Sampled-data implementation of extended PID control using delays

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Abstract
We study the sampled-data implementation of extended PID control using delays for the nth-order stochastic nonlinear systems. The derivatives are approximated by finite differences giving rise to a delayed sampled-data controller. An appropriate Lyapunov–Krasovskii (L-K) method is presented to derive linear matrix inequalities (LMIs) for the exponential stability of the resulting closed-loop system. We show that with appropriately chosen gains, the LMIs are always feasible for small enough sampling period and stochastic perturbation. We further employ an event-triggering condition that allows to reduce the number of sampled control signals used for stabilization and provide $L_2$-gain analysis. Finally, three numerical examples illustrate the efficiency of the presented approach.

KEYWORDS
$L_1$-gain analysis, event-triggered control, PID control, sampled-data control, stochastic perturbations

1 | INTRODUCTION

Proportional-integral-derivative (PID) control is widely used in many industrial processes. Many results on the classical PID control have been established, for example, for the second-order systems and for the nth-order systems. The PID control depends on the output derivative that cannot be measured in practice. Instead, the derivative can be approximated by the finite-difference leading to a delayed feedback. The delay-induced stability was studied, for example, in Niculescu and Michiel and Ramirez et al. using frequency-domain technique. Alternatively, it can be studied using the LMI-based method that allows to cope with, for example, certain types of nonlinearities and stochastic perturbations although being conservative.

Modern control usually employs digital technology for controller implementation, that is, sampled-data control. Moreover, sampled-data controller uses the sampled output only which is more practical. Thus, for practical application of PID control, its sampled-data implementation is important. By using consecutive sampled outputs, sampled-data implementation of PD control was presented for the nth-order deterministic and stochastic systems. Sampled-data implementation of PID control for the second-order deterministic systems was studied in Selivanov and Fridman. However, the idea of using consecutive sampled outputs has not been studied yet for extended PID control of the nth-order deterministic $(n \geq 3)$ or stochastic $(n \geq 2)$ systems.

In this present paper, we study extended PID control of the nth-order stochastic nonlinear systems. Differently from Zhao and Guo with the full knowledge of the system state, we consider sampled-data implementation of extended PID control by using the sampled outputs only. Following the improved approximation method with consecutive sampled outputs, we approximate the extended PID controllers depending on the output and its derivatives up to the order...
$n - 1$ as delayed sampled-data controllers. Extension to PID control of the $n$th-order stochastic systems is far from being straightforward for the following reasons:

(i) Comparatively to the models under the PD control\textsuperscript{13} or the PID control,\textsuperscript{15,16} we have additional errors to be compensated by employing additional terms in the corresponding Lyapunov functionals.

(ii) The Lyapunov functionals of Selivanov and Fridman\textsuperscript{13,15,16} are not applicable in the stochastic case. This is because a solution of a stochastic system does not have a derivative.\textsuperscript{12,14} Thus, we propose novel Lyapunov functionals depending on the deterministic and stochastic parts of the system that lead to LMI-based stability conditions.

We show that the LMIs are always feasible for small enough sampling period and stochastic perturbation if the extended PID controller that employs the full-state stabilizes the system. Moreover, we employ an event-triggering condition\textsuperscript{17-19} that allows to reduce the number of sampled control signals used for stabilization and provide $L_2$-gain analysis. Finally, three numerical examples are presented to illustrate the efficiency of the presented approach.

1.1 Notations and useful inequalities

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $I_n$ is the identity $n \times n$ matrix, the superscript $T$ stands for matrix transposition. $\mathbb{R}^n$ denotes the $n$ dimensional Euclidean space with Euclidean norm $\cdot \| \cdot \|$. $\mathbb{R}^{n \times m}$ denotes the set of all $n \times m$ real matrices with the induced matrix norm $\| \cdot \|$. Denote by $\text{diag}\{ \ldots \}$ and $\text{col}\{ \ldots \}$ block-diagonal matrix and block-column vector, respectively. $P > 0$ implies that $P$ is a positive definite symmetric matrix. $C^i$ is a class of $i$ times continuously differentiable functions.

We now present some useful inequalities:

**Lemma 1.** (Extended Jensen’s inequality\textsuperscript{20}) Denote $G = \int_a^b f(x) dx$, where $f : [a, b] \to \mathbb{R}$, $x : [a, b] \to \mathbb{R}$ and the integration concerned is well defined. Then for any $n \times n$ matrix $R > 0$ the following inequality holds:

$$G^T R G \leq \int_a^b |f(s)| ds \int_a^b |f(s)| x^T(s) R x(s) ds.$$

**Lemma 2.** (Exponential Wirtinger’s inequality\textsuperscript{21}) Let $x(t) : (a, b) \to \mathbb{R}^n$ be absolutely continuous with $x \in L_2(a, b)$ and $x(a) = 0$ or $x(b) = 0$. Then the following inequality holds:

$$\int_a^b e^{2a} x^T(s) W x(s) ds \leq e^{2b} (b - a)^2 \pi^2 \int_a^b e^{2a} x^T(s) W x(s) ds,$$

for any $a \in \mathbb{R}$ and $n \times n$ matrix $W > 0$.

2 Extended PID Control of Stochastic Nonlinear Systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration is a family $(\mathcal{F}_t, t \geq 0)$ of nondecreasing sub-$\sigma$-algebras of $\mathcal{F}$, that is, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$ and $\mathbb{P}(\cdot)$ is the probability of an event enclosed in the brackets. The mathematical expectation $\mathbb{E}$ of a random variable $\xi = \xi(w)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is defined as $\mathbb{E} \xi = \int_\Omega \xi(w) d\mathbb{P}(w)$. The scalar standard Wiener process (also called Brownian motion) is a stochastic process $w(t)$ with normal distribution satisfying $w(0) = 0$, $\mathbb{E} w(t) = 0$ ($t > 0$) and $\mathbb{E} w^2(t) = t$ ($t > 0$).

Consider the $n$th-order stochastic nonlinear system

$$d y^{(0)}(t) = \left[ \sum_{i=0}^{n-1} a_i y^{(i)}(t) + bu(t) + g(t, y^{(0)}(t), \ldots, y^{(n-1)}(t)) \right] dt + \sum_{i=0}^{n-1} d y^{(i)}(t) dw(t).$$

(1)

Here $y(t) = y^{(0)}(t) \in \mathbb{R}^p$ is the output, $y^{(i)}(t)$ $(i = 1, \ldots, n - 1)$ is the $i$th derivative of $y(t)$, $a_i, b_i \in \mathbb{R}^{p \times p}$ and $b \in \mathbb{R}^{p \times q}$ are constant matrices and $g : \mathbb{R} \times \mathbb{R}^p \times \ldots \times \mathbb{R}^p \to \mathbb{R}^p$ is a locally Lipschitz continuous in arguments from the second to the
last and satisfies for all \( t \geq 0 \) the inequality

\[
|g(t, x_0, \ldots, x_{n-1})|^2 \leq \sum_{i=0}^{n-1} x_i^T M_i x_i \quad \forall x_i \in \mathbb{R}^p, \quad i = 0, \ldots, n - 1,
\]

with some matrices \( 0 < M_i \in \mathbb{R}^{p \times p} \) \((i = 0, \ldots, n - 1)\).

