

Delay-Dependent H_∞ Control of Uncertain Discrete Delay Systems*

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A delay-dependent solution is given for state-feedback H_∞ control of linear discrete-time systems with unknown constant or time-varying delays and with polytopic or norm-bounded uncertainties. Sufficient conditions are obtained for stability and for achieving design specifications which are based on Lyapunov–Krasovskii functionals via a descriptor representation of the system. Similarly to the corresponding continuous-time results these conditions provide an efficient tool for analysis and synthesis of linear systems with time delay. The advantage of the new approach is demonstrated via a simple example.

Keywords:

1. Introduction

In this paper, a H_∞ control problem for discrete-time systems with unknown constant or time-varying delays is considered. The control problems for continuous-time delay systems have been extensively investigated in the last decade (see, e.g. [5,7,9,12,16] and the references therein). Delay-independent and, less conservative, delay-dependent sufficient conditions for H_∞ control in terms of Riccati or linear matrix inequalities (LMIs) have been derived by using Lyapunov–Krasovskii functionals. Delay-dependent conditions are based on different model transformations. The most recent one – a descriptor

representation of the system [2,3] – leads to less conservative sufficient conditions. Moreover, it allows to treat by Lyapunov–Krasovskii approach the important case of time-varying delay, where no bounds on the derivative of the delay are given [5]. Note that previously the stability conditions for such systems were derived only via Lyapunov–Razumikhin functions. The Razumikhin approach leads to more conservative conditions and it seems to be inapplicable to the case of H_∞ control.

Less attention has been given to the corresponding results for discrete-time delay systems [6,8,10,13,14]. Such systems can be transformed into augmented systems without delay, but for large delays this augmentation suffers from the ‘curse of dimension’. Moreover, the augmentation of the system is inappropriate for systems with unknown delays or systems with delays that are time-varying (such systems appear, e.g. in the field of communication networks). Delay-dependent conditions for stability and H_∞ control have been obtained by Song et al. [13] for the case of time-varying delays. The LMI conditions obtained there are conservative since in the case when the upper-bound on the delay is $\bar{h} = 1$ these conditions coincide with the well-known delay-independent conditions (see, e.g. [6]). Delay-dependent stability conditions for the case of constant delays have been obtained by Lee and Kwon [8] via a model transformation similar to the one in [11].

Recently, a descriptor approach has been applied to stability analysis of discrete delay systems with norm-bounded uncertainties [4]. The advantages of this

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approach in the continuous-time case are also evident for discrete delay systems. In the present paper, we further develop this approach to the H_∞ control problem for systems with either polytopic type or norm-bounded uncertainties. Note that the stability conditions of [4,8] are not affine in the system's matrices and they are thus inapplicable to the case of polytopic type uncertainty. For the cases of constant and time-varying delays, we derive bounded real lemmas (BRLs) in terms of LMIs which are affine in the matrices of the systems. Unlike the continuous-time case, the derivation of the BRLs in the present paper encounters additional technical difficulties (in comparison with the stability conditions). This is due to the fact that the cross terms that have to be over-bounded contain the disturbances. A state-feedback controller that stabilizes the system and satisfies prespecified requirements is found by applying P–K iterations. A simple example illustrates the efficiency of the new method.

1.1. Notation

Throughout the paper the superscript ‘T’ stands for matrix transposition, \mathcal{R}^n denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathcal{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathcal{R}^{n \times n}$ means that P is symmetric and positive definite. By ℓ_2 we denote the space of sequences $\{x_k\}$, $k = 0, 1, \dots$ with the norm $\|x_k\|_2^2 \triangleq \sum_{k=0}^{\infty} x_k^T x_k < \infty$.

2. Problem Statement

We consider the following discrete-time state-delayed system

$$\begin{aligned} x_{k+1} &= Ax_k + A_1 x_{k-h_k} + B_1 w_k + B_2 u_k, \\ x_k &= \phi_k, \quad -h \leq k \leq 0, \\ z_k &= Lx_k + Du_k, \end{aligned} \quad (1a,b)$$

where $x_k \in \mathcal{R}^n$ is the state vector, $w_k \in \mathcal{R}^q$ is the disturbance input which is assumed to be of bounded energy, $u_k \in \mathcal{R}^p$ is the control input, $z_k \in \mathcal{R}^m$ is the objective vector, h_k is a positive number representing the delay, $h_k \leq h$ and A, A_1, B_1, B_2, L and D are constant matrices of appropriate dimensions.

For simplicity only we consider the case of a single delay. The results may be easily generalized to the case of multiple delays. It is assumed, in the analysis part, that

A0. The eigenvalues of $A + A_1$ are all of absolute value less than 1.

We address first the following two analysis problems.

Problem 1. For $h_k = h$ that is an unknown number satisfying

$$0 \leq h \leq \bar{h}, \quad (2)$$

for $\{u_k\} \equiv 0$ and for a given scalar γ find whether the system is asymptotically stable and the following holds:

$$\begin{aligned} J &= \|z_k\|_2^2 - \gamma^2 \|w_k\|_2^2 < 0, \quad \forall 0 \neq \{w_k\} \in \ell_2 \\ &\text{for } \phi_k = 0, \quad -\bar{h} \leq k \leq 0. \end{aligned} \quad (3)$$

Problem 2. For all time-varying h_k that satisfy (2) find whether the system (1) with $\{u_k\} \equiv 0$ is asymptotically stable and (3) is satisfied for a given scalar γ .

