Disturbance Compensation with Finite Spectrum Assignment for Plants with Input Delay

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Abstract—This paper presents a method for compensation of unknown bounded smooth disturbances for LTI plants with known parameters in the presence of constant and known input delay. The suggested control law is a sum of the classical predictor suggested by A.Z. Manitius and A.W. Olbrot for finite spectrum assignment and a disturbance compensator. The disturbance compensator is a novel control law based on the auxiliary loop for disturbance extraction and on the disturbance prediction. A numerical implementation of the integral terms in the predictor-based control law is studied and sufficient conditions in terms of linear matrix inequalities (LMIs) are provided for an estimate on the maximum delay that preserves the stability. Numerical examples illustrate the efficiency of the method.

Index Terms—input delay, predictor, disturbance compensation, stabilization, numerical implementation.

I. INTRODUCTION

One of the central problems in the control theory is control of systems affected by unknown disturbances. This problem becomes especially complicated in the presence of input delays that are typical for process control, remote control, chemical technologies, etc. (see e.g. [2]–[5]). Delay may prevent a designer from using high gain controllers for disturbance attenuation. The first approach to control of systems with input delay was proposed by O. Smith for stable plants [6]. For unstable plants, A. Manitius and A. Olbrot suggested prediction with finite spectrum assignment in [7]. In the presence of disturbance, the predictor of [7] achieves disturbance attenuation [3], [8]. The mentioned above papers did not take into account the structure of disturbances. The next step was done in [9], where a method for compensation of a finite number of sinusoidal disturbances was proposed.

In [3], [7]–[9], integral representations of state predictors were used without considering their numerical implementations. If the prediction horizon \( h \) (\( h \) is the value of input delay) is too large, the numerical implementation may destabilize the system [11]–[14]. A necessary condition for a bound on \( h \) that preserves the stability was provided in [13]. However, sufficient conditions for \( h \) preserving the stability under numerical implementations are missing.

In the present paper, a more general than in [9] class of \((r+1)\) continuously differentiable disturbances with uniformly bounded \((r+1)\) derivatives is considered. We suggest a control law which is a sum of the classical predictor of [7] and a disturbance compensator. The disturbance compensator is a novel control law based on the auxiliary loop for disturbance extraction and on the disturbance prediction. Note that recently (when this paper was under review), for the same class of disturbances, a similar idea of a control law that predicted disturbances with horizon \( h \) and allowed to compensate their influence on the system was suggested in [10]. The disturbance prediction in [10] was based on the current values of the disturbance and its derivatives till \( r \)th order that led to an \((r+1)\)th-order observer for the disturbance and its \( r \) derivatives. The numerical implementation issues were not considered in [10].

We propose a disturbance prediction which is based on the current and the delayed values of the disturbance. The latter allows to design a predictor-based control law that employs a simple scalar observer (the so-called dirty derivative filter as considered e.g. in [15]). We study the numerical implementation of the predictor-based control law and provide, for the first time, sufficient conditions in terms of LMIs for an estimate on the maximum delay that preserves the practical stability (meaning that the solutions of the closed-loop system are ultimately bounded with a small enough bound). The efficiency of the presented method is illustrated by two examples.

II. PROBLEM FORMULATION

Consider the following system:

\[
\dot{x}(t) = Ax(t) + Bu(t-h) + Bf(t), \quad t \geq 0, \\
u(s) = 0, \quad s < 0,
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R} \) is the control, \( f(t) \in \mathbb{R} \) is an unknown and matched disturbance, \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^n \) are constant known matrices, \( h > 0 \) is known and constant time-delay. Note that our results can be easily extended to the case of multi inputs provided \( B \) is full rank (see Remark 2 below).

We assume that

A1. The function \( f : \mathbb{R}_+ \to \mathbb{R} \) is \((r+1)\) times continuously differentiable. Moreover, the unknown disturbance is uniformly bounded together with its \((r+1)\)th derivative.
A2. The pair \((A, B)\) is controllable.
The classical predictor suggested in [7] guarantees the input-to-state stability of (1) leading to ultimate bound
\[ \lim_{t \to \infty} \sup_{t \geq 0} |x(t)| \leq \delta, \]  
where \( \delta = O \left( \sup_{t \geq 0} |f(t)| \right) \). Here \( | \cdot | \) is the Euclidean norm of a vector and \( O(\chi) \) for \( \chi \in R \) means that \( \lim_{\chi \to 0} \frac{O(\chi)}{\chi} = C \), where \( C \) is a constant.

