Available online at www.sciencedirect.com
ScienceDirect
Systems & Control Letters 55 (2006) 1041 – 1053
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Input–output approach to stability and L_2-gain analysis of systems with time-varying delays

Emilia Fridman*, Uri Shaked

Department of Electrical Engineering—Systems, Tel-Aviv University, Tel-Aviv 69978, Israel

Received 14 December 2004; received in revised form 13 September 2005; accepted 3 July 2006
Available online 5 September 2006

Abstract

Stability and L_2 (l_2)-gain of linear (continuous-time and discrete-time) systems with uncertain bounded time-varying delays are analyzed under the assumption that the nominal delay values are not equal to zero. The delay derivatives (in the continuous-time) are not assumed to be less than $q < 1$. An input–output approach is applied by introducing a new input–output model, which leads to effective frequency domain and time domain criteria. The new method significantly improves the existing results for delays with derivatives not greater than 1, which were treated in the past as fast-varying delays (without any constraints on the delay derivatives). New bounded real lemmas (BRLs) are derived for systems with state and objective vector delays and norm-bounded uncertainties. Numerical examples illustrate the efficiency of the new method.

Keywords: Time-varying delay; Small-gain theorem; Lyapunov–Krasovskii functional; Norm-bounded uncertainties; H_∞-control

1. Introduction

The stability and control of continuous-time and discrete-time systems with uncertain time-delay is a subject of recurring interest. Most of the works consider delays with zero nominal values and apply different types of Lyapunov–Krasovskii Functionals (LKFs) (see e.g. [8,14–20,11,2]). Only few papers study systems with *non-zero nominal delay* values [14,11,20]. In the existing literature the uncertain time-varying delay (for continuous-time systems) has been divided into two types: the *slowly varying* delay (with delay derivative less than $q < 1$) and the *fast-varying delay* (without any constraint on the delay derivative). Systems with fast-varying delays have been usually treated via the Razumikhin approach [10]. For the first time, such systems were analyzed by the LKF techniques via the descriptor method [8], where the derivative of the LKF along the trajectories of the system depended on the state and the *state derivative*.

Robust stability has been studied also via the *input–output* approach, which reduces the stability analysis of the uncertain system to the analysis of a class of systems with the same nominal part but with additional inputs and outputs. This approach was introduced for constant delays in [12,1]. The stability conditions for constant delays by LKFs via the 1-st and the 3-rd model transformation (as defined in [15]) were recovered by this approach in [21]. The method of [21] has been generalized to the case of slowly varying delays in [11]. All the above works on the input–output approach consider the continuous-time case.

Frequency domain stability criteria for continuous-time and discrete-time systems with fast-varying delays have been derived in [13] in terms of transfer functions. In [5], a frequency domain stability criterion for continuous-time systems with fast-varying delays in terms of system matrices and delay bounds has been found via direct application of the Laplace transform.

In the present paper we reveal a third type of *moderately varying delay*, where the delay derivative is not greater than 1 (almost for all t). The latter delay appears in different applications (e.g. in networked control systems and in sampled-data control). This delay was treated in the past as a fast one, which led to restrictive results.

* Corresponding author. Tel.: +972 36405313; fax: +972 36408075.
E-mail address: emilia@eng.tau.ac.il (E. Fridman).

0167-6911/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.sysconle.2006.07.002

The present paper has been inspired by the recent monograph [11], where the input–output approach was developed for continuous-time systems with slowly varying delays. We develop the input–output approach to the continuous-time and discrete-time systems with moderately and fast-varying delays having non-zero nominal values. We introduce a new input–output model with an output, which explicitly depends on $\dot{x}(t)$ ($x(k+1)-x(k)$). This corresponds to the term with $\dot{x}(t)$ in the derivative of descriptor type LKF [7]. For the first time we apply the input–output approach to $L_2(l_2)$-gain analysis. As a result, new BRLs for systems with the delayed objective vector and with norm-bounded uncertainties are obtained, both in the frequency and in the time domain. The new method essentially improves the existing results for delays with derivatives not greater than 1. The time domain results are based on the application of the descriptor type LKF [7] combined with the free weighting matrices technique of [19].

Notation. Throughout the paper the superscript 'T' stands for matrix transposition, $\mathscr{R}^n$ denotes the n-dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathscr{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathscr{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. L_2 is the space of square integrable functions $v:[0,\infty)\to C^n$ with the norm $\|v\|_{L_2}=[\int_0^\infty \|v(t)\|^2\,\mathrm{d}t]^{1/2}$, l_2 is the space of square summable sequences with the norm$\|\cdot\|_{l_2}$, $\|A\|$ denotes the Euclidean norm of a $n\times n$ (real or complex) matrix A, which is equal to the maximum singular value of A. For a transfer function matrix of a stable system $G(s)$, $s\in C$

$$\|G\|_\infty=\sup_{-\infty<w<\infty}\|G(iw)\|,\quad i=\sqrt{-1}.$$

2. Stability and BRL in the frequency domain

2.1. Robust stability: continuous-time systems

We consider the following linear system with uncertain time-varying delays $\tau_i(t)$ $(i=1,2)$:

$$\dot{x}(t)=A_0x(t)+\sum_{i=1}^{2}A_ix(t-\tau_i(t)),\tag{1}$$

where $x(t)\in\mathscr{R}^n$ is the system state, $A_i,\ i=0,1,2$ are constant matrices.

