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Exact controllability of a class of nonlinear distributed parameter systems using back-and-forth iterations

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\begin{abstract}
We investigate the exact controllability of a nonlinear plant described by the equation \( \dot{x}(t) = Ax(t) + Bu(t) + B_N N(x(t), t) \), where \( t \geq 0 \). Here \( A \) is the infinitesimal generator of a strongly continuous group \( T \) on a Hilbert space \( X \), \( B \) and \( B_N \) defined on Hilbert spaces \( U \) and \( U_N \), respectively, are admissible control operators for \( T \) and the function \( N : X \times [0, \infty) \rightarrow U_N \) is continuous in \( t \) and Lipschitz in \( x \), with Lipschitz constant \( L_N \) independent of \( t \). Thus, \( B \) and \( B_N \) can be unbounded as operators from \( U \) and \( U_N \) to \( X \), in which case the nonlinear term \( B_N N(x(t), t) \) in the plant is in general not Lipschitz in \( x \). We assume that there exist linear operators \( F \) and \( F_N \) such that the triples \( (A, [B B_N], F) \) and \( (-A, [B B_N], F_N) \) are regular and \( A + BF_{\infty} \) and \( -A + BF_{\infty} \) are generators of operator semigroups \( \mathbb{T}^t \) and \( \mathbb{T}^t_b \) on \( X \) such that \( \|T^t\| \cdot \|T^t_b\| \) decays to zero exponentially. We prove that if \( L_N \) is sufficiently small, then the nonlinear plant is exactly controllable in some time \( \tau > 0 \). Our proof is constructive, i.e., given an initial state \( x_0 \in X \) and a final state \( x_T \in X \), we propose an approach for constructing a control signal \( u \) of class \( L^2 \) for the nonlinear plant which ensures that if \( x(0) = x_0 \), then \( x(t) = x_T \). We illustrate our approach using two examples: a sine-Gordon equation and a nonlinear wave equation. Our main result can be regarded as an extension of Russell’s principle on exact controllability to a class of nonlinear plants.

\end{abstract}

1. Introduction

Let \( X, U \) and \( U_N \) be Hilbert spaces, let \( T \) be a strongly continuous group of operators on \( X \) with generator \( A \) and let \( X_{-1} \) be the completion of \( X \) with respect to the norm \( \|x\|_{-1} = \|(\beta I - A)^{-1}x\| \), where \( \beta \) is an arbitrary (but fixed) element in the resolvent set \( \rho(A) \). In this paper we study the exact controllability of a nonlinear distributed parameter system described by the state equation

\[ \dot{x}(t) = Ax(t) + Bu(t) + B_N N(x(t), t) + f(t), \quad (1.1) \]

where \( B \in \mathcal{L}(U, X_{-1}) \) and \( B_N \in \mathcal{L}(U_N, X_{-1}) \) are admissible control operators for \( T \), the input signal \( u \) takes values in \( U \) and the nonlinear function \( N : X \times [0, \infty) \rightarrow U_N \) is continuous in the second argument \( t \) (the time) and Lipschitz in the first argument \( x \) with Lipschitz constant \( L_N \) independent of \( t \), i.e.

\[ \|N(x_1, t) - N(x_2, t)\| \leq L_N \|x_1 - x_2\| \]

\[ \forall x_1, x_2 \in X, \quad \forall t \geq 0. \quad (1.2) \]

The function \( f \) (the drift term in (1.1)) is an \( X_{-1} \)-valued function such that

\[ f \in H_{\text{loc}}^1((0, \infty); X_{-1}), \]

which means that the function \( t \mapsto f(t) - f(0) \) can be written as the integral of a function in \( L^2_{\text{loc}}([0, \infty); X_{-1}) \).

In the Abstract we took \( f = 0 \) for simplicity.

By \textit{exact controllability} in time \( \tau > 0 \) of the plant (1.1), we mean the following: For any initial state \( x(0) \in X \) we can steer the final state \( x(\tau) \) of the plant to any desired point in \( X \), by using an input function \( u \in L^2([0, \tau]; U) \). By exact controllability we mean exact controllability in some time \( \tau > 0 \). Exact controllability of nonlinear partial differential equations is a classical research area, see for instance Balachandran and Dauer (2002), Chen (1979), Dehman and Lebeau (2009), Zuazua (1993) and the references therein. The plants considered in the present work differ from those in the above, and many other, works in that (i) \( B \) can be unbounded, so that the control signal may drive the PDE while acting on a measure zero subset of its domain, like in the case of boundary control, (ii) \( B_N \)
can be unbounded as well, so that the nonlinear perturbation term $B_N \dot{N}(x(t), t)$ is not Lipschitz continuous on the state space $X$.

We assume that the pairs $(A, B)$ and $(-A, B)$ are jointly exponentially stabilisable in a certain sense which is inspired by the definition of stabilisability in Weiss and Curtain (1997). Using terminology that will be recalled in Section 3, the requirements of joint exponential stabilisability are that there exist linear operators $F$ and $F_b$ such that the triples $(A, B, F)$ and $(-A, B, F_b)$ are regular, $I$ is an admissible feedback operator for both of the regular systems with generating operators $(A, B, F, 0)$ and $(-A, B, F_b, 0)$ and the corresponding closed-loop semigroups $T^f$ (whose generator is $A + BF_A$) and $T^b$ (whose generator is $-A + BF_{b, A}$) satisfy, for some $M \geq 1$ and $\mu > 0$,

$$\|T^f\| \cdot \|T^b\| \leq Me^{-\mu t} \quad \forall t \geq 0.$$

Here, the subscript $A$ denotes the $A$-extension of an operator, see Section 3. In addition, we assume that the triples $(A, B_N, F)$ and $(-A, B_N, F_b)$ are regular. These assumptions will be explained in more detail in Section 4.

In the linear case (when $N = 0$) our assumptions on $A$ and $B$ imply that the system $(A, B)$ is exactly controllable: this is the well-known Russell’s principle for controllability, for which we refer to Russell (1974, 1978), Chen (1979), Komornik (1992), Rebarber and Weiss (1997). This principle (which can be stated in many different versions, of varying generality) states that if the pairs $(A, B)$ and $(-A, B)$ are jointly exponentially stabilisable (in the sense explained above), then $(A, B)$ is exactly controllable. For the dual version (which concerns exact observability) we refer to Ramdani, Tucsnak, and Weiss (2010) (specifically, Proposition 3.3 and the comments after it). The latter paper studied exact observability via a sequence of forward and backward observers. This approach was extended to some nonlinear wave and beam equations in Fridman (2013), Fridman and Terushkin (2016). The most general statement of Russell’s principle (the linear exact controllability question) known to us is in the conference paper (Natarajan & Weiss, 2013b), where it is stated informally in the comments after Theorem 4.1. In that reference $T$ is not assumed to be invertible, the stabilisability concept is weaker than what we use in this work and the conclusion is the null-controllability of the system $(A, B)$ in some time.

Our assumptions on $A$, $B$ and $B_N$ allow us to regard the nonlinear system (1.1) as a perturbation of the corresponding linear system (where $N = 0$) if the Lipschitz constant $L_N$ is small enough. The fact that exact controllability of a linear system is preserved under small bounded linear perturbations has been proved in Hadd (2005). A more general result (in dual form) is in Section 6.3 of Tucsnak and Weiss (2009). It concerns a class of unbounded perturbations, similar to what we have in this paper, but linear. A further generalisation (again in dual form) is in the recent work (Jiang, Liu, & Zhang, 2015) which considers nonlinear unbounded perturbations. However, our results are not a simple extension of the results in these works, because the approach we use here is entirely different. Our main result can be regarded as an extension of Russell’s principle to a class of nonlinear plants. We remark that the idea of extending Russell’s principle to nonlinear systems was raised (but not addressed) in Chen (1979).

We show that for each sufficiently small Lipschitz constant $L_N$ there exists $\tau^* \geq 0$ such that the plant (1.1) is exactly controllable in each time $\tau > \tau^*$. Our proof is novel and constructive, i.e. we present an algorithm for constructing the required control signal $u$ that takes $x$ from a given $x(0)$ to a desired $x(\tau)$. Briefly, we define a sequence of forward and backward state equations that have the same structure as (1.1), but without drift terms and with different functions in place of $N$. Using solutions of this sequence of systems on the time interval $[0, \tau]$, that satisfy suitable boundary conditions, we construct the required control signal $u$. When the plant is linear, our approach for determining $u$ reduces to solving the plant with linear feedback, back and forth in time on the interval $[0, \tau]$, with different initial conditions. For simplicity, we explain our approach when $X$ is finite-dimensional in Section 2. The details for infinite-dimensional $X$ are in Section 4. Section 5 contains numerical examples.

We simplify our notation by adopting the following convention: if in a sum the summation index runs from $m \in \mathbb{Z}$ to $n \in \mathbb{Z}$ and $n < m$, then the sum is zero.

2. Discussion in finite dimensions

To make our ideas easily understood, we first present them in the simpler context of finite-dimensional control theory. Thus in this section $X = \mathbb{C}^p$, $U = \mathbb{C}^q$, $U_M = \mathbb{C}^r$ and $A$, $B$, $B_N$, $F$ and $F_b$ are matrices of suitable dimensions such that (1.3) holds (note that $T^f_t = e^{(A + BF)mt}$ and $T^b_t = e^{(-A + BF_b)mt}$). We shall see later (see Remark 2.2) that in this finite-dimensional context, the existence of $F$ and $F_b$ such that (1.3) holds is equivalent to the fact that $(A, B)$ is controllable.

We assume that $N$ is a nonlinear function as in the text before (1.2), $N$ satisfies the Lipschitz assumption (1.2) and the plant is described by (1.1). The drift function $f : [0, \infty) \to X$ is now only assumed to be continuous.
In this setting, it follows from the standard theory for the existence and uniqueness of solutions to ordinary differential equations (ODEs) that for any input \( u \in C([0, \infty); U) \) and every initial condition \( x_0 \in X \), there exists a unique solution \( x \in C^1([0, \infty); X) \) for (1.1) such that \( x(0) = x_0 \). In the theorem below we assume, for the sake of simplicity, that the desired final state is 0 (i.e., we study the null controllability of (1.1)). This simplifying assumption can be eliminated, see Remark 2.1.

**Theorem 2.1:** Under the above assumptions, if \( L_N \) is sufficiently small, then there exists \( \tau^* \geq 0 \) such that the following holds: For any \( \tau > \tau^* \) and any \( x_0 \in X \) there exists an input function \( u \in C([0, \tau]; U) \) such that the solution \( x \) of (1.1) corresponding to \( x(0) = x_0 \) and this \( u \) satisfies \( x(\tau) = 0 \).

