Sampled-data implementation of extended PID control using delays

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Abstract
We study the sampled-data implementation of extended PID control using delays for the $n$th-order stochastic nonlinear systems. The derivatives are approximated by finite differences giving rise to a delayed sampled-data controller. An appropriate Lyapunov–Krasovskii (L-K) method is presented to derive linear matrix inequalities (LMIs) for the exponential stability of the resulting closed-loop system. We show that with appropriately chosen gains, the LMIs are always feasible for small enough sampling period and stochastic perturbation. We further employ an event-triggering condition that allows to reduce the number of sampled control signals used for stabilization and provide $L_2$-gain analysis. Finally, three numerical examples illustrate the efficiency of the presented approach.

KEYWORDS
$L_2$-gain analysis, event-triggered control, PID control, sampled-data control, stochastic perturbations

1 | INTRODUCTION

Proportional-integral-derivative (PID) control is widely used in many industrial processes.\(^1,2\) Many results on the classical PID control have been established, for example, for the second-order systems\(^3,4\) and for the $n$th-order systems.\(^6\) The PID control depends on the output derivative that cannot be measured in practice. Instead, the derivative can be approximated by the finite-difference leading to a delayed feedback. The delay-induced stability was studied, for example, in Niculescu and Michiel\(^7\) and Ramírez et al.\(^8\) using frequency-domain technique. Alternatively, it can be studied using the LMI-based method\(^9\) that allows to cope with, for example, certain types of nonlinearities and stochastic perturbations\(^10,12\) although being conservative.

Modern control usually employs digital technology for controller implementation, that is, sampled-data control. Moreover, sampled-data controller uses the sampled output only which is more practical. Thus, for practical application of PID control, its sampled-data implementation is important. By using consecutive sampled outputs, sampled-data implementation of PD control was presented for the $n$th-order deterministic\(^13\) and stochastic\(^14\) systems. Sampled-data implementation of PID control for the second-order deterministic systems was studied in Selivanov and Fridman.\(^15,16\) However, the idea of using consecutive sampled outputs has not been studied yet for extended PID control of the $n$th-order deterministic ($n \geq 3$) or stochastic ($n \geq 2$) systems.

In this present paper, we study extended PID control of the $n$th-order stochastic nonlinear systems. Differently from Zhao and Guo\(^6\) with the full knowledge of the system state, we consider sampled-data implementation of extended PID control by using the sampled outputs only. Following the improved approximation method\(^13\) with consecutive sampled outputs, we approximate the extended PID controllers depending on the output and its derivatives up to the order...
n − 1 as delayed sampled-data controllers. Extension to PID control of the nth-order stochastic systems is far from being straightforward for the following reasons:

(i) Comparatively to the models under the PD control\textsuperscript{13} or the PID control,\textsuperscript{15,16} we have additional errors to be compensated by employing additional terms in the corresponding Lyapunov functionals.

(ii) The Lyapunov functionals of Selivanov and Fridman\textsuperscript{13,15,16} are not applicable in the stochastic case. This is because a solution of a stochastic system does not have a derivative.\textsuperscript{12,14} Thus, we propose novel Lyapunov functionals depending on the deterministic and stochastic parts of the system that lead to LMI-based stability conditions.

We show that the LMIs are always feasible for small enough sampling period and stochastic perturbation if the extended PID controller that employs the full-state stabilizes the system. Moreover, we employ an event-triggering condition\textsuperscript{17-19} that allows to reduce the number of sampled control signals used for stabilization and provide $L_2$-gain analysis. Finally, three numerical examples are presented to illustrate the efficiency of the presented approach.

1.1 Notations and useful inequalities

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $I_n$ is the identity $n \times n$ matrix, the superscript $T$ stands for matrix transposition. $\mathbb{R}^n$ denotes the $n$ dimensional Euclidean space with Euclidean norm $\| \cdot \|$, $\mathbb{R}^{m \times n}$ denotes the set of all $n \times m$ real matrices with the induced matrix norm $\| \cdot \|$. Denote by $\text{diag}\{ \ldots \}$ and $\text{col}\{ \ldots \}$ block-diagonal matrix and block-column vector, respectively. $P > 0$ implies that $P$ is a positive definite symmetric matrix. $C^i$ is a class of $t$ times continuously differentiable functions.

We now present some useful inequalities:

**Lemma 1.** (Extended Jensen’s inequality\textsuperscript{20}). Denote $G = \int_b^a f(s) x(s) ds$, where $f : [a, b] \rightarrow \mathbb{R}$, $x : [a, b] \rightarrow \mathbb{R}^n$ and the integration concerned is well defined. Then for any $n \times n$ matrix $R > 0$ the following inequality holds:

$$G^T R G \leq \int_b^a |f(s)|^2 ds \int_b^a |f(s)| x^T(s) R x(s) ds.$$ 

**Lemma 2.** (Exponential Wirtinger’s inequality\textsuperscript{21}). Let $x(t) : (a, b) \rightarrow \mathbb{R}^n$ be absolutely continuous with $\dot{x} \in L_2(a, b)$ and $x(a) = 0$ or $x(b) = 0$. Then the following inequality holds:

$$\int_b^a e^{2\pi t} x^T(s) W x(s) ds \leq e^{2\pi (b-a)} \frac{4(b-a)^2}{\pi^2} \int_a^b e^{2\pi t} x^T(s) W x(s) ds,$$

for any $\alpha \in \mathbb{R}$ and $n \times n$ matrix $W > 0$.

2 EXTENDED PID CONTROL OF STOCHASTIC NONLINEAR SYSTEMS

Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a probability space. A filtration is a family $\{\mathcal{F}_t, t \geq 0\}$ of nondecreasing sub-$\sigma$-algebras of $\mathcal{F}$, that is, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$ and $\mathbb{P}\{\cdot\}$ be the probability of an event enclosed in the brackets. The mathematical expectation $\mathbb{E}$ of a random variable $\xi = \xi(w)$ on the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ is defined as $\mathbb{E}\xi = \int_\Omega \xi(w) d\mathbb{P}(w)$. The scalar standard Wiener process (also called Brownian motion) is a stochastic process $w(t)$ with normal distribution satisfying $w(0) = 0$, $\mathbb{E}w(t) = 0$ ($t > 0$) and $\mathbb{E}w^2(t) = t$ ($t > 0$).\textsuperscript{22}

Consider the nth-order stochastic nonlinear system

$$dy^{(n)}(t) = \left[\sum_{i=0}^{n-1} a_i y^{(i)}(t) + bu(t) + g(t, y^{(0)}(t), \ldots, y^{(n-1)}(t))\right] dt + \sum_{i=0}^{n-1} d_i y^{(i)}(t) dw(t). \quad (1)$$

Here $y(t) = y^{(0)}(t) \in \mathbb{R}^p$ is the output, $y^{(i)}(t)$ ($i = 1, \ldots, n - 1$) is the $i$th derivative of $y(t)$, $a_i, d_i \in \mathbb{R}^{p \times p}$ and $b \in \mathbb{R}^{p \times q}$ are constant matrices and $g : \mathbb{R} \times \mathbb{R}^p \times \ldots \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ is a locally Lipschitz continuous in arguments from the second to the
last and satisfies for all $t \geq 0$ the inequality

$$|g(t, x_0, \ldots, x_{n-1})|^2 \leq \sum_{i=0}^{n-1} x_i^T M_i x_i, \quad \forall x_i \in \mathbb{R}^p, \quad i = 0, \ldots, n-1,$$

(2)

with some matrices $0 < M_i \in \mathbb{R}^{p \times p}$ ($i = 0, \ldots, n-1$).