The uncertain delays $\tau_i(t)$ are *piecewise-continuous* functions of the form

$$\tau_i(t)=h_i+\eta_i(t),\quad i=1,2,\quad |\eta_i(t)|\leqslant\mu_i\leqslant h_i,\tag{2}$$

where h_i are the known nominal delay values and μ_i are the known upper bounds on the delay uncertainties.

To obtain a less conservative result in the case of sign-varying $\eta_1(t)$, we assume additionally that $t-\tau_1(t)$ is a *non-decreasing* function. The latter assumption means that $\tau_1(t)$ is differentiable almost for all $t\geqslant 0$ and $\dot{\tau}_1(t)=\dot{\eta}_1(t)\leqslant 1$ almost for all $t\geqslant 0$. Note that this derivative constraint is weaker than $\dot{\tau}_1\leqslant q<1$ of [11]. Moreover, $\tau_1(t)$ may include such delays that $t-\tau_1(t)$ is piecewise constant. This kind of delay appears in sampled-data control $u(t)=Kx(t_k),\ t\in[t_k,t_{k+1})$ with the sampling times $0=t_0\leqslant t_1\leqslant\cdots$ satisfying $t_{k+1}-t_k\leqslant 2\mu_1$, where $x(t_k)$ can be represented as $x(t_k)=x(t-\mu_1-\eta_1(t))$ with $\eta_1(t)=t-\mu_1-t_k,\ t\in[t_k,t_{k+1})$ [6] .

It is noted that there is no constraint on the rate of change of $\tau_2(t)$.

We assume

$A1$: Given the nominal values of the delays $h_1>0$ and $h_2>0$, the nominal system

$$\dot{x}(t)=A_0x(t)+A_1x(t-h_1)+A_2x(t-h_2),\tag{3}$$

is asymptotically stable.

The results are easily generalized to the case of any finite number of the delays.

We represent (1) in the form:

$$\dot{x}(t)=A_0x(t)+\sum_{i=1}^{2}A_ix(t-h_i)-\sum_{i=1}^{2}A_i\int_{-h_i-\eta_i}^{-h_i}\dot{x}(t+s)\,\mathrm{d}s.\tag{4}$$

Following the idea of [12,21,11] to embed the perturbed system (4) into a class of systems with additional inputs and outputs, the stability of which guarantees the stability of (4), we introduce the following auxiliary system:

$$\dot{x}(t)=A_0x(t)+\sum_{i=1}^{2}A_ix(t-h_i)+\sum_{i=1}^{2}\mu_iA_iu_i(t),$$
$$y_1(t)=\dot{x}(t),\quad y_2(t)=\sqrt{2}\dot{x}(t),\tag{5a–d}$$

with the feedback

$$u_1(t) = -\frac{1}{\mu_1}\int_{-h_1-\eta_1}^{-h_1} y_1(t+s)\,\mathrm{d}s, \quad u_2(t) = -\frac{1}{\sqrt{2}\mu_2}\int_{-h_2-\eta_2}^{-h_2} y_2(t+s)\,\mathrm{d}s. \tag{6}$$

Substitution of (6) in (5) readily leads to (4). Note that $y_1(t)$ and $y_2(t)$ differ from the output of [12,21,11], and correspond to the term with $\dot{x}(t)$ in $\dot{V}_n$ in the descriptor approach [8].

Let $u^{\mathrm{T}} = [u_1^{\mathrm{T}}\ u_2^{\mathrm{T}}]$, $y^{\mathrm{T}} = [y_1^{\mathrm{T}}\ y_2^{\mathrm{T}}]$. Then the auxiliary system (5) can be written as $y = Gu$ with transfer matrix

$$G(s) = [sI\ \ \sqrt{2}sI]^{\mathrm{T}}\left(sI - A_0 - \sum_{i=1}^{2} A_i \mathrm{e}^{-h_i s}\right)^{-1}[\mu_1 A_1\ \ \mu_2 A_2]. \tag{7}$$

Assume that $y_i(t) = 0,\ \forall t \leqslant 0,\ i = 1, 2$.

Lemma 2.1. *The following holds*:

$$\|u_i\|_{L_2} \leqslant \|y_i\|_{L_2}, \quad i = 1, 2. \tag{8}$$

Proof. For $i = 1$ we have by Jensen (Cauchy–Schwartz) inequality ([11, p. 322]) for all $t \geqslant 0$

$$\mu_1^2\|u_1(t)\|^2 = \left\|\int_{-h_1-\eta_1(t)}^{-h_1} y_1(t+s)\,\mathrm{d}s\right\|^2 \leqslant \eta_1(t)\int_{t-h_1-\eta_1(t)}^{t-h_1} \|y_1(s)\|^2\,\mathrm{d}s. \tag{9}$$

Note that in the case where $\eta_1(t) \leqslant 0$, the integral in the right side of (9) will also be non-positive so that the term in the right side of (9) will be non-negative. Integrating (9) in t from 0 to ∞, we find that

$$\mu_1^2\|u_1\|_{L_2}^2 \leqslant \int_0^\infty \eta_1(t)\int_{t-h_1-\eta_1(t)}^{t-h_1} \|y_1(s)\|^2\,\mathrm{d}s\,\mathrm{d}t.$$