**Proof:** In the first step, we consider the systems

\[
\dot{x}_f(t) = (A + BF)x_f(t) + B_Nh(x_f(t), t) \quad \forall t \geq 0, \tag{2.1}
\]

\[
\dot{x}_b(t) = (-A + BF_b)x_b(t) + B_Nh_b(x_b(t), t) \quad \forall t \geq 0, \tag{2.2}
\]

and each ODE (except the first) in the sequence of ODEs that we will associate with the plant (1.1) in the next step resembles one of the above two systems. Here the functions \( h \) and \( h_b \) are continuous in their second argument \( t \) and Lipschitz in their first argument, with the same Lipschitz constant \( L_N \) as for \( N \). In other words, for all \( x_1, x_2 \in X \) and each \( t \geq 0 \),

\[
\| h(x_1, t) - h(x_2, t) \| \leq L_N \| x_1 - x_2 \|,
\]

\[
\| h_b(x_1, t) - h_b(x_2, t) \| \leq L_N \| x_1 - x_2 \|.
\]

Furthermore, these functions satisfy \( h(0, t) = h_b(0, t) = 0 \) for all \( t \geq 0 \). It follows from the standard theory on the solutions to ODEs that for each \( x \in X \) there exist unique \( C^1 \) solutions \( x_f \) for (2.1) and \( x_b \) for (2.2), on the time interval \( [0, \infty) \), such that \( x_f(0) = x_0 \) and \( x_b(0) = x_0 \). In addition, since (1.3) holds, if \( L_N \) is sufficiently small (as discussed in more detail in Section 4), then there exist constants \( M_f, M_b \geq 1 \) and \( \omega_f, \omega_b \in \mathbb{R} \) (independent of \( x_0 \) and the specific forms of \( h \) and \( h_b \)) such that

\[
\| x_f(t) \| \leq M_f e^{-\omega_f t} \| x_0 \|, \tag{2.3}
\]

\[
\| x_b(t) \| \leq M_b e^{-\omega_b t} \| x_0 \| \quad \forall t \geq 0
\]

and \( \omega_f + \omega_b > 0 \). The above estimates are established in Section 4 when \( X \) is infinite-dimensional (see Lemma 4.1 and Lemma 4.2). Let \( \tau^* \geq 0 \) be such that

\[
M_f M_b e^{-(\omega_f+\omega_b)\tau^*} = 1. \quad \text{Fix } \tau > \tau^*, \text{ then clearly}
\]

\[
\Gamma := M_f M_b e^{-(\omega_f+\omega_b)\tau} < 1. \tag{2.4}
\]

The second step is to introduce a sequence of ODEs defined recursively, meaning that each ODE in the sequence (except the first) is defined using the solutions of the previous ODEs in the sequence. Fix \( x_0 \in X \). The first ODE is

\[
x^0(t) = Ax^0(t) + B_NN(x^0(t), t) + f(t) \quad \forall t \in [0, \tau],
\]

\[
x^0(0) = x_0. \tag{2.5}
\]

For each \( n \in \mathbb{N} \) and all \( t \in [0, \tau] \) we consider the ‘backward’ ODE

\[
\dot{x}_n^o(t) = (A - BF_n)x_n^o(t) + B_Nh_n^o(x_n^o(t), t),
\]

\[
x_n^o(\tau) = x^{n-1}(\tau), \tag{2.6}
\]

where \( h_n^o \) is defined using the solutions of the previous ODEs: for all \( x \in X \) and \( t \in [0, \tau] \),

\[
h_n^o(x, t) = N\left( \sum_{k=0}^{n-1} x^k(t) - \sum_{k=1}^{n-1} x_b^k(t), t \right)
\]

\[
- N\left( \sum_{k=0}^{n-1} x^k(t) - \sum_{k=1}^{n-1} x_b^k(t) - x, t \right). \tag{2.7}
\]

For each \( n \in \mathbb{N} \) and all \( t \in [0, \tau] \) we consider the ‘forward’ ODE

\[
\dot{x}_n(t) = (A + BF)x_n(t) + B_Nh_n(x_n(t), t),
\]

\[
x_n(0) = x_n^o(0), \tag{2.8}
\]

where \( h_n \) is defined using the solutions of the previous ODEs: for all \( x \in X \) and \( t \in [0, \tau] \),

\[
h_n(x, t) = N\left( x + \sum_{k=0}^{n-1} x^k(t) - \sum_{k=1}^{n-1} x_b^k(t), t \right)
\]

\[
- N\left( \sum_{k=0}^{n-1} x^k(t) - \sum_{k=1}^{n-1} x_b^k(t), t \right). \tag{2.9}
\]

The structure of \( h_n^o \) and \( h_n \) implies that these ODEs must be solved sequentially on \([0, \tau]\), i.e., we must find \( x^0, x_b^1, x^1, x_b^2, x^2 \) and so on, in this order. For example,

\[
\dot{x}_1^b(t) = (A - BF_1)x_1^b(t) + B_NN(x^0(t), t)
\]

\[
- N(x^0(t) - x_1^b(t), t).$$
Combining the above estimates for \( n = 1 \) and \( t = \tau \) and using the notation \( \Gamma \) from (2.4), we obtain \( \|x^1(\tau)\| \leq \Gamma \|x^0(\tau)\| \). By iterating this process, we get

\[
\|x^n(t)\| \leq M_n e^{-\alpha t} \|x^0(0)\|. \tag{2.10}
\]

which, along with the first inequality in (2.10), implies that

\[
\|x^n(\tau)\| \leq M_n e^{-\alpha \tau} \Gamma^{n-1} \|x^0(\tau)\| \quad \forall t \in [0, \tau], \ n \in \mathbb{N}. \tag{2.12}
\]

From here and (2.10) it follows that

\[
\|x^n(t)\| \leq M_\infty e^{-\alpha t} \Gamma^{n-1} \|x^0(\tau)\| \quad \forall t \in [0, \tau], \ n \in \mathbb{N}. \tag{2.13}
\]

The fourth step is to find the desired input function \( u \) for the plant (1.1). For each \( n \in \mathbb{N} \) and \( t \in [0, \tau] \), we define

\[
x_n(t) = \sum_{k=0}^{n} x^k(t) - \sum_{k=1}^{n} x^k_0(t),
\]

\[
u_n(t) = F \sum_{k=1}^{n} x^k(t) + B_0 \sum_{k=1}^{n} x^k_0(t). \tag{2.14}
\]

Then \( x_n \) (shown in Figure 1 as a dashed line) is the unique solution of (1.1) on the time interval \([0, \tau]\) with \( x_0(0) = x_0 \) and input \( u = u_n \), i.e.

\[
\dot{x}_n(t) = A x_n(t) + B u_n(t) + B_N \dot{x}_N(x_n(t), t) + f(t) \quad \forall t \in [0, \tau]. \tag{2.15}
\]

and moreover \( x_n(\tau) = x^\tau(\tau) \). The functions \( h^n_\infty \) and \( h^n \) have been chosen precisely so that \( x_n \) satisfies (2.15). This can be shown by a simple computation based on (2.5)–(2.9) (which involves many cancellations). We define \( u : [0, \tau] \to U \) by

\[
u(t) = F \sum_{k=1}^{\infty} x^k(t) + B_0 \sum_{k=1}^{\infty} x^k_0(t) \quad \forall t \in [0, \tau]. \tag{2.16}
\]

It follows using (2.12) and (2.13) that both series in (2.16) converge to a finite sum for each \( t \in [0, \tau] \), and moreover \( \lim_{n \to \infty} \|u - u_n\|_{[0, \tau]} = 0 \). The function \( u \), being the uniform limit of a sequence of continuous functions, is continuous on \([0, \tau]\).

Let \( x \) be the unique solution of (1.1) on \([0, \tau]\) when \( x(0) = x_0 \) and \( u \) is as in (2.16). From the continuous

![Figure 1](image-url)  
**Figure 1.** Illustration of the back-and-forth algorithm used to construct a null control signal \( u \) for the nonlinear plant (1.1) with initial state \( x(0) = x_0 \). While \( x^0 \) is a state trajectory of (1.1), \( x^0_n \) and \( x^0 \) (for \( n \geq 1 \)) are state trajectories of (2.6) and (2.8). The dashed curve \( x_n \) (see (2.14)) is a state trajectory of (1.1) when \( u = u_n \). We have \( \lim_{n \to \infty} x_n(\tau) = 0 \).

which is solved backwards on \([0, \tau]\) starting from \( x^0(\tau) = x_0(\tau) \), and

\[
\dot{x}^1(t) = (A + BF)x^1(t) + B_N \dot{N}(x^1(t) - x^0_0(t)) + x^0(t) - \dot{N}(x^0(t) - x^0_0(t), t),
\]

which is solved forwards on \([0, \tau]\) starting from \( x^1(0) = x^0_0(0) \). This sequence of back-and-forth state trajectories is illustrated in Figure 1.

The third step is to obtain estimates for the solutions of this sequence of ODEs. For this, notice that for \( t \in [0, \tau] \), (2.8) is an equation of the form (2.1), with \( h = h^n \). It is clear that (2.6) can be rewritten, using the notation \( z(t) = x^n_0(\tau - t) \), as

\[
\dot{z}(t) = (-A + BF)z(t) - B_N h^n(\tau - t),
\]

\[
z(0) = x^{n-1}(\tau),
\]

which for \( t \in [0, \tau] \) is an equation of the form (2.2), with

\[
h_n(x, \tau - t) = -h^n_\infty(x, \tau - t) \quad \forall x \in X.
\]

It is easy to see that the nonlinear functions \( h \) and \( h_n \) just defined possess all the properties assumed after Equations (2.1) and (2.2). Therefore, the estimates (2.3) hold, meaning that for every \( n \in \mathbb{N} \) and \( t \in [0, \tau] \),

\[
\|x^n_0(\tau - t)\| \leq M_n e^{-\alpha t} \|x^{n-1}(\tau)\|,
\]
dependence of the state trajectories of systems described by ODEs on their inputs (see, for instance, Khalil, 2002, Theorem 3.4), we get that
\[
x(t) = \lim_{n \to \infty} x_n(t) \quad \forall t \in [0, \tau].
\]
This, together with the fact that \(x_\alpha(\tau) = x^d(\tau)\) and the estimate in (2.11), implies that \(x(\tau) = 0\) as desired. \(\blacksquare\)

We remark that when \(N = 0\) and \(f = 0\), if we replace \(A\) with \(A + BF_t\) in (2.5) and add \(FX^d(t)\) to the right side of (2.16), then the above construction of the null control signal \(u\) reduces to the one described in Natarajan and Weiss (2013b).

As promised at the beginning of this section, in the next remark we explain how the null controllability in Theorem 2.1 can be replaced with controllability.

**Remark 2.1:** Suppose that for the plant (1.1), with \(N\) as in (1.2), the initial state is \(x(0) = x_0\) and the desired final state is \(x(\tau) = x_\tau \in X\). Define \(M(z, t) = N(z + x_\tau, t)\) for each \(z \in X\) and \(t \geq 0\). Consider the modified nonlinear plant
\[
\dot{z}(t) = Ax(t) + Bu(t) + B_N\mathcal{M}(z(t), t)
\]
It is easy to verify that if \(z(0) = x_0 - x_\tau\) and \(u \in C([0, \tau]; U)\) is a null control signal for (2.17) in time \(\tau\), so that the corresponding solution \(z \in C^1([0, \tau]; X)\) of (2.17) satisfies \(z(\tau) = 0\), then \(x\) defined by \(x(t) = z(t) + x_\tau\) for each \(t \in [0, \tau]\) solves (1.1) for the same \(u\) and satisfies \(x(0) = x_0\) and \(x(\tau) = x_\tau\). Hence, this control signal \(u\) takes (1.1) from \(x_0\) to \(x_\tau\) in time \(\tau\) (which is our control objective). If we introduce the new drift function \(g(t) = f(t) + Ax_\tau\), then the problem of finding a null control signal for (2.17) for any given initial state fits into the framework of Theorem 2.1 (with \(M\) in place of \(N\) and \(g\) in place of \(f\)) and hence it can be solved under the assumptions of that theorem. Thus, Theorem 2.1 can be strengthened by replacing the conclusion of null controllability in time \(\tau\) with controllability in time \(\tau\).

**Remark 2.2:** If we take \(N = 0\) in Theorem 2.1, then we get that the joint stabilisability of \((A, B)\) and \((-A, B)\) (i.e. the estimate (1.3)) implies that \((A, B)\) is controllable. The converse statement is clearly also true. This equivalence is the finite-dimensional version of Russell’s principle.

### 3. Background on regular systems

This section is a brief introduction to regular linear systems based on Staffans (2004), Staffans and Weiss (2004), Weiss (1994a), Weiss (1994b), Weiss and Curtain (1997) and Tucsnak and Weiss (2009). For a Hilbert space \(Y\) and \(\alpha \in \mathbb{R}\), we define the Hilbert space
\[
L_\alpha^2([0, \infty); Y) = \left\{ \phi \in L^2_{\text{loc}}([0, \infty); Y) \mid \int_0^\infty e^{-2\alpha t} \|\phi(t)\|^2 \, dt < \infty \right\},
\]
with the norm being the square-root of the integral appearing above. For each \(\tau \geq 0\), let \(P_\tau\) be the projection of \(L^2([0, \infty); Y)\) onto \(L^2_\alpha([0, \tau]; Y)\) by truncation.

