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Sampled-data implementation of extended PID control using delays

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Abstract

We study the sampled-data implementation of extended PID control using delays for the nth-order stochastic nonlinear systems. The derivatives are approximated by finite differences giving rise to a delayed sampled-data controller. An appropriate Lyapunov–Krasovskii (L-K) method is presented to derive linear matrix inequalities (LMIs) for the exponential stability of the resulting closed-loop system. We show that with appropriately chosen gains, the LMIs are always feasible for small enough sampling period and stochastic perturbation. We further employ an event-triggering condition that allows to reduce the number of sampled control signals used for stabilization and provide L_2-gain analysis. Finally, three numerical examples illustrate the efficiency of the presented approach.

KEYWORDS

L_2-gain analysis, event-triggered control, PID control, sampled-data control, stochastic perturbations

1 | INTRODUCTION

Proportional-integral-derivative (PID) control is widely used in many industrial processes.[1,2] Many results on the classical PID control have been established, for example, for the second-order systems[3-5] and for the nth-order systems.[6] The PID control depends on the output derivative that cannot be measured in practice. Instead, the derivative can be approximated by the finite-difference leading to a delayed feedback. The delay-induced stability was studied, for example, in Niculescu and Michiel[7] and Ramírez et al.[8] using frequency-domain technique. Alternatively, it can be studied using the LMI-based method[9] that allows to cope with, for example, certain types of nonlinearities and stochastic perturbations[10-12] although being conservative.

Modern control usually employs digital technology for controller implementation, that is, sampled-data control. Moreover, sampled-data controller uses the sampled output only which is more practical. Thus, for practical application of PID control, its sampled-data implementation is important. By using consecutive sampled outputs, sampled-data implementation of PD control was presented for the nth-order deterministic[13] and stochastic[14] systems. Sampled-data implementation of PID control for the second-order deterministic systems was studied in Selivanov and Fridman.[15,16] However, the idea of using consecutive sampled outputs has not been studied yet for extended PID control of the nth-order deterministic ($n \geq 3$) or stochastic ($n \geq 2$) systems.

In this present paper, we study extended PID control of the nth-order stochastic nonlinear systems. Differently from Zhao and Guo[6] with the full knowledge of the system state, we consider sampled-data implementation of extended PID control by using the sampled outputs only. Following the improved approximation method[13] with consecutive sampled outputs, we approximate the extended PID controllers depending on the output and its derivatives up to the order

$n-1$ as delayed sampled-data controllers. Extension to PID control of the nth-order stochastic systems is far from being straightforward for the following reasons:

(i) Comparatively to the models under the PD control[13] or the PID control,[15,16] we have additional errors to be compensated by employing additional terms in the corresponding Lyapunov functionals.
(ii) The Lyapunov functionals of Selivanov and Fridman[13,15,16] are not applicable in the stochastic case. This is because a solution of a stochastic system does not have a derivative.[12,14] Thus, we propose novel Lyapunov functionals depending on the deterministic and stochastic parts of the system that lead to LMI-based stability conditions.

We show that the LMIs are always feasible for small enough sampling period and stochastic perturbation if the extended PID controller that employs the full-state stabilizes the system. Moreover, we employ an event-triggering condition[17-19] that allows to reduce the number of sampled control signals used for stabilization and provide L_2-gain analysis. Finally, three numerical examples are presented to illustrate the efficiency of the presented approach.

1.1 | Notations and useful inequalities

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$, I_n is the identity $n\times n$ matrix, the superscript T stands for matrix transposition. $\mathbb{R}^n$ denotes the n dimensional Euclidean space with Euclidean norm $|\cdot|$, $\mathbb{R}^{n\times m}$ denotes the set of all $n\times m$ real matrices with the induced matrix norm $||\cdot||$. Denote by diag$\{\dots\}$ and col$\{\dots\}$ block-diagonal matrix and block-column vector, respectively. $P>0$ implies that P is a positive definite symmetric matrix. C^i is a class of i times continuously differentiable functions.

We now present some useful inequalities:

Lemma 1. *(Extended Jensen's inequality[20]). Denote $G=\int_b^a f(s)x(s)ds$, where $f:[a,b]\to\mathbb{R}$, $x:[a,b]\to\mathbb{R}^n$ and the integration concerned is well defined. Then for any $n\times n$ matrix $R>0$ the following inequality holds:*

$$G^TRG \le \int_b^a |f(s)|ds \int_b^a |f(s)|x^T(s)Rx(s)ds.$$

Lemma 2. *(Exponential Wirtinger's inequality[21]). Let $x(t):(a,b)\to\mathbb{R}^n$ be absolutely continuous with $\dot{x}\in L_2(a,b)$ and $x(a)=0$ or $x(b)=0$. Then the following inequality holds:*

$$\int_b^a e^{2\alpha t}x^T(s)Wx(s)ds \le e^{2|\alpha|(b-a)}\frac{4(b-a)^2}{\pi^2}\int_a^b e^{2\alpha t}\dot{x}^T(s)W\dot{x}(s)ds,$$

for any $\alpha\in\mathbb{R}$ and $n\times n$ matrix $W>0$.

2 | EXTENDED PID CONTROL OF STOCHASTIC NONLINEAR SYSTEMS

Let $\{\Omega,\mathfrak{F},\mathbf{P}\}$ be a probability space. A filtration is a family $\{\mathfrak{F}_t, t\ge 0\}$ of nondecreasing sub-σ-algebras of $\mathfrak{F}$, that is, $\mathfrak{F}_s\subset\mathfrak{F}_t$ for $s<t$ and $\mathbf{P}\{\cdot\}$ be the probability of an event enclosed in the brackets. The mathematical expectation $\mathbf{E}$ of a random variable $\xi=\xi(w)$ on the probability space $\{\Omega,\mathfrak{F},\mathbf{P}\}$ is defined as $\mathbf{E}\xi=\int_\Omega \xi(w)d\mathbf{P}(w)$. The scalar standard Wiener process (also called Brownian motion) is a stochastic process $w(t)$ with normal distribution satisfying $w(0)=0$, $\mathbf{E}w(t)=0$ $(t>0)$ and $\mathbf{E}w^2(t)=t$ $(t>0)$.[22]

Consider the nth-order stochastic nonlinear system

$$dy^{(n)}(t) = \left[\sum_{i=0}^{n-1} a_i y^{(i)}(t) + bu(t) + g(t, y^{(0)}(t), \dots, y^{(n-1)}(t))\right]dt + \sum_{i=0}^{n-1} d_i y^{(i)}(t)dw(t). \tag{1}$$

Here $y(t)=y^{(0)}(t)\in\mathbb{R}^p$ is the output, $y^{(i)}(t)$ $(i=1,\dots,n-1)$ is the ith derivative of $y(t)$, $a_i, d_i\in\mathbb{R}^{p\times p}$ and $b\in\mathbb{R}^{p\times q}$ are constant matrices and $g:\mathbb{R}\times\mathbb{R}^p\times\dots\times\mathbb{R}^p\to\mathbb{R}^p$ is a locally Lipschitz continuous in arguments from the second to the last and satisfies for all $t \geq 0$ the inequality

$$|g(t, x_0, \dots, x_{n-1})|^2 \leq \sum_{i=0}^{n-1} x_i^T M_i x_i \quad \forall x_i \in \mathbb{R}^p, \quad i = 0, \dots, n-1, \tag{2}$$

with some matrices $0 < M_i \in \mathbb{R}^{p\times p}$ $(i = 0, \dots, n-1)$.