In Zhao and Guo,\(^6\) an extended PID controller was designed as follows

$$u(t) = \left[ \bar{K}_p y(t) + \bar{K}_d \int_0^t y(s) ds + \sum_{i=1}^{n-1} \bar{K}_d y_i(t) \right],$$

(3)

where $\bar{K}_p$, $\bar{K}_d$, and $\bar{K}_d$ \(\in \mathbb{R}^{q \times p}\) ($i = 1, \ldots, n-1$) are the controller gains. Differently from Zhao and Guo\(^6\) with the full knowledge of the system state (i.e., $y_i(t)$, $i = 0, \ldots, n-1$), we consider the output-feedback control, where $y_i(t)$, $i = 1, \ldots, n-1$ in (3) are not available. Moreover, for the practical implementation we assume that the output $y(t)$ is available only at the discrete-time instants $t_k = kh$, where $k \in \mathbb{N}_0$ and $h > 0$ is the sampling period. As in Selivanov and Fridman,\(^15\) we suggest the following approximations for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}_0$:

$$y(t) = y(t) \approx \bar{y}(t_k), \quad \int_0^t y(s) ds \approx \int_0^{t_k} \bar{y}(s) ds \approx h \sum_{j=0}^{k-1} \bar{y}(t_j), \quad y_i(t) \approx \bar{y}_i(t_k) = \bar{y}_i(t_k) = y(t), \quad i = 1, \ldots, n-1,$$

(4)

where we used $\int_0^{t_k} \bar{y}(s) ds = \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \bar{y}(s) ds \approx \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} y(t_j) ds = h \sum_{j=0}^{k-1} y(t_j)$ for the approximation of the integral and applied the finite-difference method for $\bar{y}_i(t_k)$ ($i = 1, \ldots, n-1$) with

$$\bar{y}_i(t) = \frac{\bar{y}_i(t)-\bar{y}_i(t-h)}{h}, \quad i = 1, \ldots, n-1, \quad \bar{y}_0(t) = \bar{y}_0(t) = y(t),$$

(5)

and $y(t) = y(0)$ for $t < 0$. It is clear that via (5) we can compute $\bar{y}_i(t_k)$ (and thus, $\bar{y}_i(t_k)$, $i = 1, \ldots, n-1$).

Thus, we design in this paper the following sampled-data controller

$$u(t) = \bar{K}_p \bar{y}(t_k) + h \bar{K}_d \sum_{j=0}^{k-1} \bar{y}(t_j) + \sum_{i=1}^{n-1} \bar{K}_d \bar{y}_i(t_k), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0.$$

(6)

In order to study the stability of system (1) under the sampled-data controller (6), we first present the approximation errors $\bar{y}(t_k) - y(t)$ and $\bar{y}_i(t_k) - y_i(t)$ ($i = 1, \ldots, n-1$), where $t \in [t_k, t_{k+1}), k \in \mathbb{N}_0$, in a convenient form suitable for the later analysis via L-K functionals:

**Proposition 1.** If $y \in C^1$ and $y_i(t)$ is absolutely continuous with $i = 1, \ldots, n$, then $\bar{y}(t_k)$ and $\bar{y}_i(t_k)$ ($i = 1, \ldots, n-1$) defined by (5) satisfy for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}_0$

$$\bar{y}(t_k) = y(t) - \int_{t_k}^t y(s) ds,$$

(7)

$$\bar{y}_i(t_k) = y_i(t) - \int_{t_{k-h}}^{t} \varphi_1(t-s) y_i(s) ds - \int_{t_k}^{t} \varphi_1(t-s) y_i(s) ds, \quad i = 1, \ldots, n-1,$$

(8)

where

$$\varphi_1(v) = \frac{h-v}{h}, \quad v \in [0, h],$$

$$\varphi_i(v) = \begin{cases} \frac{v}{h} \int_0^v \varphi_1(\lambda) d\lambda + \frac{h-v}{h}, & v \in [0, h], \\ \frac{1}{h} \int_h^{v-h} \varphi_1(\lambda) d\lambda, & v \in (h, ih), \quad i = 1, \ldots, n-2. \\ \frac{1}{h} \int_{ih-h}^h \varphi_1(\lambda) d\lambda, & v \in [ih, ih+h], \end{cases}$$

(9)
Proof. We first introduce the errors due to the sampling:

\[
y(t_k) = y(t) - \int_{t_k}^{t} \dot{y}(s) ds, \quad \tilde{y}^{(i)}(t_k) = \tilde{y}^{(i)}(t) - \int_{t_k}^{t} \dot{y}^{(i)}(s) ds, \quad i = 1, \ldots, n - 1.
\]  

Taking into account \( y(t_k) = \tilde{y}(t_k) \) in (4), together with the first equality in (10) we obtain (7). Then following arguments for the error \( \tilde{y}^{(i)}(t) - y^{(i)}(t) \) \( (i = 1, \ldots, n - 1) \) in Proposition 1 of Selivanov and Fridman,\(^{13}\) that is,

\[
\tilde{y}^{(i)}(t) = y^{(i)}(t) - \int_{t-ih}^{t} \varphi_i(t-s)y^{(i)}(s) ds, \quad i = 1, \ldots, n - 1,
\]

where \( \varphi_i(\cdot) \) \( (i = 1, \ldots, n - 1) \) are defined by (9), we arrive at (8).

The functions \( \varphi_i(\cdot) \) \( (i = 1, \ldots, n - 1) \) have the following properties (see the proof in Selivanov and Fridman\(^{13}\)):

**Proposition 2.** The functions \( \varphi_i(\cdot) \) \( (i = 1, \ldots, n - 1) \) in (9) satisfy

1. \( \varphi_i(0) = 1, \quad \varphi_i(ih) = 0; \quad 2) \ 0 \leq \varphi_i(v) \leq 1; \quad 3) \ \frac{d}{dv} \varphi_i(v) \in \left[ -\frac{1}{n}, 0 \right]; \quad 4) \ \int_{0}^{ih} \varphi_i(v) dv = \frac{ih}{2}.
\]

By noting that \( y(t_j) = \tilde{y}(t_j) \) \( (j = 0, \ldots, k - 1) \), via (7) and (8) the sampled-data controller (6) can be presented as

\[
u(t) = \frac{1}{\kappa_p} \left\{ y(t) - \int_{t_k}^{t} \dot{y}(s) ds + h\kappa_i \sum_{j=0}^{k-1} y(t_j) + \sum_{i=1}^{n-1} \kappa_{D_i}(\delta_i(t) + \kappa_i(t)) \right\} = \frac{1}{\kappa_p}[y(t) - \int_{t_k}^{t} \dot{y}(s) ds + \sum_{i=1}^{n-1} \kappa_i(t) \delta_i(t)], \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0,
\]

where

\[
x(t) = \text{col} \left\{ y(t), y^{(1)}(t), \ldots, y^{(n-1)}(t), (t - t_k)y(t_k) + h \sum_{j=0}^{k-1} y(t_j) \right\},
\]

\[
K = [\kappa_p, \kappa_{D_1}, \ldots, \kappa_{D_{n-1}}, \kappa_1], \quad \delta_0(t) = -\int_{t_k}^{t} H_0 ds, \quad \delta_i(t) = -\int_{t_k}^{t} \dot{y}^{(i)}(s) ds, \quad \kappa_i(t) = -\int_{t-ih}^{t} \varphi_i(t-s)H_i \dot{x}(s) ds, \quad i = 1, \ldots, n - 1,
\]

\[
H_i = \{0, p_0, I_p, 0_{p \times (n-i)p}\}, \quad i = 0, \ldots, n.
\]

Using (13) and (14), the system (1), (6) has the form

\[
dx(t) = f(t) dt + D\dot{x}(t) dw(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0,
\]

where

\[
f(t) = (A + BK)x(t) + A_1 \delta_0(t) + \sum_{i=1}^{n-1} B\kappa_{D_i}(\delta_i(t) + \kappa_i(t)) + H_i^{T}g(t, H_0x(t), \ldots, H_{n-1}x(t)),
\]

\[
A = \begin{bmatrix}
0 & I_p & 0 & \cdots & 0 & 0 \\
0 & 0 & I_p & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I_p & 0 \\
a_0 & a_1 & a_2 & \cdots & a_{n-1} & 0 \\
I_p & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}, \quad A_1 = \begin{bmatrix}
0_{(n-1) \times p} & 0_{(n-1) \times p} \\
b\kappa_p & b\kappa_1 \\
I_p & 0_{p \times p}
\end{bmatrix}, \quad B = \text{col} \{0_{(n-1) \times p}, b, 0_{p \times q}\}, \quad D = \text{col} \{0_{(n-1) \times p}, \overline{D}, 0_{p \times p}\}, \quad \overline{D} = \{d_0, \ldots, d_{n-1}, 0\}.
\]
Remark 1. In (15), we follow the transformation of Zhang and Fridman\(^{23}\) that allowed to avoid an additional non-zero term \(y^{(n-1)}(t_k) - y^{(n-1)}(t) = -\int_{t}^{t_k} H_{n-1} f(s) ds - \Pi\) with \(\Pi = \int_{t}^{t_k} H_{n-1} D x dw(s)\). Note that the term \(\Pi\) has to be compensated by additional terms in the Lyapunov functional. Hence, the transformation in (15) (comparatively to Selivanov and Fridman\(^{15,16}\)) significantly simplifies the analysis in the stochastic case.