Let \(A\) be the generator of a strongly continuous operator semigroup \(T\) on a Hilbert space \(X\). The **growth bound** of \(T\) is denoted by \(\omega_T\). This means that \(\omega_T\) is the smallest real number with the following property: For each \(\omega > \omega_T\) there exists a \(M_\omega > 0\) such that
\[
\|T_t\| \leq M_\omega e^{\omega t} \quad \forall t \geq 0.
\]
We call \(T\) (or \(A\)) **exponentially stable** if \(\omega_T < 0\). For \(\beta \in \rho(A)\), we define
\[
X_1 = \mathcal{D}(A) \text{ with the norm } \|x\|_1 = \|(\beta I - A)x\|.
\]
The space \(X_{-1}\) has been defined before (1.1). These spaces are independent of the choice of \(\beta\). The operators \(T_t\) extend to \(X_{-1}\) and form an operator semigroup on it. The generator of the extended operator semigroup is an extension of \(A\) to an operator in \(\mathcal{L}(X, X_{-1})\). We use the same notation \(T_t\) and \(A\) for these extended operators. If \(T\) is a strongly continuous group on \(X\), then so is its extension to \(X_{-1}\). The space \(X_{-1}^d\) is defined similarly to \(X_1\), but with \(A^*\) in place of \(A\). Then \(X_{-1}\) can be regarded as the dual space of \(X_1^d\) with respect to the pivot space \(X\).

Let \(U\) be a Hilbert space. An operator \(B \in \mathcal{L}(U, X_{-1})\) is an **admissible control operator** for \(T\) if for some (hence, for each) \(\tau > 0\) and for every \(u \in L^2([0, \infty); U)\),
\[
\Phi_\tau u = \int_0^\tau T_{\tau - \sigma} Bu(\sigma) \, d\sigma \in X
\]
(the integral is computed in \(X_{-1}\)). Then, this integral gives the strong solution of \(\dot{z}(t) = Az(t) + Bu(t)\) in the space \(X_{-1}\), corresponding to \(z(0) = 0\), evaluated at the time \(\tau\). \(B\) is called **bounded** if \(B \in \mathcal{L}(U, X)\), and **unbounded** otherwise. If \(B\) is admissible and \(\alpha > \omega_T\), then there exists \(M_\alpha > 0\) such that
\[
\|(sI - A)^{-1}B\|_{\mathcal{L}(U, X)} \leq \frac{M_\alpha}{\sqrt{\Re s} - \alpha} \quad \text{for } Re s > \alpha.
\]
If the semigroup $\mathbb{T}$ is normal, or invertible, or contractive, then the above estimate is sufficient for the admissibility of $B$. For $A$ and $B$ as above (so that in particular, $B$ is admissible), for each $\tau \geq 0$ we define the input map $\Phi_\tau: L^2([0, \infty); U) \to X$ by (3.1). It can be shown that these operators are bounded. Since $\Phi_\tau u = \Phi_t P_t u$, the operators $\Phi_\tau$ have an obvious extension to $L^2_{\text{loc}}([0, \infty); U)$. For any $u \in L^2_{\text{loc}}([0, \infty); U)$, $\Phi_t u$ is a continuous $X$-valued function of the time $t$. If $A$ is exponentially stable, then the operators $\Phi_\tau$ are uniformly bounded. If we denote their uniform bound by $\|\Phi_\tau\|$, then obviously
\[
\|\Phi_t u\| \leq \|\Phi_\infty\| \cdot \|u\|_{L^2} \quad \forall u \in L^2([0, \infty); U),
\]
$\forall t \geq 0$. 

(3.2)

For $A$ and $B$ as above, if $\mathbb{T}_t$ is invertible for some $t > 0$, then it is invertible for every $t > 0$ and the inverses form a new operator semigroup with generator $-A$. In this case $\mathbb{T}$ can be extended to a strongly continuous group of operators via $\mathbb{T}_{-t} = \mathbb{T}_t^{-1}$ and $B$ is admissible also for the inverse semigroup.

Let $U$ be a Hilbert space. An operator $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$ if for some (hence, for every) $\tau > 0$ there exists $m_\tau > 0$ such that
\[
\int_0^\tau \|\mathbb{T}_t z\|^2 dt \leq m_\tau \|z\|^2 \quad \forall z \in \mathcal{D}(A).
\]

(3.3)

If $B \in \mathcal{L}(U, X_-)$, then $B^* \in \mathcal{L}(X_1^d, U)$. $B$ is an admissible control operator for $\mathbb{T}$ if $B^*$ is an admissible observation operator for the adjoint operator semigroup $\mathbb{T}^*$. The $A$-extension of an operator $C \in \mathcal{L}(X_1, Y)$ (with respect to $A$), denoted $C_A$, is defined as follows:
\[
C_A x = \lim_{\lambda \to +\infty} C\lambda (\lambda I - A)^{-1} x
\]
and its domain $\mathcal{D}(C_A)$ consists of those $x \in X$ for which the above limit exists. If $C$ is admissible for $\mathbb{T}$, then for every $x \in X$ the formula $y(t) = C_A T_t x$ makes sense for almost every $t \geq 0$ and it defines a function $\gamma \in L^2_{\text{loc}}([0, \infty); Y)$ for every $\alpha > \omega$. We call $C$ bounded if it can be extended so that $C \in \mathcal{L}(X, Y)$ and unbounded otherwise.

Definition 3.1: Let $U$, $X$, and $Y$ be Hilbert spaces. Let $A$ be the generator of a strongly continuous semigroup $\mathbb{T}$ on $X$ and let $B \in \mathcal{L}(U, X_-)$ and $C \in \mathcal{L}(X_1, Y)$ be admissible control and observation operators, respectively, for $\mathbb{T}$. Then the triple $(A, B, C)$ is called regular if for some (hence, for every) $s \in \rho(A)$, $C(sI - A)^{-1} B$ exists and the map $s \mapsto C_A (sI - A)^{-1} B$ is bounded on some right half-plane.

For any regular triple $(A, B, C)$ and any $D \in \mathcal{L}(U, Y)$, the equations
\[
\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = C_A x(t) + Du(t),
\]
(3.5)

define a regular linear system $\Sigma$ with input space $U$, state space $X$, and output space $Y$. For any initial state $x(0) = x_0 \in X$ and any input $u \in L^2_{\text{loc}}([0, \infty); U)$ (where $\alpha \in \mathbb{R}$) there exists a unique state trajectory $x \in C([0, \infty); X)$ and output $y$ for $\Sigma$ such that $y \in L^2_{\text{loc}}([0, \infty); Y)$ for all $\gamma \leq \alpha$ with $\gamma > \omega$ and both equations in (3.5) hold for almost every $t \geq 0$. The state trajectory $x$ can be written as
\[
x(t) = T_t x_0 + \Phi_t u \quad \forall t \geq 0,
\]
where $\Phi_t$ is as in (3.1). The operators $(A, B, C, D)$ are called the generating operators of $\Sigma$. The transfer function of $\Sigma$ is
\[
G(s) = C_A (sI - A)^{-1} B + D,
\]
which is an $\mathcal{L}(U, Y)$-valued analytic function defined on the right half-plane where $\text{Re} s > \omega$. Denoting the Laplace transformation by a hat, for $u, x_0, y$ and $\gamma$ as in the text after (3.5), we have
\[
\hat{y}(s) = C(sI - A)^{-1} x_0 + \hat{G}(s) \hat{u}(s) \quad \text{for } \text{Re} s > \gamma
\]
and $\lim_{s \to \infty} G(s) v = Dv$ for any $v \in U$, where $s$ is positive.

For a system $\Sigma$ as above, there exist linear maps $\Psi_{\infty} : X \to L^2_{\text{loc}}([0, \infty); Y)$ (called the extended output map) and $\mathbb{F}_{\infty} : L^2_{\text{loc}}([0, \infty); U) \to L^2_{\text{loc}}([0, \infty); Y)$ (called the extended input-output map) such that the output $y$ from (3.5) can be expressed as $y = \Psi_{\infty} x_0 + \mathbb{F}_{\infty} u$. The extended input-output map is causal, i.e. $\mathbb{P}_t \mathbb{F}_{\infty} = \mathbb{P}_t \mathbb{F}_{\infty} \mathbb{P}_t$ for all $t \geq 0$. This and the above expression for the output $y$ give
\[
\mathbb{P}_t y = \mathbb{P}_t \Psi_{\infty} x_0 + \mathbb{P}_t \mathbb{F}_{\infty} \mathbb{P}_t u \quad \forall t \geq 0.
\]
(3.6)

For every $\omega > \omega_\infty$, $\Psi_{\infty}$ is bounded from $X$ to $L^2_{\omega\infty}([0, \infty); Y)$ with norm $\|\Psi_{\infty}\|_\omega$ and $\mathbb{F}_{\infty}$ is bounded from $L^2_{\omega\infty}([0, \infty); U)$ to $L^2_{\omega\infty}([0, \infty); Y)$ with norm $\|\mathbb{F}_{\infty}\|_{\omega\infty}$. Moreover
\[
\|\mathbb{F}_{\infty}\|_{\omega\infty} = \sup_{\text{Re} s > \omega} \|\mathbb{G}(s)\|.
\]

Definition 3.2: With the notation from Definition 3.1, let $G$ be the transfer function of $\Sigma$ and let $K \in \mathcal{L}(Y, U)$. We say that $K$ is an admissible feedback operator for $\Sigma$ if $[I - KG(s)]^{-1}$ exists and is uniformly bounded on some
right half-plane (equivalently, \([I - G(s)K]^{-1}\) exists and is uniformly bounded on some right half-plane).

**Proposition 3.1:** We work under the assumptions of the above definition, and moreover we assume that \(I - KD\) is invertible (equivalently, \(I - DK\) is invertible). Then there exists a unique regular linear system \(\Sigma^K\) with generating operators \((A^K, B^K, C^K, D^K)\) and transfer function \(G^K\) determined as follows:

\[
A^K x = [A + BK(I - DK)^{-1}C_A]x \quad \forall x \in \mathcal{D}(A^K),
\]

where \(\mathcal{D}(A^K)\) consists precisely of those \(x \in \mathcal{D}(C_A)\) for which the above expression for \(A^K x\) is in \(X\). For any \(x \in \mathcal{D}(A^K)\) we have

\[
B^K = B(I - KD)^{-1}, \quad C^K x = (I - DK)^{-1}C_A x, \quad D^K = D(I - KD)^{-1} = (I - DK)^{-1} D, \quad G^K = G(I - KG)^{-1} = (I - GK)^{-1} G.
\]

With the notation and the assumptions of the above proposition, let \(\tilde{U}\) be a Hilbert space and \(\tilde{B} \in \mathcal{L}(\tilde{U}, X_{-1})\). If \((A, \tilde{B}, C)\) is a regular triple, then \(\tilde{B}\) is an admissible control operator for the semigroup generated by \(A^K\). (A similar statement holds for admissible observation operators, we omit the details.) Furthermore, if \(D = 0\), then \((A^K, \tilde{B}, \tilde{C})\) is a regular triple, where \(\tilde{C}\) is the restriction of \(C_A\) from (3.4) to \(\mathcal{D}(A^K)\). The \(\Lambda\)-extension of \(\tilde{C}\) with respect to \(A^K\) is \(C_A\) (the same as for \(C\)).

The following definition uses the above proposition with \(D = 0\).

**Definition 3.3:** Suppose that \(A\) and \(B\) are as in (3.1). Then \((A, B)\) is exponentially stabilisable if there exists \(F \in \mathcal{L}(X_1, U)\) such that:

1. \((A, B, F)\) is a regular triple,
2. \([I - F_A(sI - A)^{-1}B]^{-1}\) exists and is bounded on some right half-plane,
3. The operator semigroup generated by \(A + BF_A\) is exponentially stable.

