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H_∞ sampled data control of systems with time-delays

V. Suplin, E. Fridman* and U. Shaked

School of Electrical Engineering, Tel Aviv University, Tel Aviv, 69978 Israel

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Sampled-data H_∞ control of linear systems with constant state, control and measurement delays is considered. The sampling of the controlled input and of the measured output is not assumed to be uniform. The system is modelled as a continuous-time one, where the controlled input and the measurement output have piecewise-continuous delays. The input–output approach to stability and L_2-gain analysis is applied to the resulting system. The discretised Lyapunov functional method is extended to the case of multiple delays, where the Lyapunov functional is complete in one of the delays (in the state) and is simple in the other delays (those in the input and in the output), which are constant. Solutions to the state-feedback and the output-feedback H_∞ control problems are derived in terms of linear matrix inequalities (LMIs).

Keywords: sampled data control; H_∞ control; time-delay; discretised Lyapunov functional; LMI

1. Introduction

Two main approaches have been applied to the sampled-data H_∞ control problem. The first is based on the lifting technique; see e.g. Yamamoto (1990) and Bamieh, Pearson, Francis and Tannenbaum (1991), in which the problem is transformed to equivalent finite-dimensional discrete problem. This approach does not work in cases with uncertain sampling times or uncertain system matrices and it seems to be inapplicable to systems with state delay.

The second approach is based on the representation of the system in the form of a hybrid discrete/continuous model. Application of this approach to linear systems leads to necessary and sufficient conditions for stability and L_2-gain analysis in the form of differential equations (or inequalities) with jumps; see e.g., Sivashankar and Khargonekar (1994). This approach was applied to systems with state-delay, where it led to complicated conditions in terms of partial differential equations with jumps; see e.g. Fridman and Shaked (2000).

Modelling of sampled-data continuous-time systems as continuous systems with time-varying delayed control input and the analysis of such systems via the Lyapunov–Krasovskii method was introduced in Fridman, Seuret and Richard (2004). The sampled control law may be represented as a delayed control as follows.

$$u(t) = u_d(t_k) = u_d(t-(t-t_k)) = u_d(t-\tau(t)),\\ t_k \le t < t_{k+1}, \quad \tau(t) = t - t_k, \tag{1}$$

where u_d is a discrete-time control signal and the time-varying delay $\tau(t) = t - t_k$ is piecewise linear with derivative $\dot{\tau}(t) = 1$ for $t \neq t_k$. Moreover, $\tau < t_{k+1} - t_k$. Recently, the latter approach allowed us to introduce a novel performance index for H_∞ control of linear systems with non-uniform uncertain sampling (Suplin, Fridman and Shaked 2007). The latter index takes into consideration the energy of the measurement noise.

It is the objective of the present paper to extend this approach to systems with state, input and output delays. The stability and the L_2-gain of systems with time-varying delays are analysed via the input-output approach to stability: see Gu, Kharitonov and Chen (2003) and the references therein. The discretised Lyapunov functional method of Gu (1997) together with the descriptor method of Fridman (2001) is found efficient in the solution of design problems (Fridman 2006). We extend the descriptor discretised Lyapunov functional method to the case of multiple delays. For the first time, we derive a solution of the output-feedback H_∞ control problem via discretised Lyapunov functional method. In the present paper the Lyapunov functional is complete (i.e. corresponds to necessary and sufficient stability conditions) in one of the delays (in the state) and it is simple in the other delays (in the input and in the output). The latter two delays are constant and either known (in the case of output-feedback control) or unknown with known upper-bounds (in the case of state-feedback control). These delays occur due to the delays that are encountered in the communication networks.

*Corresponding author. Email: emilia@eng.tau.ac.il

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It is the purpose of the present paper to derive simple unified conditions for the stability and the performance of the closed-loop system, for both the state and the output feedback control problems. In order to also treat cases where the system with zero state delay is unstabilisable, the 'simple' Lyapunov functional cannot be applied to the delay in the state. This is why we resort to the discretised method when dealing with the latter delay. On the other hand, since the delays in the input and the output channels usually deteriorate the stability and the performance of the closed-loop system, we treat these delays by the simple Lyapunov functional and we shall thus be able to achieve trackable LMIs for the solution. In the state-feedback case, the latter approach readily copes also with unknown, but bounded, delays in the input and the state measurement channels.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by *. L_2 is the space of square integrable functions $v:[0,\infty)\to R^n$ with the norm $\|v\|_{L_2}=[\int_0^\infty \|v(t)\|^2 \mathrm{d}t]^{1/2}$.

2. Problem formulation

We consider the following linear system:

$$\left.\begin{aligned}\dot{x}(t)&=A_0x(t)+A_1x(t-h_1)+B_2u(t)+B_1\omega(t),\\ y(t)&=C_2x(t)+D_{21}n(t),\\ z(t)&=C_1(t)x(t)+D_{12}u(t),\end{aligned}\right\}\qquad\text{(2a–c)}$$

where $x(t)\in\mathcal{R}^n$ is the state, $u(t)\in\mathcal{R}^m$ is the control input, $y(t)\in\mathcal{R}^q$ is the measured output, $z(t)\in\mathcal{R}^p$ is the controlled output, $\omega(t)\in\mathcal{R}^r$ is the disturbance, $n(t)=\mathcal{R}^l$ is the noise. The matrices A_0, A_1, B_1, B_2, C_1, C_2, D_{21} and D_{12} are constant matrices of the appropriate dimensions. The scalar h_1 is a known, constant, time-delay. It is assumed that ω, $n\in L_2$.

The controller and the plant are assumed to be connected through a network. The network is modelled as a switch. When the switch is closed, a network packet (containing the controller sampled input $u(t_k)$ or a sampled plant measurement output $y(\sigma_l)$) is transmitted; when the switch is open, the output of the switch is held at the last value. The sampling may not be uniform but it is assumed that the sampling times

$$0=t_0<t_1<\cdots<t_k<\cdots,\quad \lim_{k\to\infty}t_k=\infty,$$
$$0=\sigma_0<\sigma_1<\cdots<\sigma_k<\cdots,\quad \lim_{k\to\infty}\sigma_k=\infty$$

are bounded by the following positive scalars

$$t_{k+1}-t_k\le 2\mu_2,\quad \sigma_{k+1}-\sigma_k\le 2\mu_3.\qquad(3)$$

The measurement output may be corrupted by a noise signal

$$y(\sigma_k)=C_2x(\sigma_k)+D_{21}n(\sigma_k),\quad k=0,1,\ldots\qquad(4)$$

We may thus consider the following piecewise-constant measurement

$$\bar{y}(t)=C_2x(\sigma_k)+D_{21}n(\sigma_k),\quad \sigma_k\le t<\sigma_{k+1}.\qquad(5)$$

We assume that the network induces two constant delays, that may be unknown (in the case where state-feedback can be applied). The first delay is upper bounded by $h_2-\mu_2$ and is encountered in the channel from the controller to the actuator. The second is a delay from the measurement sensor to the controller which is upper bounded by $h_3-\mu_3$.

Defining

$$\left.\begin{aligned}&\tau_1(t)=h_1,\\ &\tau_2(t)=t-t_k,\ t_k\le t-h_2+\mu_2<t_{k+1},\\ &\tau_3(t)=t-\sigma_k,\ \sigma_k\le t-h_3+\mu_3<\sigma_{k+1}.\end{aligned}\right\}\qquad\text{(6a–c)}$$

it follows that $|\tau_i(t)-h_i|\le\mu_i$, $i=2,3$. It is also found from (6) that $(\mathrm{d}/\mathrm{d}t)\tau_i=1$ almost for all $t>0$.

The definitions in (3) and (6) aim at achieving symmetric intervals for the delays: $\tau_i\in[h_i-\mu_i,\ h_i+\mu_i)$, $i=2,3$.

In the present paper we consider the state-feedback and the output-feedback H_∞ control problems.