Comparatively to the system model (see e.g., (27) in Selivanov and Fridman\(^{13}\)) under PD control, the system (15) includes additional term \(A_1\delta(t)\) (due to the additional I control) that will be compensated by the additional term \(V_{\delta_0}\) defined below (20). Note also that Lyapunov functional of Selivanov and Fridman\(^{13}\) depends on the nth-order derivative, and, thus, is not applicable in the stochastic case. This is because a solution of a stochastic system does not have a derivative.\(^{12,14}\) We will present LMI conditions via novel Lyapunov functional that depends on the deterministic and stochastic parts of the system:

**Theorem 1.** Consider the stochastic nonlinear system (1) under the sampled-data controller (6). Given \(\overline{K}_P, \overline{K}_I,\) and \(\overline{K}_D_i\) \((i = 1, \ldots, n - 1)\) let the extended PID controller (3) exponentially stabilizes (1), where \(d_i = 0\) \((i = 0, \ldots, n - 1)\) and \(g \equiv 0\), with a decay rate \(\bar{\alpha} > 0\).

(i) Given tuning parameters \(h > 0\), \(\alpha \in (0, \bar{a})\) and \(p \times p\) matrices \(M_i\) \((i = 0, \ldots, n - 1)\), let there exist \((n + 1)p \times (n + 1)p\) matrix \(P > 0\), \(2p \times 2p\) matrix \(W_0 > 0\), \(p \times p\) matrices \(W_i > 0, R_i > 0\) \((i = 1, \ldots, n - 1)\), \(Q > 0, F_1 > 0\) and \(F_2 > 0\) and scalar \(\lambda > 0\) that satisfy

\[
\Phi = \begin{bmatrix}
\Phi_{11} & PA_1 & \Phi_{13} & \Phi_{14} & 0 & PH_1^T & h(A + BK)^TH_1^T & hH_1^T & h[0, [P, 0]^T]W_0 \\
* & -\frac{\pi^2}{4} e^{-2\alpha h} W_0 & 0 & 0 & 0 & 0 & hA_1^TH_1^T & h[0, [P, 0]^T]W_0 \\
* & * & \Phi_{33} & 0 & 0 & 0 & h\Phi_{37} & 0 \\
* & * & * & \Phi_{44} & 0 & 0 & h\Phi_{47} & 0 \\
* & * & * & * & -e^{-2\alpha(n-1)h}(R_{n-1} + F_2) & 0 & 0 & 0 \\
* & * & * & * & * & -\lambda I_p & hR_{n-1}H_{n-1}^T & 0 \\
* & * & * & * & * & * & -\Xi & 0 \\
* & * & * & * & * & * & -W_0 \\
\end{bmatrix} < 0. (17)
\]

\[
\Psi = \begin{bmatrix}
W_{n-1} - Q \\
W_{n-1} - \frac{(n-1)}{2} e^{-2\alpha(n-1)h} F_2 \\
\end{bmatrix} < 0, (18)
\]

where

\[
\Phi_{11} = PA(A + BK) + (A + BK)^TP + 2\alpha P + \sum_{i=0}^{n-2} h^2 e^{2\alpha h} H_{i+1}^T W_i H_{i+1} + \sum_{i=0}^{n-2} \frac{(ih)^2}{4} H_{i+1}^T R_i H_{i+1} \\
+ D^T PD + \frac{(n-1)^2}{2} D^T H_{n-1}(F_1 + F_2) H_{n-1} + \lambda \sum_{i=0}^{n-1} H_i^T M_i H_i,
\]

\[
\Phi_{13} = \Phi_{14} = PB[\overline{K}_D, \ldots, \overline{K}_{D_{n-1}}], \quad \Phi_{33} = -\frac{\pi^2}{4} e^{-2\alpha h} \text{diag}(W_1, \ldots, W_{n-1}),
\]

\[
\Phi_{44} = -\text{diag}(e^{-2\alpha h} R_1, \ldots, e^{-2\alpha(n-1)h} R_{n-1}), \quad \Phi_{45} = [0, -e^{-2\alpha(n-1)h} R_{n-1}]^T,
\]

\[
\Phi_{37} = \Phi_{47} = [\overline{K}_{D_1}, \ldots, \overline{K}_{D_{n-1}}]^T B^TH_{n-1}^T \Xi, \quad \Xi = \frac{(n-1)^2}{4} R_{n-1} + e^{2\alpha(n-1)h} Q,
\]

with \(A, B, A_1\) and \(D\) given by (16), and \(K\) and \(H_i\) \((i = 0, \ldots, n)\) given by (14). Then the sampled-data controller (6) exponentially mean-square stabilizes (1) with a decay rate \(\alpha\).

(ii) Given any \(\alpha \in (0, \bar{a})\), LMI (17) is always feasible for small enough \(h > 0\), \(\|D\|\) and \(\|M_i\|\) \((i = 0, \ldots, n-1)\) (meaning that the sampled-data controller (6) exponentially mean-square stabilizes (1) with a decay rate \(\alpha\).

**Proof.** (i) We consider the functional

\[
V = V_0 + V_{\bar{\delta}_0} + \sum_{i=1}^{n-1} (V_{\delta_i} + V_{\bar{\delta}_i} + V_{\nu_i}) + V_{\bar{\delta}_n} + V_{F_1} + V_{F_2}, (20)
\]
where

\[ V_0(x(t)) = x^T(t)Px(t), \]

\[
V_\delta(t, \dot{x}_0) = \begin{cases} 
  h^2 \int_{t-h}^t e^{-2\alpha(t-s)} x^T(s) \begin{bmatrix} H_0 \\ H_n \end{bmatrix}^T W_0 \begin{bmatrix} H_0 \\ H_n \end{bmatrix} x(s) ds - \frac{\alpha^2}{4} e^{-2\alpha h} \int_{t-h}^t e^{-2\alpha(t-s)} \delta_0^T(s) W_0 \delta_0(s) ds, & i = 0, \\
  h^2 \int_{t-h}^t e^{-2\alpha(t-s)} \begin{bmatrix} \tilde{y}^{(i)}(s) \\ \tilde{y}^{(0)}(s) \end{bmatrix} \begin{bmatrix} W_i \\ \tilde{y}^{(0)}(s) \end{bmatrix} ds - \frac{\alpha^2}{4} e^{-2\alpha h} \int_{t-h}^t e^{-2\alpha(t-s)} \delta_i^T(s) W_i \delta_i(s) ds, & i = 1, \ldots, n - 1,
\end{cases}
\]

\[ V_{\tilde{y}}(x_i) = h^2 e^{2\alpha h} \int_{t-lh}^t e^{-2\alpha(t-s)} \phi_i(t-s)x^T(s)H_{i+1}^T W_i H_{i+1} x(s) ds, \quad i = 1, \ldots, n - 2, \]

\[ V_{\tilde{y}, \delta}(f_i) = h^2 e^{2\alpha(n-1)h} \int_{t-(n-1)lh}^t e^{-2\alpha(t-s)} \phi_{n-1}(t-s)f^T(s)H_{n-1}^T QH_{n-1} f(s) ds, \]

\[ V_{\tilde{y}, \delta}(f_i) = \frac{ih}{2} \int_{t-lh}^t e^{-2\alpha(t-s)} \phi_i(t-s)x^T(s)H_{i+1}^T R_i H_{i+1} x(s) ds, \quad i = 1, \ldots, n - 2, \]

\[ V_{\tilde{y}, \delta}(f_i) = \frac{(n-1)h}{2} \int_{t-(n-1)lh}^t e^{-2\alpha(t-s)} \phi_{n-1}(t-s)f^T(s)H_{n-1}^T R_{n-1} H_{n-1} f(s) ds, \]

\[ V_{\tilde{y}, \delta}(f_i) = \frac{(n-1)h}{2} \int_{t-(n-1)lh}^t e^{-2\alpha(t-s)} \phi_{n-1}(t-s)x^T(s)D^T H_{n-1}^T F_i H_{n-1} D x(s) ds, \]

\[ V_{\tilde{y}, \delta}(f_i) = \frac{1}{2} \int_{t-lh}^t e^{-2\alpha(t-s)} \phi_{n-1}(t-s)x^T(s)D^T H_{n-1}^T F_i H_{n-1} D x(s) ds \]

with \( P > 0, W_i > 0 (i = 0, \ldots, n - 1), R_i > 0 (i = 1, \ldots, n - 1), Q > 0, F_1 > 0, F_2 > 0 \) and

\[ \phi_i(\nu) = \int_0^\nu \phi_i(\lambda) d\lambda, \quad i = 1, \ldots, n - 1. \]