Once solutions are obtained to the above problems, the problem of finding a state-feedback control law which stabilizes the system and achieves (3) for a prescribed γ will be considered.

3. Delay-Dependent and Delay-Independent BRLs

3.1. Descriptor Model Transformation

Assume A0. We consider in this section the case where $B_2 = D = 0$. Denoting

$$y_k = x_{k+1} - x_k \quad (4)$$

the system (1) can be represented by the following descriptor form:

$$\begin{bmatrix} x_{k+1} \\ 0 \end{bmatrix} = \begin{bmatrix} y_k + x_k \\ -y_k + Ax_k - x_k + A_1 x_{k-h_k} + B_1 w_k \end{bmatrix}.$$

Since $x_{k-h_k} = x_k - \sum_{j=k-h_k}^{k-1} y_j$ it follows that

$$E\bar{x}_{k+1} = \bar{A}\bar{x}_k - \begin{bmatrix} 0 \\ A_1 \end{bmatrix} \sum_{j=k-h_k}^{k-1} y_j + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} w_k, \quad (5)$$

where

$$\begin{aligned} x_0 &= \phi_0, \quad y_0 = (A - I)\phi_0 - A_1 \phi_{-h_0}, \\ y_k &= \phi_{k+1} - \phi_k, \quad k = -\bar{h}, \dots, -1, \end{aligned} \quad (6)$$

and where

$$\begin{aligned} \bar{A} &= \begin{bmatrix} I_n & I_n \\ A + A_1 - I_n & -I_n \end{bmatrix}, \\ E &= \text{diag}\{I_n, 0\}, \quad \bar{x}_k \triangleq \begin{bmatrix} x_k \\ y_k \end{bmatrix}. \end{aligned} \quad (7)$$

Thus if, for a specific w_k , x_k is a solution of (1), then $\{x_k, y_k\}$, where y_k is defined by (4), is a solution of (5), (6) and vice versa.

Denoting

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_3 \end{bmatrix} \quad \text{and} \quad E = \text{diag}\{I_n, 0\}, \quad (8a,b)$$

we consider the following Lyapunov–Krasovskii functional:

$$V_k = V_{1,k} + V_{2,k} + V_{3,k}, \quad (9a)$$

where

$$\begin{aligned} V_{1,k} &= x_k^\top P_1 x_k = \bar{x}_k^\top EPE \bar{x}_k, & 0 < P_1 \\ V_{2,k} &= \sum_{m=-\bar{h}}^{-1} \sum_{j=k+m}^{k-1} y_j^\top [R + Q] y_j, & 0 < R, \quad 0 < Q, \\ V_{3,k} &= \sum_{j=k-\bar{h}}^{k-1} x_k^\top S x_k, & 0 < S. \end{aligned} \quad (9b-d)$$

Note that $V_{1,k}$ corresponds to necessary and sufficient conditions for stability of discrete descriptor systems without delay [15], $V_{2,k}$ is typical for delay-dependent criteria, while $V_{3,k}$ corresponds to delay-independent stability conditions [14].

3.2. The Case of Constant Delay (Problem 1)

Theorem 1. Consider the system (1) with the constant time delay that satisfies (2), with $B_2 = D = 0$. If there exist $P, Z \in \mathcal{R}^{2n \times 2n}$, $0 < S, R, Q \in \mathcal{R}^{n \times n}$ and $Y \in \mathcal{R}^{2n \times n}$, $X \in \mathcal{R}^{q \times n}$, $W \in \mathcal{R}^{q \times q}$ that satisfy the following LMIs:

$$\begin{aligned} \Gamma < 0, \quad \begin{bmatrix} Z & Y \\ Y^\top & R \end{bmatrix} \geq 0, \quad \begin{bmatrix} W & X \\ X^\top & Q \end{bmatrix} \geq 0 \\ \text{and} \quad [I \quad 0] P \begin{bmatrix} I \\ 0 \end{bmatrix} > 0, \end{aligned} \quad (10a-d)$$

where

$$\Gamma \triangleq \begin{bmatrix} \mathcal{A}^\top P \mathcal{A} - EPE + \begin{bmatrix} S & 0 \\ 0 & \bar{h}[R + Q] \end{bmatrix} + \bar{h}Z + Y[I \quad 0] + \begin{bmatrix} I \\ 0 \end{bmatrix} Y^\top & * & * & * \\ * & * & * & * \\ * & * & * & * \\ \mathcal{A} = \begin{bmatrix} I & I \\ \mathcal{A} - I & -I \end{bmatrix}, & & & \end{bmatrix} \quad (10f)$$

then (1) is asymptotically stable and for a prescribed scalar γ (3) is satisfied.