H_∞ output-feedback control problem

In the output-feedback formulation we assume that τ_2 and τ_3 are known. This assumption is justified later (see Remark 1). In the sequel we consider an output-feedback controller of the following structure:

$$\left.\begin{aligned}\dot{x}_c(t)&=A_{c0}x_c(t)+\sum_{i=1}^{3}A_{ci}x_c(t-\tau_i(t))+B_cy(t-\tau_3(t)),\\ \bar{u}(t)&=C_cx_c(t),\\ x_c(t)&=0,\ t\le 0,\end{aligned}\right\}\qquad\text{(7a–c)}$$

where $x_c\in\mathcal{R}^n$, and where $\bar{u}$ is the output of the controller, namely, $u(t)=\bar{u}(t-\tau_2(t))$.

Defining $\xi = \mathrm{col}\{x, x_c\}$, we obtain the following augmented model.

$$\dot{\xi}(t) = \bar{A}_0\xi(t) + \sum_{i=1}^{3}\bar{A}_i\xi(t-\tau_i(t)) + \bar{B}_0 w(t) + \bar{B}_3 n(t-\tau_3(t))$$
$$\xi(t) = 0,\ t \le 0,$$
$$\bar{z}(t) = \bar{L}_0\xi(t) + \bar{L}_2\xi(t-\tau_2(t)). \qquad \text{(8a–c)}$$

The matrices of the above model are

$$\bar{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & A_{c0} \end{bmatrix},\quad \bar{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & A_{c1} \end{bmatrix},$$
$$\bar{A}_2 = \begin{bmatrix} 0 & B_2C_c \\ 0 & A_{c2} \end{bmatrix},\quad \bar{A}_3 = \begin{bmatrix} 0 & 0 \\ B_cC_2 & A_{c3} \end{bmatrix}, \qquad (9)$$
$$\bar{B}_0 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},\quad \bar{B}_3 = \begin{bmatrix} 0 \\ B_cD_{21} \end{bmatrix},$$
$$\bar{L}_0 = [C_1 \quad 0],\quad \bar{L}_2 = [0 \quad D_{12}C_c].$$

The structure of (8a–c) is described in Figure 1.

Given $\gamma > 0$, find a controller of the form (7) that asymptotically stabilises the system (8) and satisfies

$$J_\infty = \int_0^\infty [\bar{z}^T(t)\bar{z}(t) - \gamma^2 w^T(t)w(t)]\mathrm{d}t - \gamma^2\sum_{k=0}^{\infty}(\sigma_{k+1}-\sigma_k)n^T(\sigma_k)n(\sigma_k) < 0, \qquad (10)$$

for the zero initial condition and $\forall w \neq 0 \in L_2$, $\forall n \neq 0 \in l_2$. The performance index J_∞ has been introduced in Suplin et al. (2007), where the last summation is a rectangular approximation of the energy entailed in the measurement noise.

Remark 1: It is assumed that the clocks in the plant and the controller are synchronized via GPS and that the signal from the measurement to the controller is supplemented with a time stamp, the time when the signal was generated. It follows that τ_3 is available to the controller. Since the controller has a full knowledge of the sampling rate of its output, the delay $\tau_2(t)$ can be directly applied in its dynamics.

The H_∞ filtering problem

Consider the system (2) with $B_2 = 0$ and $D_{12} = -I$. It is assumed that (2) is asymptotically stable. For a prescribed scalar $\gamma > 0$, find a filter of the form (7). Namely, find a filter that asymptotically stabilises the system (8), where $\bar{A}_2 = \bar{L}_2 = 0$, $\bar{L}_0 = [C_1 \ \ -C_c]$, and satisfies

$$\int_0^\infty [\bar{z}^T(t)\bar{z}(t) - \gamma^2 w^T(t)w(t)]\mathrm{d}t - \gamma^2\sum_{k=0}^{\infty}(\sigma_{k+1}-\sigma_k)n^T(\sigma_k)n(\sigma_k) < 0,$$

$\forall w \not\equiv 0 \in L_2$ and $\forall n(\sigma_l) \not\equiv 0 \in l_2$.

The state-feedback H_∞ control problem

Consider the system (2a,c). We assume that the measurement and control delays are unknown constant delays that are bounded by h_3 and h_2, respectively. The delayed state $x(t-h_3)$ is measured (here we have no sampling thus $\mu_3 = 0$) and that sampling is encountered in the control loop with uncertain sampling times that satisfy $t_{k+1} - t_k \le 2\mu_2$ with known μ_2. For a prescribed scalar $\gamma > 0$, we seek a state-feedback gain matrix K_{sf} such that the control law of the form

$$\left.\begin{aligned} &u(t) = K_{sf}x(t-\tau_s(t)), \quad \tau_s = h_s + \mu_2 + (t + t_k),\\ &h_s = h_2 + h_3,\\ &t_k \le t - h_s + \mu_2 < t_{k+1}, \end{aligned}\right\} \qquad (11)$$

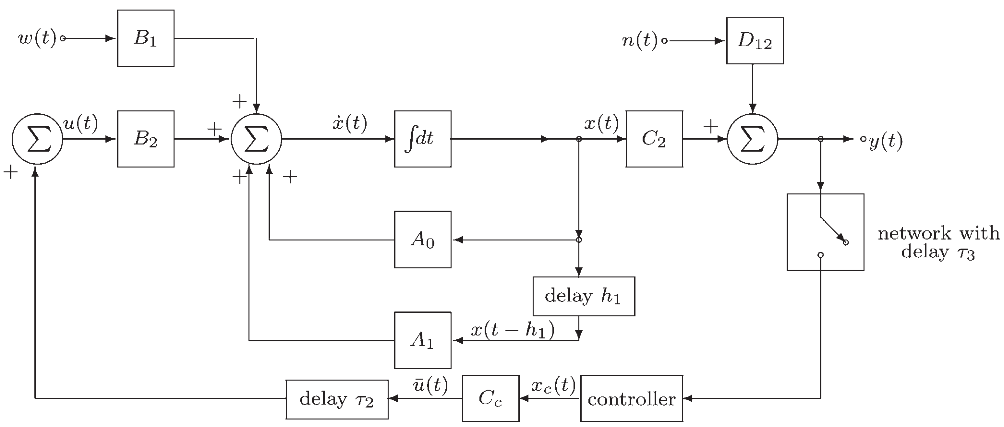

Figure 1. The structure of the closed-loop system.

asymptotically stabilises the system (2a) and achieves (10), where $n \equiv 0$.

We note that dissimilar to the output-feedback control, the state-feedback controller is not required to know the delay from the controller to the plant actuator precisely.

3. Stability and L_2-gain analysis via discretised Lyapunov functional

Consider the linear system

$$\left.\begin{aligned} \dot{\xi}(t) &= \bar{A}_0\xi(t) + \sum_{i=1}^{3}\bar{A}_i\xi(t-\tau_i(t)) \\ &\quad + \bar{B}_0\omega(t) + \bar{B}_3 n(t-\tau_3(t)), \\ \bar{z}(t) &= \bar{L}_0\xi(t) + \bar{L}_2\xi(t-\tau_2(t)), \end{aligned}\right\} \tag{12a–c}$$

where $\xi(t) \in R^{2n}$. Representing

$$\left.\begin{aligned} \xi(t-\tau_i(t)) &= \xi(t-h_i) - \int_{-h_i-\eta_i(t)}^{-h_i}\dot{\xi}(t+s)\mathrm{d}s, \\ & i = 2, 3, \\ \eta_i(t) &= \tau_i(t) - h_i \end{aligned}\right\} \tag{13}$$

and applying the input-output approach (see Gu et al. (2003) and the references therein), we consider the following system see, e.g. Fridman and Shaked (2006):

$$\left.\begin{aligned} \dot{\xi}(t) &= \bar{A}_0\xi(t) + \sum_{i=1}^{3}\bar{A}_i\xi(t-h_i) + \sum_{i=2}^{3}\mu_i\bar{A}_i v_i(t) \\ &\quad + \bar{B}_0\omega(t) + \bar{B}_3 n(t-\tau_3(t)), \\ \bar{z}(t) &= \bar{L}_0\xi(t) + \bar{L}_2\xi(t-h_2) + \mu_2\bar{L}_2 v_2(t), \\ y_\xi(t) &= \dot{\xi}(t), \end{aligned}\right\} \tag{14a–c}$$

with the feedback

$$v_i(t) = -\frac{1}{\mu_i}\int_{-h_i-\eta_i(t)}^{-h_i} y_\xi(t+s)\mathrm{d}s, \quad i = 2, 3. \tag{15}$$