We change further the order of integration in the above double integral, taking into account that $y_1(s) = 0,\ s \leqslant 0$. Notice that the double integration domain lies in the strip $t - h_1 - \mu_1 \leqslant s \leqslant t - h_1 + \mu_1,\ t \geqslant 0$ and is bounded by the plots of $s = t - h_1$ and of $s = p(t) \stackrel{\Delta}{=} t - h_1 - \eta_1(t)$. Since $p(t)$ is a non-decreasing function, the set of segments $t \in [t_1, t_2]$, where $s = p(t)$ is constant, is countable, while out of these segments $s = p(t)$ is increasing. Hence, for almost all s (for those s, where $s = p(t)$ is increasing) the inverse $t = p^{-1}(s) = q(s)$ is well-defined and satisfies $s + h_1 - q(s) = -\eta_1(q(s))$. We thus find that

$$\begin{aligned}\mu_1^2\|u_1\|_{L_2}^2 \leqslant \int_0^\infty \eta_1(t)\int_{p(t)}^{t-h_1} \|y_1(s)\|^2\,\mathrm{d}s\,\mathrm{d}t &= \left|\int_0^\infty \int_{q(s)}^{s+h_1} \eta_1(q(s))\|y_1(s)\|^2\,\mathrm{d}t\,\mathrm{d}s\right| \\ &= \left|\int_0^\infty (s + h_1 - q(s))\eta_1(q(s))\|y_1(s)\|^2\,\mathrm{d}s\right| \\ &= \int_0^\infty \eta_1^2(q(s))\|y_1(s)\|^2\,\mathrm{d}s \leqslant \mu_1^2\|y_1\|_{L_2}^2.\end{aligned}$$

For $i = 2$

$$2\mu_2^2\|u_2(t)\|^2 \leqslant \mu_2\int_{-h_2-\mu_2}^{-h_2+\mu_2} \|y_1(t+s)\|^2\,\mathrm{d}s$$

and the result follows after integration in t and changing the order of integration. □

From Lemma 2.1 it follows by the small gain theorem (see e.g. [11]) that the system (1) is input–output stable (and thus asymptotically stable, since the nominal system is time-invariant) if

$$\|G\|_\infty < 1. \tag{10}$$

Theorem 2.1. *Consider* (1) *with delays given by* (2), *where* $\eta_i(t),\ i = 0, 1$ *are piecewise-continuous functions and* $\dot{\eta}_1(t) \leqslant 1$ *for almost all* $t \geqslant 0$. *Under A*1 *the system is asymptotically stable if* (10) *holds*, *where G is given by* (7).

Remark 2.1. The conditions of Theorem 2.1 (without $\sqrt{2}$ in G) coincide with [5], where delays of the type of τ_2 given by (2) with $\eta_2 \geqslant 0$ were considered. The stability interval $\tau_2(t) \in [h_2, h_2 + \mu_2]$ guaranteed by [5] is thus ($2/\sqrt{2} = \sqrt{2}$ times) smaller than the corresponding interval $\tau_2(t) \in [h_2 - \mu_2, h_2 + \mu_2]$ of Theorem 2.1.

Since for small enough μ_i (10) is always satisfied we have

Corollary 2.1. *Under* A1, (1) *is asymptotically stable for all small enough delay uncertainties* η_i.

Remark 2.2. A stronger result may be obtained by scaling G:

$$G_X(s) = \mathrm{diag}\{X_1, X_2\}G(s)\,\mathrm{diag}\,\{X_1^{-1}, X_2^{-1}\}, \tag{11}$$

where X_i, $i = 1, 2$ are non-singular $n \times n$ matrices. Hence, under A1, (1) is asymptotically stable for all delays satisfying (2) if there exist X_i such that $\|G_X\|_\infty < 1$.

2.2. BRL: continuous-time systems

We consider the following linear system with uncertain coefficients and uncertain time-varying delays $\tau_i(t)$ (i = 1,2) as above:

$$\begin{aligned}\dot{x}(t) &= (A_0 + H\Delta E_0)x(t) + \sum_{i=1}^{2}(A_i + H\Delta E_i)x(t-\tau_i(t)) + (B_1 + H\Delta E_3)w(t),\\ z(t) &= C_0x(t) + \sum_{i=1}^{2} C_i x(t-\tau_i(t)), \quad x(s) = 0, \quad s \leqslant 0,\end{aligned} \tag{12}$$

where $x(t) \in R^n$ is a state vector, $w(t) \in \mathscr{R}^q$ is an arbitrary disturbance vector in $L_2[0\ \infty)$ and $z(t) \in \mathscr{R}^p$ is the objective vector, $A_i,\ E_i, C_i,\ i = 0, 1, 2$ and H are constant matrices of appropriate dimensions and $\Delta(t)$ is a time-varying uncertain $n \times n$ matrix that satisfies

$$\Delta^{\mathrm{T}}(t)\Delta(t) \leqslant I_n \quad \forall t \geqslant 0. \tag{13}$$