In the present paper our objective is to design a controller that decreases \( \delta \) achieving \( \delta = O \left( h^{r+1} \sup_{t \geq 0} |f(r+1)(t)| \right) \) for the class of disturbances with small enough \( h^{r+1} \sup_{t \geq 0} |f(r+1)(t)| \). Sufficient conditions for this objective are given below in Theorems 1 (under integral predictor-based laws) and 2 (under numerical implementations of the integral terms of the predictors). The proofs of Theorems 1 and 2 are given in Appendixes A and B respectively. The novel control law that we propose is based on the disturbance extraction and its prediction.

### III. Predictive Disturbance Compensation Control Scheme

We choose a vector \( K^T \in R^n \) such that the matrix \( A+BK \) is Hurwitz and suggest the control law in the form of the sum:
\[ u(t) = u_1(t) + u_2(t), \]  
where
\[ u_1(t) = K \left[ e^{Ah}x(t) + \int_{t-h}^{t} e^{A(t-\theta)}Bu_1(\theta)d\theta \right] \]  
is a classical predictor for finite spectrum assignment [7]. The novel control law \( u_2 \) that will be designed below is aimed for disturbances compensation. We illustrate the design procedure in Fig. 1 (see “State predictor” and “Disturbance compensator”).

![Control System Diagram](image)

In order to extract the disturbance \( f \) from the closed-loop system (1), (3) we use the method of [16]. We introduce an auxiliary loop in the form
\[ \dot{x}_a(t) = Ax_a(t) + Bu_1^a(t-h) + Bu_2(t-h), \]  
\[ x_a(0) = 0, \]  
\[ u_1^a(t) = K \left[ e^{Ah}x_a(t) + \int_{t-h}^{t} e^{A(t-\theta)}Bu_1^a(\theta)d\theta \right]. \]  

Defining the error function \( \varepsilon = x - x_a \), from (1), (3) and (5) we arrive at the error equation
\[ \dot{\varepsilon}(t) = A\varepsilon(t) + B(u_1(t-h) - u_1^a(t-h)) + Bf(t). \]  
Denote \( B = \text{col}\{b_1, b_2, ..., b_n\} \). Choosing any \( k \) \( (k = 1, \ldots, n) \) with \( b_k \neq 0 \), we rewrite the \( k \)th equation of system (6) in the form
\[ \dot{\varepsilon}_k(t) = a_k^T\varepsilon(t) + b_k(u_1(t-h) - u_1^a(t-h)) \]  
where \( a_k \) is the \( k \)th row of the matrix \( A \). From (7) we obtain
\[ f(t) = b_k^{-1} \left[ \dot{\varepsilon}_k(t) - a_k^T\varepsilon(t) - b_k(u_1(t-h) - u_1^a(t-h)) \right]. \]  

Note that the signal \( \dot{\varepsilon}_k \) is not available in (8). In order to find its estimate \( \dot{\varepsilon}_k \) we can use any existing observer (see e.g. [15]-[18]). We suggest the following simple dirty derivative filter [15] (see “Filter” in Fig. 1):
\[ \mu \dot{\varepsilon}_k(t) + \hat{\varepsilon}_k(t) = \dot{\varepsilon}_k(t), \quad \hat{\varepsilon}_k(0) = 0, \]  
where \( \mu > 0 \) is a small enough number. Thus, the resulting estimate \( \hat{f} \) of \( f \) (see “Disturbance estimator” in Fig. 1) has a form
\[ \hat{f}(t) = b_k^{-1} \left[ \dot{\hat{\varepsilon}}_k(t) - a_k^T\varepsilon(t) - b_k(u_1(t-h) - u_1^a(t-h)) \right]. \]  

In order to construct the disturbance compensator \( u_2 \) we approximate \( \hat{f}(t) \) by its past values \( \hat{f}(t-h), ..., \hat{f}(t-(r+1)h) \) via the mean value theorem [19]:
\[ \hat{f}(t) = \sum_{j=1}^{r+1} (-1)^{j-1} C_{r+1}^j \hat{f}(t-jh) + \hat{E}(t). \]  
Here the remainder \( \hat{E}(t) \) is given by
\[ \hat{E}(t) = h^{r+1} \hat{f}^{(r+1)} (t-(r+1)\theta h), \quad 0 < \theta < 1. \]  