In Zhao and Guo,[6] an extended PID controller was designed as follows

$$u(t) = \left[\overline{K}_P y(t) + \overline{K}_I \int_0^t y(s)ds + \sum_{i=1}^{n-1} \overline{K}_{D_i} y^{(i)}(t) \right], \tag{3}$$

where $\overline{K}_P$, $\overline{K}_I$, and $\overline{K}_{D_i} \in \mathbb{R}^{q\times p}$ $(i = 1, \dots, n-1)$ are the controller gains. Differently from Zhao and Guo[6] with the full knowledge of the system state (i.e., $y^{(i)}(t)$, $i = 0, \dots, n-1$), we consider the output-feedback control, where $y^{(i)}(t)$, $i = 1, \dots, n-1$ in (3) are not available. Moreover, for the practical implementation we assume that the output $y(t)$ is available only at the discrete-time instants $t_k = kh$, where $k \in \mathbb{N}_0$ and $h > 0$ is the sampling period. As in Selivanov and Fridman,[15] we suggest the following approximations for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}_0$:

$$y(t) = \overline{y}(t) \approx \overline{y}(t_k), \quad \int_0^t y(s)ds \approx \int_0^{t_k} \overline{y}(s)ds \approx h\sum_{j=0}^{k-1} \overline{y}(t_j), \quad y^{(i)}(t) \approx \overline{y}^{(i)}(t) \approx \overline{y}^{(i)}(t_k), \quad i = 1, \dots, n-1, \tag{4}$$

where we used $\int_0^{t_k} \overline{y}(s)ds = \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \overline{y}(s)ds \approx \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \overline{y}(t_j)ds = h\sum_{j=0}^{k-1} \overline{y}(t_j)$ for the approximation of the integral and applied the finite-difference method for $\overline{y}^{(i)}(t_k)$ $(i = 1, \dots, n-1)$ with

$$\overline{y}^{(i)}(t) = \frac{\overline{y}^{(i-1)}(t) - \overline{y}^{(i-1)}(t-h)}{h}, \quad i = 1, \dots, n-1, \quad \overline{y}^{(0)}(t) = \overline{y}(t) = y(t), \tag{5}$$

and $y(t) = y(0)$ for $t < 0$. It is clear that via (5) we can compute $\overline{y}^{(1)}(t_k)$ (and thus, $\overline{y}^{(i)}(t_k)$, $i = 2, \dots, n-1$).

Thus, we design in this paper the following sampled-data controller

$$u(t) = \overline{K}_P \overline{y}(t_k) + h\overline{K}_I \sum_{j=0}^{k-1} \overline{y}(t_j) + \sum_{i=1}^{n-1} \overline{K}_{D_i} \overline{y}^{(i)}(t_k), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0. \tag{6}$$

In order to study the stability of system (1) under the sampled-data controller (6), we first present the approximation errors $\overline{y}(t_k) - y(t)$ and $\overline{y}^{(i)}(t_k) - y^{(i)}(t)$ $(i = 1, \dots, n-1)$, where $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}_0$, in a convenient form suitable for the later analysis via L-K functionals:

Proposition 1. *If $y \in C^i$ and $y^{(i)}$ is absolutely continuous with $i = 1, \dots, n$, then $\overline{y}(t_k)$ and $\overline{y}^{(i)}(t_k)$ $(i = 1, \dots, n-1)$ defined by (5) satisfy for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}_0$*

$$\overline{y}(t_k) = y(t) - \int_{t_k}^t \dot{y}(s)ds, \tag{7}$$

$$\overline{y}^{(i)}(t_k) = y^{(i)}(t) - \int_{t-ih}^t \varphi_i(t-s)\dot{y}^{(i)}(s)ds - \int_{t_k}^t \dot{\overline{y}}^{(i)}(s)ds, \quad i = 1, \dots, n-1, \tag{8}$$

where

$$\varphi_1(v) = \frac{h-v}{h}, \quad v \in [0, h],$$
$$\varphi_{i+1}(v) = \begin{cases} \frac{1}{h}\int_0^v \varphi_i(\lambda)d\lambda + \frac{h-v}{h}, & v \in [0, h] \\ \frac{1}{h}\int_{v-h}^v \varphi_i(\lambda)d\lambda, & v \in (h, ih). \\ \frac{1}{h}\int_{v-h}^{ih} \varphi_i(\lambda)d\lambda, & v \in [ih, ih+h], \end{cases} \quad i = 1, \dots, n-2. \tag{9}$$

Proof. We first introduce the errors due to the sampling:

$$y(t_k) = y(t) - \int_{t_k}^{t} \dot{y}(s)ds, \quad \overline{y}^{(i)}(t_k) = \overline{y}^{(i)}(t) - \int_{t_k}^{t} \dot{\overline{y}}^{(i)}(s)ds, \quad i = 1, \dots, n-1. \tag{10}$$

Taking into account $y(t_k) = \overline{y}(t_k)$ in (4), together with the first equality in (10) we obtain (7). Then following arguments for the error $\overline{y}^{(i)}(t) - y^{(i)}(t)$ $(i = 1, \dots, n-1)$ in Proposition 1 of Selivanov and Fridman,[13] that is,

$$\overline{y}^{(i)}(t) = y^{(i)}(t) - \int_{t-ih}^{t} \varphi_i(t-s)\dot{y}^{(i)}(s)ds, \quad i = 1, \dots, n-1, \tag{11}$$

where $\varphi_i(\cdot)$ $(i = 1, \dots, n-1)$ are defined by (9), we arrive at (8). ■

The functions $\varphi_i(\cdot)$ $(i = 1, \dots, n-1)$ have the following properties (see the proof in Selivanov and Fridman[13]):

Proposition 2. *The functions* $\varphi_i(\cdot)$ $(i = 1, \dots, n-1)$ *in (9) satisfy*

$$1)\ \varphi_i(0) = 1, \quad \varphi_i(ih) = 0; \quad 2)\ 0 \le \varphi_i(v) \le 1; \quad 3)\ \frac{d}{dv}\varphi_i(v) \in \left[-\frac{1}{h}, 0\right); \quad 4)\ \int_0^{ih} \varphi_i(v)dv = \frac{ih}{2}. \tag{12}$$

By noting that $y(t_j) = \overline{y}(t_j)$ $(j = 0, \dots, k-1)$, via (7) and (8) the sampled-data controller (6) can be presented as

$$\begin{aligned} u(t) &= \overline{K}_P \left[y(t) - \int_{t_k}^{t} \dot{y}(s)ds\right] + h\overline{K}_I \sum_{j=0}^{k-1} y(t_j) + \sum_{i=1}^{n-1} \overline{K}_{D_i} \left[y^{(i)}(t) - \int_{t-ih}^{t} \varphi_i(t-s)\dot{y}^{(i)}(s)ds - \int_{t_k}^{t} \dot{\overline{y}}^{(i)}(s)ds\right], \\ &= Kx(t) + [\overline{K}_P, \overline{K}_I]\delta_0(t) + \sum_{i=1}^{n-1} \overline{K}_{D_i}(\delta_i(t) + \kappa_i(t)), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0, \end{aligned} \tag{13}$$

where

$$\begin{aligned} x(t) &= \mathrm{col}\left\{ y(t), y^{(1)}(t), \dots, y^{(n-1)}(t), (t-t_k)y(t_k) + h\sum_{j=0}^{k-1} y(t_j) \right\}, \\ K &= [\overline{K}_p, \overline{K}_{D_1}, \dots, \overline{K}_{D_{n-1}}, \overline{K}_I], \quad \delta_0(t) = -\int_{t_k}^{t} \begin{bmatrix} H_0 \\ H_n \end{bmatrix} \dot{x}(s)ds, \\ \delta_i(t) &= -\int_{t_k}^{t} \dot{\overline{y}}^{(i)}(s)ds, \quad \kappa_i(t) = -\int_{t-ih}^{t} \varphi_i(t-s)H_i\dot{x}(s)ds, \quad i = 1, \dots, n-1, \\ H_i &= [0_{p\times ip}, I_p, 0_{p\times(n-i)p}], \quad i = 0, \dots, n. \end{aligned} \tag{14}$$

Using (13) and (14), the system (1), (6) has the form

$$dx(t) = f(t)dt + Dx(t)dw(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0, \tag{15}$$

where

$$\begin{aligned} f(t) &= (A + BK)x(t) + A_1\delta_0(t) + \sum_{i=1}^{n-1} B\overline{K}_{D_i}(\delta_i(t) + \kappa_i(t)) + H_{n-1}^T g(t, H_0x(t), \dots, H_{n-1}x(t)), \\ A &= \begin{bmatrix} 0 & I_p & 0 & \dots & 0 & 0 \\ 0 & 0 & I_p & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I_p & 0 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} & 0 \\ I_p & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad \begin{aligned} A_1 &= \begin{bmatrix} 0_{(n-1)p\times p} & 0_{(n-1)p\times p} \\ b\overline{K}_P & b\overline{K}_I \\ I_p & 0_{p\times p} \end{bmatrix}, \\ B &= \mathrm{col}\{0_{(n-1)p\times q}, b, 0_{p\times q}\}, \\ D &= \mathrm{col}\{0_{(n-1)p\times p}, \overline{D}, 0_{p\times p}\}, \\ \overline{D} &= [d_0, \dots, d_{n-1}, 0]. \end{aligned} \end{aligned} \tag{16}$$

Remark 1. In (15), we follow the transformation of Zhang and Fridman[23] that allowed to avoid an additional non-zero term $y^{(n-1)}(t_k) - y^{(n-1)}(t) = -\int_{t_k}^{t} H_{n-1}f(s)ds - \Pi$ with $\Pi = \int_{t_k}^{t} H_{n-1}Dx(s)dw(s)$. Note that the term Π has to be compensated by additional terms in Lyapunov functional. Hence, the transformation in (15) (comparatively to Selivanov and Fridman[15,16]) significantly simplifies the analysis in the stochastic case.