In Zhao and Guo, an extended PID controller was designed as follows

\[
u(t) = \left[ \bar{K}_p y(t) + \bar{K}_d \int_0^t y(s) ds + \sum_{i=1}^{n-1} \bar{K}_D_i y^{(i)}(t) \right], \tag{3}
\]

where \( \bar{K}_p, \bar{K}_d, \) and \( \bar{K}_D_i \in \mathbb{R}^{q \times p} \) \((i = 1, \ldots, n - 1)\) are the controller gains. Differently from Zhao and Guo with the full knowledge of the system state \((i.e., y^{(i)}(t), i = 0, \ldots, n - 1)\), we consider the output-feedback control, where \( y^{(i)}(t), i = 1, \ldots, n - 1 \) in (3) are not available. Moreover, for the practical implementation we assume that the output \( y(t) \) is available only at the discrete-time instants \( t_k = kh \), where \( k \in \mathbb{N}_0 \) and \( h > 0 \) is the sampling period. As in Selivanov and Fridman, we suggest the following approximations for \( t \in [t_k, t_{k+1}] \), \( k \in \mathbb{N}_0 \):

\[
y(t) = \bar{y}(t) \approx y(t_k), \quad \int_0^t y(s) ds \approx \int_0^{t_k} \bar{y}(s) ds \approx h \sum_{j=0}^{k-1} \bar{y}(t_j), \quad y^{(i)}(t) \approx \bar{y}^{(i)}(t_k), \quad i = 1, \ldots, n - 1, \tag{4}
\]

where we used \( \int_0^t \bar{y}(s) ds = \sum_{j=0}^{k-1} \int_{t_j}^{t_j+h} \bar{y}(s) ds = \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \bar{y}(s) ds = h \sum_{j=0}^{k-1} \bar{y}(t_j) \) for the approximation of the integral and applied the finite-difference method for \( \bar{y}^{(i)}(t_k) \) \((i = 1, \ldots, n - 1)\) with

\[
\bar{y}^{(i)}(t_k) = \frac{\bar{y}^{(i-1)}(t_k) - \bar{y}^{(i-1)}(t_k - h)}{h}, \quad i = 1, \ldots, n - 1, \quad \bar{y}^{(0)}(t_k) = \bar{y}(t_k) = y(t_k). \tag{5}
\]

and \( y(t) = y(0) \) for \( t < 0 \). It is clear that via (5) we can compute \( \bar{y}^{(i)}(t_k) \) (and thus, \( \bar{y}^{(i)}(t_k), i = 2, \ldots, n - 1 \)).

Thus, we design in this paper the following sampled-data controller

\[
u(t) = \bar{K}_p \bar{y}(t_k) + h \bar{K}_d \sum_{j=0}^{k-1} \bar{y}(t_j) + \sum_{i=1}^{n-1} \bar{K}_D_i \bar{y}^{(i)}(t_k), \quad t \in [t_k, t_{k+1}], \quad k \in \mathbb{N}_0. \tag{6}
\]

In order to study the stability of system (1) under the sampled-data controller (6), we first present the approximation errors \( \bar{y}(t_k) - y(t) \) and \( \bar{y}^{(i)}(t_k) - y^{(i)}(t) \) \((i = 1, \ldots, n - 1)\), where \( t \in [t_k, t_{k+1}], k \in \mathbb{N}_0 \), in a convenient form suitable for the later analysis via L-K functionals.

**Proposition 1.** If \( y \in C^i \) and \( y^{(i)} \) is absolutely continuous with \( i = 1, \ldots, n \), then \( \bar{y}(t_k) \) and \( \bar{y}^{(i)}(t_k) \) \((i = 1, \ldots, n - 1)\) defined by (5) satisfy for \( t \in [t_k, t_{k+1}] \), \( k \in \mathbb{N}_0 \)

\[
\bar{y}(t_k) = y(t) - \int_{t_k}^t \bar{y}(s) ds, \tag{7}
\]

\[
\bar{y}^{(i)}(t_k) = y^{(i)}(t) - \int_{t_{k-ih}}^{t_i} \varphi_i(t-s) y^{(i)}(s) ds - \int_{t_{k-ih}}^{t_{k-ih+h}} \bar{y}(s) ds, \quad i = 1, \ldots, n - 1, \tag{8}
\]

where

\[
\varphi_i(v) = \frac{h - v}{h}, \quad v \in [0, h],
\]

\[
\varphi_i(v) = \begin{cases} \frac{1}{h} \int_0^v \varphi_i(\lambda) d\lambda + \frac{h-v}{h}, & v \in [0, h] \\ \frac{1}{h} \int_{v-h}^v \varphi_i(\lambda) d\lambda, & v \in (h, ih), \\ \frac{1}{h} \int_{v-h}^{ih} \varphi_i(\lambda) d\lambda, & v \in [ih, ih + h]. \end{cases}, \quad i = 1, \ldots, n - 2. \tag{9}
\]
Proof. We first introduce the errors due to the sampling:

\[ y(t_k) = y(t) - \int_{t_k}^{t} \bar{y}(s) \, ds, \quad \bar{y}^0(t_k) = \bar{y}^0(t) - \int_{t_k}^{t} \bar{y}^0(s) \, ds, \quad i = 1, \ldots, n - 1. \]  

(10)

Taking into account \( y(t_k) = \bar{y}(t_k) \) in (4), together with the first equality in (10) we obtain (7). Then following arguments for the error \( \bar{y}^0(t) - y^0(t) \) \( i = 1, \ldots, n - 1 \) in Proposition 1 of Selivanov and Fridman, \(^{13}\) that is,

\[ \bar{y}^0(t) = y^0(t) - \int_{t_i}^{t} \varphi_i(t - s) y^0(s) \, ds, \quad i = 1, \ldots, n - 1, \]  

(11)

where \( \varphi_i(\cdot) (i = 1, \ldots, n - 1) \) are defined by (9), we arrive at (8). \( \blacksquare \)

The functions \( \varphi_i(\cdot) (i = 1, \ldots, n - 1) \) have the following properties (see the proof in Selivanov and Fridman\(^ {13}\)):

**Proposition 2.** The functions \( \varphi_i(\cdot) (i = 1, \ldots, n - 1) \) in (9) satisfy

1) \( \varphi_i(0) = 1 \), \( \varphi_i(ih) = 0 \); 2) \( 0 \leq \varphi_i(\nu) \leq 1 \); 3) \( \frac{d}{dv} \varphi_i(v) \in \left[ \frac{1}{h}, 0 \right] \); 4) \( \int_0^{ih} \varphi_i(v) \, dv = \frac{i h}{2} \).

(12)

By noting that \( y(t_j) = \bar{y}(t_j) \) \( j = 0, \ldots, k - 1 \), via (7) and (8) the sampled-data controller (6) can be presented as

\[
\begin{align*}
    u(t) &= \bar{K}_p \left[ y(t) - \int_{t_k}^{t} \bar{y}(s) \, ds \right] + \bar{h} \sum_{j=0}^{k-1} y(t_j) + \sum_{i=1}^{n-1} \bar{K}_D_i \left[ y^0(t) - \int_{t_i}^{t} \varphi_i(t - s) y^0(s) \, ds - \int_{t_i}^{t} y^0(s) \, ds \right], \\
    &= \bar{K} x(t) + [\bar{K}_p, \bar{K}_D, \delta_0(t), \sum_{i=1}^{n-1} \bar{K}_D_i (\delta_i(t) + \kappa_i(t))] + [\kappa(t) + \delta_0(t)], \\
    &= \kappa(t) + \delta_0(t) + \sum_{i=1}^{n-1} \bar{K}_D_i (\delta_i(t) + \kappa_i(t)), \quad t \in [t_k, t_{k+1}], \quad k \in \mathbb{N}_0,
\end{align*}
\]

(13)

where

\[
\begin{align*}
    x(t) &= \text{col} \left\{ y(t), y^0(t), \ldots, y^{(n-1)}(t), (t - t_k) y(t_k) + h \sum_{j=0}^{k-1} y(t_j) \right\}, \\
    K &= [\bar{K}_p, \bar{K}_D_1, \ldots, \bar{K}_D_{n-1}, \bar{K}_f], \\
    \delta_0(t) &= - \int_{t_k}^{t} \left[ \begin{array}{c} H_0 \cr H_n \end{array} \right] x(s) \, ds, \\
    \delta_i(t) &= - \int_{t_i}^{t} \bar{y}^0(s) \, ds, \quad \kappa_i(t) = - \int_{t_i}^{t} \varphi_i(t - s) H_i x(s) \, ds, \quad i = 1, \ldots, n - 1, \\
    H_i &= \left[ 0_{pxp}, I_p, 0_{px(n-1)p} \right], \quad i = 0, \ldots, n.
\end{align*}
\]

(14)

Using (13) and (14), the system (1), (6) has the form

\[
dx(t) = f(t) dt + D x(t) dw(t), \quad t \in [t_k, t_{k+1}], \quad k \in \mathbb{N}_0.
\]

(15)

where

\[
f(t) = (A + BK) x(t) + A_1 \delta_0(t) + \sum_{i=1}^{n-1} b \bar{K}_D_i (\delta_i(t) + \kappa_i(t)) + H_{n-1}^{T} g(t, H_0 x(t), \ldots, H_{n-1} x(t)).
\]

\[
A_1 = \begin{bmatrix}
0 & I_p & 0 & \ldots & 0 & 0 \\
0 & 0 & I_p & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & I_p & 0 \\
a_0 & a_1 & a_2 & \ldots & a_{n-1} & 0 \\
I_p & 0 & 0 & \ldots & 0 & 0
\end{bmatrix},
\]