Here \( x_0(\theta) = x(t + \theta), \theta \in [-h, 0]. \) Since \( \delta_0(t) = -[H_0^T, H_n^T]^T \dot{x}(t), \delta_i(t) = -\tilde{y}^{(i)}(t) (i = 1, \ldots, n - 1) \) and \( \delta_{k_1}(t) = 0 (i = 0, \ldots, n - 1), \) Lemma 2 implies \( V_\delta \geq 0 \) for \( i = 0, \ldots, n - 1. \) Due to \( \phi_i(\cdot) \geq 0 \) and \( \phi_i(\cdot) \geq 0 \) we have the positivity of functional \( V(t) \) in (20). Note that the terms \( V_\delta (i = 1, \ldots, n - 1), V_{\tilde{y}} \) and \( V_{\tilde{y}, \delta} \) are from Selivanov and Fridman,\(^{13}\) whereas the novel terms \( V_{\tilde{y}, \delta}, V_{\tilde{y}, \delta}, V_{\tilde{y}, \delta}, \) and \( V_{\tilde{y}, \delta} \) are stochastic extensions of Lyapunov functionals that depend on \( \dot{x}(t). \)

Let \( L \) be the generator (see e.g., Shaikhet\(^{22}\) and Mao\(^{24}\)). We have along (15)

\[ LV_0 + 2aV_0 = 2x^T(t) Pf(t) + x^T(t) D^T P D x(t) + 2ax^T(t) Px(t). \]

Moreover, we have

\[ LV_\delta + 2aV_\delta = \begin{cases} 
  h^2 \dot{x}^T(t) \begin{bmatrix} H_0 \\ H_n \end{bmatrix}^T W_0 \begin{bmatrix} H_0 \\ H_n \end{bmatrix} \dot{x}(t) - \frac{\alpha^2}{4} e^{-2\alpha h} \delta_0^T(t) W_0 \delta_0(t), & i = 0, \\
  h^2 \begin{bmatrix} \tilde{y}^{(i)}(t) \\ \tilde{y}^{(0)}(t) \end{bmatrix} \begin{bmatrix} W_i \\ \tilde{y}^{(0)}(t) \end{bmatrix} - \frac{\alpha^2}{4} e^{-2\alpha h} \delta_i^T(t) W_i \delta_i(t), & i = 1, \ldots, n - 1.
\end{cases} \]

The terms \( V_{\tilde{y}}, i = 1, \ldots, n - 2 \) are introduced to compensate \( h^2 \begin{bmatrix} \tilde{y}^{(i)}(t) \\ \tilde{y}^{(0)}(t) \end{bmatrix} \begin{bmatrix} W_i \\ \tilde{y}^{(0)}(t) \end{bmatrix}, i = 1, \ldots, n - 2 \) in (22). By using Lemma 1, via (12) we have

\[ LV_{\tilde{y}} + 2aV_{\tilde{y}} = h^2 e^{2\alpha h} x^T(t) H_{i+1}^T W_i H_{i+1} x(t) - h^2 e^{2\alpha h} \int_{t-lh}^t e^{-2\alpha(t-s)} \frac{d}{ds} \phi_i(t-s) x^T(s) H_{i+1}^T W_i H_{i+1} x(s) ds \]

\[ \int_{t-lh}^t e^{-2\alpha(t-s)} \phi_i(t-s) x^T(s) H_{i+1}^T W_i H_{i+1} x(s) ds \]

\[ \int_{t-lh}^t e^{-2\alpha(t-s)} \phi_i(t-s) x^T(s) H_{i+1}^T W_i H_{i+1} x(s) ds \]
\[ \leq h^2 e^{2\alpha h} x^T(t) H^T_{i+1} W_i H_{i+1} x(t) - h^2 \left( \int_{t-i \Delta t}^{t} d\phi_i(t-s) \right) \int_{t-i \Delta t}^{t} \left[ \frac{d}{ds} \phi_i(t-s) \right] x^T(s) H^T_{i+1} ds W_i \int_{t-i \Delta t}^{t} \left[ \frac{d}{ds} \phi_i(t-s) \right] H_{i+1} x(s) ds, \quad i = 1, \ldots, n - 2. \]

From (8), it follows that
\[ \bar{y}^{(i)}(t) = \bar{y}^{(i)}(t) - \int_{t-i \Delta t}^{t} \phi_i(t-s) \bar{y}^{(i)}(s) ds, \quad i = 1, \ldots, n - 1. \]

Via (12) the latter implies
\[ \bar{y}^{(i)}(t) = \int_{t-i \Delta t}^{t} \left[ \frac{d}{ds} \phi_i(t-s) \right] \bar{y}^{(i)}(s) ds = \int_{t-i \Delta t}^{t} \left[ \frac{d}{ds} \phi_i(t-s) \right] H_i \bar{x}(s) ds, \quad i = 1, \ldots, n - 1. \]  

Noting that \( \int_{t-i \Delta t}^{t} d\phi_i(t-s) = \phi_i(0) - \phi_i(ih) = 1 \) and \( H_i \bar{x}(s) = H_{i+1} x(s) \) \( i = 0, \ldots, n - 2 \), from (23) and (24) we have
\[ LV_{\delta_i} + 2a V_{\delta_i} \leq h^2 e^{2\alpha h} x^T(t) H^T_{i+1} W_i H_{i+1} x(t) - h^2 \left[ \bar{y}^{(i)}(t) \right]^T W_i \left[ \bar{y}^{(i)}(t) \right], \quad i = 1, \ldots, n - 2. \]

Then the terms \(-h^2 \left[ \bar{y}^{(i)}(t) \right]^T W_i \left[ \bar{y}^{(i)}(t) \right] \) \( i = 1, \ldots, n - 2 \) in the above expression will cancel the positive term of \( LV_{\delta_i} + 2a V_{\delta_i} \) \( i = 1, \ldots, n - 2 \). Note that the term \( \bar{y}^{(i)}(t) \) with \( i = n - 1 \) in (24) has the following form:
\[ \bar{y}^{(n-1)}(t) = \int_{t-(n-1) \Delta t}^{t} \left[ \frac{d}{ds} \phi_{n-1}(t-s) \right] H_{n-1} \bar{x}(s) ds \]
where \( \rho_1(t) = \int_{t-(n-1) \Delta t}^{t} \left[ \frac{d}{ds} \phi_{n-1}(t-s) \right] H_{n-1} f(s) ds \), \( \rho_2(t) = \int_{t-(n-1) \Delta t}^{t} \left[ \frac{d}{ds} \phi_{n-1}(t-s) \right] H_{n-1} D x(s) dw(s) \).

Thus
\[ LV_{\delta_{n-1}} + 2a V_{\delta_{n-1}} = h^2 \left[ \rho_1(t) + \rho_2(t) \right]^T W_{n-1} \left[ \rho_1(t) + \rho_2(t) \right] - \frac{\pi^2}{4} e^{-2\alpha h} \delta_{n-1}(t) W_{n-1} \delta_{n-1}(t). \]

To compensate \( \rho_1(t) \), we employ the term \( V_{Y_{n-1}} \), that is,
\[ LV_{Y_{n-1}} + 2a V_{Y_{n-1}} = h^2 e^{2\alpha(n-1)h} f^T(t) H^T_{n-1} Q H_{n-1} f(t) - h^2 e^{2\alpha(n-1)h} \int_{t-(n-1) \Delta t}^{t} e^{2\alpha(t-s)} \left[ \frac{d}{ds} \phi_{n-1}(t-s) \right] f^T(s) H^T_{n-1} Q H_{n-1} f(s) ds \]
\[ \leq h^2 e^{2\alpha(n-1)h} f^T(t) H^T_{n-1} Q H_{n-1} f(t) - h^2 \rho_1^T(t) Q \rho_1(t), \]

where we applied Lemma 1 with (12). Note that (12) implies
\[ \phi_i(0) = \int_0^{ih} \phi_i(\lambda) d\lambda = \frac{ih}{2}, \quad \phi_i(ih) = 0, \quad i = 1, \ldots, n - 1. \]