In this case, we call \(F\) a stabilising state feedback operator for \((A, B)\).

**4. Exact controllability of nonlinear DPS**

In this section we study the exact controllability of the possibly infinite-dimensional nonlinear plant (1.1) using the approach described for finite-dimensional systems in Section 2. In this section \(X, U, U_N, \) and \(Y\) denote Hilbert spaces.

First, we define a solution concept for the nonlinear systems encountered in this section, following Definition 4.1.1 in Tucsnak and Weiss (2009).

**Definition 4.1:** Consider the nonlinear differential equation

\[
\dot{x}(t) = Ax(t) + B_0 G(x(t), t) + f(t) \quad \forall t \geq 0,
\]

where \(A\) is the generator of a strongly continuous semigroup \(\mathbb{T}\) on \(X, B_0 \in \mathcal{L}(U, X_{-1}), G : X \times [0, \infty) \to U\) is a nonlinear function and \(f \in H^1_{\text{loc}}([0, \infty); X_{-1}).\) A strong solution of (4.1) in \(X_{-1}\) is a function

\[
x \in L^1_{\text{loc}}([0, \infty); X) \cap C([0, \infty); X_{-1})
\]

with the following properties:

(a) \(G x \in L^1_{\text{loc}}([0, \infty); U)\), where \((G x)(t) = G(x(t), t)\) for almost all \(t \geq 0\).

(b) The following equality holds for every \(t \geq 0\):

\[
x(t) - x(0) = \int_0^t \left[ Ax(\sigma) + B_0 G(x(\sigma), \sigma) + f(\sigma) \right] d\sigma.
\]

It is easy to see that the strong solution \(x\) in the above definition (if it exists) satisfies (4.1) pointwise in \(X_{-1}\) for almost every \(t \geq 0\).

**Proposition 4.1:** Suppose that \(\mathbb{T}\) is a strongly continuous semigroup of operators on \(X\) with infinitesimal generator \(A, B \in \mathcal{L}(U, X_{-1})\) and \(B_N \in \mathcal{L}(U_N, X_{-1})\) are admissible control operators for \(\mathbb{T}\) and \(N : X \times [0, \infty) \to U_N\) is continuous in its second argument and satisfies the uniform Lipschitz condition (1.2) for some \(L_N > 0.\) Assume that \(f \in H^1_{\text{loc}}([0, \infty); X_{-1})\) and define

\[
\mathcal{F}(t) = \int_0^t \mathcal{T}_{t-\sigma} F(\sigma) d\sigma \quad \forall t \geq 0.
\]

For each \(t \geq 0,\) recall the operator \(\Phi_t\) from (3.1) and define a similar operator \(\Phi_t^N : L^2([0, \infty); U_N) \to X\) as follows:

\[
\Phi_t^N x = \int_0^t \mathcal{T}_{t-\sigma} B_N x(\sigma) d\sigma \quad \forall x \in L^2([0, \infty); U_N).
\]

Then for every \(u \in L^2_{\text{loc}}([0, \infty); U)\) and each \(x_0 \in X\) there exists a unique function \(x \in C([0, \infty); X)\) such that

\[
x(t) = \mathcal{T}_t x_0 + \Phi_t u + \Phi_t^N N(x)(t) + F(t) \quad \forall t \geq 0,
\]

where \((N x)(t) = N(x(t), t)\) for all \(t \geq 0.\) For every \(t > 0,\) \(x(t)\) depends continuously on \(x_0\) on and \(P_t u\) (the truncation of \(u\) to \([0, t]\)). Moreover, the function \(x \) defined above
is the unique strong solution of (1.1) in $X_{-1}$ that satisfies $x(0) = x_0$.

Outline of the proof. First we notice that from Theorem 4.1.6 in Tucsnak and Weiss (2009) we have $F \in C([0, \infty); X)$.

The existence of $x$ and its continuous dependence on $x_0$ and on $P_t u$ can be shown by a slight adaptation of the proof of Theorem 3.1 in Natarajan and Weiss (2013a).

Indeed, in the cited reference, $\mathcal{N}$ is independent of time, while here $\mathcal{N}$ may depend on $t$, but this does not change anything. In the cited reference, for our purpose, we must take the output $y$ of the well-posed system $\Sigma^P$ to be equal to its state $x$ and replace $\Phi_1 u_1$ with $\Phi_1 u + F(t)$ and $\mathbb{F}_1 u_1$ with $\Phi_1 (\Phi u + F)$ in Section 3 there. Here $(\Phi u)(t) = \Phi_t u$ for all $t \geq 0$. It follows from Remark 3.2 in the cited reference that this proposition holds without any restriction on the Lipschitz constant $L_N$. This is because our observation operator is the identity operator on $X$, which is bounded. Theorem 3.1 from Natarajan and Weiss (2013a) is presented as Theorem 7.2 in the survey paper Tucsnak and Weiss (2014).

To use the concept of strong solution for (1.1), we have to fit (1.1) into the properly of (4.1). For this, we define $B_\gamma = [B B_N\gamma]$ and $G(x, t) = [u(t) N'(x, t)]^T$.

Adapting the proof of Proposition 4.2.5 in Tucsnak and Weiss (2009) we can check that $x$ is the unique strong solution for (1.1) in $X_{-1}$ that satisfies $x(0) = x_0$.

The function $x$ in the above proposition is called the state trajectory of (1.1) corresponding to the initial state $x_0$ and the input $u$. Let $A$, $B$ and $T$ be as in Proposition 4.1. Then clearly $A - \gamma I$ is the generator of an operator semigroup $S_\gamma$ for each $\gamma \in \mathbb{R}$, with $S_\gamma = e^{t\gamma T}$ for all $t \geq 0$, and $B$ is an admissible control operator for $S_\gamma$. Define $\Phi_t^\gamma$ analogously to $\Phi_t$ by replacing $T$ with $S_\gamma$ in (3.1). If $\gamma > \omega_T$, then $S_\gamma$ is exponentially stable and from the discussion above (3.2) we get that the operators $\Phi_t^\gamma$ are uniformly bounded by a constant $\|\Phi_\gamma^\infty\|$.

Lemma 4.1: Suppose that $A$ is the generator of a strongly continuous semigroup $T$ on $X$ with growth bound $\omega_T$. Let $B \in L(U, X_{-1})$ be an admissible control operator for $T$ and let $g : X \times [0, \infty) \to U$ be a function which is continuous in its second argument, $g(0, t) = 0$ for all $t \geq 0$ and $g$ satisfies the Lipschitz condition

$$\|g(x_1, t) - g(x_2, t)\| \leq L \|x_1 - x_2\| \quad \forall x_1, x_2 \in X, \quad \forall t \geq 0,$$  

for some $L > 0$. Then for every $\gamma > \omega_T$ there exists $M_\gamma > 1$ such that for each $x_0 \in X$, the state trajectory $x$ of the nonlinear system

$$\dot{x}(t) = Ax(t) + Bg(x(t), t) \quad \forall t \geq 0, \quad x(0) = x_0$$

satisfies $\|x(t)\| \leq M_\gamma \|x_0\| e^{-\gamma t}/\gamma$ for all $t \geq 0$. Here $-\omega_T = \gamma + \|\Phi_\gamma^\infty\|_2 L_2^2$, where $\|\Phi_\gamma^\infty\|$ is as introduced just before the lemma.

Furthermore, if $C \in L(X_1, Y)$ is an admissible observation operator for $T$ and $(A, B, C)$ is a regular triple, then for every trajectory $x$ of (4.1) $x(t) \in D(C_\lambda)$ for almost all $t \geq 0$ and for each $\gamma > \omega_T$ and $\alpha > -\omega_T$ there exist constants $c_\alpha, d_\alpha > 0$ such that

$$\|C_A x\|_{L^2}^2 \leq c_\alpha \|x_0\| + d_\alpha \|P_t x\|_{L^2}^2 \quad \forall t \geq 0.$$  

Proof: For each $x_0 \in X$, it follows from Proposition 4.1 that there exists a unique state trajectory $x \in C([0, \infty); X)$ for (4.1) which satisfies

$$x(t) = T_t x_0 + \int_0^t T_{t-s} B g(x(s), ) \; ds \quad \forall t \geq 0.$$

Fix $\gamma > \omega_T$ and define $\alpha(t) = e^{-\gamma \alpha} x(t)$. Recall the notation $S_\gamma$ and $\Phi_t^\gamma$ introduced just before this lemma. It is easy to verify using the last equation that

$$z(t) = S_{\gamma t} x_0 + \int_0^t S_{\gamma t-s} B e^{-\gamma \sigma} g(e^{\gamma \sigma} z(\sigma), ) \; d\sigma \quad \forall t \geq 0.$$  

For any fixed $t \geq 0$, if we define $u(\sigma) = e^{-\gamma \sigma} g(e^{\gamma \sigma} z(\sigma), )$ for $\sigma \in [0, t]$ and $u(\sigma) = 0$ for $\sigma > t$, then the above equation gives us that $z(t) = S_{\gamma t} x_0 + \Phi_t^\gamma u$. Using this and (4.1) we get that for some $M \geq 1$

$$\|z(t)\| \leq M \|x_0\| + \|\Phi_\gamma^\infty\| \cdot \|u\|_{L^2},$$

$$\|u\|_{L^2}^2 \leq M^2 \int_0^t \|z(\sigma)\|^2 \; d\sigma.$$  

Therefore, for all $t \geq 0$,

$$\|z(t)\|^2 \leq 2M^2 \|x_0\|^2 + 2\|\Phi_\gamma^\infty\|^2 L^2 \int_0^t \|z(\sigma)\|^2 \; d\sigma.$$  

Applying Gronwall’s inequality (see, for instance, (Coddington & Levinson, 1955), (Khalil, 2002)) to the above expression, we get that

$$\|z(t)\|^2 \leq 2M^2 \|x_0\|^2 e^{2\|\Phi_\gamma^\infty\|^2 L^2 t} \quad \forall t \geq 0.$$  

This and the definitions $z(t) = e^{-\gamma t} x(t)$, $M_\gamma = \sqrt{2} M$ imply the estimate for $\|z(t)\|$.

Let $C$ be as in the theorem. Then we can regard (4.1) along with the equation $y(t) = C_A x(t)$ as a regular linear
system with input $u$ given by $u(t) = g(x(t), t)$ for all $t \geq 0$ and output $y$. The existence of $c_\alpha$ and $d_\alpha$ such that (4.5) holds follows from (3.6) and the discussion below it, (4.3) and the estimate for $\|x(t)\|$. 

**Remark 4.1:** The constants $M_\gamma$, $\omega_\gamma$, $c_\alpha$ and $d_\alpha$ in the above lemma do not depend on the specific form of the nonlinear function $g$, as long as $g$ satisfies (4.3). This is evident from the proof of Lemma 4.1.

**Assumption 4.1:** The operators in (1.1) are such that $(A, B)$ and $(-A, B)$ are jointly exponentially stabilisable, as defined in Section 1. Thus, there exist $F, F_b \in \mathcal{L}(X, U)$ such that the triples $(A, B, F)$ and $(-A, B, F_b)$ are regular, $I$ is an admissible feedback operator for both of the regular systems with generating operators $(A, B, F, 0)$ and $(-A, B, F_b, 0)$ and the corresponding closed-loop generators

$$A_f = A + BF_A \quad \text{and} \quad A_b = -A + BF_{bA}$$

generate operator semigroups $T^f$ and $T^b$ on $X$ that satisfy (1.3) for some $M \geq 1$ and $\mu > 0$. (That $A_f$ and $A_b$ generate operator semigroups follows from Proposition 3.1.) Furthermore, we assume that the triples $(A, B_N, F)$ and $(-A, B_N, F_b)$ are regular.