Given $\gamma > 0$, we seek a condition which guarantees that L_2-gain of (12) is less than γ, i.e. that the following inequality holds:

$$\|z\|_{L_2}^2 < \gamma^2\|w\|_{L_2}^2 \quad \forall\ 0 \neq w \in L_2. \tag{14}$$

Consider an auxiliary system

$$\begin{aligned}\dot{x}(t) &= (A_0 + H\Delta E_0)x(t) + \sum_{i=1}^{2}(A_i + H\Delta E_i)x(t-\tau_i(t)) + \gamma^{-1}(B_1 + H\Delta E_3)\bar{w}(t),\\ z(t) &= C_0x(t) + \sum_{i=1}^{2} C_i x(t-\tau_i(t)),\\ x(t) &= 0, \quad t \leqslant 0.\end{aligned} \tag{15}$$

It is clear that

$$\|z\|_{L_2}^2 < \|\bar{w}\|_{L_2}^2 \quad \forall\ 0 \neq \bar{w} \in L_2 \tag{16}$$

for (15) is equivalent to (14) for (12).

To derive 'scaled conditions' consider the following auxiliary system:

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + \sum_{i=1}^{2} A_i x(t-h_i) + \sum_{i=1}^{2}\sqrt{\mu_i}A_iX_i^{-1}u_i(t) + \rho^{-1}Hu_3(t) + \gamma^{-1}B_1\bar{w}(t),\\ y_1(t) &= \sqrt{\mu_1}X_1\dot{x}(t), \quad y_2(t) = \sqrt{2\mu_2}X_2\dot{x}(t),\\ y_3(t) &= \rho[E_0x(t) + \sum_{i=1}^{2} E_i x(t-h_i) + \sum_{i=1}^{2}\sqrt{\mu_i}E_iX_i^{-1}u_i(t) + \gamma^{-1}E_3\bar{w}(t)],\\ z(t) &= C_0x(t) + \sum_{i=1}^{2} C_i x(t-h_i) + \sum_{i=1}^{2}\sqrt{\mu_i}C_iX_i^{-1}u_i(t),\end{aligned} \tag{17a–d}$$

with the feedback of (6) and $u_3(t) = \Delta y_3(t)$. Note that the inequality $\|u_i\|_{L_2} \leqslant \|y_i\|_{L_2}$ holds for $i = 1, 2, 3$. Eq. (17) is scaled by $\sqrt{\mu_i}$ so that for $\mu_i = 0$ (17) corresponds to the case of known constant delays $\tau_i \equiv h_i$ and norm-bounded uncertainties.

The auxiliary system (17) can be written as

$$\begin{bmatrix} y \\ z \end{bmatrix} = G_{\gamma X} \begin{bmatrix} u \\ \bar{w} \end{bmatrix}, \quad u^{\mathrm{T}} = [u_1^{\mathrm{T}}\ u_2^{\mathrm{T}}\ u_3^{\mathrm{T}}], \quad y^{\mathrm{T}} = [y_1^{\mathrm{T}}\ y_2^{\mathrm{T}}\ y_3^{\mathrm{T}}], \tag{18}$$

with transfer matrix given by

$$G_{\gamma X}(s) = \mathrm{diag}\,\{X_1,\ X_2, \rho I_n, I_p\} G_\gamma\, \mathrm{diag}\,\{X_1^{-1}, X_2^{-1}, \rho^{-1} I_n, I_p\},$$

$$G_\gamma = \begin{bmatrix} \sqrt{\mu_1} s I_n \\ \sqrt{2\mu_2} s I_n \\ E_0 + \sum_{i=1}^{2} E_i \mathrm{e}^{-h_i s} \\ C_0 + \sum_{i=1}^{2} C_i \mathrm{e}^{-h_i s} \end{bmatrix} \left(s I_n - A_0 - \sum_{i=1}^{2} A_i \mathrm{e}^{-h_i s} \right)^{-1} \begin{bmatrix} \sqrt{\mu_1} A_1 & \sqrt{\mu_2} A_2 & H & \dfrac{B_1}{\gamma} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\mu_1} E_1 & \sqrt{\mu_2} E_2 & 0 & \dfrac{1}{\gamma} E_3 \\ \sqrt{\mu_1} C_1 & \sqrt{\mu_2} C_2 & 0 & 0 \end{bmatrix}. \tag{19}$$

We obtain the following result:

Theorem 2.2. *Assume A*1. *Given* $\gamma > 0$, (12) *is internally stable and has* L_2*-gain less than* γ *for all delays satisfying* (2), *if there exist non-singular* $n \times n$*-matrices* X_1, X_2 *and a scalar* $\rho \neq 0$ *such that*

$$\|G_{\gamma X}\|_\infty < 1. \tag{20}$$

Proof. Eqs. (18) and (20) imply that

$$\|y\|_{L_2}^2 + \|z\|_{L_2}^2 < \|u\|_{L_2}^2 + \|\bar{w}\|_{L_2}^2.$$

The latter inequality together with $\|u\|_{L_2}^2 \leqslant \|y\|_{L_2}^2$ yield (16) and (14). □