Proof. We apply the Lyapunov–Krasovskii method and require that $V_{k+1} - V_k$ is strictly negative to guarantee the asymptotic stability of the system and that $V_{k+1} - V_k + z_k^\top z_k - \gamma^2 \omega_k^\top \omega_k$ is strictly negative in order to satisfy (3). We obtain that

$$\begin{aligned} V_{1,k+1} - V_{1,k} &= \bar{x}_{k+1}^\top EPE \bar{x}_{k+1} - \bar{x}_k^\top EPE \bar{x}_k \\ &= \left\{ \bar{x}_k^\top \bar{\mathcal{A}}^\top - \left(\sum_{j=k-\bar{h}}^{k-1} y_j^\top \right) [0 \quad A_1^\top] + \omega_k^\top [0 \quad B_1^\top] \right\} \\ &\quad \times P \left\{ \bar{\mathcal{A}} \bar{x}_k - \begin{bmatrix} 0 \\ A_1 \end{bmatrix} \sum_{j=k-\bar{h}}^{k-1} y_j + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \omega_k \right\} - \bar{x}_k^\top EPE \bar{x}_k \\ &= \bar{x}_k^\top [\bar{\mathcal{A}}^\top P \bar{\mathcal{A}} - EPE] \bar{x}_k + \mu_k + \eta_k + \zeta_k + \nu_k, \end{aligned} \quad (11a)$$

where $\bar{\mathcal{A}}$ is defined in (7) and

$$\begin{aligned} \mu_k &= (x_k^\top - x_{k-\bar{h}}^\top) [0 \quad A_1^\top] P \begin{bmatrix} 0 \\ A_1 \end{bmatrix} (x_k - x_{k-\bar{h}}), \\ \eta_k &= -2 \sum_{j=k-\bar{h}}^{k-1} \bar{x}_k^\top \bar{\mathcal{A}}^\top P \begin{bmatrix} 0 \\ A_1 \end{bmatrix} y_j, \end{aligned} \quad (11b,c)$$

$$\begin{aligned} \zeta_k &= 2\omega_k^\top [0 \quad B_1^\top] P \bar{\mathcal{A}} \bar{x}_k + \omega_k^\top [0 \quad B_1^\top] P \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \omega_k, \\ \nu_k &= -2 \sum_{j=k-\bar{h}}^{k-1} \omega_k^\top [0 \quad B_1^\top] P \begin{bmatrix} 0 \\ A_1 \end{bmatrix} y_j, \end{aligned} \quad (11d,e)$$

$$\begin{aligned} V_{2,k+1} - V_{2,k} &= \bar{h} y_k^\top [R + Q] y_k - \sum_{j=k-\bar{h}}^{k-1} y_j^\top [R + Q] y_j \\ &= \bar{x}_k^\top \begin{bmatrix} 0 & 0 \\ 0 & \bar{h}[R + Q] \end{bmatrix} \bar{x}_k \\ &\quad - \sum_{j=k-\bar{h}}^{k-1} y_j^\top [R + Q] y_j, \end{aligned} \quad (12)$$

$$\begin{aligned} V_{3,k+1} - V_{3,k} &= x_k^\top S x_k - x_{k-\bar{h}}^\top S x_{k-\bar{h}} \\ &= \bar{x}_k^\top \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \bar{x}_k - x_{k-\bar{h}}^\top S x_{k-\bar{h}}, \end{aligned} \quad (13)$$

$$\begin{bmatrix} \mathcal{A}^\top P \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - Y & \mathcal{A}^\top P \begin{bmatrix} 0 \\ B_1 \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} X^\top & \begin{bmatrix} L^\top \\ 0 \end{bmatrix} \\ -S + [0 \quad A_1^\top] P \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & [0 \quad A_1^\top] P \begin{bmatrix} 0 \\ B_1 \end{bmatrix} - X^\top & 0 \\ * & -\gamma^2 I + \bar{h}W + [0 \quad B_1^\top] P \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & 0 \\ * & * & -I \end{bmatrix}, \quad (10e)$$

and thus

$$V_{k+1} - V_k = \bar{x}_k^T \Gamma_1 \bar{x}_k - x_{k-h}^T S x_{k-h} - \sum_{j=k-h}^{k-1} y_j^T [R + Q] y_j + \mu_k + \eta_k, \quad (14)$$

where

$$\Gamma_1 = \bar{A}^T P \bar{A} - EPE + \begin{bmatrix} S & 0 \\ 0 & \bar{h}[R + Q] \end{bmatrix}.$$

By [11], for any $a \in \mathcal{R}^n, b \in \mathcal{R}^{2n}, N \in \mathcal{R}^{2n \times n}, R \in \mathcal{R}^{n \times n}, Y \in \mathcal{R}^{n \times 2n}, Z \in \mathcal{R}^{2n \times 2n}$, the following holds

$$-2b^T N a \leq \begin{bmatrix} b \\ a \end{bmatrix}^T \begin{bmatrix} Z & Y - N \\ Y^T - N^T & R \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix},$$

$$\text{where } \begin{bmatrix} Z & Y \\ Y^T & R \end{bmatrix} \geq 0. \quad (15)$$

Applying the latter on the expression we have obtained above for η_k , taking: $N = \bar{A}^T P \begin{bmatrix} 0 \\ A_1 \end{bmatrix}$, $a = y_j$ and $b = \bar{x}_k$, we have the following:

$$\eta_k \leq \sum_{j=k-h}^{k-1} [\bar{x}_k^T \ y_j^T] \begin{bmatrix} Z & Y - \bar{A}^T P \begin{bmatrix} 0 \\ A_1 \end{bmatrix} \\ * & R \end{bmatrix} \begin{bmatrix} \bar{x}_k \\ y_j \end{bmatrix},$$

$$\begin{bmatrix} Z & Y \\ * & R \end{bmatrix} \geq 0.$$

Similarly

$$\nu_k \leq \sum_{j=k-h}^{k-1} [\omega_k^T \ y_j^T] \begin{bmatrix} W & X - [0 \ B_1^T] P \begin{bmatrix} 0 \\ A_1 \end{bmatrix} \\ * & Q \end{bmatrix} \begin{bmatrix} \omega_k \\ y_j \end{bmatrix},$$