3.1 *Systems with constant delays*

We first analyse the stability of the system

$$\dot{\xi}(t) = \bar{A}_0\xi(t) + \sum_{i=1}^{3}\bar{A}_i\xi(t-h_i) \tag{16}$$

by the following mixed descriptor LKF, which is complete in h_1 and simple (as defined in Gu et al. (2003)) in h_2, h_3.

$$\begin{aligned} V(\xi_t) &= \xi^T(t)P_1\xi(t) + 2\xi^T(t)\int_{-h_1}^{0} Q(s)\xi(t+s)\mathrm{d}s \\ &\quad + \int_{-h_1}^{0}\int_{-h_1}^{0}\xi^T(t+s)R(s,\theta)\mathrm{d}s\xi(t+\theta)\mathrm{d}\theta \\ &\quad + \int_{-h_1}^{0}\xi^T(t+s)S(s)\xi(t+s)\mathrm{d}s \\ &\quad + \sum_{i=2}^{3}\left[\int_{-h_i}^{0}\int_{t+\theta}^{t}\dot{\xi}^T(s)R^{(i)}\dot{\xi}(s)\mathrm{d}s\,\mathrm{d}\theta \right. \\ &\qquad \left. + \int_{t-h_i}^{t}\xi^T(s)S^{(i)}\xi(s)\mathrm{d}s\right], \\ & R^{(i)} > 0, \quad S^{(i)} > 0, \ P_1 > 0. \end{aligned} \tag{17}$$

A necessary condition for the application of a simple LKF in h_2, h_3 is the asymptotic stability of the system with $h_2 = 0$, $h_3 = 0$:

$$\dot{\xi}(t) = (\bar{A}_0 + \bar{A}_2 + \bar{A}_3)\xi(t) + A_1\xi(t-h_1). \tag{18}$$

The existence of a complete LKF (with $S \equiv 0$) is a necessary and sufficient condition for the asymptotic stability of (18); see, e.g. Gu et al. (2003).

Differentiating (17) along (16) we have

$$\begin{aligned} \dot{V}(\xi_t) &= 2\dot{\xi}^T(t)\left[P_1\xi(t) + \int_{-h_1}^{0} Q(s)\xi(t+s)\mathrm{d}s\right] \\ &\quad + 2\xi^T(t)\int_{-h_1}^{0} Q(s)\dot{\xi}(t+s)\mathrm{d}s \\ &\quad + 2\int_{-h_1}^{0}\int_{-h_1}^{0}\dot{\xi}^T(t+s)R(s,\theta)\mathrm{d}s\xi(t+\theta)\mathrm{d}\theta \\ &\quad + 2\int_{-h_1}^{0}\dot{\xi}^T(t+s)S(s)\xi(t+s)\mathrm{d}s \\ &\quad + \sum_{i=2}^{3}\left[h_i\dot{\xi}^T(t)R^{(i)}\dot{\xi}(t) + \int_{-h_i}^{0}\dot{\xi}^T(t+\theta)R^{(i)}\dot{\xi}(t+\theta)\mathrm{d}\theta \right. \\ &\quad \left. + \xi^T(t)S^{(i)}\xi(t) - \xi^T(t-h_i)S^{(i)}\xi(t-h_i)\right]. \end{aligned} \tag{19}$$

Adding to $\dot{V}(x_t)$ the right part of the expression

$$0 = 2[\xi^T(t)P_2^T + \dot{\xi}^T(t)P_3^T]\left[\bar{A}_0\xi(t) - \dot{\xi}(t) + \sum_{i=1}^{3}A_i\xi(t-h_i)\right], \tag{20}$$

where P_2 and P_3 are $2n \times 2n$-matrices, which is equivalent to descriptor model transformation of Fridman (2001), we integrate by parts in (19). We find

$$\begin{aligned}\dot V(\xi_t) &= \zeta^T\Xi\zeta + 2\dot\xi^T(t)\int_{-h_1}^{0} Q(s)\xi(t+s)\mathrm{d}s\\ &+\int_{-h_1}^{0}\int_{-h_1}^{0}\xi^T(t+s)\\ &\times\left(\frac{\partial}{\partial s}R(s,\theta)+\frac{\partial}{\partial\theta}R(s,\theta)\right)\xi(t+\theta)\mathrm{d}\theta\,\mathrm{d}s\\ &+2\xi^T(t)\int_{-h_1}^{0}[-\dot Q(s)+R(0,s)]\xi(t+s)\mathrm{d}s\\ &-2\xi^T(t-h_1)\int_{-h_1}^{0}R(-h_1,\theta)\xi(t+\theta)\mathrm{d}\theta\\ &-\int_{-h_1}^{0}\xi^T(t+s)\dot S(s)\xi(t+s)\mathrm{d}s\\ &-\left[\sum_{i=2}^{3}\int_{-h_i}^{0}\dot\xi^T(t+\theta)R^{(i)}\dot\xi(t+\theta)\mathrm{d}\theta\right],\end{aligned}\tag{21}$$

where

$$\left.\begin{aligned}&\zeta=\mathrm{col}\{\xi(t),\ \dot\xi(t),\ \xi(t-h_1),\ \xi(t-h_2),\ \xi(t-h_3)\},\\ &\Xi=\begin{bmatrix}\Psi & P^T\begin{bmatrix}0\\ \bar A_1\end{bmatrix}-\begin{bmatrix}Q(-h_1)\\ 0\end{bmatrix} & P^T\begin{bmatrix}0\\ \bar A_2\end{bmatrix} & P^T\begin{bmatrix}0\\ \bar A_3\end{bmatrix}\\ * & -S(-h_1) & 0 & 0\\ * & * & -S^{(2)} & 0\\ * & * & * & -S^{(3)}\end{bmatrix},\\ &\Psi=P^T\begin{bmatrix}0 & I\\ \bar A_0 & -I\end{bmatrix}+\begin{bmatrix}0 & \bar A_0^T\\ I & -I\end{bmatrix}P\\ &\quad+\begin{bmatrix}Q(0)+Q^T(0)+S(0)+\sum_{i=2}^{3}S^{(i)} & 0\\ 0 & \sum_{i=2}^{3}h_iR^{(i)}\end{bmatrix},\\ &P=\begin{bmatrix}P_1 & 0\\ P_2 & P_3\end{bmatrix}.\end{aligned}\right\}\tag{22a–d}$$

Following Wu, He, She and Liu (2004), we add the left part of the equality

$$2\left[\xi^T(t)\ \dot\xi^T(t)\right]Y_i^T\left[\xi(t)+\int_{t-h_i}^{t}\dot\xi(s)\mathrm{d}s-\xi(t-h_i)\right]=0,\quad i=2,3\tag{23}$$

to $\dot V$. Applying further Jensen's inequality (see, e.g. Gu et al. (2003))

$$\int_{-h_i}^{0}\dot\xi^T(t+\theta)\,R^{(i)}\dot\xi(t+\theta)\mathrm{d}\theta\ge 1/h_i\int_{t-h_i}^{t}\dot\xi^T(s)\mathrm{d}s\,R^{(i)}\int_{t-h_i}^{t}\dot\xi(s)\mathrm{d}s,$$

we find

$$\begin{aligned}\dot V(\xi_t)&\le\bar\zeta^T\bar\Xi\bar\zeta+2\dot\xi^T(t)\int_{-h_1}^{0}Q(s)\xi(t+s)\mathrm{d}s\\ &-\int_{-h_1}^{0}\int_{-h_1}^{0}\xi^T(t+s)\Big(\frac{\partial}{\partial s}R(s,\theta)\\ &\qquad+\frac{\partial}{\partial\theta}R(s,\theta)\Big)\xi(t+\theta)\mathrm{d}\theta\,\mathrm{d}s\\ &+2\xi^T(t)\int_{-h_1}^{0}[-\dot Q(s)+R(0,s)]\xi(t+s)\mathrm{d}s\\ &-2\xi^T(t-h_1)\int_{-h_1}^{0}R(-h_1,\theta)\xi(t+\theta)\mathrm{d}\theta\\ &-\int_{-h_1}^{0}\xi^T(t+s)\dot S(s)\xi(t+s)\mathrm{d}s,\end{aligned}\tag{24}$$