2.3. Extension to the discrete-time delay systems

We consider the following linear discrete system with uncertain coefficients and uncertain time-varying delays $\tau_i(k)$ $(i = 1, 2)$:

$$x(k+1) = (A_0 + H\Delta E_0)x(k) + \sum_{i=1}^{2} (A_i + H\Delta E_i)x(k - \tau_i(k)) + (B_1 + H\Delta E_3)w(k),$$
$$z(k) = C_0 x(k) + \sum_{i=1}^{2} C_i x(k - \tau_i(k)), \quad x(l) = 0, \ \ l \leqslant 0, \tag{21}$$

where $x(k) \in \mathscr{R}^n$ is the system state, $A_1,\ A_2, H,\ E_i$ and B_1 are constant matrices of appropriate dimensions and $\Delta(k)$ is a time-varying uncertain matrix that satisfies

$$\Delta^{\mathrm{T}}(k)\Delta(k) \leqslant I \quad \forall k \geqslant 0. \tag{22}$$

The uncertain delays $\tau_i(k)$ are supposed to have the following form:

$$\tau_i(k) = h_i + \eta_i(k), \ \ i = 1, 2, \quad -h_i \leqslant -\mu_{i-} \leqslant \eta_i(k) \leqslant \mu_{i+} \leqslant h_i, \ \ |\mu_{i-} - \mu_{i+}| \leqslant 1, \tag{23}$$

with the known bounds $\mu_{i+} \geqslant 0$ and $\mu_{i-} \geqslant 0$. Note that similarly to the continuous-time case we choose h_i in the 'middle' of the delay interval. Denote $\mu_i = \max\{\mu_{i-}, \mu_{i+}\}$, $i = 1, 2$.

We assume additionally that $k-\tau_1(k)$ is an increasing function, i.e. that τ_1 satisfies the following constraint: $\tau_1(k+1)-\tau_1(k)\leqslant 0$. Note that the constraint on τ_1 is more restrictive, than in the continuous-time case, where $t-\tau_1$ is supposed to be non-decreasing.

We assume

*A*1*d*: Given the nominal values of the delays $h_1>0$ and $h_2>0$, the nominal system

$$x(k+1)=A_0x(k)+A_1x(k-h_1)+A_2x(k-h_2), \tag{24}$$

is asymptotically stable.

Given $\gamma>0$, we are seeking a condition which guarantees that (21) is internally stable (i.e. asymptotically stable for $w=0$) and has l_2-gain less than γ, i.e. that the following inequality holds:

$$\|z\|_{l_2}^2<\gamma^2\|w\|_{l_2}^2 \quad \forall\, 0\neq w\in l_2. \tag{25}$$

We represent

$$x(k-\tau_i(k))=x(k-h_i)+\sum_{j=k-h_i-\eta_i}^{k-h_i-1}(x(j+1)-x(j)),$$

where for $\eta_i\leqslant 0$

$$\sum_{j=k-h_i-\eta_i}^{k-h_i-1}(x(j+1)-x(j))=\begin{cases}0 & \text{if } \eta_i=0,\\ -\sum_{j=k-h_i}^{k-h_i-\eta_i-1}(x(j+1)-x(j)) & \text{if } \eta_i<0.\end{cases}$$

Then (21) takes the form:

$$\begin{aligned}x(k+1)={}&A_0x(k)+\sum_{i=1}^{2}A_ix(k-h_i)-\sum_{i=1}^{2}A_i\sum_{j=k-h_i-\eta_i}^{k-h_i-1}(x(j+1)-x(j))\\ &+H\Delta\left[E_0x(k)+\sum_{i=1}^{2}E_ix(k-h_i)-\sum_{i=1}^{2}E_i\sum_{k-h_i-\eta_i}^{k-h_i-1}(x(j+1)-x(j))\right]+(B_1+H\Delta E_3)w(k),\\ z(k)={}&C_0x(k)+\sum_{i=1}^{2}C_ix(k-h_i)-\sum_{i=1}^{2}C_i\sum_{k-h_i-\eta_i}^{k-h_i-1}(x(j+1)-x(j)).\end{aligned} \tag{26}$$

Consider the following auxiliary system:

$$\begin{aligned}x(k+1)&=A_0x(k)+\sum_{i=1}^{2}A_ix(t-h_i)+\sum_{i=1}^{2}\sqrt{\mu_i}A_iX_i^{-1}u_i(k)+\rho^{-1}Hu_3(k)+\gamma^{-1}B_1w(k),\\ y_1(k)&=\sqrt{\mu_1}X_1[x(k+1)-x(k)],\quad y_2(k)=\sqrt{\mu_{2-}+\mu_{2+}}X_2[x(k+1)-x(k)],\\ y_3(k)&=\rho[E_0x(k)+\sum_{i=1}^{2}E_ix(k-h_i)+\sum_{i=1}^{2}\sqrt{\mu_i}E_iX_i^{-1}u_i(k)+\gamma^{-1}E_3w(k)],\\ z(k)&=C_0x(k)+\sum_{i=1}^{2}C_ix(k-h_i)+\sum_{i=1}^{2}\sqrt{\mu_i}C_iX_i^{-1}u_i(k),\end{aligned} \tag{27a–d}$$

with the feedback

$$\begin{aligned}u_1(k)&=-\frac{1}{\mu_1}\sum_{j=k-h_1-\eta_1}^{k-h_1-1}y_1(j),\\ u_2(k)&=-\frac{1}{\sqrt{(\mu_{2-}+\mu_{2+})\mu_2}}\sum_{j=k-h_2-\eta_2}^{k-h_2-1}y_2(j),\\ u_3(k)&=\Delta y_3(k).\end{aligned} \tag{28}$$