Comparatively to the system model (see e.g., (27) in Selivanov and Fridman[13]) under PD control, the system (15) includes additional term $A_1\delta_0(t)$ (due to the additional I control) that will be compensated by the additional term V_{δ_0} defined below (20). Note also that Lyapunov functional of Selivanov and Fridman[13] depends on the nth-order derivative, and, thus, is not applicable in the stochastic case. This is because a solution of a stochastic system does not have a derivative.[12,14] We will present LMI conditions via novel Lyapunov functional that depends on the deterministic and stochastic parts of the system:

Theorem 1. *Consider the stochastic nonlinear system (1) under the sampled-data controller (6). Given $\overline{K}_P$, $\overline{K}_I$, and $\overline{K}_{D_i}$ $(i = 1, \dots, n-1)$ let the extended PID controller (3) exponentially stabilizes (1), where $d_i = 0$ $(i = 0, \dots, n-1)$ and $g \equiv 0$, with a decay rate $\overline{\alpha} > 0$.*

(i) Given tuning parameters $h > 0$, $\alpha \in (0, \overline{\alpha})$ and $p \times p$ matrices M_i $(i = 0, \dots, n-1)$, let there exist $(n+1)p \times (n+1)p$ matrix $P > 0$, $2p \times 2p$ matrix $W_0 > 0$, $p \times p$ matrices $W_i > 0$, $R_i > 0$ $(i = 1, \dots, n-1)$, $Q > 0$, $F_1 > 0$ and $F_2 > 0$ and scalar $\lambda > 0$ that satisfy

$$\Phi = \begin{bmatrix} \Phi_{11} & PA_1 & \Phi_{13} & \Phi_{14} & 0 & PH_{n-1}^T & h(A+BK)^T H_{n-1}^T \Xi & h[H_1^T, H_0^T]W_0 \\ * & -\frac{\pi^2}{4}e^{-2\alpha h}W_0 & 0 & 0 & 0 & 0 & hA_1^T H_{n-1}^T \Xi & h[0, [I_p, 0]^T]W_0 \\ * & * & \Phi_{33} & 0 & 0 & 0 & h\Phi_{37} & 0 \\ * & * & * & \Phi_{44} & \Phi_{45} & 0 & h\Phi_{47} & 0 \\ * & * & * & * & -e^{-2\alpha(n-1)h}(R_{n-1}+F_2) & 0 & 0 & 0 \\ * & * & * & * & * & -\lambda I_p & hH_{n-1}H_{n-1}^T \Xi & 0 \\ * & * & * & * & * & * & -\Xi & 0 \\ * & * & * & * & * & * & * & -W_0 \end{bmatrix} < 0, \quad (17)$$

$$\Psi = \begin{bmatrix} W_{n-1} - Q & W_{n-1} \\ * & W_{n-1} - \frac{(n-1)}{2}e^{-2\alpha(n-1)h}F_2 \end{bmatrix} < 0, \quad (18)$$

where

$$\Phi_{11} = P(A+BK) + (A+BK)^T P + 2\alpha P + \sum_{i=0}^{n-2} h^2 e^{2\alpha i h} H_{i+1}^T W_i H_{i+1} + \sum_{i=1}^{n-2} \frac{(ih)^2}{4} H_{i+1}^T R_i H_{i+1}$$
$$+ D^T PD + \frac{(n-1)h}{2} D^T H_{n-1}^T (F_1 + F_2) H_{n-1} D + \lambda \sum_{i=0}^{n-1} H_i^T M_i H_i,$$
$$\Phi_{13} = \Phi_{14} = PB[\overline{K}_{D_1}, \dots, \overline{K}_{D_{n-1}}], \qquad \Phi_{33} = -\frac{\pi^2}{4}e^{-2\alpha h}\mathrm{diag}\{W_1, \dots, W_{n-1}\},$$
$$\Phi_{44} = -\mathrm{diag}\{e^{-2\alpha h}R_1, \dots, e^{-2\alpha(n-1)h}R_{n-1}\}, \qquad \Phi_{45} = [0, -e^{-2\alpha(n-1)h}R_{n-1}]^T,$$
$$\Phi_{37} = \Phi_{47} = [\overline{K}_{D_1}, \dots, \overline{K}_{D_{n-1}}]^T B^T H_{n-1}^T \Xi, \qquad \Xi = \frac{(n-1)^2}{4}R_{n-1} + e^{2\alpha(n-1)h}Q, \quad (19)$$

with A, B, A_1 and D given by (16), and K and H_i $(i = 0, \dots, n)$ given by (14). Then the sampled-data controller (6) exponentially mean-square stabilizes (1) with a decay rate α.

(ii) Given any $\alpha \in (0, \overline{\alpha})$, LMI (17) is always feasible for small enough $h > 0$, $\|D\|$ and $\|M_i\|$ $(i = 0, \dots, n-1)$ (meaning that the sampled-data controller (6) exponentially mean-square stabilizes (1) with a decay rate α).

Proof. (i) We consider the functional

$$V = V_0 + V_{\delta_0} + \sum_{i=1}^{n-1} \left(V_{\delta_i} + V_{\overline{y}_i} + V_{\kappa_i}\right) + V_{\delta_n} + V_{F_1} + V_{F_2}, \quad (20)$$

where

$$V_0(x(t)) = x^T(t)Px(t),$$

$$V_{\delta_i}(t,\dot{x}_t) = \begin{cases} h^2\int_{t_k}^t e^{-2\alpha(t-s)}\dot{x}^T(s)\begin{bmatrix} H_0 \\ H_n \end{bmatrix}^T W_0 \begin{bmatrix} H_0 \\ H_n \end{bmatrix}\dot{x}(s)ds - \frac{\pi^2}{4}e^{-2\alpha h}\int_{t_k}^t e^{-2\alpha(t-s)}\delta_0^T(s)W_0\delta_0(s)ds, & i=0, \\ h^2\int_{t_k}^t e^{-2\alpha(t-s)}\left[\dot{\bar{y}}^{(i)}(s)\right]^T W_i \left[\dot{\bar{y}}^{(i)}(s)\right]ds - \frac{\pi^2}{4}e^{-2\alpha h}\int_{t_k}^t e^{-2\alpha(t-s)}\delta_i^T(s)W_i\delta_i(s)ds, & i=1,\dots,n-1, \end{cases}$$

$$V_{\bar{y}_i}(x_t) = h^2e^{2\alpha ih}\int_{t-ih}^t e^{-2\alpha(t-s)}\varphi_i(t-s)x^T(s)H_{i+1}^T W_iH_{i+1}x(s)ds, \quad i=1,\dots,n-2,$$

$$V_{\bar{y}_{n-1}}(f_t) = h^2e^{2\alpha(n-1)h}\int_{t-(n-1)h}^t e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)f^T(s)H_{n-1}^T QH_{n-1}f(s)ds,$$

$$V_{\kappa_i}(x_t) = \frac{ih}{2}\int_{t-ih}^t e^{-2\alpha(t-s)}\phi_i(t-s)x^T(s)H_{i+1}^T R_iH_{i+1}x(s)ds, \quad i=1,\dots,n-2,$$

$$V_{\kappa_{n-1}}(f_t) = \frac{(n-1)h}{2}\int_{t-(n-1)h}^t e^{-2\alpha(t-s)}\phi_{n-1}(t-s)f^T(s)H_{n-1}^T R_{n-1}H_{n-1}f(s)ds,$$

$$V_{F_1}(x_t) = \frac{(n-1)h}{2}\int_{t-(n-1)h}^t e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)x^T(s)D^TH_{n-1}^T F_1H_{n-1}Dx(s)ds,$$

$$V_{F_2}(x_t) = \int_{t-(n-1)h}^t e^{-2\alpha(t-s)}\phi_{n-1}(t-s)x^T(s)D^TH_{n-1}^T F_2H_{n-1}Dx(s)ds$$