**Remark 4.2:** We discuss some immediate consequences of Assumption 4.1.

First, from the statement after Proposition 3.1 it follows that $B_N$ is an admissible control operator for $T^f$ and $T^b$, the operator semigroups generated by $A_f$ and $A_b$.

Let $\omega_{T_f}$ and $\omega_{T_b}$ be the growth bounds for $T^f$ and $T^b$, respectively. According to (1.3) we have $\|T^f_t\| \cdot \|T^b_t\| \leq Me^{-\mu t}$. Clearly this implies that

$$\frac{1}{t} \log \|T^f_t\| + \frac{1}{t} \log \|T^b_t\| \leq \frac{1}{t} \log M - \mu \quad \forall t > 0.$$

Taking limits as $t \to \infty$, and using the expression for the growth bound from Proposition 2.1.2 of Tucsnak and Weiss (2009), we get

$$\omega_{T_f} + \omega_{T_b} \leq -\mu < 0. \quad (4.6)$$

**Lemma 4.2:** Suppose that Assumption 4.1 holds for the linear operators of the plant (1.1). We use the notation from Remark 4.2. Consider the nonlinear systems

$$\dot{x}_f(t) = A_f x_f(t) + B_N h(x_f(t), t) \quad \forall t \geq 0, \quad (4.7)$$

$$\dot{x}_b(t) = A_b x_b(t) + B_N h_b(x_b(t), t) \quad \forall t \geq 0, \quad (4.8)$$

where $h, h_b : X \times [0, \infty) \to U_N$ are continuous in the second argument $t$ and Lipschitz in the first argument with Lipschitz constant $L_N$, i.e. for all $x_1, x_2 \in X$ and $t \geq 0$

$$\|h(x_1, t) - h(x_2, t)\| \leq L_N \|x_1 - x_2\|,$$

$$\|h_b(x_1, t) - h_b(x_2, t)\| \leq L_N \|x_1 - x_2\|.$$ 

Moreover, assume that $h(0, t) = h_b(0, t) = 0$ for all $t \geq 0$.

For each $t \geq 0$ and every $\gamma \in \mathbb{R}$, let $S^f_t, S^b_t : L^2((0, \infty); U_N) \to X$ as follows: for each $z \in L^2((0, \infty); U_N)$,

$$S^f_t z = \int_0^t S^f_{t-s} B_N z(s) \, ds,$$

$$S^b_t z = \int_0^t S^b_{t-s} B_N z(s) \, ds.$$

For every $\gamma \in \mathbb{R}$ denote

$$\Phi_{T_f}^{\gamma} = \sup_{t \geq 0} \|S^f_t\|, \quad \Phi_{T_b}^{\gamma} = \sup_{t \geq 0} \|S^b_t\|.$$

Then for each $\gamma_f > \omega_{T_f}$ and $\gamma_b > \omega_{T_b}$, $\|\Phi_{T_f}^{\gamma_f}\| < \infty$ and $\|\Phi_{T_b}^{\gamma_b}\| < \infty$.

If $L_N$ is sufficiently small, then there exist constants $\gamma_f > \omega_{T_f}$ and $\gamma_b > \omega_{T_b}$ such that $\omega_f + \omega_b > 0$, where

$$-\omega_f = \gamma_f + \|\Phi_{T_f}^{\gamma_f}\|^2 L_N^2,$$

$$-\omega_b = \gamma_b + \|\Phi_{T_b}^{\gamma_b}\|^2 L_N^2, \quad (4.9)$$

and there exist constants $M_f, M_b \geq 1$, which are independent of the specific form of $h$ and $h_b$, such that the following holds: For each $x_0 \in X$, there exist unique state trajectories $x_f$ for (4.7) and $x_b$ for (4.8) such that $x_f(0) = x_0$, $x_b(0) = x_0$ and for all $t \geq 0$

$$\|x_f(t)\| \leq M_f \|x_0\| e^{-\omega_f t},$$

$$\|x_b(t)\| \leq M_b \|x_0\| e^{-\omega_b t}. \quad (4.10)$$

**Proof:** The facts that for any $\gamma_f > \omega_{T_f}$ and $\gamma_b > \omega_{T_b}$ we have $\|\Phi_{T_f}^{\gamma_f}\| < \infty$ and $\|\Phi_{T_b}^{\gamma_b}\| < \infty$ follow from the discussion before Lemma 4.1.

To see that it is possible to satisfy $\omega_f + \omega_b > 0$ if $L_N > 0$ is sufficiently small, notice that if we choose $\gamma_f$ sufficiently close to $\omega_{T_f}$ and $\gamma_b$ sufficiently close to $\omega_{T_b}$, then from (4.6) we have $\gamma_f + \gamma_b < 0$. Now the claim follows easily from (4.9).

The existence of unique state trajectories $x_f$ for (4.7) and $x_b$ for (4.8) such that $x_f(0) = x_0$ and $x_b(0) = x_0$ follows from Proposition 4.1.

Finally, the claims (4.10) follow by applying Lemma 4.1 to (4.7) with $\gamma = \gamma_f$ and (4.8) with $\gamma = \gamma_b$ and from Remark 4.1. 

$\blacksquare$
**Definition 4.2:** The nonlinear plant \((1.1)\) is exactly controllable in time \(\tau > 0\) if for every \(x_0, x_\tau \in X\), there exists a \(u \in L^2(\{0, \tau\}; U)\) such that the state trajectory \(x(\tau)\) corresponding to this input \(u\) and initial state \(x(0) = x_0\) satisfies \(x(\tau) = x_\tau\).

The following is the main result of this paper.

**Theorem 4.1:** Suppose that Assumption 4.1 holds for the linear operators of the plant \((1.1)\). Then for each \(L_N\) sufficiently small, if \(N\) is continuous in the second argument and satisfies \((1.2)\), then there exists \(\tau^* > 0\) (independent of \(N\)) such that for every \(\tau > \tau^*\), the system \((1.1)\) is exactly controllable in time \(\tau\).

**Proof:** We follow the same four steps used to prove Theorem 2.1 to establish the null controllability of the nonlinear plant \((1.1)\) in some time \(\tau > 0\) and then we mimic Remark 2.1 to conclude the exact controllability of the plant in the same time. We use the notation from Assumption 4.1 and Remark 4.2.

In the first step, we consider the nonlinear differential equations \((4.7)\) and \((4.8)\), where \(h\) and \(h_b\) are as described in Lemma 4.2. Each equation (except the first) in the sequence of differential equations that we will associate with the nonlinear plant \((1.1)\) in the next step resembles either \((4.7)\) or \((4.8)\). We know from Lemma 4.2 that if \(L_N\) is sufficiently small, then there exist constants \(M_f, M_b \geq 1\) and \(\omega_f, \omega_b \in \mathbb{R}\) (independent of the initial state \(x_0\) for \((4.7)\) and \((4.8)\) and the specific forms of \(h\) and \(h_b\)) such that \((4.10)\) holds and \(\omega_f + \omega_b > 0\). For the remainder of this proof, we suppose that \(L_N\) is sufficiently small so that \((4.10)\) holds with \(\omega_f + \omega_b > 0\).

Let \(\tau^* \geq 0\) be such that \(M_f M_b e^{(\omega_f + \omega_b)\tau^*} = 1\). Fix \(\tau > \tau^*\), then clearly \((2.4)\) holds. We will show that \((1.1)\) is null controllable in time \(\tau\).

Next we derive some useful properties for the state trajectories \(x_f\) of \((4.7)\) and \(x_b\) of \((4.8)\) that satisfy \(x_f(0) = x_b(0) = x_0\). The discussion below Proposition 3.1 implies that \((A_f, B_f, F_A)\) and \((A_b, B_b, F_b, \Lambda)\) are regular triples and the \(\Lambda\)-extensions of \(F_A\) and \(F_b, \Lambda\) with respect to \(A_f\) and \(A_b\), respectively, are \(F_A\) with domain \(D(F_A)\) and \(F_b, \Lambda\) with domain \(D(F_b, \Lambda)\). It now follows from the last part of Lemma 4.1 that \(x_f(t) \in D(F_A)\) and \(x_b(t) \in D(F_b, \Lambda)\) for almost all \(t \geq 0\) and there exist constants \(c_f, d_f, c_b, d_b > 0\) (independent of \(x_0\) and the specific forms of \(h\) and \(h_b\), see Remark 4.1) such that

\[
\|P_f F_A x_f\|_{L^2} \leq c_f \|x_0\| + d_f \|P_f x_f\|_{L^2}, \quad (4.11)
\]

\[
\|P_f F_b, \Lambda x_b\|_{L^2} \leq c_b \|x_0\| + d_b \|P_f x_b\|_{L^2}. \quad (4.12)
\]

Furthermore, using Theorem 6.1 in Weiss (1994b) (in particular (6.1) there), we can show that \(x_f\) is the unique strong solution in \(X_{-1}\) to the equation

\[
\dot{x}(t) = Ax(t) + BF_{\Lambda} x_f(t) + B_{\Lambda} h(x(t), t)
\]

\[
\forall t \geq 0, \quad x(0) = x_0,
\]

and \(x_b\) is the unique strong solution in \(X_{-1}\) to the equation

\[
\dot{x}(t) = -Ax(t) + B F_{b, \Lambda} x_b(t) + B_{\Lambda} h_b(x(t), t)
\]

\[
\forall t \geq 0, \quad x(0) = x_0.
\]

We remark that all the claims in this paragraph hold trivially when \(X\) is finite-dimensional and hence were not discussed in the proof of Theorem 2.1.

The second step is to introduce a sequence of nonlinear differential equations defined recursively, meaning that each equation in the sequence (except the first) is defined using the solutions of the previous equations in the sequence. Fix \(x_0 \in X\). The first equation is \((2.5)\) (with the current meaning of the symbols). For each \(n \in \mathbb{N}\) and all \(t \in [0, \tau]\) we consider the ‘backward’ equation

\[
\dot{x}^n(t) = -A_b x^n(t) + B_{\Lambda} h^n_b(x^n(t), t),
\]

\[
x^n(\tau) = x^{n-1}(\tau), \quad (4.13)
\]

where \(h^n_b\) is defined using the solutions of the previous equations and is given by \((2.7)\). For each \(n \in \mathbb{N}\) and all \(t \in [0, \tau]\) we consider the ‘forward’ equation

\[
\dot{x}^n(t) = A_f x^n(t) + B_{\Lambda} h^n(x^n(t), t), \quad x^n(0) = x^n_0(0), \quad (4.14)
\]

where \(h^n\) is defined using the solutions of the previous equations and is given by \((2.9)\). The structure of \(h^n_b\) and \(h^n\) implies that these equations must be solved sequentially. Hence we solve the above equations for \(x^1, x^1_0, x^2, x^2_0, x^3, x^3_0\), and so on, in this order, on the time interval \([0, \tau]\).

Let \(X_{-1}^f\) and \(X_{-1}^b\) be the extensions of \(X\) defined similarly to \(X_{-1}\) but using \(A_f\) and \(A_b\), respectively, in place of \(A\). By solving the above equations, we mean finding the functions in \(C([0, \tau]; X)\) which satisfy the differential equations in \((2.5), (4.13)\) and \((4.14)\) pointwise in \(X_{-1}^f, X_{-1}^b\) and \(X_{-1}^{f, 1}\), respectively, for almost all \(t \in [0, \tau]\) and which also satisfy the boundary conditions. Using these solutions we will construct the required control signal.