The auxiliary system (27) can be written as

$$\begin{bmatrix} y \\ z \end{bmatrix} = G_{dX} \begin{bmatrix} u \\ \bar{w} \end{bmatrix}, \quad u^{\mathrm{T}} = [u_1^{\mathrm{T}}\ u_2^{\mathrm{T}}\ u_3^{\mathrm{T}}], \quad y^{\mathrm{T}} = [y_1^{\mathrm{T}}\ y_2^{\mathrm{T}}\ y_3^{\mathrm{T}}], \tag{29}$$

with the transfer matrix given by

$$G_{dX}(z) = \mathrm{diag}\{X_1,\ X_2, \rho I_n, I_p\} G_d(z)\, \mathrm{diag}\,\{X_1^{-1},\ X_2^{-1}, \rho^{-1} I_n, I_p\},$$

$$G_d(z) = \begin{bmatrix} \sqrt{\mu_1}(z-1) \\ \sqrt{\mu_{2-} + \mu_{2+}}(z-1) \\ E_0 + \sum_{i=1}^{2} E_i z^{-h_i} \\ C_0 + \sum_{i=1}^{2} C_i z^{-h_i} \end{bmatrix} \left(zI - A_0 - \sum_{i=1}^{2} A_i z^{-h_i} \right)^{-1} \begin{bmatrix} \sqrt{\mu_1} A_1 & \sqrt{\mu_2} A_2 & H & \dfrac{B_1}{\gamma} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\mu_1} E_1 & \sqrt{\mu_2} E_2 & 0 & \dfrac{1}{\gamma} E_3 \\ \sqrt{\mu_1} C_1 & \sqrt{\mu_2} C_2 & 0 & 0 \end{bmatrix}. \tag{30}$$

Theorem 2.3. *Assume A1d. Given* $\gamma > 0$, (21) *is internally stable and has l_2-gain less than γ for all delays satisfying* (23) *and* $\tau_1(k+1) - \tau_1(k) \leqslant 0$, *if there exist non-singular* $n \times n$*-matrices* $X_i,\ i = 1, 2$ *and a scalar* $\rho \neq 0$ *such that*

$$\|G_{dX}\|_\infty < 1. \tag{31}$$

Proof. The proof is similar to the one for the continuous-time case, where integration is replaced by summation. Thus, in order to show that $\|u_1\|_{l_2} \leqslant \|y_1\|_{l_2}$, we apply the Cauchy–Schwartz inequality

$$\mu_1^2 \|u_1(k)\|^2 = \left\| \sum_{j=k-h_1-\eta_1}^{k-h_1-1} y_1(j) \right\|^2 \leqslant \eta_1 \sum_{j=k-h_1-\eta_1}^{k-h_1-1} \|y_1(j)\|^2, \quad k \geqslant 0. \tag{32}$$

The function $j = p(k) \stackrel{\Delta}{=} k - h_1 - \eta_1(k)$ is strongly increasing. Hence, the inverse $k = p^{-1}(j) = q(j)$ is well-defined and satisfies $|q(j) - j - h_1| \leqslant \mu_1$. Then, summing (32) in k, changing the order of the summation and taking into account that $y_1(k) = 0,\ k \leqslant 0$, we find that

$$\mu_1^2 \|u_1\|_{l_2}^2 \leqslant \sum_{k=1}^{\infty} \eta_1 \sum_{j=k-h_1-\eta_1}^{k-h_1-1} \|y_1(j)\|^2 = \left| \sum_{j=1}^{\infty} \sum_{k=q(j)}^{j+h_1-1} \eta_1 \|y_1(j)\|^2 \right| \leqslant \mu_1^2 \|y_1\|_{l_2}^2. \qquad \square$$

3. Stability and BRL in the time domain

In the continuous-time case, let V_n be a LKF, which guarantees the stability of the nominal system (3). It is well-known that the following condition along (17):

$$\mathscr{W} \stackrel{\Delta}{=} \dot{V}_n(t) + \|y(t)\|^2 + \|z(t)\|^2 - \|u(t)\|^2 - \|\bar{w}(t)||^2 < -\varepsilon(\|x(t)\|^2 + \|u(t)\|^2 + \|\bar{w}(t)\|^2), \quad \varepsilon > 0 \tag{33}$$

guarantees that the H_∞-norm of (17) is less than 1. Therefore, (33) is a sufficient condition for the feasibility of the frequency domain condition (20) of Theorem 2.2.

In the discrete-time case the corresponding condition along (27) has the form

$$\mathscr{W}_d \stackrel{\Delta}{=} V_n(k+1) - V_n(k) + \|y(k)\|^2 + \|z(k)\|^2 - \|u(k)\|^2 - \|\bar{w}(k)\|^2 < -\varepsilon(\|x(k)\|^2 + \|u(k)\|^2 + \|\bar{w}(k)\|^2). \tag{34}$$

We choose the descriptor type V_n [7].