where

$$\left.\begin{aligned}&\bar\Xi=\begin{bmatrix}\bar\Psi & P^T\begin{bmatrix}0\\ \bar A_1\end{bmatrix}-\begin{bmatrix}Q(-h_1)\\ 0\end{bmatrix} & P^T\begin{bmatrix}0\\ \bar A_2\end{bmatrix}-Y_2^T & P^T\begin{bmatrix}0\\ \bar A_3\end{bmatrix}-Y_3^T & -h_2Y_2^T & -h_3Y_3^T\\ * & -S(-h_1) & 0 & 0 & 0 & 0\\ * & * & -S^{(2)} & 0 & 0 & 0\\ * & * & * & -S^{(3)} & 0 & 0\\ * & * & * & * & -h_2R^{(2)} & 0\\ * & * & * & * & * & -h_3R^{(3)}\end{bmatrix},\\ &\bar\zeta=\begin{bmatrix}\zeta(t)\\ \frac{1}{h_2}\int_{t-h_2}^{t}\dot\xi(s)\mathrm{d}s\\ \frac{1}{h_3}\int_{t-h_3}^{t}\dot\xi(s)\mathrm{d}s\end{bmatrix},\\ &\bar\Psi=\Psi+\sum_{i=2}^{3}\left[\,Y_i^T\quad 0\,\right].\end{aligned}\right\}\tag{25a–c}$$

We apply next the discretisation method of Gu (1997). We divide the delay interval $[-h_1, 0]$ into N segments $[\theta_p,\ \theta_{p-1}]$, $p=1,\dots,N$ of equal length $h=h_1/N$, where $\theta_p=-ph$. This divides the square $[-h_1, 0]\times[-h_1, 0]$ into $N\times N$ small squares $[\theta_p,\ \theta_{p-1}]\times[\theta_q,\ \theta_{q-1}]$. Each small square is further divided into two triangles.

The continuous matrix functions $Q(s)$ and $S(s)$ are chosen to be linear within each segment and the continuous matrix function $R(s,\theta)$ is chosen to be linear within each triangular

$$\left.\begin{aligned}&Q(\theta_p+\alpha h)\\ &\quad=(1-\alpha)Q_p+\alpha Q_{p-1},\\ &S(\theta_p+\alpha h)\\ &\quad=(1-\alpha)S_p+\alpha S_{p-1},\quad \alpha\in[0,1],\\ &R(\theta_p+\alpha h,\theta_q+\beta h)\\ &\quad=\begin{cases}(1-\alpha)R_{pq}+\beta R_{p-1,q-1}+(\alpha-\beta)R_{p-1,q}, & \alpha\ge\beta,\\ (1-\beta)R_{pq}+\alpha R_{p-1,q-1}+(\beta-\alpha)R_{p,q-1}, & \alpha<\beta.\end{cases}\end{aligned}\right\}\tag{26}$$

Thus, the LKF is completely determined by P_1, Q_p, S_p, R_{pq}, $p,q=0,1,\dots,N$.

The LKF condition $V(\xi_t) \geq \varepsilon\|\xi(t)\|^2,\ \varepsilon > 0$ is satisfied (Gu et al. (2003, p. 185)) if $S_p > 0$, $p=0,1,\ldots,N$ and

$$\begin{bmatrix} P_1 & \tilde{Q} \\ * & \tilde{R}+\tilde{S} \end{bmatrix} > 0, \tag{27}$$

where

$$\left.\begin{aligned} \tilde{Q} &= [Q_0 \quad Q_1 \quad \cdots \quad Q_N], \\ \tilde{S} &= \mathrm{diag}\{1/hS_0, 1/hS_1, \ldots, 1/hS_N\}, \\ \tilde{R} &= \begin{bmatrix} R_{00} & R_{01} & \cdots & R_{0N} \\ R_{10} & R_{11} & \cdots & R_{1N} \\ \cdots & \cdots & \cdots & \cdots \\ R_{N0} & R_{N1} & \cdots & R_{NN} \end{bmatrix}. \end{aligned}\right\} \tag{28}$$

To derive the LKF derivative condition we note that

$$\begin{aligned} \dot{S}(\xi) &= 1/h(S_{p-1}-S_p), \\ \dot{Q}(\xi) &= 1/h(Q_{p-1}-Q_p), \\ \frac{\partial}{\partial\xi}R(\xi,\theta)+\frac{\partial}{\partial\theta}R(\xi,\theta) &= 1/h(R_{p-1,q-1}-R_{pq}). \end{aligned} \tag{29}$$

Equations (21), (24), (29) imply (cf. Gu et al. (2003) (5.146)–(5.164))

$$\begin{aligned} \dot{V}(x_t) \leq{}& \bar{\zeta}^T\Xi_d\bar{\zeta} - \int_0^1 \phi^T(\alpha)S_d\phi(\alpha)\mathrm{d}\alpha \\ &- \int_0^1\left[\int_0^1 \phi^T(\alpha)R_d\phi(\beta)\mathrm{d}\alpha\right]\mathrm{d}\beta \\ &+ 2\zeta^T\int_0^1 [D^s+(1-2\alpha)D^a]\phi(\alpha)\mathrm{d}\alpha, \end{aligned} \tag{30}$$

where $\bar{\zeta}$ is given by (25b) and

$$\left.\begin{aligned} &\phi^T(\alpha) = [\xi^T(t-h+\alpha h)\ \xi^T(t-2h+\alpha h)\cdots\xi^T(t-Nh+\alpha h)], \\ &\Xi_d = \begin{bmatrix} \bar{\Psi} & P^T\begin{bmatrix}0\\ \bar{A}_1\end{bmatrix} - \begin{bmatrix}Q_N\\0\end{bmatrix} & P^T\begin{bmatrix}0\\ \bar{A}_2\end{bmatrix} - Y_2^T & P^T\begin{bmatrix}0\\ \bar{A}_3\end{bmatrix} - Y_3^T & -h_2Y_2^T & -h_3Y_3^T \\ * & -S_N & 0 & 0 & 0 & 0 \\ * & * & -S^{(2)} & 0 & 0 & 0 \\ * & * & * & -S^{(3)} & 0 & 0 \\ * & * & * & * & -h_2R^{(2)} & 0 \\ * & * & * & * & * & -h_3R^{(3)} \end{bmatrix}, \\ &\Psi_d = P^T\begin{bmatrix}0 & I\\ \bar{A}_0 & -I\end{bmatrix} + \begin{bmatrix}0 & \bar{A}_0^T\\ I & -I\end{bmatrix}P + \begin{bmatrix} Q_0+Q_0^T+S_0+\sum_{i=2}^3 S^{(i)} & 0 \\ 0 & \sum_{i=2}^3 h_iR^{(i)} \end{bmatrix} + \sum_{i=2}^3[\,Y_i^T \quad 0\,], \\ &S_d = \mathrm{diag}\{S_0-S_1, S_1-S_2,\ldots,S_{N-1}-S_N\}, \quad R_d = \begin{bmatrix} R_{d11} & R_{d12} & \cdots & R_{d1N} \\ R_{d21} & R_{d22} & \cdots & R_{d2N} \\ \cdots & \cdots & \cdots & \cdots \\ R_{dN1} & R_{dN2} & \cdots & R_{dNN} \end{bmatrix}, \quad R_{dpq} = h(R_{p-1,q-1}-R_{pq}), \\ &D^s = [D_1^s\ D_2^s\ \cdots\ D_N^s], \quad D^a = [D_1^a\ D_2^a\ \cdots\ D_N^a], \quad D_p^s = \begin{bmatrix} h/2(R_{0,p-1}+R_{0p})-(Q_{p-1}-Q_p) \\ h/2(Q_{p-1}+Q_p) \\ -h/2(R_{N,p-1}+R_{Np}) \end{bmatrix}, \quad D_p^a = \begin{bmatrix} -h/2(R_{0,p-1}-R_{0p}) \\ -h/2(Q_{p-1}-Q_p) \\ h/2(R_{N,p-1}-R_{Np}) \end{bmatrix} \end{aligned}\right\} \tag{31a–i}$$

Applying Gu et al. (2003, Proposition 5.21) to (30) we conclude that $\dot{V}(\xi_t) < 0$ if the following LMI holds

$$\begin{bmatrix} \Xi_d & \begin{bmatrix}D^s\\0\end{bmatrix} & \begin{bmatrix}D^a\\0\end{bmatrix} \\ * & -R_d-S_d & 0 \\ * & * & -3S_d \end{bmatrix} < 0. \tag{32}$$

Moreover, (32) implies that $S_0 > S_1 > \ldots S_N > 0$; see Gu et al. (2003, Proposition 5.22). Hence, (32) guarantees $V(\xi_t) \geq \varepsilon\|\xi(t)\|^2,\ \varepsilon > 0$. We thus proved the following.