with $P>0$, $W_i>0$ $(i=0,\dots,n-1)$, $R_i>0$ $(i=1,\dots,n-1)$, $Q>0$, $F_1>0$, $F_2>0$ and

$$\phi_i(v) = \int_v^{ih}\varphi_i(\lambda)d\lambda, \quad i=1,\dots,n-1.$$

Here $x_t(\theta)=x(t+\theta)$, $\theta\in[-h,0]$. Since $\dot{\delta}_0(t) = -[H_0^T,H_n^T]^T\dot{x}(t)$, $\delta_i(t) = -\dot{\bar{y}}^{(i)}(t)$ $(i=1,\dots,n-1)$ and $\delta_i(t_k)=0$ $(i=0,\dots,n-1)$, Lemma 2 implies $V_{\delta_i}\ge 0$ for $i=0,\dots,n-1$. Due to $\phi_i(\cdot)\ge 0$ and $\varphi_i(\cdot)\ge 0$ we have the positivity of functional $V(t)$ in (20). Note that the terms V_{δ_i} $(i=1,\dots,n-1)$, $V_{\bar{y}_i}$ and V_{κ_i} $(i=1,\dots,n-2)$ are from Selivanov and Fridman,[13] whereas the novel terms $V_{\bar{y}_{n-1}}$, $V_{\kappa_{n-1}}$, V_{F_1}, and V_{F_2} are stochastic extensions of Lyapunov functionals that depend on $\dot{x}(t)$.

Let L be the generator (see e.g., Shaikhet[22] and Mao[24]). We have along (15)

$$LV_0 + 2\alpha V_0 = 2x^T(t)Pf(t) + x^T(t)D^TPDx(t) + 2\alpha x^T(t)Px(t). \tag{21}$$

Moreover, we have

$$LV_{\delta_i} + 2\alpha V_{\delta_i} = \begin{cases} h^2\dot{x}^T(t)\begin{bmatrix} H_0 \\ H_n \end{bmatrix}^T W_0 \begin{bmatrix} H_0 \\ H_n \end{bmatrix}\dot{x}(t) - \frac{\pi^2}{4}e^{-2\alpha h}\delta_0^T(t)W_0\delta_0(t), & i=0, \\ h^2\left[\dot{\bar{y}}^{(i)}(t)\right]^T W_i\left[\dot{\bar{y}}^{(i)}(t)\right] - \frac{\pi^2}{4}e^{-2\alpha h}\delta_i^T(t)W_i\delta_i(t), & i=1,\dots,n-1. \end{cases} \tag{22}$$

The terms $V_{\bar{y}_i}$, $i=1,\dots,n-2$ are introduced to compensate $h^2\left[\dot{\bar{y}}^{(i)}(t)\right]^T W_i\left[\dot{\bar{y}}^{(i)}(t)\right]$, $i=1,\dots,n-2$ in (22). By using Lemma 1, via (12) we have

$$LV_{\bar{y}_i} + 2\alpha V_{\bar{y}_i} = h^2e^{2\alpha ih}x^T(t)H_{i+1}^T W_iH_{i+1}x(t) - h^2e^{2\alpha ih}\int_{t-ih}^t e^{-2\alpha(t-s)}\left[\frac{d}{ds}\varphi_i(t-s)\right]x^T(s)H_{i+1}^T W_iH_{i+1}x(s)ds$$

$$
\begin{aligned}
&\le h^2 e^{2\alpha ih} x^T(t) H_{i+1}^T W_i H_{i+1} x(t) \\
&- h^2\left(\int_{t-ih}^t d\varphi_i(t-s)\right)^{-1} \int_{t-ih}^t \left[\frac{d}{ds}\varphi_i(t-s)\right] x^T(s) H_{i+1}^T ds W_i \int_{t-ih}^t \left[\frac{d}{ds}\varphi_i(t-s)\right] H_{i+1} x(s) ds, \quad i = 1, \dots, n-2.
\end{aligned} \tag{23}
$$

From (8), it follows that

$$
\bar{y}^{(i)}(t) = y^{(i)}(t) - \int_{t-ih}^t \varphi_i(t-s)\dot{y}^{(i)}(s)ds, \quad i = 1, \dots, n-1.
$$

Via (12) the latter implies

$$
\dot{\bar{y}}^{(i)}(t) = \int_{t-ih}^t \left[\frac{d}{ds}\varphi_i(t-s)\right] \dot{y}^{(i)}(s)ds = \int_{t-ih}^t \left[\frac{d}{ds}\varphi_i(t-s)\right] H_i \dot{x}(s)ds, \quad i = 1, \dots, n-1. \tag{24}
$$

Noting that $\int_{t-ih}^t d\varphi_i(t-s) = \varphi_i(0) - \varphi_i(ih) = 1$ and $H_i\dot{x}(s) = H_{i+1}x(s)$ $(i = 0, \dots, n-2)$, from (23) and (24) we have

$$
LV_{\bar{y}_i} + 2\alpha V_{\bar{y}_i} \le h^2 e^{2\alpha ih} x^T(t) H_{i+1}^T W_i H_{i+1} x(t) - h^2 \left[\dot{\bar{y}}^{(i)}(t)\right]^T W_i \left[\dot{\bar{y}}^{(i)}(t)\right], \quad i = 1, \dots, n-2. \tag{25}
$$

Then the terms $-h^2 \left[\dot{\bar{y}}^{(i)}(t)\right]^T W_i \left[\dot{\bar{y}}^{(i)}(t)\right]$ $(i = 1, \dots, n-2)$ in the above expression will cancel the positive term of $LV_{\delta_i} + 2\alpha V_{\delta_i}$ $(i = 1, \dots, n-2)$. Note that the term $\dot{\bar{y}}^{(i)}(t)$ with $i = n-1$ in (24) has the following form:

$$
\dot{\bar{y}}^{(n-1)}(t) = \int_{t-(n-1)h}^t \left[\frac{d}{ds}\varphi_{n-1}(t-s)\right] H_{n-1}\dot{x}(s)ds \overset{(15)}{=} \rho_1(t) + \rho_2(t), \tag{26}
$$

where

$$
\rho_1(t) = \int_{t-(n-1)h}^t \left[\frac{d}{ds}\varphi_{n-1}(t-s)\right] H_{n-1} f(s)ds, \quad \rho_2(t) = \int_{t-(n-1)h}^t \left[\frac{d}{ds}\varphi_{n-1}(t-s)\right] H_{n-1} D x(s) dw(s).
$$

Thus

$$
LV_{\delta_{n-1}} + 2\alpha V_{\delta_{n-1}} \overset{(22)}{=} h^2 [\rho_1(t) + \rho_2(t)]^T W_{n-1} [\rho_1(t) + \rho_2(t)] - \frac{\pi^2}{4} e^{-2\alpha h} \delta_{n-1}^T(t) W_{n-1} \delta_{n-1}(t). \tag{27}
$$

To compensate $\rho_1(t)$, we employ the term $V_{\bar{y}_{n-1}}$, that is,

$$
\begin{aligned}
LV_{\bar{y}_{n-1}} + 2\alpha V_{\bar{y}_{n-1}} &= h^2 e^{2\alpha(n-1)h} f^T(t) H_{n-1}^T Q H_{n-1} f(t) - h^2 e^{2\alpha(n-1)h} \int_{t-(n-1)h}^t e^{-2\alpha(t-s)} \left[\frac{d}{ds}\varphi_{n-1}(t-s)\right] f^T(s) H_{n-1}^T Q H_{n-1} f(s) ds \\
&\le h^2 e^{2\alpha(n-1)h} f^T(t) H_{n-1}^T Q H_{n-1} f(t) - h^2 \rho_1^T(t) Q \rho_1(t),
\end{aligned} \tag{28}
$$

where we applied Lemma 1 with (12). Note that (12) implies

$$
\phi_i(0) = \int_0^{ih} \varphi_i(\lambda)d\lambda = \frac{ih}{2}, \quad \phi_i(ih) = 0, \quad i = 1, \dots, n-1. \tag{29}
$$