$$\begin{bmatrix} W & X \\ * & Q \end{bmatrix} \geq 0.$$

Hence,

$$\eta_k \leq \sum_{j=k-h}^{k-1} y_j^T R y_j + \bar{h} \bar{x}_k^T Z \bar{x}_k + 2\bar{x}_k^T \begin{bmatrix} Y - \bar{A}^T P \begin{bmatrix} 0 \\ A_1 \end{bmatrix} \\ * \end{bmatrix} x_k - 2\bar{x}_k^T \begin{bmatrix} Y - \bar{A}^T P \begin{bmatrix} 0 \\ A_1 \end{bmatrix} \\ * \end{bmatrix} x_{k-h},$$

$$\nu_k \leq \sum_{j=k-h}^{k-1} y_j^T Q y_j + \bar{h} \omega_k^T W \omega_k + 2\omega_k^T \begin{bmatrix} X - [0 \ B_1^T] P \begin{bmatrix} 0 \\ A_1 \end{bmatrix} \\ * \end{bmatrix} (x_k - x_{k-h}). \quad (16)$$

The asymptotic stability of the system follows from (10) by [4]. The performance requirement of (3) is satisfied if, defining $\xi_k = \text{col}\{\bar{x}_k, x_{k-h}, \omega_k, x_k\}$ we require that

$$V_{k+1} - V_k + x_k^T L^T L x_k - \gamma^2 \omega_k^T \omega_k \leq \xi_k^T \Gamma \xi_k < 0,$$

since summation in the latter inequality from $k = 0$ till $k = \infty$ implies (3). \square

Remark 1. If the LMIs of Theorem 1 are feasible, then (1) is asymptotically stable for $h = 0$ and thus A0 holds.

The result of Theorem 1 depends on the delay bound \bar{h} . The corresponding criterion for asymptotic stability which is delay-independent can be readily derived as a special case of Theorem 1. Choosing $Z = 0, W = 0, R = Q = \rho I_n, Y = 0, X = 0$ where ρ is a positive scalar and letting ρ tend to zero we obtain the following.

Corollary 1. The system (1) is asymptotically stable and satisfies (3) independently of the delay if there exist $P \in \mathcal{R}^{2n \times 2n}$ and $0 < S \in \mathcal{R}^{n \times n}$ that satisfy (10d) and the following LMI

$$\Gamma_{\text{ind}} = \begin{bmatrix} \mathcal{A}^T P \mathcal{A} - EPE + \begin{bmatrix} I \\ 0 \end{bmatrix} S [I \ 0] & \mathcal{A}^T P \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & \mathcal{A}^T P \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & \begin{bmatrix} L^T \\ 0 \end{bmatrix} \\ * & -S + [0 \ A_1^T] P \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & [0 \ A_1^T] P \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & 0 \\ * & * & -\gamma^2 I + [0 \ B_1^T] P \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & 0 \\ * & * & * & -I \end{bmatrix} < 0. \quad (17)$$

It follows from (10b) that if R and Q are taken to be positive-definite, then $Z \geq YR^{-1}Y^T$, $W \geq XQ^{-1}X^T$ and thus there exists a solution to (10a–d) with $R > 0$, $Q > 0$ iff there exists a solution to (10a,d) where Z is replaced by $YR^{-1}Y^T$. A sufficient BRL condition is thus the following.

Lemma 1. Consider the system (1) with the constant delay that satisfies (2), with $B_2 = D = 0$. This system is asymptotically stable and for a prescribed scalar γ (3) is satisfied if there exist $P \in \mathcal{R}^{2n \times 2n}$, of the structure (8), $0 < S$ and $R \in \mathcal{R}^{n \times n}$ and $Y \in \mathcal{R}^{2n \times n}$ that satisfy the following LMIs.

$$\hat{\Gamma} < 0 \quad \text{and} \quad P_1 > 0, \quad (18a,b)$$

where

$$\hat{\Gamma} \triangleq \begin{bmatrix} \begin{bmatrix} A^T \\ 0 & A_1^T \\ 0 & B_1^T \end{bmatrix} P \begin{bmatrix} A & 0 & 0 \\ & A_1 & B_1 \end{bmatrix} + \text{diag}\{S - P_1, \bar{h}[R + Q], -S, -\gamma^2 I\} + \begin{bmatrix} Y \\ 0 \\ X \end{bmatrix} \begin{bmatrix} I \\ 0 \\ -I \end{bmatrix}^T + \begin{bmatrix} I \\ 0 \\ -I \end{bmatrix} \begin{bmatrix} Y^T & 0 & X^T \end{bmatrix} \begin{bmatrix} L^T \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} Y \\ 0 \\ X \end{bmatrix} \begin{bmatrix} I \\ 0 \\ -I \end{bmatrix}^T & \begin{bmatrix} I \\ 0 \\ -I \end{bmatrix} \begin{bmatrix} Y^T & 0 & X^T \end{bmatrix} \begin{bmatrix} L^T \\ 0 \\ 0 \end{bmatrix} \\ * & \bar{h} \begin{bmatrix} Y \\ 0 \\ 0 \end{bmatrix} & \bar{h} \begin{bmatrix} 0 \\ 0 \\ X \end{bmatrix} \\ * & 0 & 0 \\ * & -\bar{h}R & 0 \\ * & * & -\bar{h}Q \end{bmatrix} \quad (18c)$$