Lemma 1: *The system* (16) *is asymptotically stable if there exist* $2n\times 2n$ *matrices* $P_1 > 0$, P_2, P_3, $S_p = S_p^T$, Q_p, $R_{pq} = R_{qp}^T$, $p=0,1,\ldots,N$, $q=0,1,\ldots,N$, $R^{(i)}$, $S^{(i)}$, $i=2,\ 3$ *and* $2n\times 4n$*-matrices* Y_i *such that LMIs* (27), (32) *are satisfied with notations defined in* (28) *and* (31).

3.2 *Stability of systems with uncertain time-varying delays*

Consider the input–output model (14), (15). Assuming that $y_\xi(t)=0$ for $\forall t\leq 0$, the following holds for $\dot{\tau}_i \leq 1$ and for some $2n\times 2n$ matrices $R_{ia} > 0$ (see Fridman and Shaked (2006)):

$$\int_0^\infty v_i^T(t)\mu_iR_{ia}v_i(t)\mathrm{d}t \leq \int_0^\infty y_\xi^T(t)\mu_iR_{ia}y_\xi(t)\mathrm{d}t. \tag{33}$$

Then the following condition along (14)

$$\mathcal{W}(t) \triangleq \Delta\dot{V}(\xi_t) + \sum_{i=2}^{3}\mu_i y_\xi^T(t)R_{ia}y_\xi(t) - \sum_{i=2}^{3}\mu_i v_i^T(t)R_{ia}v_i(t) < -\varepsilon(\|\xi(t)\|^2 + \|\dot{\xi}(t)\|^2 + \|v(t)\|^2), \quad \varepsilon > 0 \tag{34}$$

guarantees the asymptotic stability of (12) Gu et al. (2003).

Differentiating $V(\xi_t)$ along the trajectories of (14) we obtain that $\dot{V}$ is given by (19), where

$$2\dot{\xi}^T(t)P_1\xi(t) = 2\begin{bmatrix}\xi(t)\\ \dot{\xi}(t)\end{bmatrix}^T P^T\left[\begin{pmatrix}0 & I\\ \bar{A}_0 & -I\end{pmatrix}\begin{pmatrix}\xi(t)\\ \dot{\xi}(t)\end{pmatrix} + \sum_{i=1}^{3}\begin{pmatrix}0\\ \bar{A}_i\end{pmatrix}\xi(t-h_i) + \sum_{i=2}^{3}\mu_i\begin{pmatrix}0\\ \bar{A}_i\end{pmatrix}v_i(t)\right]. \tag{35}$$

Therefore, choosing Q, S and R to be piecewise-linear of the form (26), we find, similarly to the previous section, that

$$\mathcal{W}(t) = \zeta_v^T\Xi_v\zeta_v - \int_0^1 \phi^T(\alpha)S_d\phi(\alpha)\mathrm{d}\alpha - \int_0^1\left[\int_0^1 \phi^T(\alpha)R_d\phi(\beta)\mathrm{d}\alpha\right]\mathrm{d}\beta + 2\zeta^T\int_0^1[D^s + (1-2\alpha)D^a]\phi(\alpha)h\,\mathrm{d}\alpha, \tag{36}$$

with the notations defined in (31) and

$$\zeta_v^T = [\bar{\zeta}^T \; v_2^T(t) \; v_3^T(t)],$$

$$\Xi_v = \begin{bmatrix} \hat{\Xi} & \mu_2\begin{bmatrix}P^T\\0\\0\end{bmatrix}\begin{bmatrix}0\\ \bar{A}_2\end{bmatrix} & \mu_3\begin{bmatrix}P^T\\0\\0\end{bmatrix}\begin{bmatrix}0\\ \bar{A}_3\end{bmatrix}\\ * & -\mu_2R_{2a} & 0\\ * & * & -\mu_3R_{3a}\end{bmatrix},$$

where

$$\hat{\Xi} = \Xi_d + \mathrm{diag}\left\{0_{2n\times 2n}, \sum_{i=2}^{3}\mu_iR_{ia}, 0\right\}. \tag{37}$$

and where $\bar{\zeta}$ is given by (25a).

Applying Gu et al. (2003, Proposition 5.21) to (36), and Schur complements to the last three terms of Ξ_v, we conclude that $\mathcal{W}(t) < 0$ if the following LMI holds:

$$\Xi_r = \begin{bmatrix} \hat{\Xi} & \bar{D}^s & \bar{D}^a & \mu_2\Phi_2 & \mu_3\Phi_3\\ * & -R_d - S_d & 0 & 0 & 0\\ * & * & -3S_d & 0 & 0\\ * & * & * & -\mu_2R_{2a} & 0\\ * & * & * & * & -\mu_3R_{3a}\end{bmatrix} < 0, \tag{38}$$

where

$$\Phi_i = \begin{bmatrix}P^T\\0\\0\end{bmatrix}\begin{bmatrix}0\\ \bar{A}_i\end{bmatrix}, \; i = 2, 3, \; \bar{D}^s = \begin{bmatrix}D^s\\ \underline{0}\end{bmatrix}, \quad \bar{D}^a = \begin{bmatrix}D^a\\ \underline{0}\end{bmatrix}.$$

We thus obtained the following

Lemma 2: *The system* (12) *is asymptotically stable for all the delays satisfying* (6), *if there exist* $2n\times 2n$ *matrices* $0 < P_1, \; P_2, \; P_3, \; R_{ia}, \; i = 2, 3, \; S_p = S_p^T, \; Q_p, \; R_{pq} = R_{qp}^T, \; p = 0, 1, \ldots, N, \; q = 0, 1, \ldots, N, \; R^{(i)}, \; S^{(i)}, \; i = 2, 3$ *and* $2n \times 4n$*-matrices* Y_i *such that the LMIs* (27), (38) *are satisfied with notations defined in* (28) *and* (31b–i).

3.3 L_2-gain analysis

Define the following function

$$\mathrm{d}J(t) = \mathcal{W}(t) + \bar{z}^T(t)\bar{z}(t) - \gamma^2w^T(t)w(t) - \gamma^2n^T(t-\tau_3(t))n(t-\tau_3(t)). \tag{39}$$

We find, similarly to the previous sections, that $\mathrm{d}J(t) < 0$ if the following holds.

$$\begin{bmatrix} \hat{\Xi} & \bar{D}^s & \bar{D}^a & \mu_2\Phi_2 & \mu_3\Phi_3 & \Phi_4 & \Phi_5 & \Phi_6\\ * & -R_d - S_d & 0 & 0 & 0 & 0 & 0 & 0\\ * & * & -3S_d & 0 & 0 & 0 & 0 & 0\\ * & * & * & -\mu_2R_{2a} & 0 & 0 & 0 & \mu_2\bar{L}_2^T\\ * & * & * & * & -\mu_3R_{3a} & 0 & 0 & 0\\ * & * & * & * & 0 & -\gamma^2I_r & 0 & 0\\ * & * & * & * & 0 & 0 & -\gamma^2I_l & 0\\ * & * & * & * & 0 & 0 & 0 & -I_p\end{bmatrix} < 0, \tag{40}$$

where

$$\Phi_i = \begin{bmatrix} P^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \bar{A}_i \end{bmatrix}, \quad i = 2, 3$$

$$\Phi_4 = \begin{bmatrix} P^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \bar{B}_0 \end{bmatrix}, \quad \Phi_5 = \begin{bmatrix} P^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \bar{B}_3 \end{bmatrix},$$

$$\Phi_6 = \begin{bmatrix} \bar{L}_0^T \\ 0 \\ \bar{L}_2^T \\ 0 \end{bmatrix}.$$

It follows from (39) and $\mathrm{d}J(t) < 0$ that one has:

$$\int_0^\infty \mathcal{W}(t)\mathrm{d}t < \int_0^\infty [-\bar{z}^T(t)\bar{z}(t) + \gamma^2 w(t)^T w(t)]\mathrm{d}t + \gamma^2 \sum_{k=0}^{\infty} (\sigma_{k+1} - \sigma_k) n^T(\sigma_k) n(\sigma_k).$$