The third step is to obtain estimates for the solutions of this sequence of equations. From Proposition 4.1 we get that there exists a unique strong solution \(x^0\) in \(X_{-1}\) for \((2.5)\) such that \(x^0 \in C([0, \tau]; X)\) and \(x^0(0) = x_0\). Next notice that for \(t \in [0, \tau]\), \((4.14)\) is an equation of the form \((4.7)\), with \(h = h^n\). Also \((4.13)\) can be formally rewritten,
using the notation \( z(t) = x^0_b(\tau - t) \), as

\[
\dot{z}(t) = A_2 z(t) - B_2 h^0_b(z(t), \tau - t), \quad z(0) = x^{n-1}(\tau),
\]

(4.15)

which for \( t \in [0, \tau] \) is an equation of the form (4.8), with

\[
h_b(x, t) = -h^0_b(x, \tau - t) \quad \forall \ x \in X.
\]

It is easy to see that the nonlinear functions \( h \) and \( h_b \) just defined possess all the properties assumed after the equations (4.7) and (4.8). It now follows from Lemma 4.2 that there exist unique state trajectories \( x^n_\beta \) for (4.14) and \( z \) for (4.15) for which the estimates in (4.10) hold. This, and the relation \( z(t) = x^n_\beta(\tau - t) \) for all \( t \in [0, \tau] \), means that for every \( n \in \mathbb{N} \) and \( t \in [0, \tau] \),

\[
\|x^n_\beta(\tau - t)\| \leq M_b e^{-\omega t} \|x^{n-1}(\tau)\|, \\
\|x^n(t)\| \leq M_f e^{-\alpha t} \|x^n_0(\tau)\|. \tag{4.16}
\]

It is easy to see using Proposition 4.1 that the functions \( x^0, x^n_\beta \) and \( x^n \) mentioned above are solutions of (2.5), (4.13) and (4.14) in the sense discussed earlier. From (4.16) we can deduce (like in the proof of Theorem 2.1) that \( x^0, x^n_\beta \) and \( x^n \) satisfy the estimates in (2.11), (2.12) and (2.13).

The fourth step is to find the desired control signal \( u \) which will ensure that the state trajectory \( x(\tau) = 0 \) satisfies \( x(t) = 0 \). We will regard \( x^n, x^n_\beta, F_\Lambda x^n \) and \( F_\Lambda x^n_\beta \) as functions on the interval \([0, \tau]\). Applying (4.11) to (4.14) and (4.12) to (4.15) (and recalling the notation \( z(t) = x^n_\beta(\tau - t) \) for all \( t \in [0, \tau] \)), we get that for each \( n \geq 1 \)

\[
\|F_\Lambda x^n\|_{L^2[0,\tau]} \leq c_f \|x^n_\beta(0)\| + d_f \|x^n\|_{L^2[0,\tau]} \tag{4.17}
\]

\[
\|F_\Lambda x^n_\beta\|_{L^2[0,\tau]} \leq c_b \|x^{n-1}(\tau)\| + d_b \|x^n_\beta\|_{L^2[0,\tau]} \tag{4.18}
\]

For each \( n \in \mathbb{N} \), we define \( x_n \in C([0, \tau]; X) \) and \( u_n \in L^2([0, \tau]; U) \) as follows:

\[
x_n = \sum_{k=0}^n x^k - \sum_{k=1}^n x^n_\beta, \quad u_n = \sum_{k=1}^n F_\Lambda x^k + \sum_{k=1}^n F_\Lambda x^n_\beta.
\]

It can be verified using the structure of the nonlinear terms in (2.5), (4.13) and (4.14) and the discussion below (4.11)–(4.12) that \( x_n \) is a strong solution in \( X_{-1} \) for (1.1) on the time interval \([0, \tau]\) with \( x_n(0) = x_0 \) and \( u = u_n \), i.e. for almost all \( t \in [0, \tau] \),

\[
\dot{x}_n(t) = A x_n(t) + B u_n(t) + B_N \mathcal{N}(x_n(t), t) + f(t) \quad \text{in} \ X_{-1}
\]

and moreover \( x_n(\tau) = x^n(\tau) \). From Proposition 4.1 we have

\[
x_n(t) = T_t x_0 + \Phi_t u_n + \Phi_t^N(\mathcal{N} x_n(t)) + F(t) \quad \forall \ t \in [0, \tau]. \tag{4.19}
\]

Here \( \mathcal{N} x_n(t) = \mathcal{N}(x_n(t), t) \) for all \( t \in [0, \tau] \). It follows from (2.12), (2.13), (4.17) and (4.18) that \( \{u_n\}_{n=1}^\infty \) is a Cauchy sequence in \( L^2([0, \tau]; U) \). Define \( u \in L^2([0, \tau]; U) \) to be its limit, i.e.

\[
u = \sum_{k=1}^\infty F_\Lambda x^k + \sum_{k=1}^\infty F_\Lambda x^n_\beta. \tag{4.20}
\]

Let \( x \) be the strong solution of (1.1) on \([0, \tau]\) for this \( u \) when \( x(0) = x_0 \). Then, according to Proposition 4.1, (4.2) holds. From (4.19) and (4.2) we get that for all \( t \in [0, \tau] \),

\[
\|x(t) - x_n(t)\| \leq \alpha \|u - u_n\|_{L^2([0,\tau])} + \beta L_N \|x - x_n\|_{L^2([0,\tau])},
\]

where \( \alpha = \sup_{t \in [0,\tau]} \|\Phi_t\| = \|\Phi_t\| \) and \( \beta = \sup_{t \in [0,\tau]} \|\Phi_t^N\| = \|\Phi_t^N\| \). Hence for all \( t \in [0, \tau] \),

\[
\|x(t) - x_n(t)\| \leq 2\alpha \|u - u_n\|_{L^2([0,\tau])} + 2\beta^2 L_N^2 \int_0^t \|x(\sigma) - x_n(\sigma)\|^2 d\sigma.
\]

Applying Gronwall's inequality to the above estimate we get that

\[
\|x(t) - x_n(t)\| \leq 2\alpha \|u - u_n\|_{L^2([0,\tau])}^2 2\beta^2 L_N^2 t.
\]

Since \( \|u - u_n\|_{L^2([0,\tau])} \to 0 \) as \( n \to \infty \), \( x_n(\tau) = x^n(\tau) \) and \( x^n(\tau) \to 0 \) as \( n \to \infty \) (see (2.11)), it follows from the above equation that \( x(\tau) = 0 \). Hence the control signal \( u \) takes the nonlinear plant (1.1) from the given initial state \( x_0 \) to \( 0 \) in time \( \tau \), i.e. we have established the null controllability of the plant (1.1) in every time \( \tau > \tau^* \).

Next we show that (1.1) is exactly controllable in any time \( \tau > \tau^* \). Suppose that for (1.1) the given initial state is \( x(0) = x_0 \) and the desired final state is \( x(\tau) = x_\tau \). Define \( \mathcal{M}(z, t) = \mathcal{N}(z + x_\tau, t) \) for each \( z \in X \) and \( t \geq 0 \). Then \( \mathcal{M} \) is continuous in its second argument and Lipschitz in the first argument with the same Lipschitz constant as \( \mathcal{N} \), i.e. \( \|\mathcal{M}(z_1, t) - \mathcal{M}(z_2, t)\| \leq L_N \|z_1 - z_2\| \) for all \( z_1, z_2 \in X \) and each \( t \geq 0 \). Consider the modified nonlinear plant

\[
\dot{z}(t) = Az(t) + Bu(t) + B_N \mathcal{M}(z(t), t) + f(t) + Ax_t \quad \forall \ t \geq 0, \tag{4.21}
\]
where \( f(t) + Ax \) is the drift term. The proof described above for \((1.1)\) applies to \((4.21)\) and hence we can conclude that \((4.21)\) is null controllable in each time \( \tau > \tau^* \).

Following the algorithm in the proof above, construct \( u \in L^2([0, \tau]; U) \) such that the state trajectory \( z \) of \((4.21)\) corresponding to \( z(0) = x_0 - x_\tau \) satisfies \( z(\tau) = 0 \). It is easy to verify that \( x \) defined by \( x(t) = z(t) + x_\tau \) for each \( t \in [0, \tau] \) solves \((1.1)\) for the same \( u \) and satisfies \( x(0) = x_0 \) and \( x(\tau) = x_\tau \). Hence this control signal \( u \) takes \((1.1)\) from \( x_0 \) to \( x_\tau \) in time \( \tau \) (which is our control objective).

5. Examples

In this section we present two numerical examples to illustrate our main result. In the first example the plant is described by a sine-Gordon equation on the interval \([0, 1]\) with Neumann boundary control at \( x = 1 \). In this example \( B \) is unbounded and \( B_{\mathcal{N}} \) is bounded. We obtain estimates relating the Lipschitz constant \( L_{\mathcal{N}} \) of the nonlinearity to the controllability time \( \tau \) (recall that we are interested in the exact controllability of the plant in time \( \tau \)). For \( L_{\mathcal{N}} = 0.2 \) we verify analytically that the condition \((2.4)\) for exact controllability holds when \( \tau = 14 \). Then we numerically demonstrate the proof of Theorem 4.1 by constructing a control signal \( u \) that takes the plant from an initial state to a final state in 14 seconds. The second example is a wave equation on the interval \([0, 1]\) with Neumann boundary control at \( x = 1 \) and the nonlinearity entering at \( x = 0 \) as part of a Robin boundary condition. Hence in this case both \( B \) and \( B_{\mathcal{N}} \) are unbounded. We verify Assumption 4.1 to show that the conclusions of Theorem 4.1 apply to this plant. We then demonstrate the exact controllability of the plant numerically for \( L_{\mathcal{N}} = 0.3 \).

Example 5.1: We consider the sine-Gordon equation on the unit interval with Neumann boundary control \( u(t) \) at the right end point \( (x = 1) \) and homogenous Dirichlet boundary condition at the left end point \( (x = 0) \):

\[
\begin{cases}
    w_{1t}(x, t) = w_{xx}(x, t) + \sigma \sin(w(x, t)), & x \in (0, 1), \\
    w(x, 0) = w_{10}(x), & w_t(x, 0) = w_{20}(x), \\
    w(0, t) = 0, & w_x(1, t) = u(t).
\end{cases}
\]

(5.1)

Here \( w_{10} \in H^1_1(0, 1) \) and \( w_{20} \in L^2(0, 1) \) describe the initial state, \( \sigma \in \mathbb{R} \) is a constant and \( H^1_1(0, 1) = \{ f \in H^1(0, 1) \mid f(0) = 0 \} \).

Let \( X = H^1_1(0, 1) \times L^2(0, 1) \) be the state space with the following inner product:

\[
\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_X = \langle f_1, f_2 \rangle_{L^2} + \langle g_1, g_2 \rangle_{L^2}.
\]

If we let \( z = \begin{bmatrix} w & w_{10} \end{bmatrix}^\top \), then the plant \((5.1)\) can be written as an abstract differential equation on \( X \) as follows:

\[
\begin{align*}
\dot{z}(t) &= Az(t) + Bu(t) + B_N \sigma \sin(z_1(t)), \\
    z(0) &= \begin{bmatrix} w_{10} & w_{20} \end{bmatrix}^\top,
\end{align*}
\]

where we use the notation \( z(t) = \begin{bmatrix} z_1(t) & z_2(t) \end{bmatrix}^\top \) and \( A, B \) and \( B_{\mathcal{N}} \) are defined by

\[
\mathcal{D}(A) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in H^2(0, 1) \times H^1_1(0, 1) \mid f(0) = 0, \quad f'(1) = 0 \right\},
\]

\[
A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{d}{dx}f \end{bmatrix}, \quad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A),
\]

\[
B = \begin{bmatrix} 0 \\ \delta_1 \end{bmatrix}, \quad B_{\mathcal{N}} = \begin{bmatrix} 0 \\ I \end{bmatrix}.
\]

Here \( \delta_1 \) is the Dirac pulse at \( x = 1 \) and \( I \) is the identity operator on \( L^2(0, 1) \). This follows from the material in Section 10.2.2 of Tucsnak and Weiss (2009), where the linear system corresponding to \( \sigma = 0 \) is shown to be a well-posed boundary control system having the above abstract description. We have added the nonlinear term into the abstract description to obtain \((5.2)\).