Defining

$$\hat{J} = \text{diag} \left\{ \begin{bmatrix} I & 0 & 0 & 0 \\ A - I & -I & A_1 - I & B_1 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, I_m, I_n, I_n \right\} \quad (19)$$

it is readily obtained that

$$\hat{J}^T \hat{\Gamma} \hat{J} = \begin{bmatrix} \Psi + \bar{h} \begin{bmatrix} A^T - I \\ -I \\ A_1^T - I \\ B_1^T \end{bmatrix} [R + Q] \begin{bmatrix} A^T - I \\ -I \\ A_1^T - I \\ B_1^T \end{bmatrix}^T & \begin{bmatrix} L^T \\ 0 \\ 0 \\ 0 \end{bmatrix} & \bar{h} \tilde{\mathcal{A}}^T Y & \bar{h} \begin{bmatrix} 0 \\ 0 \\ 0 \\ X \end{bmatrix} \\ * & -I & 0 & 0 \\ * & * & -\bar{h}R & 0 \\ * & * & * & -\bar{h}Q \end{bmatrix}, \quad (20a)$$

where

$$\begin{aligned} \Psi &= \begin{bmatrix} A^T & 0 \\ -I & I \\ A_1^T - I & I \\ B_1^T & 0 \end{bmatrix} P \begin{bmatrix} A^T & 0 \\ -I & I \\ A_1^T - I & I \\ B_1^T & 0 \end{bmatrix}^T + \text{diag}\{S - P_1, 0, -S, -\gamma^2 I\} \\ &+ \begin{bmatrix} \tilde{\mathcal{A}}^T Y + \begin{bmatrix} 0 \\ 0 \\ 0 \\ X \end{bmatrix} \begin{bmatrix} I \\ 0 \\ -I \\ 0 \end{bmatrix}^T \\ \begin{bmatrix} I \\ 0 \\ -I \\ 0 \end{bmatrix} \begin{bmatrix} Y^T \tilde{\mathcal{A}} + [0 \ 0 \ 0 \ X^T] \end{bmatrix} \end{bmatrix}, \\ \tilde{\mathcal{A}} &= \begin{bmatrix} I & 0 & 0 & 0 \\ A - I & -I & A_1 - I & B_1 \end{bmatrix}. \end{aligned} \quad (20b,c)$$

The latter can be used, together with Theorem 1, to derive a sufficient condition for stability and prescribed H_∞ -norm bound in the case of polytopic uncertainty. Denoting

$$\Omega = [A \quad A_1 \quad B_1 \quad L],$$

assuming that $\Omega \in \mathcal{Co}\{\Omega_j, j = 1, \dots, N\}$, namely

$$\Omega = \sum_{j=1}^N f_j \Omega_j \quad \text{for some } 0 \leq f_j \leq 1, \quad \sum_{j=1}^N f_j = 1, \quad (21)$$

where the vertices of the polytope are described by $\Omega_j = [A^{(j)} \quad A_1^{(j)} \quad B_1^{(j)} \quad L^{(j)}]$, $j = 1, 2, \dots, N$, and denoting

is satisfied, over the entire polytope, if there exist $P^{(j)} \in \mathcal{R}^{2n \times 2n}$, of the structure (22a), $0 < S^{(j)}$, R , $Q \in \mathcal{R}^{n \times n}$, $X^{(j)} \in \mathcal{R}^{q \times n}$ and $Y^{(j)} \in \mathcal{R}^{2n \times n}$, of the structure (22b), $j = 1, \dots, N$ that satisfy the following set of N LMIs.

$$\bar{\Gamma} = \begin{bmatrix} \tilde{\Psi}^{(j)} & \bar{\mathcal{A}}^{(j)\text{T}} P_1 & \bar{\mathcal{A}}^{(j)\text{T}} \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{h}[R+Q] & \bar{h} \bar{\mathcal{A}}^{(j)\text{T}} Y^{(j)} & \bar{h} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ X^{(j)} \end{bmatrix} & \begin{bmatrix} L^{(j)\text{T}} \\ 0 \end{bmatrix} \\ * & -P_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\bar{h}[R+Q] & 0 & 0 & 0 & 0 \\ * & * & * & -\bar{h}R & 0 & 0 & 0 \\ * & * & * & * & -\bar{h}Q & 0 & 0 \\ * & * & * & * & * & -I & 0 \end{bmatrix} < 0, \quad j=1, \dots, N, \quad (23a)$$

where

$$\begin{aligned} \tilde{\Psi}^{(j)} = & \bar{\mathcal{A}}^{(j)\text{T}} P_2 \begin{bmatrix} 0 & I & I & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix} P_2^{\text{T}} \bar{\mathcal{A}}^{(j)} + \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix} P_3^{(j)} \begin{bmatrix} 0 & I & I & 0 \end{bmatrix} + \text{diag}\{S^{(j)} - P_1, 0, -S^{(j)}, -\gamma^2 I\} \\ & + \begin{bmatrix} \tilde{\mathcal{A}}^{(j)\text{T}} Y^{(j)} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ X^{(j)} \end{bmatrix} \end{bmatrix} \begin{bmatrix} I & 0 & -I & 0 \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ -I \\ 0 \end{bmatrix} \begin{bmatrix} Y^{(j)\text{T}} \tilde{\mathcal{A}}^{(j)} + \begin{bmatrix} 0 & 0 & 0 & X^{(j)\text{T}} \end{bmatrix} \end{bmatrix} \end{aligned} \quad (23b)$$

and where

$$\tilde{\mathcal{A}}^{(j)} = \begin{bmatrix} I & 0 & 0 & 0 \\ A^{(j)} - I & -I & A_1^{(j)} - I & B_1^{(j)} \end{bmatrix}. \quad (23c)$$