(16)

\[
A = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 \\
a_0 & a_1 & a_2 & \ldots & a_{n-1} \\
I_p & 0 & 0 & \ldots & 0
\end{bmatrix},
\]

\[
B = \text{col} \left[ 0_{(n-1)pxp}, b, 0_{pxp} \right], \\
D = \text{col} \left[ 0_{(n-1)pxp}, \bar{D}, 0_{pxp} \right], \\
\bar{D} = [d_0, \ldots, d_{n-1}, 0].
\]
Remark 1. In (15), we follow the transformation of Zhang and Fridman\textsuperscript{23} that allowed to avoid an additional non-zero term \( y(n-1)(t_k) - y(n-1)(t) = - \int_{t_k}^t H_{n-1} f(s)ds - \Pi \) with \( \Pi = \int_{t_k}^t H_{n-1} D x(s)dw(s) \). Note that the term \( \Pi \) has to be compensated by additional terms in Lyapunov functional. Hence, the transformation in (15) (comparatively to Selivanov and Fridman\textsuperscript{15,16}) significantly simplifies the analysis in the stochastic case.

Comparatively to the system model (see e.g., (27) in Selivanov and Fridman\textsuperscript{13}) under PD control, the system (15) includes additional term \( A \delta(t) \) (due to the additional I control) that will be compensated by the additional term \( V_{\delta_0} \) defined below (20). Note also that Lyapunov functional of Selivanov and Fridman\textsuperscript{13} depends on the \( n \)-th order derivative, and, thus, is not applicable in the stochastic case. This is because a solution of a stochastic system does not have a derivative.\textsuperscript{12,14} We will present LMI conditions via novel Lyapunov functional that depends on the deterministic and stochastic parts of the system:

**Theorem 1.** Consider the stochastic nonlinear system (1) under the sampled-data controller (6). Given \( \overline{K}_P, \overline{K}_I, \) and \( \overline{K}_D \), \( (i = 1, \ldots, n - 1) \) let the extended PID controller (3) exponentially stabilizes (1), where \( d_i = 0 \) \( (i = 0, \ldots, n - 1) \) and \( g \equiv 0 \) with a decay rate \( \bar{\alpha} > 0 \).

(i) Given tuning parameters \( h > 0, \alpha \in (0, \overline{\alpha}) \) and \( p \times p \) matrices \( M_i \) \( (i = 0, \ldots, n - 1) \), let there exist \( (n + 1)p \times (n + 1)p \) matrix \( P > 0, 2p \times 2p \) matrix \( W_0 > 0, p \times p \) matrices \( W_I > 0, R_I > 0 \) \( (i = 1, \ldots, n - 1) \), \( Q > 0, F_I > 0 \) and \( F_2 > 0 \) and scalar \( \lambda > 0 \) that satisfy

\[
\Phi = \begin{bmatrix}
\Phi_{11} & PA_1 & \Phi_{14} & 0 & PH_{n-1}^T & h(A + BK)^TH_{n-1}^T & h(0, [I, 0])^TW_0 \\
- \frac{\alpha}{4} e^{-2\alpha h}W_0 & 0 & 0 & 0 & hA_1^TH_{n-1}^T & h[0, [I, 0]]^TW_0 \\
* & \Phi_{33} & 0 & 0 & hA_1^TH_{n-1}^T & 0 \\
* & * & \Phi_{44} & \Phi_{45} & 0 & hA_1^TH_{n-1}^T \\
* & * & * & \Phi_{45} & 0 & hA_1^TH_{n-1}^T \\
* & * & * & * & -e^{-2\alpha(n-1)h} (R_{n-1} + F_2) & 0 \\
* & * & * & * & -\lambda I_p & hA_1^TH_{n-1}^T \\
* & * & * & * & * & -\Xi \\
* & * & * & * & * & -W_0
\end{bmatrix} < 0, \quad (17)
\]

\[
\Psi = \begin{bmatrix}
W_{n-1} & W_{n-1} \\
W_{n-1} & W_{n-1} - \frac{(n-1)^2}{2} e^{-2\alpha(n-1)h} F_2
\end{bmatrix} < 0, \quad (18)
\]

where

\[
\Phi_{11} = P(A + BK) + (A + BK)^TP + 2\alpha P + \sum_{i=0}^{n-2} h_i e^{2\alpha h} H_{i+1}^T W_I H_{i+1} + \sum_{i=1}^{n-2} \frac{(ih)^2}{4} H_{i+1}^T R_I H_{i+1}
\]

\[
+ D^TPD + \frac{(n-1)h}{2} D^TH_{n-1}(F_I + F_2)H_{n-1}D + \lambda \sum_{i=0}^{n-1} H_{i+1}^T M_i H_i,
\]

\[
\Phi_{13} = \Phi_{14} = PB[\overline{K}_{D_1}, \ldots, \overline{K}_{D_{n-1}}], \quad \Phi_{33} = - \frac{\alpha^2}{4} e^{-2\alpha h} \text{diag}(W_I, \ldots, W_{n-1}),
\]

\[
\Phi_{44} = -\text{diag}(e^{-2\alpha h} R_1, \ldots, e^{-2\alpha(n-1)h} R_{n-1}), \quad \Phi_{45} = [0, -e^{-2\alpha(n-1)h} R_{n-1}]^T,
\]

\[
\Phi_{37} = \Phi_{47} = [\overline{K}_{D_1}, \ldots, \overline{K}_{D_{n-1}}]^T B^T H_{n-1}^T, \quad \Xi = \frac{(n-1)^2}{4} R_{n-1} + e^{2\alpha(n-1)h} Q,
\]

with \( A, B, A_1 \) and \( D \) given by (16), and \( K \) and \( H_i \) \( (i = 0, \ldots, n) \) given by (14). Then the sampled-data controller (6) exponentially mean-square stabilizes (1) with a decay rate \( \alpha \).

(ii) Given any \( \alpha \in (0, \overline{\alpha}) \), LMI (17) is always feasible for small enough \( h > 0, \|D\| \) and \( \|M_i\| \) \( (i = 0, \ldots, n - 1) \) (meaning that the sampled-data controller (6) exponentially mean-square stabilizes (1) with a decay rate \( \alpha \)).

**Proof.** (i) We consider the functional

\[
V = V_0 + V_{\delta_0} + \sum_{i=1}^{n-1} (V_{\delta_i} + V_{\gamma_i} + V_{\eta_i}) + V_{\delta_n} + V_{F_1} + V_{F_2}, \quad (20)
\]
where

\[
V_0(x(t)) = x^T(t)Px(t),
\]

\[
V_\delta_i(t, \dot{x}_i) = \begin{cases} 
\begin{bmatrix} h^2 \int_{t_i}^t e^{-2\alpha(t-s)} \dot{x}^T(s) & H_0^T \\ H_0 \\ H_n \\ H_n \\ H_n \\ H_n \end{bmatrix} \begin{bmatrix} H_0 \\ H_0 \\ H_n \\ H_n \\ H_n \\ H_n \end{bmatrix} \dot{x}(s)ds - \frac{\alpha}{4} e^{-2\alpha \delta_0(t)} \delta_0(t)W_0 \dot{x}(s)ds, & i = 0, \\
\begin{bmatrix} h^2 \int_{t_i}^t e^{-2\alpha(t-s)} \dot{y}^T(s) & \chi(s) \end{bmatrix} \begin{bmatrix} \gamma(s) \\ \gamma(s) \end{bmatrix} ds - \frac{\alpha}{4} e^{-2\alpha \delta_i(t)} \delta_i(t)W_i \dot{x}(s)ds, & i = 1, \ldots, n - 1,
\end{cases}
\]

\[
V_\delta_1(x_i) = h^2 e^{2\alpha \delta_1} \int_{t_i}^t e^{-2\alpha(t-s)} \phi_1(t-s)x^T(s)H_{i+1}^T W_i H_{i+1} x(s)ds, & i = 1, \ldots, n - 2,
\]