For the \( \rho_2(t) \)-term, by using Itô isometry (see, e.g., Shaikhet\textsuperscript{22} and Mao\textsuperscript{24}), via (12) we have for any \( p \times p \) matrix \( F_i > 0 \)
\[ e^{-2\alpha(n-1)h} h \mathbb{E} \rho_2^T(t) F_i \rho_2(t) = e^{-2\alpha(n-1)h} h \mathbb{E} \int_{t-(n-1) \Delta t}^{t} \left[ \frac{d}{ds} \phi_{n-1}(t-s) \right] x^T(s) D^T H^T_{n-1} F_i H_{n-1} x(s) ds \]
\[ \leq \mathbb{E} \int_{t-(n-1) \Delta t}^{t} e^{-2\alpha(t-s)} \left[ \frac{d}{ds} \phi_{n-1}(t-s) \right] x^T(s) D^T H^T_{n-1} F_i H_{n-1} x(s) ds. \]
The latter together with (29) leads to
\[
\mathbb{E} LV_{F_1} + 2a \mathbb{E} V_{F_1} = \frac{(n - 1)h}{2} \mathbb{E} x^T(t)D^T H_{n-1}^T F_1 H_{n-1} \, dx(t)
- \frac{(n - 1)h}{2} \mathbb{E} \int_{t-(n-1)h}^t e^{-2 \alpha t(s-t)} \left[ \frac{d}{ds} \phi_{n-1}(t - s) \right] x^T(s)D^T H_{n-1}^T F_2 H_{n-1} \, dx(s) \, ds
\leq \frac{(n - 1)h}{2} \mathbb{E} x^T(t)D^T H_{n-1}^T F_1 H_{n-1} \, dx(t) - \frac{(n - 1)h^2}{2} e^{-2 \alpha(n-1)h} \mathbb{E} \rho_2^T(t) F_1 \rho_2(t).
\]
(30)

By using Lemma 1, via (29) we have
\[
LV_{\kappa_i} + 2a V_{\kappa_i} = \frac{(ih)^2}{4} x^T(t)H_{i+1}^T R_i H_{i+1} x(t) - \frac{ih}{2} \int_{t-(n-1)h}^t e^{-2 \alpha t(s-t)} \phi_{n-1}(t - s) x^T(s)H_{i+1}^T R_i H_{i+1} x(s) \, ds
\leq \frac{(ih)^2}{4} x^T(t)H_{i+1}^T R_i H_{i+1} x(t) - e^{-2 \alpha ih} \kappa_i^T(t) R_i \kappa_i(t), \quad i = 1, \ldots, n-2.
\]
(31)

\[
LV_{\kappa_{n-1}} + 2a V_{\kappa_{n-1}} \leq \frac{(n - 1)h^2}{4} f^T(t)H_{n-1}^T R_{n-1} H_{n-1} f(t)
- e^{-2 \alpha(n-1)h} \left[ \int_{t-(n-1)h}^t \phi_{n-1}(t - s) f^T(s)H_{n-1}^T \, ds \right] R_{n-1} \left[ \int_{t-(n-1)h}^t \phi_{n-1}(t - s) H_{n-1} f(s) \, ds \right]
= \frac{(n - 1)h^2}{4} f^T(t)H_{n-1}^T R_{n-1} H_{n-1} f(t) - e^{-2 \alpha(n-1)h} \left[ \kappa_{n-1}(t) + \rho_3(t) \right]^T R_{n-1} \left[ \kappa_{n-1}(t) + \rho_3(t) \right],
\]
(32)

where
\[
\rho_3(t) = \int_{t-(n-1)h}^t \phi_{n-1}(t - s) H_{n-1} \, dx(s) \, dw(s).
\]

To compensate \(\rho_3(t)\), we employ the term \(V_{F_1}\) that leads to
\[
\mathbb{E} LV_{F_1} + 2a \mathbb{E} V_{F_1} \leq \frac{(n - 1)h}{2} \mathbb{E} x^T(t)D^T H_{n-1}^T F_1 H_{n-1} \, dx(t) - \int_{t-(n-1)h}^t e^{-2 \alpha t(s-t)} \phi_{n-1}(t - s) x^T(s)D^T H_{n-1}^T F_2 H_{n-1} \, dx(s) \, ds
\leq \frac{(n - 1)h}{2} \mathbb{E} x^T(t)D^T H_{n-1}^T F_1 H_{n-1} \, dx(t) - e^{-2 \alpha(n-1)h} \mathbb{E} \rho_2^T(t) F_1 \rho_2(t).
\]
(33)

where we applied Itô isometry with (12). From (2), we have
\[
|g(t, H_0 x(t), \ldots, H_{n-1} x(t))|^2 \leq \sum_{i=0}^{n-1} x^T(t)H_i^T M_i H_i x(t).
\]
(34)

Hence, the following inequality holds:
\[
\lambda \left[ \sum_{i=0}^{n-1} x^T(t)H_i^T M_i H_i x(t) - |g(t, H_0 x(t), \ldots, H_{n-1} x(t))|^2 \right] \geq 0.
\]
(35)

for some constant \(\lambda > 0\).

In view of (21), (22), (25), (27), (28), and (30)–(33), taking into account the relations \(H_0 \tilde{x}(t) = H_1 x(t)\) and \(H_0 \tilde{x}(t) = y(t) = H_0 x(t) + [I, 0] \delta_0(t)\) and applying S-procedure with (35) we obtain
\[
\mathbb{E} LV + 2a \mathbb{E} V \leq \mathbb{E} \tilde{x}^T(t) \bar{D}_x^2(t) + h^2 \mathbb{E} \eta^T(t) \bar{D}_{\eta}^2(t) + h^2 \mathbb{E} \tilde{e}_i^T(t) H_i^T \left[ \frac{(n - 1)^2}{4} R_{n-1} + e^{2 \alpha(n-1)h} Q \right] H_{n-1} f(t)
+ h^2 \mathbb{E} \left[ H_0 x(t) + [I, 0] \delta_0(t) \right]^T \mathbb{E} \left[ H_1 x(t) \right] \left[ \begin{array}{c}
H_0 x(t) + [I, 0] \delta_0(t)
\end{array} \right]
\]
where $\Phi$ is obtained from $\Phi$ in (17) by taking away the last two block-columns and block-rows, $\Psi$ is given by (18) and

$$
\xi(t) = \col \{ x(t), \delta_0(t), \ldots, \delta_{n-1}(t), \kappa_0(t), \ldots, \kappa_{n-1}(t), \rho_0(t), \ldots, \rho_{n-1}(t), \mu(t), \ldots, \mu_{n-1}(t) \}, \quad \eta(t) = \col \{ \rho_0(t), \rho_2(t) \}.
$$

Substituting (16) for $f(t)$ and further applying Schur complement, we deduce that $\Phi < 0$ given by (17) guarantees $\mathbf{ELV} + 2\alpha \mathbf{EV} \leq 0$ implying that the sampled-data controller (6) exponentially mean-square stabilizes (1) with a decay rate $\alpha$.

(ii) The system (1), (3) has the form

$$
dx_c(t) = \left[ (A + BK)x_c(t) + H^T_{n-1}g(t, H_0x(t), \ldots, H_{n-1}x(t)) \right] dt + Dx(t) dw(t),
$$

where $A, B, D$ are given by (16) and $K$ is given by (14). If the PID controller (3) exponentially stabilizes (1), where $g \equiv 0$ and $d_i = 0 (i = 0, \ldots, n - 1)$ (and thus, $D = 0$), with a decay rate $\alpha > 0$, then there exists $0 < P \in \mathbb{R}^{(n+1)p \times (n+1)p}$ such that $P(A + BK) + (A + BK)^T P + 2\alpha P < 0$ for any $\alpha \in (0, \bar{\alpha})$.

Thus,

$$
P(A + BK) + (A + BK)^T P + 2\alpha P + D^T PD < 0,
$$

for small enough $|D|$. We choose in LMI (17) $W_0 = \frac{1}{\sqrt{h}} I_{2p}, R_1 = W_1 = Q = F_1 = F_2 = \frac{1}{\sqrt{h}} I_p (i = 1, \ldots, n - 1)$ and $\lambda = \frac{1}{\sqrt{h}}$.