For the $\rho_2(t)$-term, by using Itô isometry (see, e.g., Shaikhet[22] and Mao[24]), via (12) we have for any $p \times p$ matrix $F_1 > 0$

$$
\begin{aligned}
e^{-2\alpha(n-1)h} h \mathbf{E} \rho_2^T(t) F_1 \rho_2(t) &= e^{-2\alpha(n-1)h} h \mathbf{E} \int_{t-(n-1)h}^t \left[\frac{d}{ds}\varphi_{n-1}(t-s)\right]^2 x^T(s) D^T H_{n-1}^T F_1 H_{n-1} D x(s) ds \\
&\le \mathbf{E} \int_{t-(n-1)h}^t e^{-2\alpha(t-s)} \left[\frac{d}{ds}\varphi_{n-1}(t-s)\right] x^T(s) D^T H_{n-1}^T F_1 H_{n-1} D x(s) ds.
\end{aligned}
$$

The latter together with (29) leads to

$$\begin{aligned}
\mathbf{E}LV_{F_1}+2\alpha\mathbf{E}V_{F_1} &= \frac{(n-1)h}{2}\mathbf{E}x^T(t)D^TH_{n-1}^TF_1H_{n-1}Dx(t) \\
&\quad -\frac{(n-1)h}{2}\mathbf{E}\int_{t-(n-1)h}^{t} e^{-2\alpha(t-s)}\left[\frac{d}{ds}\varphi_{n-1}(t-s)\right]x^T(s)D^TH_{n-1}^TF_2H_{n-1}Dx(s)ds \\
&\le \frac{(n-1)h}{2}\mathbf{E}x^T(t)D^TH_{n-1}^TF_1H_{n-1}Dx(t)-\frac{(n-1)h^2}{2}e^{-2\alpha(n-1)h}\mathbf{E}\rho_2^T(t)F_1\rho_2(t).
\end{aligned} \tag{30}$$

By using Lemma 1, via (29) we have

$$\begin{aligned}
LV_{\kappa_i}+2\alpha V_{\kappa_i} &= \frac{(ih)^2}{4}x^T(t)H_{i+1}^TR_iH_{i+1}x(t)-\frac{ih}{2}\int_{t-ih}^{t}e^{-2\alpha(t-s)}\varphi_i(t-s)x^T(s)H_{i+1}^TR_iH_{i+1}x(s)ds \\
&\le \frac{(ih)^2}{4}x^T(t)H_{i+1}^TR_iH_{i+1}x(t)-e^{-2\alpha ih}\kappa_i^T(t)R_i\kappa_i(t), \quad i=1,\dots,n-2.
\end{aligned} \tag{31}$$

$$\begin{aligned}
LV_{\kappa_{n-1}}+2\alpha V_{\kappa_{n-1}} &\le \frac{(n-1)^2h^2}{4}f^T(t)H_{n-1}^TR_{n-1}H_{n-1}f(t) \\
&\quad -e^{-2\alpha(n-1)h}\left[\int_{t-(n-1)h}^{t}\varphi_{n-1}(t-s)f^T(s)H_{n-1}^Tds\right]^TR_{n-1}\left[\int_{t-(n-1)h}^{t}\varphi_{n-1}(t-s)H_{n-1}f(s)ds\right] \\
&= \frac{(n-1)^2h^2}{4}f^T(t)H_{n-1}^TR_{n-1}H_{n-1}f(t)-e^{-2\alpha(n-1)h}[\kappa_{n-1}(t)+\rho_3(t)]^TR_{n-1}[\kappa_{n-1}(t)+\rho_3(t)],
\end{aligned} \tag{32}$$

where

$$\rho_3(t)=\int_{t-(n-1)h}^{t}\varphi_{n-1}(t-s)H_{n-1}Dx(s)dw(s).$$

To compensate $\rho_3(t)$, we employ the term V_{F_2} that leads to

$$\begin{aligned}
\mathbf{E}LV_{F_2}+2\alpha\mathbf{E}V_{F_2} &\le \frac{(n-1)h}{2}\mathbf{E}x^T(t)D^TH_{n-1}^TF_2H_{n-1}Dx(t)-\int_{t-(n-1)h}^{t}e^{-2\alpha(t-s)}\varphi_{n-1}(t-s)x^T(s)D^TH_{n-1}^TF_2H_{n-1}Dx(s)ds \\
&\le \frac{(n-1)h}{2}\mathbf{E}x^T(t)D^TH_{n-1}^TF_2H_{n-1}Dx(t)-e^{-2\alpha(n-1)h}\mathbf{E}\rho_3^T(t)F_2\rho_3(t).
\end{aligned} \tag{33}$$

where we applied Itô isometry with (12). From (2), we have

$$|g(t,H_0x(t),\dots,H_{n-1}x(t))|^2\le\sum_{i=0}^{n-1}x^T(t)H_i^TM_iH_ix(t). \tag{34}$$

Hence, the following inequality holds:

$$\lambda\left[\sum_{i=0}^{n-1}x^T(t)H_i^TM_iH_ix(t)-|g(t,H_0x(t),\dots,H_{n-1}x(t))|^2\right]\ge 0, \tag{35}$$

for some constant $\lambda>0$.

In view of (21), (22), (25), (27), (28), and (30)–(33), taking into account the relations $H_0\dot{x}(t)=H_1x(t)$ and $H_n\dot{x}(t)=y(t_k)=H_0x(t)+[I_P,0]\delta_0(t)$ and applying S-procedure with (35) we obtain

$$\begin{aligned}
\mathbf{E}LV+2\alpha\mathbf{E}V &\le \mathbf{E}\xi^T(t)\overline{\Phi}\xi(t)+h^2\mathbf{E}\eta^T(t)\Psi\eta(t)+h^2\mathbf{E}f^T(t)H_{n-1}^T\left[\frac{(n-1)^2}{4}R_{n-1}+e^{2\alpha(n-1)h}Q\right]H_{n-1}f(t) \\
&\quad +h^2\mathbf{E}\begin{bmatrix}H_1x(t)\\ H_0x(t)+[I_P,0]\delta_0(t)\end{bmatrix}^TW_i\begin{bmatrix}H_1x(t)\\ H_0x(t)+[I_P,0]\delta_0(t)\end{bmatrix}
\end{aligned}$$

$$\overset{(18)}{\leq} \mathbf{E}\xi^T(t)\overline{\Phi}\xi(t) + h^2\mathbf{E}f^T(t)H_{n-1}^T\left[\frac{(n-1)^2}{4}R_{n-1} + e^{2\alpha(n-1)h}Q\right]H_{n-1}f(t)$$
$$+ h^2\mathbf{E}\begin{bmatrix} H_1x(t) \\ H_0x(t)+[I_P,0]\delta_0(t)\end{bmatrix}^T W_i \begin{bmatrix} H_1x(t) \\ H_0x(t)+[I_P,0]\delta_0(t)\end{bmatrix}, \tag{36}$$

where $\overline{\Phi}$ is obtained from Φ in (17) by taking away the last two block-columns and block-rows, Ψ is given by (18) and

$$\xi(t) = \mathrm{col}\{x(t), \delta_0(t), \dots, \delta_{n-1}(t), \kappa_1(t), \dots, \kappa_{n-1}(t), \rho_3(t), g(t, H_0x(t), \dots, H_{n-1}x(t))\}, \quad \eta(t) = \mathrm{col}\{\rho_1(t), \rho_2(t)\}. \tag{37}$$

Substituting (16) for $f(t)$ and further applying Schur complement, we deduce that $\Phi < 0$ given by (17) guarantees $\mathbf{E}LV + 2\alpha\mathbf{E}V \leq 0$ implying that the sampled-data controller (6) exponentially mean-square stabilizes (1) with a decay rate α.