3.3. The Case of Time-Varying Delay (Problem 2)

Choosing for this case $V_{3,k} \equiv 0$ the required criterion is obtained by substituting $S = 0$ in (10). From Theorem 1 we thus obtain the following:

Theorem 3. Consider the system (1) with time-varying delay that satisfies (2), with $B_2 = D = 0$. This system is asymptotically stable and for a prescribed scalar γ (3) is satisfied if there exist $P, Z \in \mathcal{R}^{2n \times 2n}$, $W \in \mathcal{R}^{q \times q}$, $R, Q \in \mathcal{R}^{n \times n}$, $X \in \mathcal{R}^{q \times n}$ and $Y \in \mathcal{R}^{2n \times n}$ that satisfy (10), where $S = 0$.

Remark 2. Necessary conditions for feasibility of (10) with $S = 0$ are the following: $P_3 < 0$ and A_1 is non-singular. Note that the LMI of Corollary 1 (which corresponds to delay-independent conditions) is never feasible if $S = 0$. The reason is that applying Schur

complements to the second row and column of (17) with $S = 0$, one get in the (1, 1) block of the resulting LMI the element

$$\mathcal{A}^{\text{T}} \text{diag}\{P_1 - P_2 P_3^{-1} P_2^{\text{T}}, 0\} \mathcal{A} - EPE,$$

which is never negative-definite. Thus our approach does not give delay-independent solution in the time-varying delay case. Delay-independent stability conditions in the case of time-varying delay were obtained recently in [4] via Razumikhin approach.

Similarly in the case of polytopic type uncertainty we obtain

Theorem 4. Consider the system (1) with time-varying delay that satisfies (2) with $B_2 = D = 0$ and with

system matrices that reside in Ω . This system is asymptotically stable and, for a prescribed scalar γ it satisfies (3) over the entire uncertainty polytope if there exist $P^{(j)} \in \mathcal{R}^{2n \times 2n}$, of the structure (22a), $Y^{(j)} \in \mathcal{R}^{2n \times n}$, of the structure (22b), $j = 1, \dots, N$, $X^{(j)} \in \mathcal{R}^{q \times n}$ and $R, Q \in \mathcal{R}^{n \times n}$ that satisfy the LMIs of (23), where $S^{(j)} = 0$.

Example 1 (Stability analysis). We consider the system (1) where

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.97 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} \quad \text{and} \\ B_1 = B_2 = 0. \quad (24)$$

Assuming that h is constant, we seek the maximum value of \bar{h} for which the asymptotic stability of the system is guaranteed. We compare three methods: The criterion of [13], Theorem 1 in [8] and Theorem 1 above. It is found that the method of [13] does not provide a solution even for $\bar{h} = 1$. The maximum

value of \bar{h} , achievable by the method of [8], is 12, whereas a value of $\bar{h} = 16$ was obtained by applying Theorem 1 of the present paper. Using augmentation it is found that the system considered is asymptotically stable for all $h \leq 18$.

Allowing h to be time-varying we apply Theorem 3. We obtain that asymptotic stability is guaranteed for all $h \leq 8$.

4. Stabilization and H_∞ Control via State-Feedback

4.1. The Case of Polytopic Uncertainty

Considering next the case where in (1) B_2 and D are not zero and the uncertainty polytope is given by

$$\bar{\Omega} = [A \quad A_1 \quad B_1 \quad B_2 \quad L \quad D],$$

assuming that $\bar{\Omega} \in \mathcal{Co}\{\bar{\Omega}_j, j = 1, \dots, \hat{N}\}$, namely

$$\Omega = \sum_{j=1}^{\hat{N}} f_j \bar{\Omega}_j \quad \text{for some } 0 \leq f_j \leq 1, \quad \sum_{j=1}^{\hat{N}} f_j = 1, \quad (25)$$

where the vertices of the polytope are described by

$$\bar{\Omega}_j = [A^{(j)} \quad A_1^{(j)} \quad B_1^{(j)} \quad B_2^{(j)} \quad L^{(j)} \quad D^{(j)}], \quad j = 1, 2, \dots, \hat{N}.$$

We seek a control law

$$u_k = Kx_k, \quad (26)$$

which stabilizes the system and achieves a prescribed bound on the H_∞ -norm of the closed-loop over $\bar{\Omega}$. Replacing $A^{(j)}$ and $L^{(j)}$ in Theorem 2 with $A^{(j)} + B_2^{(j)}K$ and $L^{(j)} + D^{(j)}K$, respectively, we obtain matrix inequalities which are not affine in the matrices P and K . Similar to the state-feedback design of systems without delay, the adjoint of the system (1) may be considered and the stabilizability and the H_∞ -norm requirements can be achieved for the adjoint system restricting the resulting P_2 $[0I]Y$ and R to be proportional to P_1 . This procedure may be quite conservative and it will involve, anyhow, a search for three scalar parameters. Alternatively, a P–K iteration method may be used which, starting from any stabilizing solution for the required \bar{h} will minimize iteratively the obtained value of γ . The stabilizing solution for the required \bar{h} can be obtained by using a P–K iteration where, starting from $\bar{h} = 0$ the stabilizing feedback gain is found while sequentially increasing the value of

\bar{h} . The latter procedure was successfully applied to many examples.