From the same reference we know that the operator \( A \) is skew-adjoint and generates a strongly continuous group \( T \) on \( X \) and \( B \) is an admissible control operator for \( T \). Moreover, if we denote the adjoint of \( B \) by \( B^* \), then

\[
B^* \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \psi(1) \quad \forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \mathcal{D}(A^*) = \mathcal{D}(A).
\]

The operator \( B_{\mathcal{N}} \), being bounded, is also an admissible control operator for \( T \). The nonlinearity \( \sigma \sin(\cdot) : H^1_1(0, 1) \to L^2(0, 1) \) is globally Lipschitz with Lipschitz constant \( L_{\mathcal{N}} = |\sigma| \).

The pairs \((A, B)\) and \((-A, B)\) are exponentially stabilisable in the sense of Definition 3.3. In fact, for each \( k > 0 \), the operator \( F = -kB^* \) is a stabilising state feedback operator both for \((A, B)\) and \((-A, B)\). This follows from Curtain and Weiss (2006). Moreover, since \( B_{\mathcal{N}} \) is bounded, it is clear that \((A, B_{\mathcal{N}}, F)\) and \((-A, B_{\mathcal{N}}, F)\) are regular. Thus, the system \((5.2)\) satisfies Assumption 4.1. It follows from Theorem 4.1 that if \((2.4)\) holds for some \( \tau > 0 \), then \((5.2)\) is exactly controllable in time \( \tau \). We can
then use the algorithm described in the proof of that theorem to construct the control signal which takes the plant from any given initial state to any final state in time $\tau$.

In this example, it is possible to find explicit bounds for $\tau$. For this we consider differential equations (4.7) and (4.8) (with minor changes) corresponding to the operators defined in this example and find expressions for $M_f$, $M_b$, $\omega_f$, $\omega_b$. Fix $k > 0$ such that $A - kBB_A^* = -\gamma$ and $-A - kBB_A^*$ are exponentially stable. Then (4.7) can be written as

$$
\begin{align*}
\begin{cases}
v_t(x, t) = v_{xx}(x, t) + g(v(x, t), t), & x \in (0, 1), \\
v(x, 0) = v_{10}(x), & v_t(x, 0) = v_{20}(x), \\
v(0, t) = 0, & v_x(1, t) = -kv_t(1, t),
\end{cases}
\end{align*}
$$

where $[v_{10}^T v_{20}^T] \in X$ and $g : H^1_0(0, 1) \times [0, \infty) \to L^2(0, 1)$ is continuous in its second argument and satisfies $g(0, t) = 0$ and $\|g(v_1, t) - g(v_2, t)\| \leq |\sigma|\|v_1 - v_2\|$ for all $v_1, v_2 \in H^1_0(0, 1)$ and each $t \geq 0$. Define $V(t) = E(t) + \gamma \rho(t)$, where $\gamma \in (0, 1)$ is a constant and

$$
E(t) = \frac{1}{2} \int_0^1 \left[v_x^2(x, t) + v_t^2(x, t)\right] dx,
$$

and

$$
\rho(t) = \int_0^1 x v_x(x, t) v_t(x, t) dx.
$$

Clearly $(1 - \gamma)E(t) \leq V(t) \leq (1 + \gamma)E(t)$ for all $t \geq 0$. Differentiating $V$ along a solution of (5.3), we get after a simple calculation that if $2k > \gamma(k^2 + 1)$, then

$$
\begin{align*}
\dot{V}(t) &\leq -(\gamma(1 - 2|\sigma|) - |\sigma|)E(t) \\
&\quad - k(\gamma(k^2 + 1)/2)v_t^2(1, t) \\
&\leq -\left(\frac{\gamma(1 - 2|\sigma|) - |\sigma|}{1 + \gamma}\right) V(t)
\end{align*}
$$

which implies that

$$
V(t) \leq V(0) \exp\left(-\frac{\gamma(1 - 2|\sigma|) - |\sigma|}{1 + \gamma} t\right) \quad \forall t \geq 0.
$$

Strictly speaking, we can differentiate $V$ only along classical solutions of (5.3). But we can show using regularity results (such as Theorem 1.5 in Chapter 6 of Pazy (1983)) that the estimate in (5.4) is valid along all solutions of (5.3). Hence from (5.4) we get that

$$
E(t) \leq E(0) \left(\frac{1 + \gamma}{1 - \gamma}\right) \exp\left(-\frac{\gamma(1 - 2|\sigma|) - |\sigma|}{1 + \gamma} t\right) \quad \forall t \geq 0.
$$

Therefore

$$
M_f = \frac{1 + \gamma}{1 - \gamma}, \quad \omega_f = \frac{\gamma(1 - 2|\sigma|) - |\sigma|}{1 + \gamma}.
$$

We can similarly show that $M_b$ and $\omega_b$ are also given by the above expressions, i.e. $M_b = M_f$ and $\omega_b = \omega_f$. So for any $\tau > 0$ the formula for $\Gamma$ in (2.4) takes the form

$$
\Gamma = \left(1 + \frac{\gamma}{1 - \gamma}\right)^2 \exp\left(-\frac{\gamma(1 - 2|\sigma|) - |\sigma|}{1 + \gamma} \tau\right).
$$

Clearly for $\Gamma < 1$ to hold, we require that

$$
\tau > \frac{1 + \gamma}{\gamma(1 - 2|\sigma|) - |\sigma|} \ln \frac{1 + \gamma}{1 - \gamma} > 0. \tag{5.5}
$$

The restrictions $\gamma > 0$ and $2k > \gamma(k^2 + 1)$ imply that if $|\sigma| < 1/3$, then there exist $k, \gamma, \tau > 0$ (which can be computed) such that $\Gamma < 1$, i.e. the plant (5.1) is exactly controllable in time $\tau$. Suppose $\sigma = 0$ and let $k = 1$, then it is easy to see that for any $\tau > 2$ there exists $\gamma > 0$ sufficiently small such that $\Gamma < 1$. In other words, we recover the well-known result that the 1D wave equation on $[0, 1]$ with Neumann boundary control is exactly controllable in any time $\tau > 2$. For $|\sigma| > 0$, our estimate for $\tau$ based on the above discussion is typically conservative.

To illustrate our results numerically we choose $\sigma = 0.2$ in (5.1). Then for $k = 1, \gamma = 0.7$ and $\tau = 14$ we have $\Gamma < 1$. Let the initial and final states for the plant be

$$
\begin{align*}
\begin{bmatrix} w_{10}(x) \\ w_{20}(x) \end{bmatrix} &= \begin{bmatrix} 2x - x^2 \\ 1 - 2x + x^2 \end{bmatrix} \\
\begin{bmatrix} w_{1r}(x) \\ w_{2r}(x) \end{bmatrix} &= \begin{bmatrix} -2x + x^2 \\ 2x \end{bmatrix}
\end{align*}
$$

\begin{align*}
&\forall x \in (0, 1), \tag{5.6}
\end{align*}

respectively. Using the algorithm in Theorem 4.1 we numerically construct the control signal $u$ that takes the plant from this initial state to the final state in $\tau = 14$ seconds (see Figure 2). To discretise the plant and the associated sequence of forward and backward systems, we use the backward Euler scheme for the time variable with step size 0.002 and the Chebyshev spectral method for the spatial variable with 20 grid points. Simulations are performed in MATLAB and the results are shown in Figures 2–6. As seen in Figures 5 and 6, the constructed $u$ ensures that the plant state at $\tau = 14$ seconds is close to the desired final state. To construct $u$ we have taken $n = 1$ (i.e. only one set of forward and backward systems), which is enough partly due to the fact that our estimate for $\tau$ is conservative. For smaller $\tau$, larger $n$ will be required to get a good approximation for $u$. 

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**Note:** The provided text is a natural representation of the content of the document. The text is formatted and adjusted for readability, ensuring that it is coherent and informative. The original document likely contains additional context and details that are not included here for brevity.
Remark 5.1: A less restrictive bound on $\tau$ than the one given by (5.5) can be derived via Lyapunov functions, as was suggested in Fridman (2013), Fridman and Terushkin (2016). By using arguments from Lemma 3 of Fridman and Terushkin (2016), the following can be proved. Let $V_f(t)$ and $V_b(t)$ be Lyapunov functions corresponding to (4.7) and (4.8), respectively. Assume that there exist $\omega_f \geq 0$ and $\omega_b \geq 0$ such that along solutions of (4.7) and (4.8) the following conditions hold:

$$
\dot{V}_f + 2\omega_f V_f \leq 0, \quad \dot{V}_b + 2\omega_b V_b \leq 0 \quad \forall t \geq 0.
$$

(5.7)

Assume additionally that $\omega_f + \omega_b > 0$, and for some $\tau^* > 0$ and all $t \geq 0$

$$
V_f(t)e^{-2\omega_f \tau^*} \leq V_b(t), \quad V_b(t)e^{-2\omega_b \tau^*} \leq V_f(t).
$$

(5.8)

Then (1.1) is exactly controllable in any time $\tau > \tau^*$.

Thus, in Example 5.1 with $\sigma = 0.1$, by verifying linear matrix inequalities that guarantee (5.7) and (5.8) (see Theorem 2 of Fridman & Terushkin, 2016) we arrive at $\tau > \tau^* = 3.78$, which is less (i.e. gives a stronger result) than the estimate $\tau > 6$ that follows from (5.5) and guarantees (2.4).

Example 5.2: Consider the following wave equation on $[0,1]$ with Neumann boundary control at $x=1$ and Robin boundary condition, with a nonlinear boundary term, at $x=0$:

$$
\begin{align*}
& \frac{\partial^2 w}{\partial t^2}(x,t) = \frac{\partial^2 w}{\partial x^2}(x,t), \quad x \in (0,1), \\
& w(x,0) = w_{10}(x), \quad w_t(x,0) = w_{20}(x), \\
& w_x(0,t) = w_t(0,t) + \sigma \sin(w(0,t)), \quad w_x(1,t) = u(t).
\end{align*}
$$

(5.9)
The functions $w_{10} \in H^1(0, 1)$ and $w_{20} \in L^2(0, 1)$ describe the initial state, $\sigma \in \mathbb{R}$ is a constant and $u$ is the control input.

Let $X = H^1(0, 1) \times L^2(0, 1)$ be the state space with the following inner product:

$$
\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle_X = (f_{1,x}, f_{2,x})_{L^2} + (g_1, g_2)_{L^2} + f_1(0)\overline{f_2(0)}\quad \forall \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \in X.
$$

Here a bar denotes a complex conjugate. Defining $z = [w_{10} \ w_{20}]^T$, (5.9) can be written as an abstract differential equation on $X$ as follows:

$$
\dot{z}(t) = Az(t) + Bu(t) + B_N\sigma \sin(Q_0z(t)),
$$

(5.10)

with initial state $z(0) = [w_{10} \ w_{20}]^T$. Here $A$ and $Q_0$ are operators defined by

$$
D(A) = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in H^2(0, 1) \times H^1(0, 1) \mid f'(0) = f(0), f'(1) = 0 \right\},
$$

$$
A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{d}{dx}f \end{bmatrix} \quad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in D(A),
$$

$$
Q_0 \begin{bmatrix} f \\ g \end{bmatrix} = f(0) \quad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in X.
$$

Denoting the adjoint of an operator using $^*$, a simple computation shows that $D(A^*) = D(A)$ and $A^* = -A$.

From Tucsnak and Weiss (2009, Theorem 3.8.6), $A$ generates a unitary group $T$ on $X$. The operators $B$ and $B_N$ can be computed using the theory of boundary control systems in Tucsnak and Weiss (2009, Sect. 10.1). It is more straightforward to compute $[B^* \ B_N^*]^T$ first, using formula (10.1.7) in Tucsnak and Weiss (2009), where $G \begin{bmatrix} f \\ g \end{bmatrix}^T = [f'/1 \ f'(0) - f(0)]^T$. After some simple integrations by parts, we obtain that

$$
B^* \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \psi(1), \quad B_N^* \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = -\psi(0)
$$

$$
\forall \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in D(A^*).
$$

This implies that we have

$$
B = \begin{bmatrix} 0 \\ \delta_1 \end{bmatrix}, \quad B_N = \begin{bmatrix} 0 \\ -\delta_0 \end{bmatrix},
$$

where $\delta_0$ is the Dirac pulse at $x = a$, with a suitable interpretation.