\[
V_{\delta_{i-1}}(f_i) = h^2 e^{2\alpha(n-1)\delta_0} \int_{t_{i-1}}^t e^{-2\alpha(t-s)} \phi_{n-1}(t-s)f^T(s)H_{n-1}^T QH_{n-1} f(s)ds,
\]

\[
V_{\lambda_0}(x_i) = \frac{ih}{2} \int_{t_i}^t e^{-2\alpha(t-s)} \phi_0(t-s)x^T(s)H_{i+1}^T R_i H_{i+1} x(s)ds, & i = 1, \ldots, n - 2,
\]

\[
V_{\lambda_{i-1}}(f_i) = \frac{(n-1)h}{2} \int_{t_{i-1}}^t e^{-2\alpha(t-s)} \phi_{n-1}(t-s)f^T(s)H_{n-1}^T H_{n-1} R_{n-1} H_{n-1} f(s)ds,
\]

\[
V_{P_1}(x_i) = \frac{(n-1)h}{2} \int_{t_{i-1}}^t e^{-2\alpha(t-s)} \phi_{n-1}(t-s)x^T(s)H_{i+1}^T H_{i+1} P_1 H_{i+1} D(x)ds,
\]

\[
V_{P_2}(x_i) = \int_{t_{i-1}}^t e^{-2\alpha(t-s)} \phi_{n-1}(t-s)x^T(s)H_{i+1}^T H_{i+1} P_2 H_{i+1} D(x)ds
\]

with \( P > 0, W_i > 0 \ (i = 0, \ldots, n - 1), R_i > 0 \ (i = 1, \ldots, n - 1), Q > 0, F_1 > 0, F_2 > 0 \) and

\[
\phi_i(v) = \int_{v}^{\infty} \psi_0(\lambda) d\lambda, & i = 1, \ldots, n - 1.
\]

Here \( x_i(\theta) = x(t+\theta), \ \theta \in [-h, 0] \). Since \( \delta_0(t) = -[H_0^T, H_n^T] \dot{x}(t), \ \delta_i(t) = -[\gamma_i(t)] \ (i = 1, \ldots, n - 1) \), Lemma 2 implies \( V_{\delta_i} \geq 0 \) for \( i = 0, \ldots, n - 1 \). Due to \( \phi_i(\cdot) \geq 0 \) and \( \psi_0(\cdot) \geq 0 \) we have the positivity of functional \( V(t) \) in (20). Note that the terms \( V_{\delta_1}(i = 1, \ldots, n - 1), V_{\delta_{i-1}}(i = 1, \ldots, n - 2), V_{\lambda_{i-1}}(i = 1, \ldots, n - 2) \) are from Selivanov and Fridman, whereas the novel terms \( V_{\lambda_{n-1}}, V_{\lambda_{n-1}}(i = 1, \ldots, n - 1), V_P, V_{P_2} \) are stochastic extensions of Lyapunov functionals that depend on \( \dot{x}(t) \).

Let \( L \) be the generator (see e.g., Shaikhet and Mao). We have along (15)

\[
LV_0 + 2aV_0 = 2x^T(t) Pf(t) + x^T(t) D^T浦x(t) + 2ax^T(t)P \dot{x}(t).
\]

Moreover, we have

\[
LV_\delta_i + 2aV_\delta_i = \begin{cases} 
\begin{bmatrix} h^2 \dot{x}^T(t) & H_0^T \\ H_0 \\ H_n \\ H_n \\ H_n \\ H_n \end{bmatrix} \begin{bmatrix} H_0 \\ H_0 \\ H_n \\ H_n \\ H_n \\ H_n \end{bmatrix} \dot{x}(t) - \frac{\alpha}{4} e^{-2\alpha \delta_0(t)}W_0 \dot{x}(t), & i = 0, \\
\begin{bmatrix} h^2 \chi^T(t) & \gamma(s) \end{bmatrix} \begin{bmatrix} \gamma(s) \\ \gamma(s) \end{bmatrix} ds - \frac{\alpha}{4} e^{-2\alpha \delta_i(t)}W_i \dot{x}(t), & i = 1, \ldots, n - 1
\end{cases}
\]

The terms \( V_\delta_i, i = 1, \ldots, n - 2 \) are introduced to compensate \( h^2 \begin{bmatrix} \gamma_i(t) \\ \gamma_i(t) \end{bmatrix} \begin{bmatrix} \gamma_i(t) \\ \gamma_i(t) \end{bmatrix} ds, i = 1, \ldots, n - 2 \) in (22). By using Lemma 1, via (12) we have

\[
LV_{\delta_i} + 2aV_{\delta_i} = h^2 e^{2\alpha \delta_i} x^T(t)H_{i+1}^TW_iH_{i+1}x(t) - h^2 e^{2\alpha \delta_i} \int_{t_i}^t e^{-2\alpha(t-s)} \frac{d}{ds} \phi_i(t-s) x^T(s)H_{i+1}^T W_i H_{i+1} x(s)ds
\]
\( \leq h^2 e^{2a(t)} x^T(t) H_{i+1}^T W_i H_{i+1} x(t) \)

\(- h^2 (f_{i+1} \phi(t - s) - \int_{t-i}^t \sum_{i=1}^{n-2} H_i(x(s))ds, \quad i = 1, \ldots, n-2.\)

(23)

From (8), it follows that

\( \tilde{y}^{(i)}(t) = y^{(i)}(t) - \int_{t-i}^t \phi_i(t - s) y^{(i)}(s)ds, \quad i = 1, \ldots, n-1.\)

Via (12) the latter implies

\( \tilde{y}^{(i)}(t) = \int_{t-i}^t \sum_{i=1}^{n-2} \phi_i(t - s) \tilde{y}^{(i)}(s)ds = \int_{t-i}^t \phi_i(t - s) H_i x(s)ds, \quad i = 1, \ldots, n-1.\)

(24)

Noting that \( \int_{t-i}^t \phi_i(t - s) = \phi_i(0) - \phi_i(ih) = 1 \) and \( H_i x(s) = H_{i+1} x(s) (i = 0, \ldots, n-2), \) from (23) and (24) we have

\( LV \tilde{y}^{(i)}(t) \leq h^2 e^{2a(t)} x^T(t) H_{i+1}^T W_i H_{i+1} x(t) - h^2 \tilde{y}^{(i)}(t)^T W_i \tilde{y}^{(i)}(t), \quad i = 1, \ldots, n-2.\)

(25)

Then the terms \(- h^2 \tilde{y}^{(i)}(t)^T W_i \tilde{y}^{(i)}(t) (i = 1, \ldots, n-2)\) in the above expression will cancel the positive term of \( LV \tilde{y}^{(i)}(t) \) \( (i = 1, \ldots, n-2). \)

Note that the term \( \tilde{y}^{(i)}(t) \) with \( i = n-1 \) in (24) has the following form:

\( \tilde{y}^{(n-1)}(t) = \int_{t-(n-1)h}^t \phi_{n-1}(t - s) H_{n-1} x(s)ds = \rho_1(t) + \rho_2(t),\)

(26)

where

\[ \rho_1(t) = \int_{t-(n-1)h}^t \phi_{n-1}(t - s) H_{n-1} x(s)ds, \quad \rho_2(t) = \int_{t-(n-1)h}^t \phi_{n-1}(t - s) H_{n-1} D x(s)dw(s).\]

Thus

\( LV \tilde{y}^{(n-1)}(t) \leq h^2 [\rho_1(t) + \rho_2(t)]^T W_{n-1} [\rho_1(t) + \rho_2(t)] - \pi^2 e^{-2a(t)} \delta_{n-1}^T(t) W_{n-1} \delta_{n-1}(t).\)

(27)

To compensate \( \rho_2(t) \), we employ the term \( V_{\tilde{y}^{(n-1)}} \), that is,

\[ LV \tilde{y}^{(n-1)} + 2a \phi_1 \tilde{y}^{(n-1)} = h^2 e^{2a(n-1)} f^T(t) H_{n-1}^T Q H_{n-1} f(t) - h^2 e^{2a(n-1)} h \int_{t-(n-1)h}^t \phi_{n-1}(t - s) f^T(s) H_{n-1}^T Q H_{n-1} f(s)ds \]

\[ \leq h^2 e^{2a(n-1)} f^T(t) H_{n-1}^T Q H_{n-1} f(t) - h^2 \rho_1^2(t) Q \rho_1(t), \]