This implies that

$$-J_\infty = \int_0^\infty [-\bar{z}^T(t)\bar{z}(t) + \gamma^2 w(t)^T w(t)]\mathrm{d}t + \gamma^2 \sum_{k=0}^{\infty} (\sigma_{k+1} - \sigma_k) n^T(\sigma_k) n(\sigma_k)) > 0.$$

Lemma 3: *The system* (12a–c) *is asymptotically stable and* $J_\infty < 0$ *for a prescribed positive scalar* γ, *if there exist* $2n \times 2n$ *matrices* $0 < P_1, P_2, P_3$, $S_p = S_p^T, Q_p, R_{ia}$, $i = 2, 3$, $R_{pq} = R_{qp}^T$, $p = 0, 1, \ldots, N$, $q = 0, 1, \ldots, N$, $R^{(i)}$, $S^{(i)}$, $i = 2, 3$ *and* $2n \times 4n$*-matrices* Y_i *such that the LMIs* (40), (27) *are satisfied with notations defined in* (28) *and* (31b–i).

4. Output-feedback design

In order to obtain a solution to the output-feedback control problem for the system (2) we consider (40), (27), denote the partitions:

$$P_2^T = \begin{bmatrix} X & M \\ \bar{M} & U \end{bmatrix} \quad \text{and} \quad P_2^{-T} = \begin{bmatrix} Y & K \\ \bar{K} & \bar{V} \end{bmatrix}, \tag{41}$$

where X, M, Y and K are $n \times n$-matrices, and define

$$J = \begin{bmatrix} I & Y^T \\ 0 & K^T \end{bmatrix}, \quad \bar{J} = \mathrm{diag}\{J, J, \ldots, J, J, J\}.$$

We also take $P_3 = \delta P_2$ where δ is a scalar tuning parameter. Defining:

$$[\bar{P}_1 \ \bar{Q}_p \ \bar{S}_p \ \bar{R}_{pq} \ \bar{S}^{(i)} \ \bar{R}^{(i)} \ \bar{R}_{ia}] = J^T[P_1 J \ Q_p J \ S_p J \ R_{pq} J \ S^{(i)} J \ R^{(i)} J \ R_{ia} J],$$

$$\bar{Y}_i = J^T Y_i \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}, \quad i = 2, 3$$

we multiply (40) by $\mathrm{diag}\{J^T, \ldots, J^T, I_r, I_l, I_p$, from the left, and by $\mathrm{diag}\{J, \ldots, J, I_r, I_l, I_p\}$, from the right. We also multiply (27) by $\mathrm{diag}\{J^T, \ldots, J^T\}$ and $\mathrm{diag}\{J, \ldots, J\}$, from the left and the right, respectively. We obtain the following inequalities:

$$\begin{bmatrix} \bar{P}_1 & \tilde{Q} \\ * & \tilde{R} + \tilde{S} \end{bmatrix} > 0, \tag{42}$$

$$\begin{bmatrix} \hat{\Xi} & \bar{D}^s & \bar{D}^a & \mu_2\bar{\Phi}_2 & \mu_3\bar{\Phi}_3 & \bar{\Phi}_4 & \bar{\Phi}_5 & \bar{\Phi}_6 \\ * & -R_d - S_d & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -3S_d & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\mu_2\bar{R}_{2a} & 0 & 0 & 0 & \bar{\Phi}_7 \\ * & * & * & * & -\mu_3\bar{R}_{3a} & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{43}$$

where

$$
\left.
\begin{aligned}
\hat{\Xi} &= \begin{bmatrix}
\hat{\Psi} & \begin{bmatrix} I_{2n} \\ \delta I_{2n} \end{bmatrix}\begin{bmatrix} XA_1 & \hat{A}_1 \\ A_1 & A_1 Y^T \end{bmatrix} - \begin{bmatrix} \bar{Q}_N \\ 0 \end{bmatrix} & \begin{bmatrix} I_{2n} \\ \delta I_{2n} \end{bmatrix}\begin{bmatrix} 0 & \hat{A}_2 \\ 0 & B_2\hat{C}_c \end{bmatrix} - \bar{Y}_2^T & \begin{bmatrix} I_{2n} \\ \delta I_{2n} \end{bmatrix}\begin{bmatrix} \hat{B}_c C_2 & \hat{A}_3 \\ 0 & 0 \end{bmatrix} - \bar{Y}_3^T & -h_2\bar{Y}_2^T & -h_3\bar{Y}_3^T \\
* & -\bar{S}_N & 0 & 0 & 0 & 0 \\
* & * & -\bar{S}^{(2)} & 0 & 0 & 0 \\
* & * & * & -\bar{S}^{(3)} & 0 & 0 \\
* & * & * & * & -h_2\bar{R}^{(2)} & 0 \\
* & * & * & * & * & -h_3\bar{R}^{(3)}
\end{bmatrix} \\
\hat{\Psi} &= \left[\begin{bmatrix} I_{2n} \\ \delta I_{2n} \end{bmatrix}\begin{bmatrix} XA_0 & \hat{A}_0 \\ A_0 & A_0 Y^T \end{bmatrix}\begin{bmatrix} I_{2n} \\ 0 \end{bmatrix}\bar{P}_1 - \begin{bmatrix} I_{2n} \\ \delta I_{2n} \end{bmatrix}\begin{bmatrix} X & T \\ I & Y^T \end{bmatrix}\right] + \left[\begin{bmatrix} I_{2n} \\ \delta I_{2n} \end{bmatrix}\begin{bmatrix} XA_0 & \hat{A}_0 \\ A_0 & A_0 Y^T \end{bmatrix}\begin{bmatrix} I_{2n} \\ 0 \end{bmatrix}\bar{P}_1 - \begin{bmatrix} I_{2n} \\ \delta I_{2n} \end{bmatrix}\begin{bmatrix} X & T \\ I & Y^T \end{bmatrix}\right]^T \\
&\quad + \begin{bmatrix} \bar{Q}_0 + \bar{Q}_0^T + \bar{S}_0 + \sum_{i=2}^{3}\bar{S}^{(i)} & 0 \\ 0 & \sum_{i=2}^{3} h_i\bar{R}^{(i)} + \mu_i\bar{R}_{ia} \end{bmatrix} + \sum_{i=2}^{3}\begin{bmatrix} \bar{Y}_i^T & 0 \end{bmatrix}, \\
\bar{\Phi}_2 &= \begin{bmatrix} \begin{bmatrix} I_{2n} \\ \delta I_{2n} \end{bmatrix}\begin{bmatrix} 0 & \hat{A}_2 \\ 0 & B_2\hat{C}_c \end{bmatrix} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad
\bar{\Phi}_3 = \begin{bmatrix} \begin{bmatrix} I_{2n} \\ \delta I_{2n} \end{bmatrix}\begin{bmatrix} \hat{B}_c C_2 & \hat{A}_3 \\ 0 & 0 \end{bmatrix} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad
\bar{\Phi}_4 = \begin{bmatrix} \begin{bmatrix} I_{2n} \\ \delta I_{2n} \end{bmatrix}\begin{bmatrix} XB_1 \\ B_1 \end{bmatrix} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad
\bar{\Phi}_5 = \begin{bmatrix} \begin{bmatrix} I_{2n} \\ \delta I_{2n} \end{bmatrix}\begin{bmatrix} \hat{B}_c D_{21} \\ 0 \end{bmatrix} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
\bar{\Phi}_6 &= \begin{bmatrix} \begin{bmatrix} I_{2n} \\ 0_{2n} \end{bmatrix}\begin{bmatrix} C_1^T \\ YC_1^T + \hat{C}_c^T D_{12}^T \end{bmatrix} \\ 0 \\ \begin{bmatrix} 0 \\ \hat{C}_c^T D_{12}^T \end{bmatrix} \\ 0 \end{bmatrix}, \quad
\bar{\Phi}_7 = \mu_2\begin{bmatrix} 0 \\ \hat{C}_c^T D_{12}^T \end{bmatrix},
\end{aligned}
\right\} \quad (44)
$$