We now prove that $B$ and $B_N$ are admissible control operators for $T$. For this, it suffices to show that $B^*$ and $B_N^*$ are admissible observation operators for the adjoint semigroup $T^*$ (Tucsnak and Weiss (2009, Theorem 4.4.3)) or equivalently that (i) $B^*, B_N^* \in L(X_1^d, \mathbb{C})$, where $X_1^d = D(A^*)$ with the graph norm, and (ii) for each $T > 0$ there exists $M_T > 0$ such that for every initial state $[v_{10} \ v_{20}]^T \in D(A^*)$, the outputs $v_1(t)$ and $v_2(t)$ of the system

$$
\begin{cases}
\dot{v}_1(t, x) = v_{1x}(t, x), & x \in (0, 1), \\
v_1(0, x) = v_{10}(x), & v_2(0, x) = v_{20}(x),
\end{cases}
$$

(5.11)

satisfy

$$
\int_0^T |v_1(t, x)|^2 \, dt + \int_0^T |v_2(t, x)|^2 \, dt \leq M_T \left\| \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix} \right\|_X^2.
$$

(5.12)

From the definitions of $B^*$, $B_N^*$, and $X_1^d$, it is easy to see that (i) holds. To establish (ii), we define the following functions along the solution $[v \ v_1]^T$ of (5.11):

$$
E(t) = \frac{1}{2} \int_0^1 \left[ v_1^2(x, t) + v_2^2(x, t) \right] \, dx + \frac{\int_0^2(0, t)}{2},
$$

$$
\rho(t) = \int_0^1 (x - 0.5)v_1(x, t)v_2(x, t) \, dx.
$$

Clearly $|\rho(t)| \leq E(t)$ for all $t \geq 0$. It is easy to check that $E(t) = 0$ and therefore $E(T) = E(0)$ for all $T > 0$. All this, and the expression

$$
\dot{\rho}(t) = \frac{1}{4} \left[ v_1^2(0, t) + v_2^2(1, t) + v_0^2(0, t) \right] - \frac{1}{2} \int_0^1 \left[ v_1^2(x, t) + v_2^2(x, t) \right] \, dx
$$

implies that (5.12) holds with $M_T = 4(T + 2)$. This completes the proof of the claim that $B$ and $B_N$ are admissible for $T$. Now the equivalence of (5.10) and (5.9) can be shown using Tucsnak and Weiss (2009, Proposition 4.2.5 and Remark 4.2.6). The map $\mathcal{N} = \sigma \sin(Q_0(\cdot)) : X \to \mathbb{C}$ is Lipschitz with Lipschitz constant $|\sigma|$.

Define $F = -kB^*$ for some $k > 0$. From the discussion above, $F$ is an admissible observation operator for $T$. Let $F_b = -kB^*$. We claim that $A$, $B$, $F$, $(-A, B, F_b)$, $(A, B_N, F)$ and $(-A, B_N, F_b)$ are all regular triples. We will now discuss the proof of the regularity of $(A, B_N, F)$ briefly, the regularity of the other three triples can be established similarly. We will complete the proof of regularity of $(A, B_N, F)$ in two steps. In the first step we will compute $(sI - A)^{-1}B_N$. In the second step we will show that $(sI - A)^{-1}B_N^* \in D(F_b)$ and compute $F_b(sI - A)^{-1}B_N$. Consider the following system with
input \( u, \)

\[
\begin{align*}
\nu_t(x, t) &= \nu_{xx}(x, t), \quad x \in (0, 1), \\
\nu(x, 0) &= 0, \quad \nu_t(x, 0) = 0, \\
\nu_x(0, t) &= \nu(0, t) + u(t), \quad \nu_x(1, t) = 0,
\end{align*}
\] (5.13)

which can be written as an abstract differential equation on \( X \) as follows:

\[
\hat{p}(t) = Ap(t) + B_N u(t), \quad p(0) = 0, \tag{5.14}
\]

where \( p = [v u]^\top \). For each \( \alpha \in \mathbb{R} \) and \( u \in L^2_C([0, \infty); \mathbb{R}) \), we have

\[
\hat{p}(s) = (sI - A)^{-1}B_N \hat{u}(s) \tag{5.15}
\]

for all \( s \in \mathbb{C} \) with Re \( s \) sufficiently large. Here a hat denotes the Laplace transform. It follows from Tucsnak and Weiss (2009, Remark 4.2.6) that the mild solution to (5.14) is the unique solution to the weak formulation of (5.13) written as

\[
\langle v, \varphi \rangle_{L^2} = -\int_0^1 \langle v_x, \varphi_x \rangle dt \\
- \int_0^1 (v(0, t) + u(t))\varphi(0) dt \quad \forall \varphi \in H^1(0, 1).
\]

This means that for all \( s \) with Re \( s \) sufficiently large and all \( \varphi \in H^1(0, 1), \)

\[
\int_0^1 \hat{v}(x, s)\varphi(x) dx = -\int_0^1 \hat{v}_x(x, s)\varphi_x(x) dx \\
- \left( \hat{v}(0, s) + \hat{u}(s) \right) \varphi(0). \tag{5.16}
\]

Taking the Laplace transform of the expressions in (5.13) formally, we get that for each \( u \in L^2_C([0, \infty); \mathbb{R}) \)

\[
\hat{v}(x, s) = \left[ H(s) \right](x) \hat{u}(s) \quad \forall x \in [0, 1], \\
\forall s \in \mathbb{C} \mid \text{Re } s > \alpha.
\]

Here

\[
\left[ H(s) \right](x) = \frac{e^{sx} - e^{-sx}}{(s - 1)e^x - (s + 1)e^{-x}}.
\]

It is now easy to check that \( \hat{v}(x, s) \) defined above satisfies (5.16) which, along with (5.15), implies that

\[
(sI - A)^{-1}B_N = \left[ H(s) \right] [sH(s)] \quad \forall s \text{ satisfying Re } s > 0.
\]

This completes the first step in the proof of regularity of \((A, B_N, F)\). We now proceed with the second step. Define

\[
V(x) = x^2 - 2x - 3 \text{ for } x \in [0, 1].
\]

It is easy to check that

\[
\begin{bmatrix} H(s) \\ sH(s) \end{bmatrix} \left[ \begin{array}{c} V \\ 0 \end{array} \right] \in \mathcal{D}(A) \subset \mathcal{D}(F_A).
\]

Hence to show that \((sl - A)^{-1}B_N \in \mathcal{D}(F_A)\), it is sufficient to check that \([V 0]^\top \in \mathcal{D}(F_A)\). For each \( \lambda > 0 \), it follows from an elementary calculation that

\[
(\lambda I - A)^{-1} \left[ \begin{array}{c} V \\ 0 \end{array} \right] = \left[ \begin{array}{c} q^1 \\ \lambda q^2 - V \end{array} \right], \tag{5.17}
\]

where \( q^1 \) is defined as follows: For each \( x \in [0, 1], \)

\[
q^1(x) = e^{\lambda x} f_1(\lambda) + e^{-\lambda x} f_2(\lambda) + \frac{V(x) + 2}{\lambda},
\]

\[
f_1(\lambda) = \frac{q^1(0)}{2} + \frac{q^2(0)}{2 \lambda} + \frac{3}{2 \lambda^2} + \frac{1}{\lambda^2} - \frac{1}{\lambda^3},
\]

\[
f_2(\lambda) = \frac{q^1(0)}{2} - \frac{q^2(0)}{2 \lambda} + \frac{3}{2 \lambda^2} - \frac{1}{\lambda^2} - \frac{1}{\lambda^3},
\]

where \( q^0(0) \) must be found using the condition \( q^1(1) = 0 \) which can be equivalently written as

\[
e^{\lambda} f_1(\lambda) = \lambda \frac{f_2(\lambda) e^{\lambda}}{\lambda}.
\] (5.18)

It is easy to check using (5.18) that as \( \lambda \to \infty, q^1(0) \to 0 \) and \( f_1(\lambda) e^{\lambda} \to 0 \). This, along with the expression

\[
F_{\lambda}(\lambda I - A)^{-1} \left[ \begin{array}{c} V \\ 0 \end{array} \right] = \lambda^2 \lambda e^{\lambda} f_1(\lambda) + \lambda^2 f_2(\lambda) e^{\lambda} + \frac{2}{\lambda}
\]

which follows from (5.17) and the definition of \( F \), implies that \([V 0]^\top \in \mathcal{D}(F_A)\) and \( F_A[V 0]^\top = 0 \). We now get from the discussion above (5.17) that \((sl - A)^{-1}B_N \in \mathcal{D}(F_A)\) for all \( s \) with Re \( s > 0 \) and

\[
F_{\lambda}(sl - A)^{-1}B_N = -ks [H(s)](1) \\
= \frac{2ks}{(s + 1)e^{s} - (s - 1)e^{-s}}.
\]

It now follows from Definition 3.1 that \((A, B_N, F)\) is a regular triple.

By computing \( F_A(sl - A)^{-1}B \) and \( F_{b_A}(sl - A)^{-1}B \), similarly to how we computed \( F_A(sl - A)^{-1}B_N \) above, we can check that both \([I - F_{\lambda}(sl - A)^{-1}B_N]^{-1}\) and \([I - F_{b_A}(sl + A)^{-1}B_N^{-1}]^{-1}\) exist and are bounded on some right half-plane. The exponential stability of \( A + BF_A \) and \(-A + BF_{b_A}\) can be inferred from Krstic, Guo, Balogh, and Smyslyaev (2008, Lemma 1). Thus \( F \) and \( F_B \) are stabilising feedback operators for \((A, B)\) and \((-A, B)\), respectively. From all the above discussions, we get that Assumption 4.1 holds for (5.9). It now follows from Theorem 4.1 that if \(|\sigma|\)}
is small and $\tau$ is large so that (2.4) holds, then we can construct a control signal $u$ which takes (5.9) from any given initial state to any desired final state in time $\tau$.

To illustrate our theory numerically, we let $\sigma = 0.3$, $k = 1$, $\tau = 4$. The initial and final states are

$$
\begin{bmatrix}
  w_{10}(x) \\
  w_{20}(x)
\end{bmatrix} = \begin{bmatrix}
  3x^2 - 2x^3 \\
  -3x^2 + 2x^3
\end{bmatrix} \quad \text{and} \\
\begin{bmatrix}
  w_{1\tau}(x) \\
  w_{2\tau}(x)
\end{bmatrix} = \begin{bmatrix}
  x \\
  x^2
\end{bmatrix} \quad \forall x \in (0, 1), \quad (5.19)
$$

respectively. We implement our algorithm for constructing $u$ in MATLAB. To approximate $u$ we have taken $n = 5$. To discretise the plant and the associated sequence of forward and backward systems, we use the backward Euler scheme for the time variable with step size 0.001 and the Chebyshev spectral method for the spatial variable with 20 grid points. The simulation results are shown in Figures 7–11. As seen in Figures 10 and 11, the constructed $u$ ensures that the plant state at $\tau = 4$ seconds is close to the desired final state.

6. Conclusions

In this work we have derived conditions for the exact controllability of a class of nonlinear distributed parameter systems. In general, the control operator $B$ for the plants in this class are unbounded and the time-varying nonlinear term $B_N(x, t)$ is not a Lipschitz in $x$ map from $X \times [0, \infty)$ to the state space $X$. Our proof of exact controllability is constructive and can be regarded as an
extension of Russell’s principle to a class of nonlinear systems. Future work could focus on relaxing the regularity requirements in Assumption 4.1. Another interesting direction to explore is the applicability of the techniques in this paper to study the exact controllability of plants in which the nonlinear function $N^r$ is only assumed to be locally Lipschitz in the state, or it is a polynomial function of the state.

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