Approximation of unknown signals via the mean value theorem was suggested in [20]. From (10) and (11) we find
\[ f(t) = \hat{f}(t) + b_k^{-1} \eta(t), \]  
where
\[ \eta(t) = \dot{\varepsilon}_k(t) - \dot{\hat{\varepsilon}}_k(t). \]  

Substitution of \( u(t) = u_1(t) + u_2(t) \) and (13) into (1) leads to
\[ \dot{x}(t) = Ax(t) + Bu_1(t-h) + Bu_2(t-h) + B \sum_{j=1}^{r+1} (-1)^{j-1} C_{r+1}^j \hat{f}(t-jh) + B\lambda(t), \]  
where
\[ \lambda(t) = \hat{E}(t) + b_k^{-1} \eta(t). \]  
Choosing in (15) the control law \( u_2 \) as
\[ u_2(t) = -\sum_{j=1}^{r+1} (-1)^{j-1} C_{r+1}^j \hat{f}(t-(j-1)h) \]  
(see “Disturbance compensator” in Fig. 1), we arrive at
\[ \dot{x}(t) = Ax(t) + Bu_1(t-h) + B\lambda(t). \]  
It will be shown in Appendix A that solutions of (18), (4) are ultimately bounded and their ultimate bound is of the same
order as the ultimate bound \( \Delta(\mu) := \lim_{t \to \infty} \sup_{t \geq 0} |\lambda(t)| \) of \( \lambda \) and that
\[
\lim_{\mu \to 0} \Delta(\mu) = h^{r+1} \sup_{t \geq 0} |f^{r+1}(t)|.
\] (19)

Thus, the proposed control law allows to decrease the influence of the disturbance on the solutions of the closed-loop system if \( h^{r+1} \sup_{t \geq 0} |f^{r+1}(t)| \ll \sup_{t \geq 0} |f(t)| \). This is different from [3], [7], [8], where the closed-loop systems and the corresponding bounds directly depend on \( f \).

We are in a position to formulate the main result of this section:

**Theorem 1:** Given \( K^T \in \mathbb{R}^n \) such that the matrix \( A_0 := A + BK \) is Hurwitz and given a scalar \( \alpha > 0 \), let there exist a constant \( \beta > 0 \) and an \( n \times n \) matrix \( P > 0 \) that satisfy the following LMI:
\[
Q := \begin{pmatrix} A_0^T P + PA_0 + 2\alpha P & Pe^{Ah}B \\ P & -\beta \end{pmatrix} < 0.
\] (20)

Then for all small enough \( \mu > 0 \) there exists \( \Delta(\mu) > 0 \) such that solutions of (1) under the control law \( u = u_1 \) only. In [9], the results are confined to sinusoidal signals \( f \), whereas the control law \( u_2 \) is needed for the identification of parameters of sinusoidal signals and for their compensation. The proposed control law allows to compensate a wider than in [9] class of disturbances and employs a simple scalar observer (9) (in [10] an \( (r+1) \)-order observer is used for the disturbance predictor).

**Remark 1:** In [3], the influence of the disturbance \( f \) is attenuated by the control law \( u = u_1 \) only. In [9], the results are confined to sinusoidal signals \( f \), whereas the control law \( u_2 \) is needed for the identification of parameters of sinusoidal signals and for their compensation. The proposed control law allows to compensate a wider than in [9] class of disturbances and employs a simple scalar observer (9) (in [10] an \( (r+1) \)-order observer is used for the disturbance predictor).

**Remark 2:** Our results can be easily extended to (1) with several inputs \( u(t) \in \mathbb{R}^m \) if \( B \) is full rank. In this case, there always exist \( m \) linearly independent rows of \( B \). Then, similarly to (8), \( f \) can be found from (6) by employing the corresponding to these rows equations of (6).