and where $\tilde{R}$, $\tilde{Q}$, $\tilde{S}$, D^s, D^a, R_d, S_d are given by (28) and (31) correspondingly with bars over R_{pq}, Q_p, S_p, $p=1,\ldots,N$, $q=1,\ldots,N$. In (43) we define the new variables:

$$
\left.
\begin{aligned}
\hat{A}_0 &= XA_0 Y^T + MA_{c0}K^T \\
\hat{A}_1 &= XA_1 Y^T + MA_{c1}K^T \\
\hat{A}_2 &= XB_2 C_c K^T + MA_{c2}K^T \\
\hat{A}_3 &= MB_c C_2 Y^T + MA_{c3}K^T \\
\hat{B}_c &= MB_c \\
\hat{C}_c &= C_c K^T \\
T &= XY^T + MK^T
\end{aligned}
\right\} \quad (45)
$$

and obtain the following theorem.

Theorem 1: *For a prescribed* $\gamma > 0$, *there exists a controller of the form* (7) *that achieves* $J_\infty > 0$ *for all non-zero* w, $n \in L_2[0,\infty)$ *and for* $\tau_i(t)$, $i=1,2,3$ *satisfying* (6), *if there exist* $2n \times 2n$*-matrices* $\bar{P}_1 > 0, \bar{S}^{(i)}, \bar{R}^{(i)}, \bar{R}_{ia},\ i=2,3,\ \bar{S}_p = \bar{S}_p^T,\ \bar{Q}_p,\ \bar{R}_{pq} = \bar{R}_{qp}^T$, $p=0,1,\ldots,N$, $q=0,1,\ldots,N$, $2n \times 4n$*-matrices* $\bar{Y}_i$, $n \times n$*-matrices* $\hat{A}_i$, $i=0,\ldots,3$, X, Y, T, $n \times q$*-matrix* $\hat{B}_c$ *and* $m \times n$*-matrix* $\hat{C}_c$ *that, for a tuning scalar* δ, *satisfy LMIs* (42), (43), *where* $\tilde{R}$, $\tilde{Q}$, $\tilde{S}$, D^s, D^a, R_d, S_d *are given by* (28) *and* (31), *correspondingly*, *with bars over* R_{pq}, Q_p, S_p, $p=1,\ldots,N$, $q=1,\ldots,N$.

If the corresponding LMIs have a solution, the controller matrices can be readily obtained from (45) using the following equations:

$$
\left.
\begin{aligned}
A_{c0} &= M^{-1}(\hat{A}_0 - XA_0 Y^T)K^{-T} \\
A_{c1} &= M^{-1}(\hat{A}_1 - XA_1 Y^T)K^{-T} \\
A_{c2} &= M^{-1}(\hat{A}_2 - XB_2 C_c K^T)K^{-T} \\
A_{c3} &= M^{-1}(\hat{A}_3 - MB_c C_2 Y^T)K^{-T} \\
B_c &= M^{-1}\hat{B}_c \\
C_c &= \hat{C}_c K^{-T}
\end{aligned}
\right\}
$$

Remark 2: In the above, it was inherently assumed that MK^T is nonsingular. This nonsingularity follows from the fact that $MK^T = T - XY^T$. If there existed a non-zero v such that $(T - XY^T)v = 0$ then defining $f = \mathrm{col}\{-Y^T v, v\}$ and

$$\bar{W} = \begin{bmatrix} X & T \\ I & Y^T \end{bmatrix}$$

it would be readily found that $\bar{W}f = 0$. This would contradict the fact that $\bar{W} + \bar{W}^T$ appears in the second $(2n \times 2n)$ diagonal block in $\hat{\Psi}$ of (44) and thus from (43) it should be nonsingular.

Remark 3: The solution to the filtering problem that is defined in §2 readily follows from the above results by taking in (44) $D_{12} = -I$ and $\hat{A}_2$, B_2 equal to zero, and by deleting rows and columns that correspond to the delay τ_2.

5. State-feedback design

Consider the system (2a,c) with the control law of (11). The closed-loop system has the form

$$\left.\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau_1(t)) \\ &\quad + B_2 K_{sf} x(t-\tau_s(t)) + B_1 \omega(t), \\ z(t) &= C_1(t)x(t) + D_{12} K_{sf} x(t-\tau_s(t)). \end{aligned}\right\} \quad (46)$$

We apply Lemma 3 to (46), where n is replaced by $n/2$, and $S^{(3)} = R^{(3)} = R_{3a} = 0$, $Y_3 = 0$. We take $P_3 = \delta P_2$ in the sequel, where δ is a tuning scalar. Defining

$$\left.\begin{aligned} &\bar{J} = P_2^{-1}, \\ &[\bar{P}_1, \bar{Q}_p\ \bar{S}_p\ \bar{R}_{pq}\ \bar{S}^{(2)}\ \bar{R}^{(2)}\ \bar{R}_{2a}] \\ &\quad = \bar{J}^T[P_1\bar{J}\ Q_p\bar{J}\ S_p\bar{J}\ R_{pq}\bar{J}\ S^{(2)}\bar{J}\ R^{(2)}\bar{J}\ R_{2a}\bar{J}], \\ &\bar{Y}_2 = \bar{J}^T Y_2 \begin{bmatrix} \bar{J} & 0 \\ 0 & \bar{J} \end{bmatrix}, \end{aligned}\right\} \quad (47)$$

and

$$\bar{K}_{sf} = K_{sf}\bar{J}$$

we multiply (40) by $\mathrm{diag}\{\bar{J}^T, \ldots, \bar{J}^T, I_r, I_p\}$, from the left, and by $\mathrm{diag}\{\bar{J}, \ldots, \bar{J}, I_r, I_p\}$, from the right. We multiply (27) by $\mathrm{diag}\{\bar{J}^T, \ldots, \bar{J}^T\}$ and $\mathrm{diag}\{\bar{J}, \ldots, \bar{J}\}$, from the left and the right, respectively. We obtain the following inequality:

$$\begin{bmatrix} \hat{\Xi} & \bar{D}^s & \bar{D}^a & \mu_2\Phi_3 & \Phi_4 & \Phi_6 \\ * & -R_d - S_d & 0 & 0 & 0 & 0 \\ * & * & -3S_d & 0 & 0 & 0 \\ * & * & * & -\mu_2 R_{2a} & 0 & \Phi_7 \\ * & * & * & 0 & -\gamma^2 I_r & 0 \\ * & * & 0 & 0 & 0 & -I_p \end{bmatrix} < 0, \quad (48)$$

where

$$\Phi_3 = \begin{bmatrix} B_2\bar{K}_{sf} \\ \delta B_2\bar{K}_{sf} \\ 0_{n\times n} \\ 0_{n\times n} \\ 0_{n\times n} \end{bmatrix}, \quad \Phi_4 = \begin{bmatrix} B\bar{J} \\ \delta B\bar{J} \\ 0_{n\times r} \\ 0_{n\times r} \\ 0_{n\times r} \end{bmatrix}, \quad \Phi_6 = \begin{bmatrix} 0_{n\times p} \\ \bar{J}^T\bar{L}^T \\ 0_{n\times p} \\ \bar{K}_{sf}^T D_{12}^T \\ 0_{n\times p} \end{bmatrix}.$$