(ii) The system (1), (3) has the form

$$dx_c(t) = \left[(A+BK)x_c(t) + H_{n-1}^T g(t, H_0x_c(t), \dots, H_{n-1}x_c(t))\right]dt + Dx(t)dw(t),$$
$$x_c(t) = \mathrm{col}\left\{y(t), y^{(1)}(t), \dots, y^{(n-1)}(t), \int_0^t y(s)ds\right\},$$

where A, B, D are given by (16) and K is given by (14). If the PID controller (3) exponentially stabilizes (1), where $g \equiv 0$ and $d_i = 0$ $(i = 0, \dots, n-1)$ (and thus, $D = 0$), with a decay rate $\overline{\alpha} > 0$, then there exists $0 < P \in \mathbb{R}^{(n+1)p\times(n+1)p}$ such that $P(A+BK) + (A+BK)^TP + 2\alpha P < 0$ for any $\alpha \in (0, \overline{\alpha})$. Thus,

$$P(A+BK) + (A+BK)^TP + 2\alpha P + D^TPD < 0, \tag{38}$$

for small enough $|D|$. We choose in LMI (17) $W_0 = \frac{1}{\sqrt{h}}I_{2p}$, $R_i = W_i = Q = F_1 = F_2 = \frac{1}{\sqrt{h}}I_p$ $(i = 1, \dots, n-1)$ and $\lambda = \frac{1}{\sqrt{h}}$. Applying Schur complement, $\overline{\Phi} < 0$ is equivalent to

$$P(A+BK) + (A+BK)^TP + 2\alpha P + D^TPD + \sqrt{h}(G_1 + hG_2) + \frac{1}{\sqrt{h}}\sum_{i=0}^{n-1} H_i^TM_iH_i < 0, \tag{39}$$

where

$$G_1 = (n-1)D^TH_{n-1}^TH_{n-1}D + \frac{4}{\pi^2}e^{2\alpha h}P[(A_1 + B\overline{K}_P)(A_1 + B\overline{K}_P)^T + \sum_{i=1}^{n-1} B\overline{K}_{D_i}\overline{K}_{D_i}^TB^T + B\overline{K}_I\overline{K}_I^TB^T]P$$
$$+ \sum_{i=1}^{n-2} e^{2ih}PB\overline{K}_{D_i}\overline{K}_{D_i}^TB^TP + 2e^{2(n-1)h}PB\overline{K}_{D_{n-1}}\overline{K}_{D_{n-1}}^TB^TP + PH_{n-1}^TH_{n-1}P,$$
$$G_2 = \sum_{i=0}^{n-2}\left(e^{2\alpha ih} + \frac{i^2}{4}\right)H_{i+1}^TH_{i+1}.$$

Inequality (38) implies (39) for small enough $h > 0$ and $\|M_i\|$ $(i = 0, \dots, n-1)$ since $\sqrt{h}(G_1 + hG_2) \to 0$ and $\frac{1}{\sqrt{h}}\sum_{i=0}^{n-1}H_i^TM_iH_i = \sqrt{h}\sum_{i=0}^{n-1}H_i^TH_i \to 0$ for $h \to 0$ where we choose, for example, $M_i = hI_p$ $(i = 0, \dots, n-1)$, implying the feasibility of $\overline{\Phi} < 0$ for small enough $h > 0$ and $\|M_i\|$ $(i = 0, \dots, n-1)$. Finally, applying Schur complement to the last two block-columns and block-rows of Φ given by (17), we find that $\Phi < 0$ is feasible for small enough $h > 0$ if $\overline{\Phi} < 0$ is feasible. Thus, LMI (17) is always feasible for small enough $h > 0$, $\|D\|$ and $\|M_i\|$ $(i = 0, \dots, n-1)$. ∎

For the deterministic case (i.e., the system (1) with $d_i = 0$ $(i = 0, \dots, n-1)$), we consider the functional $\tilde{V}$ that is obtained from V in (20) by setting $F_1 = F_2 = 0$ and changing $f(s)$ and Q respectively as $\dot{x}(s)$ and W_{n-1}. The latter includes additional terms V_{δ_i}, $V_{\overline{y}_i}$, V_{κ_i} $(i = 2, \dots, n-1)$ to compensate additional errors $\delta_i(t)$ and $\kappa_i(t)$ $(i = 2, \dots, n-1)$ in (15) comparatively to Selivanov and Fridman.[15,16]

Corollary 1. *Consider the deterministic nonlinear system (1) with* $d_i = 0$ $(i = 0, \dots, n-1)$ *under the sampled-data controller (6). Given* $\overline{K}_P$, $\overline{K}_I$ *and* $\overline{K}_{D_i}$ $(i = 1, \dots, n-1)$ *let the extended PID controller (3) exponentially stabilizes (1), where* $d_i = 0$ $(i = 0, \dots, n-1)$ *and* $g \equiv 0$, *with a decay rate* $\overline{\alpha} > 0$.

(i) Given tuning parameters $h > 0$, $\alpha \in (0, \overline{\alpha})$ *and* $p \times p$ *matrices* M_i $(i = 0, \dots, n-1)$, *let there exist* $(n+1)p \times (n+1)p$ *matrix* $P > 0$, $2p \times 2p$ *matrices* $W_0 > 0$ *and* $p \times p$ *matrices* $W_i > 0$ *and* $R_i > 0$ $(i = 1, \dots, n-1)$ *and scalar* $\lambda > 0$ *that satisfy*

$$\tilde{\Phi} < 0, \tag{40}$$

where $\tilde{\Phi}$ *is obtained from* Φ *in (17) by setting* $D = 0$, $F_1 = F_2 = 0$, $Q = W_{n-1}$ *and and taking away the fifth block-column and block-row. Then the sampled-data controller (6) exponentially stabilizes (1), where* $d_i = 0$ $(i = 0, \dots, n-1)$, *with a decay rate* α.

(ii) Given any $\alpha \in (0, \overline{\alpha})$, *LMI (40) is always feasible for small enough* $h > 0$ *and* $\|M_i\|$ $(i = 0, \dots, n-1)$ *(meaning that the sampled-data controller (6) exponentially stabilizes (1), where* $d_i = 0$ $(i = 0, \dots, n-1)$, *with a decay rate* α*).*

Remark 2. Note that less conservative integral inequalities were introduced e.g. in Seuret et al.[25,26] to improve the results via LMIs. However, the LMIs of Seuret et al.[25,26] cannot be guaranteed to be always feasible. By contrast, we provide in (ii) of Theorem 1 and Corollary 1 (and Theorems 2 and 3 below) the feasibility guarantee of LMIs which were obtained by using Jensen's and Wirtinger's inequalities.

3 | EVENT-TRIGGERED PID CONTROL

Event-triggered control allows to reduce the number of signals transmitted through a communication network (see e.g., Tabuada,[17] Yue et al.,[18] and Heemels et al.[19]). The idea is to transmit the signal only when it satisfies some preselected event-triggering condition. For simplicity we here introduce an event-triggering condition with respect to the control signals:[15]

$$[u(t_k) - \hat{u}_{k-1}]^T \Theta \, [u(t_k) - \hat{u}_{k-1}] > \sigma u^T(t_k)\Theta u(t_k), \tag{41}$$

where $\sigma \in [0, 1)$ and $0 < \Theta \in \mathbb{R}^{q\times q}$ are the event-triggering parameters, $u(t_k)$ is from (6) and $\hat{u}_{k-1}$ denotes the last transmitted control signal. Thus, $\hat{u}_0 = u(t_0)$ and

$$\hat{u}_k = \begin{cases} u(t_k), & \text{if (41) is true,} \\ \hat{u}_{k-1}, & \text{if (41) is false.} \end{cases} \tag{42}$$

Hence, the system (1) becomes

$$dy^{(n)}(t) = \left[\sum_{i=0}^{n-1} a_i y^{(i)}(t) + b\hat{u}_k + g(t, y^{(0)}(t), \dots, y^{(n-1)}(t))\right] dt + \sum_{i=0}^{n-1} d_i y^{(i)}(t) dw(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0, \tag{43}$$

with $\hat{u}_k$ given by (42). Introduce the event-triggering error

$$e_k = \hat{u}_k - u(t_k). \tag{44}$$

Then following the modeling in the previous section, the system (43) under the event-triggered PID control (3), (41), (42) can be presented as (cf. (15))

$$dx(t) = [f(t) + Be_k]dt + Dx(t)dw(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0. \tag{45}$$

Theorem 2. *Consider the stochastic nonlinear system (1) under the event-triggered PID controller (6), (41), (42). Given* $\overline{K}_P$, $\overline{K}_I$ *and* $\overline{K}_{D_i}$ $(i = 1, \dots, n-1)$ *let the extended PID controller (3) exponentially stabilizes (1), where* $g \equiv 0$ *and* $d_i = 0$ $(i = 0, \dots, n-1)$, *with a decay rate* $\overline{\alpha} > 0$.