(28)

where we applied Lemma 1 with (12). Note that (12) implies

\[ \phi_1(0) = \int_0^{ih} \phi_1(\lambda) d\lambda = \frac{ih}{2}, \quad \phi_1(ih) = 0, \quad i = 1, \ldots, n-1.\]

(29)

For the \( \rho_2(t) \)-term, by using Itô isometry (see, e.g., Shaikh and Mao), via (12) we have for any \( p \times p \) matrix \( F_1 > 0 \)

\[ e^{-2a(t)} h E f_2^T(t) F_1 \rho_2(t) = e^{-2a(t)} h E \int_{t-(n-1)h}^t \phi_{n-1}(t - s) f^T(s) D H_{n-1}^T F_1 H_{n-1} D x(s)ds \]

\[ \leq E \int_{t-(n-1)h}^t e^{-2a(t)} \phi_{n-1}(t - s) f^T(s) D H_{n-1}^T F_1 H_{n-1} D x(s)ds.\]
The latter together with (29) leads to

\[
\mathbf{ELV}_{F_1} + 2\alpha \mathbf{E} \mathbf{V}_{F_1} = \frac{(n - 1)h}{2} \mathbf{Ex}_T(t)D^T H_{n-1}^TF_1H_{n-1}Dx(t) \\
- \frac{(n - 1)h}{2} \mathbf{E} \int_{t-(n-1)h}^{t} e^{-2\alpha(t-s)} \left[ \frac{d}{ds} \varphi_{n-1}(t-s) \right] x^T(s)D^T H_{n-1}^TF_2H_{n-1}Dx(s)ds \\
\leq \frac{(n - 1)h}{2} \mathbf{Ex}_T(t)D^T H_{n-1}^TF_1H_{n-1}Dx(t) - \frac{(n - 1)h^2}{2} e^{-2\alpha(n-1)h} \mathbf{E} \rho_5^T(t)F_1\rho_3(t). \tag{30}
\]

By using Lemma 1, via (29) we have

\[
LV_{\kappa_i} + 2\alpha V_{\kappa_i} = \frac{(ih)^2}{4} x^T(t)H_{i+1}^TR_iH_{i+1}x(t) - \frac{ih}{2} \int_{t-(i-1)h}^{t} e^{-2\alpha(t-s)} \varphi_{i}(t-s) x^T(s)H_{i+1}^TR_iH_{i+1}x(s)ds \\
\leq \frac{(ih)^2}{4} x^T(t)H_{i+1}^TR_iH_{i+1}x(t) - e^{-2\alpha h} k_i^T(t)R_i\kappa_i(t), \quad i = 1, \ldots, n - 2. \tag{31}
\]

\[
LV_{\kappa_{n-1}} + 2\alpha V_{\kappa_{n-1}} \leq \frac{(n - 1)h^2}{4} f^T(t)H_{n-1}^TR_{n-1}H_{n-1}f(t) \\
- e^{-2\alpha(n-1)h} \left[ \int_{t-(n-1)h}^{t} \varphi_{n-1}(t-s) f^T(s)H_{n-1}^Tds \right] R_{n-1} \left[ \int_{t-(n-1)h}^{t} \varphi_{n-1}(t-s) H_{n-1}f(s)ds \right] \\
= \frac{(n - 1)h^2}{4} f^T(t)H_{n-1}^TR_{n-1}H_{n-1}f(t) - e^{-2\alpha(n-1)h} [k_{n-1}(t) + \rho_3(t)]^T R_{n-1} [k_{n-1}(t) + \rho_3(t)], \tag{32}
\]

where

\[
\rho_3(t) = \int_{t-(n-1)h}^{t} \varphi_{n-1}(t-s) H_{n-1}Dx(s)dw(s).
\]

To compensate \(\rho_3(t)\), we employ the term \(V_{F_1}\), that leads to

\[
\mathbf{ELV}_{F_1} + 2\alpha \mathbf{E} \mathbf{V}_{F_1} \leq \frac{(n - 1)h}{2} \mathbf{Ex}_T(t)D^T H_{n-1}^TF_2H_{n-1}Dx(t) - \int_{t-(n-1)h}^{t} e^{-2\alpha(t-s)} \varphi_{n-1}(t-s) x^T(s)D^T H_{n-1}^TF_2H_{n-1}Dx(s)ds \\
\leq \frac{(n - 1)h}{2} \mathbf{Ex}_T(t)D^T H_{n-1}^TF_2H_{n-1}Dx(t) - e^{-2\alpha(n-1)h} \mathbf{E} \rho_5^T(t)F_2\rho_3(t). \tag{33}
\]

where we applied Itô isometry with (12). From (2), we have

\[
|g(t, H_0x(t), \ldots, H_{n-1}x(t))|^2 \leq \sum_{i=0}^{n-1} x^T(t)H_i^TM_iH_ix(t). \tag{34}
\]

Hence, the following inequality holds:

\[
\lambda \left[ \sum_{i=0}^{n-1} x^T(t)H_i^TM_iH_ix(t) - |g(t, H_0x(t), \ldots, H_{n-1}x(t))|^2 \right] \geq 0, \tag{35}
\]

for some constant \(\lambda > 0\).

In view of (21), (22), (25), (27), (28), and (30)–(33), taking into account the relations \(H_0x(t) = H_1x(t)\) and \(H_nx(t) = y(t_k) = H_0x(t) + [I_P, 0] \delta_0(t)\) and applying S-procedure with (35) we obtain

\[
\mathbf{ELV} + 2\alpha \mathbf{E} \mathbf{V} \leq \mathbf{E} \xi^T(t)\overline{\xi}(t) + h^2 \mathbf{E} \eta^T(t)\overline{\eta}(t) + h^2 \mathbf{E} \eta^T(t)H_{n-1} \left[ \frac{(n - 1)^2}{4} R_{n-1} + e^{-2\alpha(n-1)h}Q \right] H_{n-1}f(t) \\
+ h^2 \mathbf{E} \left[ \begin{array}{c}
H_1x(t) \\
H_0x(t) + [I_P, 0] \delta_0(t)
\end{array} \right]^T W_i \left[ \begin{array}{c}
H_1x(t) \\
H_0x(t) + [I_P, 0] \delta_0(t)
\end{array} \right].
\]
\[ \Phi = \sum_{i=0}^{n-1} \Phi_i \Phi_i^T \]

(36)

where \( \Phi \) is obtained from \( \Phi \) in (17) by taking away the last two block-columns and block-rows, \( \Psi \) is given by (18) and

\[ r(t) = \{x(t), \delta_0(t), \ldots, \delta_{n-1}(t), \kappa_1(t), \ldots, \kappa_{n-1}(t), \rho_0(t), \rho_1(t), \ldots, \rho_{n-1}(t) \} \]

(37)

Substituting (16) for \( f(t) \) and further applying Schur complement, we deduce that \( \Phi < 0 \) given by (17) guarantees \( \Delta V + 2a\Delta V \leq 0 \) implying that the sampled-data controller (6) exponentially mean-square stabilizes (1) with a decay rate \( \alpha \).

(ii) The system (1), (3) has the form

\[ \begin{align*}
\dot{x}_c(t) &= \left[ (A + BK)x_c(t) + H_{n-1}^T g(t, H_0 x(t), \ldots, H_{n-1} x(t)) \right] dt + D x(t) dw(t), \\
\end{align*} \]

where \( A, B, D \) are given by (16) and \( K \) is given by (14). If the PID controller (3) exponentially stabilizes (1), where \( g \equiv 0 \) and \( d_i = 0 \) for \( i=0, \ldots, n-1 \) (and thus, \( D = 0 \)), with a decay rate \( \alpha > 0 \), then there exists \( 0 < P \in \mathbb{R}^{(n+1)p \times (n+1)p} \) such that

\[ P(A + BK) + (A + BK)^T P + 2aP < 0 \]

for any \( a \in (0, \alpha) \). Thus,

\[ P(A + BK) + (A + BK)^T P + 2aP + D^T PD < 0, \]

(38)

for small enough \( |D| \). We choose in LMI (17) \( W_0 = \frac{1}{\sqrt{h}} I_{2p}, R_i = W_i = Q = F_1 = F_2 = \frac{1}{\sqrt{h}} I_{p} \) for \( i = 1, \ldots, n-1 \) and \( \lambda = \frac{1}{\sqrt{h}} \).