3.1. Discrete-time results

We combine the discrete-time descriptor LKF (see e.g. [2]):

$$V_n(k) = x^{\mathrm{T}}(k)P_1x(k) + \sum_{i=1}^{2}\sum_{m=-h_i}^{-1}\sum_{j=k+m}^{k-1}\bar{y}(j)^{\mathrm{T}}R_i\bar{y}(j)$$

$$+\sum_{i=1}^{2}\sum_{j=k-h_i}^{k-1}x(j)^{\mathrm{T}}S_ix(j), \quad \bar{y}(k) = x(k+1)-x(k), \quad P_1>0, \quad R>0, \quad S>0. \tag{35}$$

with the free weighting matrices technique of [19].

Lemma 3.1. *The nominal system* (24) *is asymptotically stable if there exist* $n \times n$ *matrices* $0<P_1,\ P_2,\ P_3,\ S_i>0,\ Y_{i1},\ Y_{i2},\ R_i>0,\ T_i$ *such that the following LMIs are feasible*:

$$\Gamma_n = \begin{bmatrix} \Psi_n & P^{\mathrm{T}}\begin{bmatrix}0\\A_1\end{bmatrix} - Y_1^{\mathrm{T}} + \begin{bmatrix}T_1\\0\end{bmatrix} & P^{\mathrm{T}}\begin{bmatrix}0\\A_2\end{bmatrix} - Y_2^{\mathrm{T}} + \begin{bmatrix}T_2\\0\end{bmatrix} & h_1Y_1^{\mathrm{T}} & h_2Y_2^{\mathrm{T}} \\ * & -S_1-T_1-T_1^{\mathrm{T}} & 0 & -h_1T_1^{\mathrm{T}} & 0 \\ * & * & -S_2-T_2-T_2^{\mathrm{T}} & 0 & -h_2T_2^{\mathrm{T}} \\ * & * & * & -h_1R_1 & 0 \\ * & * & * & * & -h_2R_2 \end{bmatrix} < 0, \tag{36}$$

where

$$\Psi_n = P^{\mathrm{T}}\begin{bmatrix}0 & I\\ A_0-I & -I\end{bmatrix} + \begin{bmatrix}0 & A_0^{\mathrm{T}}-I\\ I & -I\end{bmatrix}P + \begin{bmatrix}\sum_{i=1}^{2}S_i & 0\\ 0 & P_1+\sum_{i=1}^{2}h_iR_i\end{bmatrix} + \sum_{i=1}^{2}\begin{bmatrix}Y_i\\0\end{bmatrix} + \sum_{i=1}^{2}\begin{bmatrix}Y_i\\0\end{bmatrix}^{\mathrm{T}},$$

$$P = \begin{bmatrix}P_1 & 0\\ P_2 & P_3\end{bmatrix}, \quad Y_i = [Y_{i1}\ Y_{i2}]. \tag{37a–c}$$

Proof. Denote $\bar{x}^{\mathrm{T}}(k) = [x^{\mathrm{T}}(k)\ \bar{y}^{\mathrm{T}}(k)]$. We have along the trajectories of (24):

$$\begin{aligned} &V_n(k+1)-V_n(k)\\ &\quad= 2x^{\mathrm{T}}(k)P_1y(k) + y^{\mathrm{T}}(k)P_1y(k) + \sum_{i=1}^{2}x^{\mathrm{T}}(k)S_ix(k)\\ &\qquad-\sum_{i=1}^{2}x^{\mathrm{T}}(k-h_i)S_ix(k-h_i) + \sum_{i=1}^{2}h_i\bar{y}^{\mathrm{T}}(k)R_i\bar{y}(k) - \sum_{i=1}^{2}\sum_{k-h_i}^{k-1}\bar{y}^{\mathrm{T}}(j)R_i\bar{y}(j)\\ &\quad= 2\bar{x}^{\mathrm{T}}(k)P^{\mathrm{T}}\begin{bmatrix}\bar{y}(k)\\0\end{bmatrix} + \bar{y}^{\mathrm{T}}(k)P_1\bar{y}(k) + \sum_{i=1}^{2}x^{\mathrm{T}}(k)S_ix(k)\\ &\qquad-\sum_{i=1}^{2}x^{\mathrm{T}}(k-h_i)S_ix(k-h_i) + \sum_{i=1}^{2}h_i\bar{y}^{\mathrm{T}}(k)R_i\bar{y}(k) - \sum_{i=1}^{2}\sum_{k-h_i}^{k-1}\bar{y}^{\mathrm{T}}(j)R_i\bar{y}(j). \end{aligned} \tag{38}$$