### IV. Numerical Implementation of the Predictive Control Scheme

Note that the integral terms in control laws (4) and (5) are supposed to be implemented numerically. For numerical implementation of these terms a cubature formula can be used:
\[
u_1(t) = K[e^{Ah}x(t)] + \sum_{p=0}^{q} m_p e^{q-p}hA Bu_1(t - q^{-1}ph),
\] (21)
\[
u_1^+(t) = K[e^{Ah}x_a(t)] + \sum_{p=0}^{q} m_p e^{q-p}hA Bu_1^+(t - q^{-1}ph),
\] (22)
where the values of \( m_p \) depend on the chosen numerical scheme, the integer \( q \) determines the approximation precision. In our consideration, we assume that the values of \( m_p \) are small enough for large enough \( q \). This is the case e.g. in the trapezoidal rule.

We will present below LMI-based sufficient conditions for finding \( h \) that preserves ultimate boundedness of solutions to (1) under the predictor-based control law with \( u_1 \) and \( u_2 \) given by (21) and (22). The idea of the Lyapunov-based analysis of the resulting closed-loop system is the following. From (18) we find
\[
Bu_1(t - q^{-1}h) = \dot{x}(t - (q^{-1}p - 1)h) - Ax(t - (q^{-1}p - 1)h) - BX(t - q^{-1}ph) - B\lambda(t - (q^{-1}p - 1)h).
\] (23)
Substitution of the right-hand side of (23) into (21) leads to
\[
\dot{x}(t) = Ax(t) + BK[e^{Ah}x(t - h) + \sum_{p=0}^{q} m_p e^{q-p}hA \lambda(t - q^{-1}ph)]
\] (24)
Solving (23) with respect to \( \dot{x} \) we arrive at the neutral type system with the input that is given by a linear combination of \( \lambda(t) \) and its delayed values. By using a simple Lyapunov functional for the resulting neutral type system, we will derive in Appendix B sufficient LMI-based conditions for its input-state stability. Then the estimate on the ultimate bound of the solutions to the closed-loop system under (21) will follow from the ultimate bound \( \Delta(\mu) \) of \( \lambda \) and from relation (19).

To formulate the main result of this section, we will use the following notations:
\[
M := I - BKm_0, \quad F_p := M^{-1}BKm_pe^{q-p}hA \quad (p = 1, \ldots, q), \quad D_j := M^{-1}BKm_re^{q-j}hA \quad (j = 1, \ldots, q - 1), \quad D_q := M^{-1}BK(e^{Ah} - m_qhA), \quad A_s := A + \sum_{p=1}^{q} D_p.
\]
**Theorem 2:** Let the matrix \( M = I - BKm_0 \) be nonsingular. Given \( h > 0 \) and a scalar \( \alpha > 0 \), let there exist a constant \( \beta > 0 \) and \( n \times n \) matrices \( P > 0 \), \( P_2 \), \( P_3 \), \( P_q > 0 \), \( R_p > 0 \), \( Q_p > 0 \), \( p = 1, 2, \ldots, q \) that satisfy the following LMI:
\[
\Psi := \begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\
\Psi_{22} & 0 & 0 & 0 \\
\Psi_{33} & 0 & 0 & 0 \\
\Psi_{44} & 0 & 0 & -\beta \end{pmatrix} < 0
\] (26)
with notations given by (25), where
\[
\Psi_{11} = P_2^TA_s + A_s^T P_2 + \sum_{p=1}^{q} S_p, \quad \Psi_{12} = P - P_2^T + A_s^T P_3, \quad \Psi_{13} = -P_3 - P_2 + \sum_{p=1}^{q} g_{pq}hR_p + \sum_{p=1}^{q} P_q, \quad \Psi_{14} = \begin{pmatrix} \Psi_{11}^* & \Psi_{12}^* & \Psi_{13}^* & \Psi_{14}^* \\
\Psi_{22}^* & 0 & 0 & 0 \\
\Psi_{33}^* & 0 & 0 & 0 \\
\Psi_{44}^* & 0 & 0 & -\beta \end{pmatrix}, \quad \Psi_{12} = -h\text{diag} \{ e^{q-1} - e^{-2a_0h}R_1, \ldots, e^{-2a_0(q-1)h}R_{q-1}, e^{-2a_0h}R_q \}, \quad \Psi_{13} = -e^{-\alpha(q-1)h}Q_1, \ldots, e^{-\alpha(q-1)h}Q_{q-1}, e^{-\alpha h}Q_q \}
\]
Then for all small enough \( \mu > 0 \) there exists \( \Delta(\mu) > 0 \) that satisfies (19) and such that solutions of (1) under the control
law (3), (5), (10), (9), (17), (21), (22) are ultimately bounded and (2) holds with \( \delta = O(\Delta(\mu)) \).
LMI (26) is always feasible for \( \alpha < \max \text{Re}(\sigma(A_0)) \) and small enough \( m_p \) and \( h \).