Applying Schur complement, \( \Phi < 0 \) is equivalent to

\[ P(A + BK) + (A + BK)^T P + 2aP + D^T PD + \sqrt{h}(G_1 + hG_2) + \frac{1}{\sqrt{h}} \sum_{i=0}^{n-1} H_i^T M_i H_i < 0, \]

(39)

where

\[ G_1 = (n-1)D^T H_{n-1}^T H_{n-1} D + \frac{4}{n^2 h^2} e^{2ah} P[(A_1 + BK)(A_1 + BK)^T] + \sum_{i=1}^{n-1} B_i K_i^T B_i^T + B_i K_i^T B_i^T P \]

\[ + \sum_{i=1}^{n-2} e^{2ih} P B_i K_i^T B_i^T + 2e^{2ih} P B_i K_i^T B_i^T + PH_i^T H_i^T H_i P. \]

\[ G_2 = \sum_{i=0}^{n-2} \left( e^{2ah} + \frac{1}{4} \right) H_{i+1}^T H_{i+1} \]

Inequality (38) implies (39) for small enough \( h > 0 \) and \( ||M_i|| \) (for \( i = 0, \ldots, n-1 \)) since \( \sqrt{h}(G_1 + hG_2) \to 0 \) and \( \frac{1}{\sqrt{h}} \sum_{i=0}^{n-1} H_i^T M_i H_i \to 0 \) for \( h \to 0 \) where we choose, for example, \( M_i = hF_i \) for \( i = 0, \ldots, n-1 \), implying the feasibility of \( \Phi < 0 \) for small enough \( h > 0 \) and \( ||M_i|| \) (for \( i = 0, \ldots, n-1 \)). Finally, applying Schur complement to the last two block-columns and block-rows of \( \Phi \) given by (17), we find that \( \Phi < 0 \) is feasible for small enough \( h > 0 \) if \( \Phi < 0 \) is feasible. Thus, LMI (17) is always feasible for small enough \( h > 0 \), \( ||D|| \) and \( ||M_i|| \) (for \( i = 0, \ldots, n-1 \)).

For the deterministic case (i.e., the system (1) with \( d_i = 0 \) for \( i = 0, \ldots, n-1 \)), we consider the functional \( \hat{V} \) that is obtained from \( V \) in (20) by setting \( F_1 = F_2 = 0 \) and changing \( f(s) \) and \( Q \) respectively as \( \hat{x}(s) \) and \( \hat{W}_{n-1} \). The latter includes additional terms \( V_s, V_3, V_{k_i} \) (for \( i = 2, \ldots, n-1 \)) to compensate additional errors \( \delta_i(t) \) and \( \kappa_i(t) \) (for \( i = 2, \ldots, n-1 \)) in (15) comparatively to Selivanov and Fridman.15,16
Corollary 1. Consider the deterministic nonlinear system (1) with $d_i = 0$ ($i = 0, \ldots, n-1$) under the sampled-data controller (6). Given $\overline{K}_P$, $\overline{K}_I$, and $\overline{K}_D_i$ ($i = 1, \ldots, n-1$) let the extended PID controller (3) exponentially stabilizes (1), where $d_i = 0$ ($i = 0, \ldots, n-1$) and $g \equiv 0$, with a decay rate $\overline{\alpha} > 0$.

(i) Given tuning parameters $h > 0$, $\alpha \in (0, \overline{\alpha})$ and $p \times p$ matrices $M_i$ ($i = 0, \ldots, n-1$), let there exist $(n+1)p \times (n+1)p$ matrix $P > 0$, $2p \times 2p$ matrices $W_0 > 0$ and $p \times p$ matrices $W_i > 0$ and $R_i > 0$ ($i = 1, \ldots, n-1$) and scalar $\lambda > 0$ that satisfy

$$\Phi < 0,$$

(40)

where $\Phi$ is obtained from $\Phi$ in (17) by setting $D = 0$, $F_1 = F_2 = 0$, $Q = W_{n-1}$ and and taking away the fifth block-column and block-row. Then the sampled-data controller (6) exponentially stabilizes (1), where $d_i = 0$ ($i = 0, \ldots, n-1$), with a decay rate $\alpha$.

(ii) Given any $\alpha \in (0, \overline{\alpha})$, LMI (40) is always feasible for small enough $h > 0$ and $\|M_i\|$ ($i = 0, \ldots, n-1$) (meaning that the sampled-data controller (6) exponentially stabilizes (1), where $d_i = 0$ ($i = 0, \ldots, n-1$), with a decay rate $\alpha$).

Remark 2. Note that less conservative integral inequalities were introduced e.g. in Seuret et al.\textsuperscript{25,26} to improve the results via LMIs. However, the LMIs of Seuret et al.\textsuperscript{25,26} cannot be guaranteed to be always feasible. By contrast, we provide in (ii) of Theorem 1 and Corollary 1 (and Theorems 2 and 3 below) the feasibility guarantee of LMIs which were obtained by using Jensen’s and Wirtinger’s inequalities.

3 | EVENT-TRIGGERED PID CONTROL

Event-triggered control allows to reduce the number of signals transmitted through a communication network (see e.g., Tabuada.\textsuperscript{17} Yue et al.\textsuperscript{18} and Héemels et al.\textsuperscript{19}). The idea is to transmit the signal only when it satisfies some preselected event-triggering condition. For simplicity we here introduce an event-triggering condition with respect to the control signals:\textsuperscript{15}

$$[u(t_k) - \hat{u}_{k-1}]^T \Theta [u(t_k) - \hat{u}_{k-1}] > \sigma u^T(t_k) \Theta u(t_k),$$

(41)

where $\sigma \in [0, 1)$ and $0 < \Theta \in \mathbb{R}^{p \times q}$ are the event-triggering parameters, $u(t_k)$ is from (6) and $\hat{u}_{k-1}$ denotes the last transmitted control signal. Thus, $\hat{u}_0 = u(t_0)$ and

$$\hat{u}_k = \begin{cases} u(t_k), & \text{if (41) is true}, \\ \hat{u}_{k-1}, & \text{if (41) is false}. \end{cases}$$

(42)

Hence, the system (1) becomes

$$dy^{(n)}(t) = \left[ \sum_{i=0}^{n-1} a_i y^{(i)}(t) + b \hat{u}_k + g(t, y^{(0)}(t), \ldots, y^{(n-1)}(t)) \right] dt + \sum_{i=0}^{n-1} d_i y^{(0)}(t) dw(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0,$$

(43)

with $\hat{u}_k$ given by (42). Introduce the event-triggering error

$$e_k = \hat{u}_k - u(t_k).$$

(44)

Then following the modeling in the previous section, the system (43) under the event-triggered PID control (3), (41), (42) can be presented as (cf. (15))

$$dx(t) = [f(t) + Be_k] dt + D x(t) dw(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0.$$

(45)

Theorem 2. Consider the stochastic nonlinear system (1) under the event-triggered PID controller (6), (41), (42). Given $\overline{K}_P$, $\overline{K}_I$, and $\overline{K}_D_i$ ($i = 1, \ldots, n-1$) let the extended PID controller (3) exponentially stabilizes (1), where $g \equiv 0$ and $d_i = 0$ ($i = 0, \ldots, n-1$), with a decay rate $\overline{\alpha} > 0$.

(i) Given tuning parameters $h > 0$, $\alpha \in (0, \overline{\alpha})$, $\sigma \in [0, 1)$ and $p \times p$ matrices $M_i$ ($i = 0, \ldots, n-1$), let there exist $(n+1)p \times (n+1)p$ matrix $P > 0$, $2p \times 2p$ matrices $W_0 > 0$, $p \times p$ matrices $W_i > 0$, $R_i > 0$ ($i = 1, \ldots, n-1$), $Q > 0$, $F_1 > 0$ and $F_2 > 0$,
where $\Phi$ and $\Xi$ are respectively given by (17) and (19), $K$ and $H_n$ are given by (14) and $B$ is given by (16). Then the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate $\alpha$.

(ii) Given any $\alpha \in (0, \overline{\alpha})$, LMI (46) is always feasible for small enough $h > 0$, $\alpha \in (0, 1)$, $\|D\|$ and $\|M_i\|$ ($i = 0, \ldots, n - 1$) (meaning that the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate $\alpha$).