(i) Given tuning parameters $h > 0$, $\alpha \in (0, \overline{\alpha})$, $\sigma \in [0, 1)$ *and* $p \times p$ *matrices* M_i $(i = 0, \dots, n-1)$, *let there exist* $(n+1)p \times (n+1)p$ *matrix* $P > 0$, $2p \times 2p$ *matrices* $W_0 > 0$, $p \times p$ *matrices* $W_i > 0$, $R_i > 0$ $(i = 1, \dots, n-1)$, $Q > 0$, $F_1 > 0$ *and* $F_2 > 0$,

$q \times q$ matrix $\Theta > 0$ and scalar $\lambda > 0$ that satisfy (18) and

$$\Phi_e = \left[\begin{array}{c|cc} & PB & \sigma K^T\Theta \\ & 0 & \sigma[\bar{K}_P, \bar{K}_I]^T\Theta \\ & 0 & \sigma[\bar{K}_{D_1}, \dots, \bar{K}_{D_{n-1}}]^T\Theta \\ & 0 & \sigma[\bar{K}_{D_1}, \dots, \bar{K}_{D_{n-1}}]^T\Theta \\ \Phi & 0 & 0 \\ & 0 & 0 \\ & h\Xi H_{n-1}B & 0 \\ & 0 & 0 \\ \hline * & -\Theta & 0 \\ & * & -\sigma\Theta \end{array}\right] < 0, \tag{46}$$

where Φ and Ξ are respectively given by (17) and (19), K and H_{n-1} are given by (14) and B is given by (16). Then the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate α.

(ii) Given any $\alpha \in (0, \overline{\alpha})$, LMI (46) is always feasible for small enough $h > 0$, $\sigma \in (0, 1)$, $\|D\|$ and $\|M_i\|$ $(i = 0, \dots, n-1)$ (meaning that the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate α).

Proof. (i) Using the triggering error (44), the event-triggering condition (41), (42) guarantees

$$0 \le \sigma u^T(t_k)\Theta u(t_k) - e_k^T\Theta e_k. \tag{47}$$

Consider the functional V from (20) with $f(t)$ changed by $f(t) + Be_k$. Following the proof of item (i) of Theorem 1, along (45) we have (cf. (36))

$$\begin{aligned} \mathbf{E}LV + 2\alpha\mathbf{E}V &\overset{(47)}{\le} \mathbf{E}LV + 2\alpha\mathbf{E}V + \sigma\mathbf{E}u^T(t_k)\Theta u(t_k) - \mathbf{E}e_k^T\Theta e_k \\ &\le \mathbf{E}\xi_e^T(t)\overline{\Phi}_e\xi_e(t) + h^2\mathbf{E}(f(t) + Be_k)^T H_{n-1}^T \left[\frac{(n-1)^2}{4}R_{n-1} + e^{2\alpha(n-1)h}Q\right] H_{n-1}(f(t) + Be_k) \\ &\quad + h^2\mathbf{E}\begin{bmatrix} H_1x(t) \\ H_0x(t) + [I_P, 0]\delta_0(t) \end{bmatrix}^T W_i \begin{bmatrix} H_1x(t) \\ H_0x(t) + [I_P, 0]\delta_0(t) \end{bmatrix} + \sigma\mathbf{E}u^T(t_k)\Omega u(t_k), \end{aligned} \tag{48}$$

where $\xi_e(t) = \mathrm{col}\{\xi(t), e_k\}$ with $\xi(t)$ given by (37), $\overline{\Phi}_e$ is obtained from Φ_e in (46) by taking away the i- and j-blocks with $i \in \{7, 8, 10\}$ or $j \in \{7, 8, 10\}$. Substituting (13) and (16), respectively, for $u(t_k)$ and $f(t)$ and further applying Schur complement, we find that $\Phi_e < 0$ given by (46) guarantees $\mathbf{E}LV + 2\alpha\mathbf{E}V \le 0$ implying that the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate α.

(ii) The proof of (ii) is similar to (ii) of Theorem 1. ■

Remark 3. To select the tuning parameters h, α, σ, M_i and d_i $(i = 0, \dots, n-1)$ we suggest the following algorithm: choose $\overline{K}_P$, $\overline{K}_I$ and $\overline{K}_{D_i}$ $(i = 1, \dots, n-1)$ via pole-placement such that the extended PID controller (1) exponentially stabilizes (13), where $g \equiv 0$ and $d_i = 0$ $(i = 0, \dots, n-1)$, with a decay rate $\overline{\alpha} > 0$. By solving the LMIs with $M_i = 0$, $d_i = 0$ $(i = 0, \dots, n-1)$, $\sigma = 0$ and small enough $h > 0$, we find a critical maximal value of α as $\alpha^* < \overline{\alpha}$. Then, by choosing $\alpha \in [0, \alpha^*]$ with $M_i = 0$, $d_i = 0$ $(i = 0, \dots, n-1)$ and small enough $h > 0$, we find a critical maximum value of σ as σ^*. The same is done for M_i, d_i $(i = 0, \dots, n-1)$ that leads to critical maximum values of M_i, d_i $(i = 0, \dots, n-1)$, respectively, as M_i^*, d_i^* $(i = 0, \dots, n-1)$. Then for $\alpha \in [0, \alpha^*]$, $\sigma \in [0, \sigma^*]$, $M_i \in [0, M_i^*]$ and $d_i \in [0, d_i^*]$ $(i = 0, \dots, n-1)$, we can obtain a critical maximal value of $h = h^*$ such that for $h > h^*$ the LMI becomes unfeasible.

4 | L_2-GAIN ANALYSIS

The direct Lyapunov method is applicable not only to the stability but also to the performance analysis,[9] for example, L_2-gain analysis. In this section, we consider L_2-gain analysis of the perturbed systems, namely (cf. (43))

$$dy^{(n)}(t) = \left[\sum_{i=0}^{n-1} a_i y^{(i)}(t) + b\hat{u}_k + b_v v(t) + g(t, y^{(0)}(t), \dots, y^{(n-1)}(t))\right] dt + \sum_{i=0}^{n-1} d_i y^{(i)}(t)dw(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0, \tag{49}$$

where $b_v \in \mathbb{R}^{p \times p_v}$ is a constant matrix and $v(t) \in \mathbb{R}^{p_v}$ is the external disturbance in $L_2[0, \infty)$.

The system (49) under the event-triggered PID control (3), (41), (42) has the form:

$$dx(t) = [f(t) + Be_k + B_v v(t)]dt + Dx(t)dw(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0, \tag{50}$$

where $x(t)$ is given by (14), $f(t)$, B and D are given by (16) and

$$B_v = \text{col}\{0_{(n-1)p \times p_v}, b_v, 0_{p \times p_v}\}. \tag{51}$$

Consider next the controlled output

$$z(t) = Cx(t) + C_v v(t), \quad z(t) \in \mathbb{R}^l, \tag{52}$$

where $C \in \mathbb{R}^{l \times (n+1)p}$ and $C_v \in \mathbb{R}^{l \times p_v}$ are constant matrices. For a prechosen $\gamma > 0$ we introduce the following performance index:

$$J = \int_0^\infty \left[z^T(t)z(t) - \gamma^2 v^T(t)v(t)\right] dt. \tag{53}$$

We seek conditions that will lead to $\mathbf{E}J \leq 0$ for all $x(t)$ satisfying (50) with the zero initial condition $x(0) = 0$ and for all $0 \neq v \in L_2[0, \infty)$. In this case the system (50), (52) has L_2-gain less than or equal to γ. Moreover, if the system (50) with $v \equiv 0$ is exponentially mean-square stable, then the system (50) is internally exponentially mean-square stable.

Lemma 3. [9] *Given $\alpha \geq 0$ and $\gamma > 0$, let for V given by (20) the following inequality holds along the solutions of (50):*

$$\mathbf{E}LV + 2\alpha\mathbf{E}V + \mathbf{E}z^T(t)z(t) - \gamma^2 v^T(t)v(t) < 0 \quad \forall 0 \neq v(t) \in \mathbb{R}^{p_v} \text{ and } \forall t \geq 0. \tag{54}$$

If (54) holds with $\alpha = 0$, then the system (50), (52) has L_2-gain less than or equal to γ. Moreover, if (54) holds with $\alpha > 0$, then the system (50) is internally exponentially mean-square stable with a decay rate α.