We assume that

A1. The system for $h = 0$ is stabilizable.

The algorithm that achieves, for a given \bar{h} , state-feedback controller that asymptotically stabilizes the system over the entire uncertainty polytope is as follows:

Algorithm 1 (*Finding a stabilizing K*).

- *Step 1:* Find a state-feedback gain matrix K that stabilizes the system for $h = 0$. Here standard stabilization methods can be used [1]. Set $h = 0$.
- *Step 2:* Set $h = h + 1$. Replace $A^{(j)}$ and $L^{(j)}$ in (23) by $A^{(j)} + B_2^{(j)}K$ and $L^{(j)} + D^{(j)}K$, respectively, $j = 1, \dots, N$ and solve for $P_i, i = 1, 2, 3, Y, X, R, Q, S$ and γ while minimizing the latter. In the time varying delay case take $S = 0$ and in the delay-independent problem take $R = Q = 0, X = 0$ and $Y = 0$.
- *Step 3:* Substitute the resulting $P_i, i = 1, 2, 3, Y, X, R, Q, S$ in the above N LMIs and solve for K and γ , again minimizing γ .
- *Step 4:* If $h = \bar{h}$ stop. Otherwise go to Step 2.

Once a stabilizing controller is found, we apply the following:

Algorithm 2 (*Minimizing γ*)

- *Step 1:* Start with any stabilizing state-feedback gain matrix K that stabilizes the system, for example the one derived in Algorithm 1.
- *Step 2:* Replace $A^{(j)}$ and $L^{(j)}$ in (23) by $A^{(j)} + B_2^{(j)}K$ and $L^{(j)} + D^{(j)}K$, respectively, $j = 1, \dots, N$ and solve for $P_i, i = 1, 2, 3, Y, X, R, Q, S$ and γ while minimizing the latter. In the time varying delay case take $S = 0$ and in the delay-independent problem take $R = 0$ and $Y = 0$.
- *Step 3:* Substitute the resulting $P_i, i = 1, 2, 3, Y, X, R, Q, S$ in the above N LMIs and solve for K and γ , again minimizing γ .
- *Step 4:* If a prescribed convergence condition on the sequence of previous values of γ is met (e.g. a decrease of not more than 1% during 5 iterations) stop. Otherwise go to Step 2.

The latter algorithm converges to some value, γ_0 , since the sequence of the obtained values for γ is nonincreasing and it is bounded from below by zero.

4.2. The Case of Norm-Bounded Uncertainties

In the above we have treated the case where the uncertain parameters reside in a polytope. The case

where these parameters are of bounded norm is treated next. We considered (1) with norm-bounded

$$\begin{bmatrix} \bar{\Gamma} + \bar{d}^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix} H_1 H_1^T [0 \quad I_m] + \bar{d} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \bar{E}^T \bar{E} [I_n \quad 0] \\ * \\ * \end{bmatrix} \mathcal{M}_1^T H \quad d \mathcal{M}^T \begin{bmatrix} -dI & 0 \\ * & -dI \end{bmatrix} < 0, \quad (31)$$

uncertainties, namely:

$$\begin{aligned} x_{k+1} &= (A + H\Delta_k \bar{E})x_k + (A_1 + H\Delta_k \bar{E}_1)x_{k-h_k} \\ &\quad + B_1 w_k + (B_2 + H\Delta_k \bar{E}_2)u_k, \quad x_k = \phi_k, \\ &\quad -h \leq k \leq 0, \\ z_k &= (L + H_1 \bar{\Delta}_k \bar{E})x_k + (D + H_1 \bar{\Delta}_k \bar{E}_2)u_k, \end{aligned} \quad (27)$$

where $x_k \in \mathcal{R}^n$ is the state vector, h_k is a positive number representing the delay, $h_k \leq \bar{h}$ and $A, A_1, H, H_1 \bar{E}, \bar{E}_1$ and \bar{E}_2 are constant matrices of appropriate dimensions and $\Delta_k \in \mathcal{R}^{r_1 \times r_2}$ and $\bar{\Delta}_k \in \mathcal{R}^{\bar{r}_1 \times \bar{r}_2}$ are time-varying uncertain matrices that satisfy

$$\Delta_k^T \Delta_k \leq I_{r_2}, \quad \bar{\Delta}_k^T \bar{\Delta}_k \leq I_{\bar{r}_2} \quad (28)$$

and we address the same stabilization and H_∞ control issues that were treated in the previous sections.

Replacing in (23a) $A^{(j)}$ in $\mathcal{A}^{(j)}$ with $A + H\Delta_k \bar{E}$, $A_1^{(j)}$ with $A_1 + H\Delta_k \bar{E}_1$ and $L^{(j)}$ with $L + H_1 \bar{\Delta}_k \bar{E}_1$, the LMI (23a) can be written as

$$\begin{aligned} \bar{\Gamma} + \mathcal{M}^T \Delta_k^T H^T \mathcal{M}_1 + \mathcal{M}_1^T H \Delta_k \mathcal{M} + [I_n \quad 0]^T \bar{E}^T \bar{\Delta}_k^T \\ \times H_1^T [0 \quad I_m] + [0 \quad I_m]^T H_1 \bar{\Delta}_k \bar{E} [I_n \quad 0] < 0, \end{aligned} \quad (29a)$$

where $\bar{\Gamma}$ is defined as in (23a) for A, A_1, B_1 and L without superscripts,

$$\begin{aligned} \mathcal{M} &= [\bar{E} \quad 0 \quad \bar{E}_1 \quad 0 \quad 0 \quad 0 \quad 0], \\ \mathcal{M}_1 &= [P_2 [0 \quad I I 0] + [0 \quad I] Y [I 0 \quad -I 0] \quad P_1 \quad \bar{h} [R + Q] \quad \bar{h} [0 \quad I] Y \quad 0]. \end{aligned} \quad (29b,c)$$