Proof. (i) Using the triggering error (44), the event-triggering condition (41), (42) guarantees

$$0 \leq \sigma u^T(t_k)\Theta u(t_k) - e^T(t_k)e(t_k).$$

Consider the functional $V$ from (20) with $f(t)$ changed by $f(t) + B e(t_k)$. Following the proof of item (i) of Theorem 1, along (45) we have (cf. (36))

$$c_1 V + 2\alpha E V \leq c_2 V + 2\alpha E V + \sigma E u^T(t_k)\Theta u(t_k) - E^T \Theta e(t_k)$$

$$\leq (1 - e^{-2\alpha(1 - \alpha)h^2})^{1/2} V + \sigma E u^T(t_k)\Theta u(t_k),$$

where $\xi(t) = \text{col}(\xi(t), e(t))$ with $\xi(t)$ given by (37), $\Phi_\xi$ is obtained from $\Phi_\xi$ in (46) by taking away the i- and j-blocks with $i \in \{7, 8, 10\}$ or $j \in \{7, 8, 10\}$. Substituting (13) and (16), respectively, for $u(t_k)$ and $f(t)$ and further applying Schur complement, we find that $\Phi_\xi < 0$ given by (46) guarantees $c_1 V + 2\alpha E V \leq 0$ implying that the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate $\alpha$.

(ii) The proof of (ii) is similar to (ii) of Theorem 1.

Remark 3. To select the tuning parameters $h$, $\alpha$, $\sigma$, $M_i$ and $d_i$ ($i = 0, \ldots, n - 1$) we suggest the following algorithm: choose $\overline{K}_p$, $\overline{K}_i$ and $\overline{K}_d$ ($i = 1, \ldots, n - 1$) via pole-placement such that the extended PID controller (1) exponentially stabilizes (13), where $g \equiv 0$ and $d_i = 0$ ($i = 0, \ldots, n - 1$), with a decay rate $\overline{\alpha} > 0$. By solving the LMIs with $M_i = 0$, $d_i = 0$ ($i = 0, \ldots, n - 1$), $\sigma = 0$ and small enough $h > 0$, we find a critical maximal value of $\alpha$ as $\alpha^* < \overline{\alpha}$. Then, by choosing $\sigma \in [0, \alpha^*]$ with $M_i = 0$, $d_i = 0$ ($i = 0, \ldots, n - 1$) and small enough $h > 0$, we find a critical maximal value of $\sigma$ as $\sigma^*$. The same is done for $M_i$, $d_i$ ($i = 0, \ldots, n - 1$) that leads to critical maximum values of $M_i$, $d_i$ ($i = 0, \ldots, n - 1$), respectively, as $M_i^*$, $d_i^*$ ($i = 0, \ldots, n - 1$). Then for $\alpha \in [0, \alpha^*]$, $\sigma \in [0, \sigma^*]$, $M_i \in [0, M_i^*]$ and $d_i \in [0, d_i^*]$ ($i = 0, \ldots, n - 1$), we can obtain a critical maximal value of $h = h^*$ such that for $h > h^*$ the LMI becomes unfeasible.
4 | \(L_2\)-GAIN ANALYSIS

The direct Lyapunov method is applicable not only to the stability but also to the performance analysis. For example, \(L_2\)-gain analysis. In this section, we consider \(L_2\)-gain analysis of the perturbed systems, namely (cf. (43))

\[
dy^{(n)}(t) = \sum_{i=0}^{n-1} a_i y^{(i)}(t) + b \dot{u} + b v(t) + g(t, y^{(0)}(t), \ldots, y^{(n-1)}(t)) \quad dt + \sum_{i=0}^{n-1} d y^{(i)}(t) d w(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0, \tag{49}
\]

where \(b_\nu \in \mathbb{R}^{p \times p}\) is a constant matrix and \(v(t) \in \mathbb{R}^p\) is the external disturbance in \(L_2[0, \infty)\).

The system (49) under the event-triggered PID control (3), (41), (42) has the form:

\[
dx(t) = [f(t) + Be_k + B v(t)] dt + D x(t) d w(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0, \tag{50}
\]

where \(x(t)\) is given by (14), \(f(t), B\) and \(D\) are given by (16) and

\[
B_\nu = \text{col}(0, 0_{1 \times p}, b_\nu, 0_{p \times p}). \tag{51}
\]

Consider next the controlled output

\[
z(t) = C x(t) + C_\nu \nu(t), \quad z(t) \in \mathbb{R}^l, \tag{52}
\]

where \(C \in \mathbb{R}^{l \times (n+1)p}\) and \(C_\nu \in \mathbb{R}^{l \times p}\) are constant matrices. For a prechosen \(\gamma > 0\) we introduce the following performance index:

\[
J = \int_0^\infty \left[ z^T(t)z(t) - \gamma^2 \nu^T(t)\nu(t) \right] dt. \tag{53}
\]

We seek conditions that will lead to \(\mathbf{E} J \leq 0\) for all \(x(t)\) satisfying (50) with the zero initial condition \(x(0) = 0\) and for all \(0 \leq \nu \in L_2[0, \infty)\). In this case the system (50), (52) has \(L_2\)-gain less than or equal to \(\gamma\). Moreover, if the system (50) with \(\nu \equiv 0\) is exponentially mean-square stable, then the system (50) is internally exponentially mean-square stable.

**Lemma 3.** Given \(\alpha \geq 0\) and \(\gamma > 0\), let for \(V\) given by (20) the following inequality holds along the solutions of (50):

\[
\mathbf{E} L V + 2 \alpha \mathbf{E} V + \mathbf{E} z^T(t) z(t) - \gamma^2 \mathbf{E} \nu^T(t) \nu(t) < 0 \quad \forall 0 \neq \nu(t) \in \mathbb{R}^p \quad \text{and} \quad \forall t \geq 0. \tag{54}
\]

If (54) holds with \(\alpha = 0\), then the system (50), (52) has \(L_2\)-gain less than or equal to \(\gamma\). Moreover, if (54) holds with \(\alpha > 0\), then the system (50) is internally exponentially mean-square stable with a decay rate \(\alpha\).

Based on Lemma 3, we now present the following NML conditions.

**Theorem 3.** Consider the stochastic nonlinear system (1) with an additive external disturbance \(v(t)\) under the event-triggered PID controller (6), (41), (42) leading to system (50), and the controlled output (52). Given \(\overline{K}_p, \overline{K}_1\) and \(\overline{K}_D, (i = 1, \ldots, n - 1)\) let the extended PID controller (3) exponentially stabilizes (1), where \(g \equiv 0\) and \(d_i = 0, (i = 0, \ldots, n - 1)\), with a decay rate \(\bar{\omega} > 0\).

(i) Given tuning parameters \(h > 0, \alpha \in (0, \bar{\omega}), \sigma \in [0, 1)\) and \(\gamma > 0\), and \(p \times p\) matrices \(M_i (i = 0, \ldots, n - 1)\), let there exist \((n + 1)p \times (n + 1)p\) matrices \(P > 0, 2p \times 2p\) matrices \(W_0 > 0, p \times p\) matrices \(W_i > 0, R_i > 0\) \((i = 1, \ldots, n - 1), Q > 0, F_1 > 0, F_2 > 0, q \times q\) and \(\Theta > 0\) and scalar \(\lambda > 0\) that satisfy (18) and

\[
\Phi_{u_1} = \begin{bmatrix}
PB_{\nu} & C^T \\
0_{(2m+2)p \times p} & 0_{(2m+2)p \times p} \\
h \Xi H_{n-1} B_{\nu} & 0_{p \times p} \\
0_{2p \times q} & 0_{2p \times q}
\end{bmatrix} < 0, \tag{55}
\]

(ii) Given tuning parameters \(h > 0, \alpha \in (0, \bar{\omega}), \sigma \in [0, 1)\) and \(\gamma > 0\), and \(p \times p\) matrices \(M_i (i = 0, \ldots, n - 1)\), let there exist \((n + 1)p \times (n + 1)p\) matrices \(P > 0, 2p \times 2p\) matrices \(W_0 > 0, p \times p\) matrices \(W_i > 0, R_i > 0\) \((i = 1, \ldots, n - 1), Q > 0, F_1 > 0, F_2 > 0, q \times q\) and \(\Theta > 0\) and scalar \(\lambda > 0\) that satisfy (18) and

\[
\Phi_{u_2} = \begin{bmatrix}
PB_{\nu} & C^T \\
0_{(2m+2)p \times p} & 0_{(2m+2)p \times p} \\
h \Xi H_{n-1} B_{\nu} & 0_{p \times p} \\
0_{2p \times q} & 0_{2p \times q}
\end{bmatrix} < 0, \tag{56}
\]