Based on Lemma 3, we now present the following LMI conditions:

Theorem 3. *Consider the stochastic nonlinear system (1) with an additive external disturbance $v(t)$ under the event-triggered PID controller (6), (41), (42) leading to system (50), and the controlled output (52). Given $\overline{K}_P$, $\overline{K}_I$ and $\overline{K}_{D_i}$ $(i = 1, \dots, n-1)$ let the extended PID controller (3) exponentially stabilizes (1), where $g \equiv 0$ and $d_i = 0$ $(i = 0, \dots, n-1)$, with a decay rate $\overline{\alpha} > 0$.*

(i) Given tuning parameters $h > 0$, $\alpha \in (0, \overline{\alpha})$, $\sigma \in [0, 1)$ and $\gamma > 0$, and $p \times p$ matrices M_i $(i = 0, \dots, n-1)$, let there exist $(n+1)p \times (n+1)p$ matrix $P > 0$, $2p \times 2p$ matrices $W_0 > 0$, $p \times p$ matrices $W_i > 0$, $R_i > 0$ $(i = 1, \dots, n-1)$, $Q > 0$, $F_1 > 0$ and $F_2 > 0$, $q \times q$ matrix $\Theta > 0$ and scalar $\lambda > 0$ that satisfy (18) and

$$\Phi_{L_2} = \left[\begin{array}{c|cc} & PB_v & C^T \\ & 0_{(2n+2)p \times p_v} & 0_{(2n+2)p \times p_l} \\ \Phi_e & h\Xi H_{n-1} B_v & 0_{p_v \times p_l} \\ & 0_{2(p+q) \times p_v} & 0_{2(p+q) \times p_l} \\ \hline * & -\gamma^2 I_v & C_v^T \\ & * & -I_l \end{array}\right] < 0, \tag{55}$$

TABLE 1 Maximum value of h via linear matrix inequalities

	Example 1			Example 2			Example 3		
d_1	0	0.2	0.5	0	0.2	0.5	0	0.01	0.02
Selivanov and Fridman[15]	0.0047	—	—	—	—	—	—	—	—
Selivanov and Fridman[16]	0.019	—	—	—	—	—	—	—	—
Corollary 1	0.019	—	—	0.105	—	—	0.084	—	—
Theorem 1	0.019	0.012	0.002	0.105	0.910	0.055	0.084	0.070	0.001

where H_{n-1}, Ξ, Φ_e and B_v are respectively given by (14), (19), (46), and (51), and C and C_v are given by (52). Then the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate α, and the system (50), (52) has L_2-gain less than or equal to γ.

(ii) Given any $\alpha \in (0, \overline{\alpha})$, LMI (55) is always feasible for small enough $h > 0$, $\sigma \in (0, 1)$, $\frac{1}{\gamma} > 0$, $\|D\|$ and $\|M_i\|$ $(i = 0, \dots, n-1)$ (meaning that the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate α).

5 | EXAMPLES

To illustrate the efficiency, we present three examples including a servo positioning system.

Example 1. Consider system (1) with

$$a_0 = 0, \quad a_1 = -8.4, \quad b = 35.71, \quad g \equiv 0. \tag{56}$$

The system is not stable if $u = 0$. The PID controller (3) with

$$\overline{K}_P = -10, \quad \overline{K}_I = -40, \quad \overline{K}_{D_1} = -0.65. \tag{57}$$

stabilizes system (1) with (56) for small enough stochastic perturbations. Let $\alpha = 5$ be the desired decay rate. In the deterministic case (i.e., $d_0 = d_1 = 0$), LMIs of Corollary 1 and Selivanov and Fridman[16] lead to the same result which is larger than that via Selivanov and Fridman.[15] In the stochastic case, LMIs of Theorem 1 with $d_0 = 0$ and different values of d_1 lead to efficient results (see Table 1).

Consider now system (1) with (56) under the event-triggered PID control. For $h = 0.005$, $d_0 = 0$ and $d_1 = 0.2$, LMI of Theorem 2 is feasible for a maximum value of $\sigma = 0.074$. Sampled-data control requires to transmit $1/h + 1 = 201$ control signals during 1 s of simulations. By performing numerical simulations with 10 randomly chosen initial conditions $\|x(0)\|_\infty \leq 1$ where we applied Euler-Maruyama method[27] using a step size $10dt$ with $dt = 10^{-6}$, the event-triggered control requires to transmit on average 63.95 control signals. Thus, the even-triggering mechanism (41), (42) reduces the number of transmitted control signals by almost 69%.

Example 2. (Chain of three integrators). Consider system (1) with

$$a_i = 0, \quad i = 0, 1, 2, \quad b = 1, \quad g \equiv 0. \tag{58}$$

Using the pole placement, we find that for (3) with

$$\overline{K}_P = -6.026, \quad \overline{K}_I = -1.716, \quad \overline{K}_{D_1} = -7.91, \quad \overline{K}_{D_2} = -4.6, \tag{59}$$

the eigenvalues of $A + BK$ are -1, -1.1, -1.2 and -1.3. Therefore, the PID controller (3) with (59) stabilizes system (1) with (58) for small enough stochastic perturbations.

Let $\alpha = 0.2$, $d_0 = d_2 = 0$. For different values of d_1, the maximum values of h that preserve the exponential stability are presented in Table 1. It is clear that LMIs of Corollary 1 and Theorem 1 lead to efficient results whereas Selivanov and Fridman[15,16] fail. For $h = 0.04$ and $d_1 = 0.2$, LMIs of Theorem 2 are feasible for a maximum value of $\sigma = 0.119$. We next perform numerical simulations with 10 randomly chosen initial conditions $\|x(0)\|_\infty \leq 1$ by using Euler–Maruyama method[27] with a step size $10dt$ and $dt = 10^{-6}$. One can find that the event-triggered control requires to transmit on average 96.8 control signals during 10 seconds. Note that the number of transmissions for the sampled-data control is given by $10/h + 1 = 251$. Thus, the event-triggering mechanism (41), (42) reduces the number of transmitted control signals by over 61%.

Example 3. Consider the servo positioning system with a stochastic perturbation[28,29]

$$\theta_1 dy^{(1)}(t) = [-\theta_4 y^{(1)}(t) + u(t) - F(y^{(1)}(t)) + b_v v(t)]dt + d_1 y^{(1)}(t)dw(t), \tag{60}$$

where $F(\dot{y}(t)) = \theta_2 \tanh(700\dot{y}(t)) + \theta_3[\tanh(15\dot{y}(t)) - \tanh(1.5\dot{y}(t))]$, $y(t)$ is the motor rotation angle, $u(t)$ is the control input and $w(t)$ is the load disturbance. Set $[\theta_1, \theta_2, \theta_3, \theta_4] = [0.0025, 0.02, 0.01, 0.205]$. Following the previous modeling, the system (60) under an event-triggered PID control can be written in the form of (45) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{\theta_4}{\theta_1} & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{\theta_1} \\ 0 \end{bmatrix}, \quad B_v = \begin{bmatrix} 0 \\ b_v \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{d_1}{\theta_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and with $g = -F(\dot{y}(t))$. Note that the latter nonlinearity satisfies (2) with $M_0 = 0$ and $M_1 = 14.13$. Moreover, the controlled output is given by (52) with $C = [1, 0, 0]$ and $C_v = 2$. The PID controller (3) with

$$\overline{K}_P = -0.4980, \quad \overline{K}_I = -0.0255, \quad \overline{K}_{D_1} = -0.270, \tag{61}$$

exponentially stabilizes the system (60).

Set $\alpha = 0.1$ and $d_0 = 0$. For different values of d_1 and $b_v = 0$, LMIs of Corollary 1 and Theorem 1 lead to efficient results in Table 1. For $h = 0.05$, $d_1 = 0.01$ and $b_v = 0$, LMIs of Theorem 2 are feasible for a maximum value of $\sigma = 0.04$. Sampled-data control requires to transmit $5/h + 1 = 101$ control signals during 5 s. By performing numerical simulations with 10 randomly chosen initial conditions $\|x(0)\|_\infty \leq 1$ where we applied Euler–Maruyama method[27] using a step size $10dt$ with $dt = 10^{-6}$, the event-triggered control requires to transmit on average 32.6 control signals. Thus, the even-triggering mechanism (41), (42) reduces the number of transmitted control signals by over 67%. Moreover, for $h = 0.02$, $d_1 = 0.01$, $b_v = 1$ and $\sigma = 0.04$, by LMIs of Theorem 3 a minimum value of $\gamma = 2.02$ is obtained.

6 | CONCLUSIONS

In this paper, sampled-data implementation of extended PID control using delays has been presented for the nth-order stochastic nonlinear systems. We have employed an event-triggering condition that allows to reduce the number of sampled control signals used for stabilization and have studied L_2-gain analysis. The suggested method may be useful for delay-induced consensus in multi-agent systems under an extended PID control. This may be a topic for the future research.

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DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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