V. EXAMPLES

**Example 1.** Consider (1) with parameters from [9], where

\[
A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad h = 0.45.
\]

Choose \( \mu = 0.01 \) in filter (9), where \( k = 2 \). Use \( r = 3 \) in the disturbance compensation control law (17), where \( \hat{f} = \hat{e}_2 - [1 \ 0]^T \varepsilon \). Control laws (21) and (22) used for numerical implementation with \( K = [-3 -3] \) and \( q = 5 \) are defined as follows:

\[
u_1(t) = -[3 3] \left[ e^{0.45A} x(t) + 0.09(0.5u_1(t)) + \sum_{p=1}^{4} e^{0.45q^{-1}pA} Bu_1(t - 0.45q^{-1}p) + 0.5e^{0.45A} B u_1(t - 0.45) \right],
\]

\[
u_1^q(t) = -[3 3] \left[ e^{0.45A} x_a(t) + 0.09(0.5u_1^q(t)) + \sum_{p=1}^{4} e^{0.45q^{-1}pA} B u_1^q(t - 0.45q^{-1}p) + 0.5e^{0.45A} B u_1^q(t - 0.45) \right].
\]

Here the trapezoidal rule is used for the approximation of the integral term. For numerical simulations we choose \( x(0) = [1 \ 2]^T \). Note, that the control laws \( u_1 \) and \( u_1^q \) are verified for \( q = 10, 30, 50 \). However, enlarging \( q \) does not affect on the stability and quality of transients in the closed-loop system [11], [12], [14].

Consider first the following disturbance: \( f = 1 + \sin 0.2t \). In Fig. 2 and Fig. 3 the plots of the state are presented for the control law [3], [7], [8] (for \( u = u_1 \)), the control law from [9] and the proposed one respectively. In figures the solid curve corresponds to \( x_1 \), the dashed curve corresponds to \( x_2 \). The simulations show that the control law of [3], [7], [8] does not compensate this disturbance. The control law of [9] ensures the exact compensation of the disturbance. Moreover, the proposed control law compensates the disturbance with the accuracy \( \delta = 0.02 \).

**Example 2.** Consider the model of the DC motor [21] in the form

\[
I \dot{\varphi}(t) = k \Phi i(t - h) - M(t),
\]

where \( \varphi \) is a rotation angle of the motor shaft, \( I \) is an inertia moment of the motor rotating part, \( i \) is a current in the armature circuit, \( k \) is a constructive constant, \( \Phi \) is a magnetic flux, \( M \) is a resistance moment depending on unknown load, \( h = 0.66 \) is a time-delay caused by remote control [21]. Denote \( x_1 = \varphi \), \( x_2 = \dot{\varphi} \), \( u = (k \Phi / I) i \) and \( \omega = (1 / I) M \). Then the model (28) can be represented in the form of (1), where \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), \( B = [0 \ 1]^T \). The goal is to design such a control law that the motor shaft angle rotates to zero and stops with some accuracy.

Let \( K = [-3 -3] \) and \( q = 5 \) in (21) and (22), where the trapezoidal rule is used. Choose \( \mu = 0.01 \) and \( k = 2 \) in filter (9). The disturbance \( \omega \) is simulated by the solution of (27), where \( \chi = 40 \). The simulations show that the control law \( u = Kx \) cannot stabilize the system with \( h > 0.41 \) and \( f = 0 \), whereas the proposed numerically implemented control law cannot stabilize the system for \( h > 0.74 \). LMI (26) is feasible for \( h \leq 0.66 \) (which is not far from 0.73 that follows from simulations).
For numerical simulations we choose \( x(0) = [1 0]^T \). In Fig. 5, Fig. 6 the plots of \( x_1 \) and \( x_2 \) are presented for the proposed control law with \( r = 0 \), \( r = 2 \) and \( r = 3 \) in (17). The simulations show that disturbances are compensated by the proposed control law with the accuracy \( \delta \) equal to 0.6, 0.3 and 0.1 for \( r \) equal to 0, 2 and 3 respectively.