$$\Phi_7 = \mu_2 \bar{K}_{sf}^T D_{12}^T, \quad \hat{\Xi} = \Xi_d + \mathrm{diag}\{0_{n\times n}, \mu_2 R_{2a}, 0_{3n\times 3n}\}$$

$$\Xi_d = \begin{bmatrix} \bar{\Psi} & \begin{bmatrix} A_1\bar{J} \\ \delta A_1\bar{J} \end{bmatrix} - \begin{bmatrix} \bar{Q}_N \\ 0 \end{bmatrix} & \begin{bmatrix} B_2\bar{K}_{sf} \\ \delta B_2\bar{K}_{sf} \end{bmatrix} - \bar{Y}_2^T & -h_s\bar{Y}_2^T \\ * & -\bar{S}_N & 0 & 0 \\ * & * & -\bar{S}^{(2)} & 0 \\ * & * & * & -h_s\bar{R}^{(2)} \end{bmatrix},$$

$$\begin{aligned} \Psi_d &= \begin{bmatrix} A_0\bar{J} & \bar{P}_1 - \bar{J} \\ \delta A_0\bar{J} & -\delta\bar{J} \end{bmatrix} + \begin{bmatrix} A_0\bar{J} & \bar{P}_1 - \bar{J} \\ \delta A_0\bar{J} & +\delta\bar{J} \end{bmatrix}^T \\ &\quad + \begin{bmatrix} \bar{Q}_0 + \bar{Q}_0^T + \bar{S}_0 + \bar{S}^{(2)} & 0 \\ 0 & h_s\bar{R}^{(2)} \end{bmatrix} \\ &\quad + \begin{bmatrix} \bar{Y}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \bar{Y}_2^T & 0 \end{bmatrix} \end{aligned} \quad (49)$$

and where $\tilde{R}$, $\tilde{Q}$, $\tilde{S}$, D^s, D^a, R_d, S_d are given by (28) and (31), correspondingly, with bars over R_{pq}, Q_p, S_p, $p = 1, \ldots, N$, $q = 1, \ldots, N$. If a solution to the LMIs (42), (48) is obtained, the state-feedback gain is given by

$$K_{sf} = \bar{K}_{sf}\bar{J}^{-1}.$$

6. Examples

Example 1: Consider the following linear system taken from the H_∞ output-feedback control example in Azuma, Ikeda and Uchida (1999):

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} x(t-1) \\ &\quad + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\omega(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\ z(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) + D_{12}u(t), \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + D_{21}n(t), \end{aligned}$$

where $D_{12} = D_{21} = 0$. In Azuma et al. (1999), an H_∞ output-feedback controller has been found for this system without considering sampling and network delays. A minimum value of $\gamma = 0.9$ is reported there.

In our solution we take $D_{12} = D_{21} = 10^{-4}$ and we assume that there are delays in the control and the measurement channels characterised by $h_2 = h_3 = 0.04$ and sampling with $\mu_2 = \mu_3 = 0.01$; see (6). We apply Theorem 1, for $N = 3$, and obtain a near minimum value of $\gamma = 0.79$, using the controller defined by the following matrices:

$$A_{c0} = \begin{bmatrix} 23.3255 & -23.6404 \\ 26.0709 & -26.3612 \end{bmatrix},$$

$$A_{c1} = \begin{bmatrix} -0.9002 & 2.8905 \\ -0.7965 & 2.5589 \end{bmatrix},$$

$$A_{c2} = \begin{bmatrix} -27.6906 & 12.0732 \\ -24.3654 & 10.6235 \end{bmatrix},$$

$$A_{c3} = \begin{bmatrix} 99.3337 & -112.0346 \\ 99.4568 & -112.1735 \end{bmatrix},$$

$$B_c = \begin{bmatrix} 4.8804 \\ 4.8865 \end{bmatrix},$$

$$C_c = \begin{bmatrix} -201.6769 & 87.9281 \end{bmatrix}.$$

Example 2: Consider the the following linear system taken from the output-feedback control example in Mahmoud and Zribi (1999):

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} -0.2 & 0 \\ -1 & -1 \end{bmatrix} x(t-0.1) + \begin{bmatrix} 0.45 \\ 0.25 \end{bmatrix} \omega(t) + \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix} u(t-0.2),$$

$$z(t) = \begin{bmatrix} 0.45 & 0.65 \end{bmatrix} x(t) + D_{12} u(t),$$

$$y(t) = \begin{bmatrix} 0.1 & 0 \end{bmatrix} x(t) + D_{21} n(t),$$

where $D_{12} = D_{21} = 0$. In Mahmoud and Zribi (1999), an H_∞ output-feedback controller has been found for this system without considering sampling delays. A minimum value of $\gamma = 3.177$ is reported there.

In our solution we assume that the control and the measurement channels are sampled with $\mu_2 = \mu_3 = 0.01$ see (6). We apply Theorem 1, for $N = 3$, and obtain a near minimum value of $\gamma = 0.0949$.

7. Conclusions

The output-feedback H_∞ control problem is solved for the case where the system contains a constant known delay in its dynamics and where the measurements and the control encounter interval time-varying delays with delay derivative not greater than 1. The obtained solution is applied to the case where the control input and the measurement are non-uniformly sampled and to the special cases of H_∞ state-feedback control and of H_∞ filtering. For the first time a solution of the output-feedback H_∞ control problem is derived via discretised Lyapunov functional method.

The results obtained are applicable to remote control schemes where the controller (master) remotely drives the plant (slave) via communication lines that encounter delays that are measured on line.

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References

Azuma, T., Ikeda, K., and Uchida, K. (1999), Infinitedimensional LMI Approach to H_∞ Control Synthesis for Linear Systems with Time Delay, in *5th European Control Conference*, Karlsruhe, 1814–1818.

Bamieh, B., Pearson, J., Francis, B., and Tannenbaum, A. (1991), 'A Lifting Technique for Linear Periodic Systems', *Systems & Control Letters*, 17, 79–88.

Fridman, E. (2001), 'New Lyapunov-Krasovskii Functionals for Stability of Linear Retarded and Neutral Type Systems', *Systems & Control Letters*, 43, 309–319.

Fridman, E. (2006), 'Descriptor Discretized Lyapunov Functional Method: Analysis and Design', *IEEE Transactions on Automation and Control*, 51(5), 890–897.

Fridman, E., Seuret, A., and Richard, J.-P. (2004), 'Robust Sampled-data Stabilisation of Linear Systems: An Input Delay Approach', *Automatica*, 40, 1441–1446.

Fridman, E., and Shaked, U. (2000), 'Sampled-data H_∞ State-feedback Control of Systems with State Delays', *International Journal of Control*, 73(12), 1115–1128.

Fridman, E., and Shaked, U. (2006), 'Input-oUtput Approach to Stability and L_2-gain Analysis of Systems with Time-varying Delays', *Systems & Control Letters*, 55(12), 1041–1053.

Gu, K. (1997), 'Discretized LMI Set in the Stability Problem of Linear Time Delay Systems', *Interantional Journal of Control*, 68, 923–934.

Gu, K., Kharitonov, V., and Chen, J. (2003), *Stability of Time-delay Systems*, Boston: Birkhauser.

Mahmoud, M., and Zribi, M. (1999), 'H_∞ - Controllers for Time-delay Systems Using Linear Matrix Inequalities', *Journal of Optimization Theory and Applications*, 100(1), 89–122.

Sivashankar, N., and Khargonekar, P. (1994), 'Characterization of the L_2-induced Norm for Linear Systems with Jumps with Applications to Sampled-data Systems', *SIAM Journal of Control and Optimization*, 32, 1128–1150.

Suplin, V., Fridman, E., and Shaked, U. (2007), 'Sampled-data H_∞ Control and Filtering: Nonuniform Uncertain Sampling', *Automatica*, 43, 1072–1083.

Wu, M., He, Y., She, J.-H., and Liu, G.-P. (2004), 'Delay-dependent Criteria for Robust Stability of Time-varying Delay Systems', *Automatica*, 40, 1435–1439.

Yamamoto, Y. (1990), New Approach to Sampled-data Control Systems A Function Space Method , in *Proceedings 29th Conference on Decision and Control*, HW: Honolulu, 1882–1887.