Applying Schur complement, $\Phi < 0$ is equivalent to

$$
P(A + BK) + (A + BK)^T P + 2\alpha P + D^T PD + \sqrt{h}(G_1 + hG_2) + \frac{1}{\sqrt{h}} \sum_{i=0}^{n-1} H^T_i M_i H_i < 0,
$$

where

$$
G_1 = (n - 1)D^T H^T_{n-1} D + \frac{4}{\pi^2} e^{2\alpha h} P\left[ (A + BK_p)(A + BK_p)^T + \sum_{i=1}^{n-1} BK_{n-i} K_{n-i} B^T + BK_{n-1} K_{n-1} B^T \right] P
$$

$$
+ \sum_{i=1}^{n-2} e^{2\alpha h} PBK_{n-i} K_{n-i} B^T P + 2e^{2(n-1)\alpha h} PBK_{n-2} K_{n-2} B^T P + PH^T_{n-1} H_{n-1} P,
$$

$$
G_2 = \sum_{i=0}^{n-2} \left( e^{2\alpha h} \frac{1}{4} \right) H^T_{i+1} H_{i+1}.
$$

Inequality (38) implies (39) for small enough $h > 0$ and $\|M_i\| (i = 0, \ldots, n - 1)$ since $\sqrt{h}(G_1 + hG_2) \to 0$ and $\frac{1}{\sqrt{h}} \sum_{i=0}^{n-1} H^T_i M_i H_i = \sqrt{h} \sum_{i=0}^{n-1} H^T_i H_i \to 0$ for $h \to 0$ where we choose, for example, $M_i = h L_p (i = 0, \ldots, n - 1)$, implying the feasibility of $\Phi < 0$ for small enough $h > 0$ and $\|M_i\| (i = 0, \ldots, n - 1)$. Finally, applying Schur complement to the last two block-columns and block-rows of $\Phi$ given by (17), we find that $\Phi < 0$ is feasible for small enough $h > 0$ if $\Phi < 0$ is feasible. Thus, LMI (17) is always feasible for small enough $h > 0, \|D\|$ and $\|M_i\| (i = 0, \ldots, n - 1)$.

For the deterministic case (i.e., the system (1) with $d_i = 0 (i = 0, \ldots, n - 1)$), we consider the functional $\tilde{V}$ that is obtained from $V$ in (20) by setting $F_1 = F_2 = 0$ and changing $f(s)$ and $Q$ respectively as $\tilde{x}(s)$ and $\tilde{W}_{n-1}$. The latter includes additional terms $V_{\delta_1}, V_{\mu_1}, V_{\mu_i} (i = 2, \ldots, n - 1)$ to compensate additional errors $\delta_i(t)$ and $\mu_i(t) (i = 2, \ldots, n - 1)$ in (15) comparatively to Selivanov and Fridman, 15,16.
Corollary 1. Consider the deterministic nonlinear system (1) with \( d_i = 0 \) \((i = 0, \ldots, n - 1)\) under the sampled-data controller (6). Given \( \overline{K}_p, \overline{K}_f \) and \( \overline{K}_d \) \((i = 1, \ldots, n - 1)\) let the extended PID controller (3) exponentially stabilizes (1), where \( d_i = 0 \) \((i = 0, \ldots, n - 1)\) and \( g \equiv 0 \), with a decay rate \( \alpha > 0 \).

(i) Given tuning parameters \( h > 0, \alpha \in (0, \overline{\alpha}) \) and \( p \times p \) matrices \( M_i \) \((i = 0, \ldots, n - 1)\), let there exist \((n + 1)p \times (n + 1)p\) matrix \( P > 0, 2p \times 2p \) matrices \( W_0 > 0 \) and \( p \times p \) matrices \( W_i > 0 \) and \( R_i > 0 \) \((i = 1, \ldots, n - 1)\), \( Q > 0, F_1 > 0 \) and \( F_2 > 0 \), with a decay rate \( \lambda > 0 \) that satisfy

\[
\Phi < 0, \quad (40)
\]

where \( \Phi \) is obtained from \( \Phi \) in (17) by setting \( D = 0, F_1 = F_2 = 0, Q = W_{n-1} \) and and taking away the fifth block-column and block-row. Then the sampled-data controller (6) exponentially stabilizes (1), where \( d_i \equiv 0 \) \((i = 0, \ldots, n - 1)\) with a decay rate \( \alpha \).

(ii) Given any \( \alpha \in (0, \overline{\alpha}) \), LMI (40) is always feasible for small enough \( h > 0 \) and \( \|M_i\| \) \((i = 0, \ldots, n - 1)\) (meaning that the sampled-data controller (6) exponentially stabilizes (1), where \( d_i \equiv 0 \) \((i = 0, \ldots, n - 1)\) with a decay rate \( \alpha \).

Remark 2. Note that less conservative integral inequalities were introduced e.g. in Seuret et al.\(^{25,26}\) to improve the results via LMIs. However, the LMIs of Seuret et al.\(^{25,26}\) cannot be guaranteed to be always feasible. By contrast, we provide in (ii) of Theorem 1 and Corollary 1 (and Theorems 2 and 3 below) the feasibility guarantee of LMIs which were obtained by using Jensen’s and Wirtinger’s inequalities.

3 | EVENT-TRIGGERED PID CONTROL

Event-triggered control allows to reduce the number of signals transmitted through a communication network (see e.g., Tabuada,\(^{17}\) Yue et al.,\(^{18}\) and Heemels et al.\(^ {19}\)). The idea is to transmit the signal only when it satisfies some preselected event-triggering condition. For simplicity we here introduce an event-triggering condition with respect to the control signals:\(^ {15}\)

\[
[u(t_k) - \hat{u}_{k-1}]^T \Theta [u(t_k) - \hat{u}_{k-1}] > \sigma u^T(t_k) \Theta u(t_k), \quad (41)
\]

where \( \sigma \in [0, 1) \) and \( 0 < \Theta \in \mathbb{R}^{q \times q} \) are the event-triggering parameters, \( u(t_k) \) is from (6) and \( \hat{u}_{k-1} \) denotes the last transmitted control signal. Thus, \( \hat{u}_0 = u(t_0) \) and

\[
\hat{u}_k = \begin{cases} u(t_k), & \text{if (41) is true,} \\ \hat{u}_{k-1}, & \text{if (41) is false.} \end{cases} \quad (42)
\]

Hence, the system (1) becomes

\[
dy(t_k) = \left[ \sum_{i=0}^{n-1} a_iy^{(i)}(t_k) + b \hat{u}_k + g(t, y^{(0)}(t_k), \ldots, y^{(n-1)}(t_k)) \right] dt + \sum_{i=0}^{n-1} dy^{(i)}(t_k) dw(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0, \quad (43)
\]

with \( \hat{u}_k \) given by (42). Introduce the event-triggering error

\[
e_k = \hat{u}_k - u(t_k). \quad (44)
\]

Then following the modeling in the previous section, the system (43) under the event-triggered PID control (3), (41), (42) can be presented as (cf. (15))

\[
dx(t) = [f(t) + B_k] dt + D x(t) dw(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0. \quad (45)
\]

Theorem 2. Consider the stochastic nonlinear system (1) under the event-triggered PID controller (6), (41), (42). Given \( \overline{K}_p, \overline{K}_f \) and \( \overline{K}_d \) \((i = 1, \ldots, n - 1)\) let the extended PID controller (3) exponentially stabilizes (1), where \( g \equiv 0 \) and \( d_i = 0 \) \((i = 0, \ldots, n - 1)\), with a decay rate \( \overline{\alpha} > 0 \).

(i) Given tuning parameters \( h > 0, \alpha \in (0, \overline{\alpha}), \sigma \in [0, 1) \) and \( p \times p \) matrices \( M_i \) \((i = 0, \ldots, n - 1)\), let there exist \((n + 1)p \times (n + 1)p\) matrix \( P > 0, 2p \times 2p \) matrices \( W_0 > 0 \) and \( p \times p \) matrices \( W_i > 0 \) and \( R_i > 0 \) \((i = 1, \ldots, n - 1)\), \( Q > 0, F_1 > 0 \) and \( F_2 > 0 \), with a decay rate \( \lambda > 0 \) that satisfy

\[
\Phi < 0, \quad (40)
\]

where \( \Phi \) is obtained from \( \Phi \) in (17) by setting \( D = 0, F_1 = F_2 = 0, Q = W_{n-1} \) and and taking away the fifth block-column and block-row. Then the sampled-data controller (6) exponentially stabilizes (1), where \( d_i \equiv 0 \) \((i = 0, \ldots, n - 1)\) with a decay rate \( \alpha \).