Differentiating (r + 1) times equation (30), we obtain

\[
z_0^{(r+2)}(t) = A_0 z_0^{(r+1)}(t) + e^{A h} B f^{(r+1)}(t).
\]

Consider the Lyapunov function \( V = z_0^{(r+1)}(t)^T P z_0^{(r+1)}(t) \) and differentiate it along (36). We have

\[
W = \dot{V} + 2\alpha V - \beta(f^{(r+1)})^2
\leq \left[ z_0^{(r+1)}(t)^T f^{(r+1)}(t) \right]^T Q \left[ z_0^{(r+1)}(t)^T f^{(r+1)}(t) \right]^T \leq 0,
\]

where the latter inequality follows from (20). Then (36) is input-to-state stable, and the uniform boundedness of \( f^{(r+1)} \) implies the ultimate boundedness of \( z_0^{(r+1)} \). Hence, \( z_0^{(r+2)} \) defined by the right-hand side of (31) is also ultimately bounded. Further, from the equation that results from the differentiation (r + 2) times of (29) we conclude that \( e^{r+2} \) is ultimately bounded.

**Step 2:** the feasibility of (20). Since \( A_0 \) is Hurwitz, the Lyapunov inequality \( A_0^2 P + P A_0 + 2\alpha P < 0 \) is always feasible for \( \alpha < \max \text{Re}(\sigma(A_0)) \). Then, by Schur complement, (20) is feasible for large enough \( \beta \).

**Step 3:** ultimate bound on \( \eta^{(r+1)} \). Differentiating (14) and substituting from (9) \( \dot{\hat{e}}_k(t) = \mu^{-1} \eta(t) \) we obtain

\[
\mu \eta(t) = -\eta(t) + \mu \dot{\hat{e}}_k(t).
\]

Differentiating (32) (r + 1) times, we have

\[
\mu \eta^{(r+2)}(t) = -\eta^{(r+1)}(t) + \mu \epsilon^{(r+2)}(t).
\]

Since \( e^{r+2} \) is ultimately bounded, then \( \eta^{(r+1)} \) is ultimately bounded and

\[
\lim_{t \to \infty} \sup_{t \geq 0} \left| \eta^{(r+1)}(t) \right| = O(\mu).
\]

**Step 4:** ultimate bound on \( \lambda \). Differentiating (r + 1) times equation (13) we find \( f^{(r+1)}(t) = f^{(r+1)}(t) - b_k^{-1} \eta^{(r+1)}(t) \). Then (12) can be presented as

\[
\dot{E}(t) = -h r^{r+1} \left[ f^{(r+1)}(t - (r + 1)\theta h) - b_k^{-1} \eta^{(r+1)}(t - (r + 1)\theta h) \right], \quad 0 < \theta < 1.
\]

Due to (33), (34) and the fact that \( f^{(r+1)} \) is uniformly bounded, it follows from (16) that \( \lambda(t) \) is ultimately bounded

\[
\Delta(\mu) := \lim_{t \to \infty} \sup_{t \geq 0} |\lambda(t)| < \infty \quad \text{and} \quad (19) \quad \text{satisfied.}
\]

**Step 5:** ultimate bound of \( \lambda \). Consider next the following change of the state \( x(t) \) in (18):

\[
z_0(t) = e^{A h} x(t) + \int_{t-h}^t e^{A(t-\theta)} B u_1(\theta)d\theta.
\]

It follows from (4) that \( u_1(\theta) = K z_1(\theta) \) and that

\[
x(t) = e^{-A h} \left[ z_0(t) - \int_{0}^h e^{A s} B K z_1(s + t) ds \right].
\]

Differentiating (35) and taking into account (18), we obtain

\[
z_1(t) = A_0 z_0(t) + e^{A h} B \lambda(t).
\]