(iii) Given tuning parameters \(h > 0, \alpha \in (0, \bar{\omega}), \sigma \in [0, 1)\) and \(\gamma > 0\), and \(p \times p\) matrices \(M_i (i = 0, \ldots, n - 1)\), let there exist \((n + 1)p \times (n + 1)p\) matrices \(P > 0, 2p \times 2p\) matrices \(W_0 > 0, p \times p\) matrices \(W_i > 0, R_i > 0\) \((i = 1, \ldots, n - 1), Q > 0, F_1 > 0, F_2 > 0, q \times q\) and \(\Theta > 0\) and scalar \(\lambda > 0\) that satisfy (18) and

\[
\Phi_{u_3} = \begin{bmatrix}
PB_{\nu} & C^T \\
0_{(2m+2)p \times p} & 0_{(2m+2)p \times p} \\
h \Xi H_{n-1} B_{\nu} & 0_{p \times p} \\
0_{2p \times q} & 0_{2p \times q}
\end{bmatrix} < 0, \tag{57}
\]
where $H_{n-1}$, $Z$, $\Phi_\sigma$ and $B_\sigma$ are respectively given by (14), (19), (46), and (51), and $C$ and $C_\sigma$ are given by (52). Then the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate $\alpha$, and the system (50), (52) has $L_2$-gain less than or equal to $\gamma$.

(ii) Given any $\alpha \in (0, \overline{\alpha})$, LMI (55) is always feasible for small enough $h > 0$, $\sigma \in (0, 1)$, $\frac{1}{\gamma} > 0$, $\|D\|$ and $\|M_i\|$ $(i = 0, \ldots, n - 1)$ (meaning that the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate $\alpha$).

5 \ EXAMPLES

To illustrate the efficiency, we present three examples including a servo positioning system.

**Example 1.** Consider system (1) with

$$a_0 = 0, \quad a_1 = -8.4, \quad b = 35.71, \quad g = 0.$$  \hspace{1cm} (56)

The system is not stable if $u = 0$. The PID controller (3) with

$$\overline{K}_p = -10, \quad \overline{K}_f = -40, \quad \overline{K}_{Di} = -0.65.$$  \hspace{1cm} (57)

stabilizes system (1) with (56) for small enough stochastic perturbations. Let $\alpha = 5$ be the desired decay rate. In the deterministic case (i.e., $d_0 = d_1 = 0$), LMIs of Corollary 1 and Selivanov and Fridman\textsuperscript{16} lead to the same result which is larger than that via Selivanov and Fridman.\textsuperscript{15} In the stochastic case, LMIs of Theorem 1 with $d_0 = 0$ and different values of $d_1$ lead to efficient results (see Table 1).

Consider now system (1) with (56) under the event-triggered PID control. For $h = 0.005$, $d_0 = 0$ and $d_1 = 0.2$, LMI of Theorem 2 is feasible for a maximum value of $\sigma = 0.074$. Sampled-data control requires to transmit $1/h + 1 = 201$ control signals during 1 s of simulations. By performing numerical simulations with 10 randomly chosen initial conditions $\|x(0)\|_\infty \leq 1$ where we applied Euler-Maruyama method\textsuperscript{27} using a step size $10dt$ with $dt = 10^{-6}$, the event-triggered control requires to transmit on average 63.95 control signals. Thus, the even-triggering mechanism (41), (42) reduces the number of transmitted control signals by almost 69%.

**Example 2.** (Chain of three integrators). Consider system (1) with

$$a_i = 0, \quad i = 0, 1, 2, \quad b = 1, \quad g \equiv 0.$$  \hspace{1cm} (58)

Using the pole placement, we find that for (3) with

$$\overline{K}_p = -6.026, \quad \overline{K}_f = -1.716, \quad \overline{K}_{Di} = -7.91, \quad \overline{K}_{Di} = -4.6.$$  \hspace{1cm} (59)

the eigenvalues of $A + BK$ are $-1$, $-1.1$, $-1.2$ and $-1.3$. Therefore, the PID controller (3) with (59) stabilizes system (1) with (58) for small enough stochastic perturbations.

Let $\alpha = 0.2$, $d_0 = d_2 = 0$. For different values of $d_1$, the maximum values of $h$ that preserve the exponential stability are presented in Table 1. It is clear that LMIs of Corollary 1 and Theorem 1 lead to efficient results whereas Selivanov
and Fridman\textsuperscript{15,16} fail. For $h = 0.04$ and $d_1 = 0.2$, LMIs of Theorem 2 are feasible for a maximum value of $\sigma = 0.119$. We next perform numerical simulations with 10 randomly chosen initial conditions $\|x(0)\|_\infty \leq 1$ by using Euler–Maruyama method\textsuperscript{27} with a step size 10$dt$ and $dt = 10^{-6}$. One can find that the event-triggered control requires to transmit on average 96.8 control signals during 10 seconds. Note that the number of transmissions for the sampled-data control is given by $10/h + 1 = 251$. Thus, the event-triggering mechanism (41), (42) reduces the number of transmitted control signals by over 61%.

**Example 3.** Consider the servo positioning system with a stochastic perturbation\textsuperscript{28,29}

$$\theta_1 dy^{(1)}(t) = [-\theta_2 y^{(1)}(t) + u(t) - F(y^{(1)}(t)) + b_y y(t)] dt + d_1 y^{(1)}(t) dw(t), \tag{60}$$

where $F(y(t)) = \theta_2 \tanh(700y(t)) + \theta_3 \tanh(15y(t))$, $y(t)$ is the motor rotation angle, $u(t)$ is the control input and $w(t)$ is the load disturbance. Set $[\theta_1, \theta_2, \theta_3, \theta_4] = [0.0025, 0.02, 0.01, 0.205]$. Following the previous modeling, the system (60) under an event-triggered PID control can be written in the form of (45) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{a_1}{\delta_1} & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{\delta_1} \\ 0 \end{bmatrix}, \quad B_e = \begin{bmatrix} b_e \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{d_1}{\delta_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and with $g = -F(y(t))$. Note that the latter nonlinearity satisfies (2) with $M_0 = 0$ and $M_1 = 14.13$. Moreover, the controlled output is given by (52) with $C = [1, 0, 0]$ and $C_v = 2$. The PID controller (3) with

$$\overline{K}_p = -0.4980, \quad \overline{K}_I = -0.0255, \quad \overline{K}_{D_1} = -0.270,$$ \tag{61}

exponentially stabilizes the system (60).

Set $\sigma = 0.1$ and $d_0 = 0$. For different values of $d_1$ and $b_e = 0$, LMIs of Corollary 1 and Theorem 1 lead to efficient results in Table 1. For $h = 0.05$, $d_1 = 0.01$ and $b_e = 0$, LMIs of Theorem 2 are feasible for a maximum value of $\sigma = 0.04$. Sampled-data control requires to transmit $5/h + 1 = 101$ control signals during 5 s. By performing numerical simulations with 10 randomly chosen initial conditions $\|x(0)\|_\infty \leq 1$ where we applied Euler–Maruyama method\textsuperscript{27} using a step size 10$dt$ with $dt = 10^{-6}$, the event-triggered control requires to transmit on average 32.6 control signals. Thus, the even-triggering mechanism (41), (42) reduces the number of transmitted control signals by over 67%. Moreover, for $h = 0.02$, $d_1 = 0.01$, $b_e = 1$ and $\sigma = 0.04$, by LMIs of Theorem 3 a minimum value of $\gamma = 2.02$ is obtained.

6 | CONCLUSIONS

In this paper, sampled-data implementation of extended PID control using delays has been presented for the $n$th-order stochastic nonlinear systems. We have employed an event-triggering condition that allows to reduce the number of sampled control signals used for stabilization and have studied $L_2$-gain analysis. The suggested method may be useful for delay-induced consensus in multi-agent systems under an extended PID control. This may be a topic for the future research.

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**DATA AVAILABILITY STATEMENT**

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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REFERENCES


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