(ii) Given any \( \alpha \in (0, \overline{\alpha}) \), LMI (40) is always feasible for small enough \( h > 0 \) and \( \|M_i\| \) \((i = 0, \ldots, n - 1)\) (meaning that the sampled-data controller (6) exponentially stabilizes (1), where \( d_i \equiv 0 \) \((i = 0, \ldots, n - 1)\) with a decay rate \( \alpha \).
Consider the functional $\Phi_e$ given by (46) and

$$
\Phi_e = \begin{bmatrix}
PB & \sigma K^T \Theta \\
0 & \sigma [\bar{K}_p, \bar{K}_l]^T \Theta \\
0 & \sigma [\bar{K}_D, \ldots, \bar{K}_{D_{n-1}}]^T \Theta \\
0 & 0 \\
0 & 0 \\
h \Xi H_{n-1} B & 0 \\
0 & 0 \\
* & -\Theta \\
* & -\sigma \Theta 
\end{bmatrix} < 0, \quad (46)
$$

where $\Phi$ and $\Xi$ are respectively given by (17) and (19), $K$ and $H_{n-1}$ are given by (14) and $B$ is given by (16). Then the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate $\alpha$.

Proof. (i) Using the triggering error (44), the event-triggering condition (41), (42) guarantees

$$
0 \leq \sigma u^T(t_k) \Theta u(t_k) - e_k^T \Theta e_k. \quad (47)
$$

Consider the functional $V$ from (20) with $f(t)$ changed by $f(t) + Be_k$. Following the proof of item (i) of Theorem 1, along (45) we have (cf. (36))

$\mathbf{ELV} + 2a \mathbf{EV} \leq \mathbf{ELV} + 2a \mathbf{EV} + \sigma \mathbf{E} u^T(t_k) \Theta u(t_k) - \mathbf{E} e_k^T \Theta e_k$

$$
\quad \leq \mathbf{E} e^T_{e}(t) \bar{\Phi}_e e(t) + h^2 \mathbf{E} f(t) + Be_k)^T H_{n-1}^{T} \left[ \frac{(n-1)^2}{4} R_{n-1} + e^{2a(n-1)h} Q \right] H_{n-1} f(t) + Be_k) \quad (48)
$$

where $\xi(t)$ is a col{\xi(t), e_k} with $\xi(t)$ given by (37), $\bar{\Phi}_e$ is obtained from $\Phi_e$ in (46) by taking away the $i$- and $j$-blocks with $i \in \{7, 8, 10\}$ or $j \in \{7, 8, 10\}$. Substituting (13) and (16), respectively, for $\theta(t)$ and $f(t)$ and further applying Schur complement, we find that $\Phi_e < 0$ given by (46) guarantees $\mathbf{ELV} + 2a \mathbf{EV} \leq 0$ implying that the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate $\alpha$.

(ii) The proof of (ii) is similar to (ii) of Theorem 1. $

\text{Remark 3.}$ To select the tuning parameters $h$, $\alpha$, $\sigma$, $M_i$ and $d_i$ ($i = 0, \ldots, n-1$) we suggest the following algorithm: choose $\bar{K}_p, \bar{K}_l$ and $\bar{K}_{D_i}$ ($i = 1, \ldots, n-1$) via pole-placement such that the extended PID controller (1) exponentially stabilizes (13), where $g \equiv 0$ and $d_i = 0$ ($i = 0, \ldots, n-1$), with a decay rate $\alpha > 0$. By solving the LMIs with $M_i = 0$, $d_i = 0$ ($i = 0, \ldots, n-1$), $\sigma = 0$ and small enough $h > 0$, we find a critical maximal value of $\alpha$ as $\alpha^* < \alpha$. Then, by choosing $\alpha \in [0, \alpha^*]$ with $M_i = 0$, $d_i = 0$ ($i = 0, \ldots, n-1$) and small enough $h > 0$, we find a critical maximum value of $\sigma$ as $\sigma^*$. The same is done for $M_i, d_i$ ($i = 0, \ldots, n-1$) that leads to critical maximum values of $M_i, d_i$ ($i = 0, \ldots, n-1$), respectively, as $M_i^*, d_i^*$ ($i = 0, \ldots, n-1$). Then for $\alpha \in [0, \alpha^*]$, $\sigma \in [0, \sigma^*]$, $M_i \in [0, M_i^*]$ and $d_i \in [0, d_i^*]$ ($i = 0, \ldots, n-1$), we can obtain a critical maximal value of $h = h^*$ such that for $h > h^*$ the LMI becomes unfeasible.
4 | L₂-GAIN ANALYSIS

The direct Lyapunov method is applicable not only to the stability but also to the performance analysis,⁹ for example, L₂-gain analysis. In this section, we consider L₂-gain analysis of the perturbed systems, namely (cf. (43))

\[ dy(t) = \left[ \sum_{i=0}^{n-1} a_i y^{(i)}(t) + b u_k + b_v v(t) + g(t, y(t), \ldots, y^{(n-1)}(t)) \right] dt + \sum_{i=0}^{n-1} d_j i(t) dw_i, \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0, \] (49)

where \( b_v \in \mathbb{R}^{p \times p} \) is a constant matrix and \( v(t) \in \mathbb{R}^p \) is the external disturbance in \( L_2[0, \infty) \).

The system (49) under the event-triggered PID control (3), (41), (42) has the form:

\[ dx(t) = [ f(t) + B e_k + B_v v(t)] dt + D x(t) dw(t), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}_0, \] (50)

where \( x(t) \) is given by (14), \( f(t) \), \( B \) and \( D \) are given by (16) and

\[ B_v = \text{col}\{0_{(n-1)p \times p}, b_v, 0_{p \times p}\}. \] (51)

Consider next the controlled output

\[ z(t) = C x(t) + C_v v(t), \quad z(t) \in \mathbb{R}^l, \] (52)

where \( C \in \mathbb{R}^{l \times (n+1)p} \) and \( C_v \in \mathbb{R}^{l \times p} \) are constant matrices. For a prechosen \( \gamma > 0 \) we introduce the following performance index:

\[ J = \int_0^\infty \left[ z^T(t) z(t) - \gamma^2 v^T(t)v(t) \right] dt. \] (53)

We seek conditions that will lead to \( EJ \leq 0 \) for all \( x(t) \) satisfying (50) with the zero initial condition \( x(0) = 0 \) and for all \( 0 \neq v \in L_2[0, \infty) \). In this case the system (50), (52) has L₂-gain less than or equal to \( \gamma \). Moreover, if the system (50) with \( v \equiv 0 \) is exponentially mean-square stable, then the system (50) is internally exponentially mean-square stable.

**Lemma 3.**⁹ Given \( \alpha \geq 0 \) and \( \gamma > 0 \), let for \( V \) given by (20) the following inequality holds along the solutions of (50):

\[ E LV + 2 \alpha EV + E z^T(t) z(t) - \gamma^2 v^T(t) v(t) < 0 \quad \forall 0 \neq v(t) \in \mathbb{R}^p \; \text{and} \; \forall t \geq 0. \] (54)

If (54) holds with \( \alpha = 0 \), then the system (50), (52) has L₂-gain less than or equal to \( \gamma \). Moreover, if (54) holds with \( \alpha > 0 \), then the system (50) is internally exponentially mean-square stable with a decay rate \( \alpha \).

Based on Lemma 3, we now present the following LMI conditions:

**Theorem 3.** Consider the stochastic nonlinear system (1) with an additive external disturbance \( v(t) \) under the event-triggered PID controller (6), (41), (42) leading to system (50), and the controlled output (52). Given \( \overline{K}_p, \overline{K}_T \) and \( \overline{K}_D \) (\( i = 1, \ldots, n-1 \)) let the extended PID controller (3) exponentially stabilizes (1), where \( g \equiv 0 \) and \( d_i = 0 \) (\( i = 0, \ldots, n-1 \)), with a decay rate \( \overline{a} > 0 \).

