEASIBILITY RESULTS: $\delta = 0.1$

Acutation	ν	N	Gain	σ	ρ	κ
Non-local	10	1	119.35	16.597	15.855	–
Non-local	39.5	2	$[3591 \quad 3417]$	11.4	1.9739	–
Boundary	10	1	$[-2.16 \quad -1041.6]$	1.8258	0.9266	3.24

as a tuning parameter leads to improvement of ρ by a factor of ≈ 90. For non-local actuation feasibility of the LMIs was obtained for a relatively large value of ρ.

For simulation of the closed-loop system we consider non-local actuation with $b(x) = \chi_{[0.3,0.9]}(x)$. Let $\nu = 10$ and $\delta = 0.1$, leading to $N = 1$. The corresponding controller gain is $K = 119.35$, whereas $\rho = 15.855$ (see Table I). We choose

$$z(x, 0) = Ax(1 - x)\cos(2x), \;\; A = 33.2. \tag{60}$$

Note that $z(\cdot, 0) \in H_0^1(0, 1)$ and $\|z_x(\cdot, 0)\| \approx 15.5$. Using modal decomposition we approximate the solution as $z(x, t) \approx \sum_{n=1}^{60} z_n(t)\phi_n(x)$, whereas $z_x(x, t) \approx \sum_{n=1}^{60} z_n(t)\phi_n'(x)$. Then, the closed-loop system (11) (with tail ODEs truncated after 60 coefficients) is simulated using MATLAB, subject to the control law (10). We choose $t \in [0, 6]$ as the simulation time. A surface plot of the solution $z(x, t)$ is given in Figure 1. The numerical simulation validates our theoretical results.

To check for conservativeness of our estimate on ρ, we further simulate the closed-loop system (11) with initial condition (60), while increasing A therein. Stability of the closed-loop system was preserved for $A = 49.2$, which corresponds to $\|z_x(\cdot, 0)\| \approx 22.47$ (to be compared with $\rho = 15.855$ in Table I). This illustrates conservatism of our estimate on ρ.

IV. Conclusion

A direct Lyapunov approach for regional state-feedback stabilization of the semilinear KSE via modal decomposition was introduced. Non-local and boundary actuation were considered. For boundary control, a novel dynamic extension was suggested that enlarged the bound on the ball of attraction in the example. A topic for future research can be optimization of the LMIs in order to improve the estimate on the radius of the ball of attraction.

References

[1] P. Christofides, *Nonlinear and Robust Control of PDE Systems: Methods and Applications to Transport Reaction Processes*. Boston, MA, USA: Birkhäuser, 2001.

[2] Y. Kuramoto and T. Tsuzuki, "On the formation of dissipative structures in reaction-diffusion systems: Reductive perturbation approach," *Progr. Theor. Phys.*, vol. 54, no. 3, pp. 687–699, 1975.

[3] G. Sivashinsky, "Nonlinear analysis of hydrodynamic instability in laminar flames—I. Derivation of basic equations," *Acta Astronautica*, vol. 4, pp. 1177–1206, Nov./Dec. 1977.

[4] B. Nicolaenko, "Some mathematical aspects of flame chaos and flame multiplicity," *Physica D Nonlinear Phenomena*, vol. 20, no. 1, pp. 109–121, 1986.

[5] P. D. Christofides and A. Armaou, "Global stabilization of the Kuramoto-Sivashinsky equation via distributed output feedback control," *Syst. Control Lett.*, vol. 39, no. 4, pp. 283–294, 2000.

[6] W.-J. Liu and M. Krstić, "Stability enhancement by boundary control in the Kuramoto-Sivashinsky equation," *Nonlinear Anal. Theory Methods Appl.*, vol. 43, no. 4, pp. 485–507, 2001.

[7] E. Cerpa, "Null controllability and stabilization of the linear Kuramoto-Sivashinsky equation," *Commun. Pure Appl. Anal.*, vol. 9, no. 1, pp. 91–102, 2010.

[8] P. Guzmán, S. Marx, and E. Cerpa, "Stabilization of the linear Kuramoto-Sivashinsky equation with a delayed boundary control," *IFAC-PapersOnLine*, vol. 52, no. 2, pp. 70–75, 2019.

[9] R. Katz and E. Fridman, "Finite-dimensional boundary control of the linear Kuramoto-Sivashinsky equation under point measurement with guaranteed L^2-gain," *IEEE Trans. Autom. Control*, early access, Oct. 19, 2021, doi: 10.1109/TAC.2021.3121234.

[10] E. Cerpa, P. Guzmán, and A. Mercado, "On the control of the linear Kuramoto-Sivashinsky equation," *ESAIM Control Optim. Calculus Variations*, vol. 23, no. 1, pp. 165–194, 2017.

[11] R. Al Jamal and K. Morris, "Linearized stability of partial differential equations with application to stabilization of the Kuramoto–Sivashinsky equation," *SIAM J. Control Optim.*, vol. 56, no. 1, pp. 120–147, 2018.

[12] W. Kang and E. Fridman, "Distributed sampled-data control of Kuramoto-Sivashinsky equation," *Automatica*, vol. 95, pp. 514–524, Sep. 2018.

[13] W. Kang and E. Fridman, "Sampled-data control of 2D Kuramoto-Sivashinsky equation," *IEEE Trans. Autom. Control*, early access, Mar. 31, 2021, doi: 10.1109/TAC.2021.3070070.

[14] C. Prieur and E. Trélat, "Feedback stabilization of a 1-D linear reaction–diffusion equation with delay boundary control," *IEEE Trans. Autom. Control*, vol. 64, no. 4, pp. 1415–1425, Apr. 2019.

[15] A. Mironchenko, C. Prieur, and F. Wirth, "Local stabilization of an unstable parabolic equation via saturated controls," *IEEE Trans. Autom. Control*, vol. 66, no. 5, pp. 2162–2176, May 2021.

[16] R. Katz and E. Fridman, "Constructive method for finite-dimensional observer-based control of 1-D parabolic PDEs," *Automatica*, vol. 122, Dec. 2020, Art. no. 109285.

[17] G. Hagen and I. Mezic, "Spillover stabilization in finite-dimensional control and observer design for dissipative evolution equations," *SIAM J. Control Optim.*, vol. 42, no. 2, pp. 746–768, 2003.

[18] R. Katz and E. Fridman, "Global stabilization of a 1D semilinear heat equation via modal decomposition and direct Lyapunov approach," submitted for publication.

[19] I. Karafyllis, "Lyapunov-based boundary feedback design for parabolic PDEs," *Int. J. Control*, vol. 94, no. 5, pp. 1247–1260, 2021.

[20] M. Renardy and R. C. Rogers, *An Introduction to Partial Differential Equations*. vol. 13. New York, NY, USA: Springer, 2006.

[21] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*. Basel, Switzerland: Birkhäuser Verlag, 2009.

[22] L. C. Evans, "Partial differential equations," *Grad. Stud. Math.*, vol. 19, no. 4, p. 7, 1998.

[23] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, vol. 840. Berlin, Germany: Springer, 2006.

[24] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44. New York, NY, USA: Springer-Verlag, 1983.

[25] E. Fridman, *Introduction to Time-Delay Systems: Analysis and Control* (Systems and Control: Foundations and Applications). Cham, Switzerland: Birkhäuser, 2014.

[26] R. Katz and E. Fridman, "Finite-dimensional control of the heat equation: Dirichlet actuation and point measurement," *Eur. J. Control*, vol. 62, pp. 158–164, Nov. 2021.