It is well known that the following holds true for any two real matrices α and β of the appropriate dimensions and for Δ_k that satisfies (28) (see, e.g. [8]):

$$\alpha \Delta_k \beta + \beta^T \Delta_k^T \alpha^T \leq d^{-1} \alpha \alpha^T + d \beta^T \beta, \quad (30)$$

where d is some positive scalar.

Choosing once $\alpha = \mathcal{M}_1^T H$ and $\beta = \mathcal{M}$ and then $\alpha = [0 \quad I_m]^T H_1$ and $\bar{E} [I_n \quad 0]$, we apply (30) to (29a) and obtain the following.

Theorem 5. Consider the system (27) with the constant time delay that satisfies (2), with $B_2 = D = 0$ and with Δ_k and $\bar{\Delta}_k$ that satisfy (28). This system is asymptotically stable and for a prescribed scalar γ (3) is

satisfied if there exist $P \in \mathcal{R}^{2n \times 2n}$, of the structure (8a), $0 < S, R \in \mathcal{R}^{n \times n}$, $Y \in \mathcal{R}^{2n \times n}$ and positive scalars d and \bar{d} that satisfy the following LMI

where $\bar{\Gamma}$ is defined as in (23a) for the system's matrices without superscripts.

The corresponding result for state-feedback control is readily obtained from Theorem 5, where A, L and \bar{E} are replaced by $A + B_2 K$, $L + DK$ and $\bar{E} + \bar{E}_2 K$, respectively. The resulting LMIs can be solved, similarly to the solutions of the LMIs in Sections 3.2 and 3.3, by applying the above P–K iteration.

4.3. Example 2: State-Feedback H_∞ Control

Consider the system of (1) where

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 \\ 0 & 1.01 + g \end{bmatrix}, \\ A_1 &= \begin{bmatrix} -0.02 & -0.005 \\ 0 & -0.01 + g \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \quad L = [1 \quad 0] \quad \text{and} \quad D = 0.1 \end{aligned} \quad (32)$$

and where the unknown parameter g satisfies $|g| \leq 0.01$. By [13] no solution is found for the nominal system ($g = 0$) with $\bar{h} = 1$. Applying the above algorithms to this problem it is found that in the case of constant delay the system is stabilizable for all values of constant delay $h \leq 71$ and for all admissible values of g . For the case of time-varying delay, where $S = 0$, no solution has been found for $\bar{h} = 1$ that is feasible for all the uncertain values of g .

For constant delay $h \leq 67$ and for all the admissible values of g , a minimum upper-bound on the disturbance attenuation level $\gamma_{\min} = 12.0598$ is achieved by the state-feedback with the gain matrix $K[10.4951 \quad -98.5152]$. The question arises what is the difference between the latter minimum value of γ , obtained for the latter K , and the upper-bound on the peak value of the Bode magnitude plots of the transfer functions between w and z that are obtained for $|g| \leq 1$. It is found that the bound on the peak value is 11.3.

One of the advantages of the methods proposed in the present paper, compared to the one that augments the system in order to incorporate the delayed states in the state vector, is that the dimension of the system treated is fixed and does not depend on the delay. In the present example a solution for a constant $h = 67$ is derived by the method of the present paper without augmenting the system to be of order 136.

The same system was considered in [8] with norm-bounded uncertainties. Taking there $H = 0.2$, $\bar{E} = \bar{E}_1 = 0.01I$, $\bar{E}_2 = 0$, $H_1 = 0$ and $g = 0$ and considering the stabilization problem only, a maximum value of $\bar{h} = 41$ was obtained for the case of constant delay. Applying our algorithms we obtain that the system with the above norm-bounded uncertainties is stabilizable for all constant $h \leq 67$. For constant $h \leq 64$ the feedback gain matrix $K = [-8.8754 \ -7.0691]$ leads to the minimum bound $\gamma_{\min} = 180.07$. Assuming that the delay is time-varying we substitute $S = 0$ in Theorem 5 and find that there is a solution to the problem of stabilizing via a state-feedback controller for all $h \leq 43$. The resulting gain matrix is $K = [-6.7766 \ -20.5924]$ and the minimum achievable bound on γ for $h = 43$ is 169.4722 (for $X = 0$ and $Q = 0$).

5. Conclusions

Delay-dependent criteria have been derived for state-feedback H_∞ control of uncertain discrete-time systems with uncertain constant or time-varying delay. Lyapunov–Krasovskii functionals via descriptor model transformation are used. Sufficient conditions for achieving prescribed disturbance attenuation level are derived in terms of LMIs for the case of systems with either polytopic or norm-bounded uncertainties. The delay-independent condition for the case of the constant delay is derived as a special case of our conditions. The H_∞ state-feedback controller is obtained by applying P–K iterations.

The method developed in this paper may be applied in the future to output-feedback H_∞ control of discrete delay systems as well as to discrete descriptor systems.

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