Under (20) the system (36) is input-to-state stable and

\[
\sigma_{\text{min}}(P) \lim_{t \to \infty} \sup_{t \geq 0} \left| z_1(t) \right|^2 \leq \lim_{t \to \infty} \sup_{t \geq 0} \left( z_1(t) P z_1(t) \right) \leq 0.5 \alpha^{-1} \beta \Delta^2(\mu).
\]
where $P, Q_p, S_p$ and $R_p$ are positive matrices. We use the descriptor method with free matrices $P_2$ and $P_3$ (see [22]), where $\hat{\zeta}$ is not substituted by the right-hand side of (43). Differentiating (44) along (43), we have

$$
\dot{V}_1 + 2aV_1 = 2\zeta^T(t)P\dot{\zeta}(t) + 2a\zeta^T(t)P\zeta(t)
+ 2\zeta^T(t)P_2^T + \zeta^T(t)P_3^T \big|_{\zeta = \zeta(t)} + A_s\zeta(t)
- \sum_{p=1}^{q} D_p \int_{t-pq^{-1}h}^{t} \zeta(s)ds
+ \sum_{p=1}^{q} F_p \zeta(t - pq^{-1}h) + Bw(f^{(r+1)}),
$$

$$
\dot{V}_2 + 2aV_2 = \zeta^T(t) \sum_{p=1}^{q} P_p \zeta(t)
- \sum_{p=1}^{q} e^{-2a(t - pq^{-1}h)} \zeta^T(t - pq^{-1}h)Q_p \zeta(t - pq^{-1}h),
$$

$$
\dot{V}_3 + 2aV_3 = \zeta^T(t) \sum_{p=1}^{q} S_p \zeta(t)
+ \sum_{p=1}^{q} e^{-2a(t - pq^{-1}h)} \zeta^T(t - pq^{-1}h) \sum_{p=1}^{q} R_p \zeta(t - pq^{-1}h) + \sum_{p=1}^{q} \zeta^T(t) R_p \zeta(t).
$$

By Jensen's inequality [25]

$$
- \int_{t-pq^{-1}h}^{t} e^{2a(s-t)} \zeta^T(t) R_p \zeta(t)ds
\leq - e^{-2a(t - pq^{-1}h)} \int_{t-pq^{-1}h}^{t} \zeta^T(t) R_p \zeta(t)ds
\leq \xi(t - (q-1)h), \zeta(t - (q-2)(n-1)h), ..., \zeta(t - h),
$$

$$
\xi = \col{\xi_1, \xi_2, \xi_3}(f^{(r+1)}).
$$

From (44)-(45) we arrive at $\dot{V} + 2aV = \beta(w(f^{(r+1)}))^2 \leq \xi^T \Psi \xi \leq 0$, where the last inequality follows from (26). Then, by comparison principle,

$$
\lim_{t \to \infty, i \geq 0} \sup \{\zeta^T(t) P \zeta(t)\} \leq 0.5\alpha^{-1} \max_{i \geq 0} (w(f^{(r+1)}))^2,
$$

i.e. $\zeta(t)$ is ultimately bounded. LMI (26) guarantees also the stability of the difference equation $\zeta(t) = \sum_{p=1}^{q} F_p \zeta(t - pq^{-1}h)$ [22]. Consider now (43) as a difference equation with respect to $\zeta(t)$, where the nonhomogeneous term is defined by $\zeta(t)$ and $w(f^{(r+1)})$. Then by using (iii) of Lemma 3.1 in [2] we conclude that $\hat{\zeta} = e^{\hat{\alpha}(t+\delta)}$ is ultimately bounded due to ultimate boundedness of $\zeta$ and uniform boundedness of $w(f^{(r+1)})$.

**Step 2: ultimate bound on $x$.** Ultimate boundedness of $\lambda$ and relation (19) follow from Steps 3 and 4 of the proof of Theorem 1.

Since the structure of (24) is similar to the one of (40), we conclude that LMI (26) imply $\lim_{t \to \infty, i \geq 0} \sup \{x^T(t) P x(t)\} \leq 0.5a^{-1} \Delta^2 (\mu) \beta$. The latter yields (2) with $\delta = O(\Delta(\mu))$.

**Step 3: the feasibility of LMI (26).** Since $m_0$ is small enough, the matrix $(I - BK m_0)$ is invertible. Additionally, since $m_p$ ($p = 1, \ldots, q$) are small enough, the matrices $F_p$ defined by (25) are small enough. The latter guarantees the stability of the difference equation $\zeta(t) = \sum_{p=1}^{q} F_p \zeta(t - pq^{-1}h)$. The stability of the difference equation and the fact that $A_s$ given by (25) is Hurwitz guarantee the feasibility of LMI (26) for small enough $m_p$ and $h$ [4], [22].
REFERENCES