(i) Given tuning parameters \( h > 0 \), \( \alpha \in (0, \overline{a}) \), \( \sigma \in (0, 1) \) and \( \gamma > 0 \), and \( p \times p \) matrices \( M_i \) (\( i = 0, \ldots, n-1 \)), let there exist \( (n+1)p \times (n+1)p \) matrix \( P > 0 \), \( 2p \times 2p \) matrices \( W_0 > 0 \), \( p \times p \) matrices \( W_i > 0 \), \( R_i > 0 \) (\( i = 1, \ldots, n-1 \)), \( Q > 0 \), \( F_1 > 0 \) and \( F_2 > 0 \), \( q \times q \) matrix \( \Theta > 0 \) and scalar \( \lambda > 0 \) that satisfy (18) and

\[
\Phi_{L_2} = \begin{bmatrix}
PB_v & C^T \\
0_{(n+2)p \times p} & 0_{(2n+2)p \times p} \\
hZ H_{n-1} B_v & 0_{p \times p} \\
0_{2(p+q) \times p} & 0_{2(p+q) \times p} \\
* & -\gamma^2 I_q & C_v^T \\
* & * & -I_l
\end{bmatrix} < 0, \] (55)
TABLE 1  Maximum value of $h$ via linear matrix inequalities

<table>
<thead>
<tr>
<th>$d_1$</th>
<th>Example 1</th>
<th>Example 2</th>
<th>Example 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>Selivanov and Fridman\textsuperscript{15}</td>
<td>0.0047</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Selivanov and Fridman\textsuperscript{16}</td>
<td>0.019</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Corollary 1</td>
<td>0.019</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Theorem 1</td>
<td>0.019</td>
<td>0.012</td>
<td>0.002</td>
</tr>
</tbody>
</table>

where $H_{n-1}$, $\Xi$, $\Phi_e$ and $B_v$ are respectively given by (14), (19), (46), and (51), and $C$ and $C_v$ are given by (52). Then the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate $\alpha$, and the system (50), (52) has $L_2$-gain less than or equal to $\gamma$.

(ii) Given any $\alpha \in (0, \alpha_{\text{max}})$, LMI (55) is always feasible for small enough $h > 0$, $\sigma \in (0, 1)$, $\frac{1}{\gamma} > 0$, $\|D\|$ and $\|M_i\|$ ($i = 0, \ldots, n - 1$) (meaning that the event-triggered PID controller (6), (41), (42) exponentially mean-square stabilizes (1) with a decay rate $\alpha$).

5  EXAMPLES

To illustrate the efficiency, we present three examples including a servo positioning system.

**Example 1.** Consider system (1) with

$$a_0 = 0, \quad a_1 = -8.4, \quad b = 35.71, \quad g \equiv 0. \quad (56)$$

The system is not stable if $u = 0$. The PID controller (3) with

$$\overline{K}_P = -10, \quad \overline{K}_I = -40, \quad \overline{K}_{D_1} = -0.65. \quad (57)$$

stabilizes system (1) with (56) for small enough stochastic perturbations. Let $\alpha = 5$ be the desired decay rate. In the deterministic case (i.e., $d_0 = d_1 = 0$), LMIs of Corollary 1 and Selivanov and Fridman\textsuperscript{16} lead to the same result which is larger than that via Selivanov and Fridman.\textsuperscript{15} In the stochastic case, LMIs of Theorem 1 with $d_0 = 0$ and different values of $d_1$ lead to efficient results (see Table 1).

Consider now system (1) with (56) under the event-triggered PID control. For $h = 0.005$, $d_0 = 0$ and $d_1 = 0.2$, LMI of Theorem 2 is feasible for a maximum value of $\sigma = 0.074$. Sampled-data control requires to transmit $1/h + 1 = 201$ control signals during 1 s of simulations. By performing numerical simulations with 10 randomly chosen initial conditions $\|x(0)\|_\infty \leq 1$ where we applied Euler-Maruyama method\textsuperscript{27} using a step size $10dt$ with $dt = 10^{-6}$, the event-triggered control requires to transmit on average 63.95 control signals. Thus, the even-triggering mechanism (41), (42) reduces the number of transmitted control signals by almost 69%.

**Example 2.** (Chain of three integrators). Consider system (1) with

$$a_i = 0, \quad i = 0, 1, 2, \quad b = 1, \quad g \equiv 0. \quad (58)$$

Using the pole placement, we find that for (3) with

$$\overline{K}_P = -6.026, \quad \overline{K}_I = -1.716, \quad \overline{K}_{D_1} = -7.91, \quad \overline{K}_{D_2} = -4.6. \quad (59)$$

the eigenvalues of $A + BK$ are $-1$, $-1.1$, $-1.2$ and $-1.3$. Therefore, the PID controller (3) with (59) stabilizes system (1) with (58) for small enough stochastic perturbations.

Let $\alpha = 0.2$, $d_0 = d_2 = 0$. For different values of $d_1$, the maximum values of $h$ that preserve the exponential stability are presented in Table 1. It is clear that LMIs of Corollary 1 and Theorem 1 lead to efficient results whereas Selivanov
and Fridman\cite{15,16} fail. For $h = 0.04$ and $d_1 = 0.2$, LMIs of Theorem 2 are feasible for a maximum value of $\sigma = 0.119$. We next perform numerical simulations with 10 randomly chosen initial conditions $\|x(0)\|_\infty \leq 1$ by using Euler–Maruyama method\cite{27} with a step size $10dt$ and $dt = 10^{-6}$. One can find that the event-triggered control requires to transmit on average 96.8 control signals during 10 seconds. Note that the number of transmissions for the sampled-data control is given by $10/h + 1 = 251$. Thus, the event-triggering mechanism (41), (42) reduces the number of transmitted control signals by over 61%.

**Example 3.** Consider the servo positioning system with a stochastic perturbation\cite{28,29}

$$
\theta_1 d\dot{y}^{(1)}(t) = [-\theta_2 y^{(1)}(t) + u(t) - F(y^{(1)}(t)) + b_v v(t)]dt + d_1 y^{(1)}(t)dw(t),
$$

where $F(y(t)) = \theta_2 \tanh(700y(t)) + \theta_3 [\tanh(15y(t)) - \tanh(1.5y(t))]$, $y(t)$ is the motor rotation angle, $u(t)$ is the control input and $w(t)$ is the load disturbance. Set $[\theta_1, \theta_2, \theta_3, \theta_4] = [0.0025, 0.02, 0.01, 0.205]$. Following the previous modeling, the system (60) under an event-triggered PID control can be written in the form of (45) with

$$
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\tfrac{\sigma_1}{\beta_1} & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ \beta_1 \end{bmatrix}, \quad B_v = \begin{bmatrix} 0 \\ b_v \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tfrac{\sigma_1}{\beta_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

and with $g = -F(y(t))$. Note that the latter nonlinearity satisfies (2) with $M_0 = 0$ and $M_1 = 14.13$. Moreover, the controlled output is given by (52) with $C = [1, 0, 0]$ and $C_v = 2$. The PID controller (3) with

$$
\overline{K}_P = -0.4980, \quad \overline{K}_I = -0.0255, \quad \overline{K}_{D_1} = -0.270
$$

exponentially stabilizes the system (60).

Set $a = 0.1$ and $d_0 = 0$. For different values of $d_1$ and $b_v = 0$, LMIs of Corollary 1 and Theorem 1 lead to efficient results in Table 1. For $h = 0.05$, $d_1 = 0.01$ and $b_v = 0$, LMIs of Theorem 2 are feasible for a maximum value of $\sigma = 0.04$. Sampled-data control requires to transmit $5/h + 1 = 101$ control signals during 5 s. By performing numerical simulations with 10 randomly chosen initial conditions $\|x(0)\|_\infty \leq 1$ where we applied Euler–Maruyama method\cite{27} using a step size $10dt$ with $dt = 10^{-6}$, the event-triggered control requires to transmit on average 32.6 control signals. Thus, the even-triggering mechanism (41), (42) reduces the number of transmitted control signals by over 67%. Moreover, for $h = 0.02$, $d_1 = 0.01$, $b_v = 1$ and $\sigma = 0.04$, by LMIs of Theorem 3 a minimum value of $\gamma = 2.02$ is obtained.

6 | CONCLUSIONS

In this paper, sampled-data implementation of extended PID control using delays has been presented for the nth-order stochastic nonlinear systems. We have employed an event-triggering condition that allows to reduce the number of sampled control signals used for stabilization and have studied $L_2$-gain analysis. The suggested method may be useful for delay-induced consensus in multi-agent systems under an extended PID control. This may be a topic for the future research.

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DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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