Following [19] we add the left part of the equality

$$2[\bar{x}^{\mathrm{T}}(k)Y_i^{\mathrm{T}}\ x^{\mathrm{T}}(k-h_i)T_i^{\mathrm{T}}]\left[x(k) - \sum_{j=k-h_i}^{k-1}\bar{y}(j) - x(k-h_i)\right] = 0, \quad i=1,2 \tag{39}$$

to $V_n(k+1)-V_n(k)$ and substitute $0=-\bar y(k)+(A_0-I)x(k)+\sum_{i=1}^2 A_i x(k-h_i)$. We find

$$\begin{aligned}&V_n(k+1)-V_n(k)\\&\quad=2\bar x^{\mathrm T}(k)P^{\mathrm T}\left[\begin{bmatrix}0 & I\\ A_0-I & -I\end{bmatrix}\bar x(k)+\sum_{i=1}^2\begin{bmatrix}0\\ A_i\end{bmatrix}x(k-h_i)\right]+\bar y^{\mathrm T}(k)P_1\bar y(k)\\&\qquad+\sum_{i=1}^2 x^{\mathrm T}(k)S_i x(k)-\sum_{i=1}^2 x^{\mathrm T}(k-h_i)S_i x(k-h_i)\\&\qquad+\sum_{i=1}^2 h_i\bar y^{\mathrm T}(k)R_i\bar y(k)-\sum_{i=1}^2\sum_{k-h_i}^{k-1}\bar y^{\mathrm T}(j)R_i\bar y(j)\\&\qquad+2\sum_{i=1}^2\left[\bar x^{\mathrm T}(k)Y_i^{\mathrm T}\quad x^{\mathrm T}(k-h_i)T_i^{\mathrm T}\right]\left[x(k)-\sum_{j=k-h_i}^{k-1}\bar y(j)-x(k-h_i)\right].\end{aligned}\tag{40}$$

Applying further the Cauchy–Schwartz inequality

$$\sum_{k-h_i}^{k-1}\bar y^{\mathrm T}(j)R_i\bar y(j)\geqslant\frac{1}{h_i}\left[\sum_{k-h_i}^{k-1}\bar y^{\mathrm T}(j)\right]R_i\left[\sum_{k-h_i}^{k-1}\bar y(j)\right],$$

and taking into consideration (36), we conclude that (24) is asymptotically stable since

$$V_n(k+1)-V_n(k)\leqslant\xi^{\mathrm T}(k)\Gamma_n\xi(k)<0,$$

where

$$\xi^{\mathrm T}(k)=\left[x^{\mathrm T}(k)\ \ \bar y^{\mathrm T}(k)\ \ x^{\mathrm T}(k-h_1)\ \ x^{\mathrm T}(k-h_2)\ \ \frac{1}{h_1}\sum_{k-h_1}^{k-1}\bar y^{\mathrm T}(j)\ \ \frac{1}{h_2}\sum_{k-h_2}^{k-1}\bar y^{\mathrm T}(j)\right].\qquad\square\tag{41}$$

Remark 3.1. For $T_i=0,\ i=1,2$, the LMIs (36) coincide with the stability conditions of [2], where bounding of the cross terms of [17] has been applied instead of using the technique of [19]. The additional degrees of freedom in (36) may improve the results for uncertain systems (see Example 2 below). Note that additional degrees of freedom may be introduced by changing the first multiplier of (39) to a product of $\xi^{\mathrm T}(k)$ with the corresponding free matrices.

We consider now the uncertain system (21). To derive BRL for this system we check the condition (34) along the trajectories of (27). We have

$$V_n(k+1)-V_n(k)\leqslant\xi^{\mathrm T}(k)\Gamma_n\xi(k)+2\bar x^{\mathrm T}(k)P^{\mathrm T}\begin{bmatrix}0\\ \sum_{i=1}^2\sqrt{\mu_i}A_iX_i^{-1}u_i(k)+\dfrac{H}{\rho}u_3(t)+\dfrac{B_1}{\gamma}\bar w(k)\end{bmatrix}.\tag{42}$$

Applying the definition (34) and denoting $\zeta^{\mathrm T}(k)=[\xi_1^{\mathrm T}(k)\ x(k-h_2)\ u_1^{\mathrm T}(k)\ u_2^{\mathrm T}(k)\ u_3^{\mathrm T}(k)\ \bar w^{\mathrm T}(k)]$ we readily obtain

$$\mathscr W_d\leqslant\zeta^{\mathrm T}(k)\Gamma\zeta(k)+\|y(k)\|^2+\|z(k)\|^2,\tag{43}$$

where

$$\Gamma=\begin{bmatrix}\Gamma_n & \sqrt{\mu_1}P^{\mathrm T}\begin{bmatrix}0_{n\times n}\\ A_1X_1^{-1}\\ 0_{4n\times n}\end{bmatrix} & \sqrt{\mu_2}P^{\mathrm T}\begin{bmatrix}0_{n\times n}\\ A_2X_2^{-1}\\ 0_{4n\times n}\end{bmatrix} & P^{\mathrm T}\begin{bmatrix}0_{n\times n}\\ \dfrac{H}{\rho}\\ 0_{4n\times n}\end{bmatrix} & P^{\mathrm T}\begin{bmatrix}0_{n\times n}\\ \dfrac{B_1}{\gamma}\\ 0_{4n\times n}\end{bmatrix}\\ * & -I_n & 0 & 0 & 0\\ * & * & -I_n & 0 & 0\\ * & * & * & -I_n & 0\\ * & * & * & * & -I_q\end{bmatrix}.\tag{44}$$

Note that the matrices $-I$ on the diagonal in (44) stem from $\|u_i(k)\|^2,\ i=1,\dots,3$ and $\|\bar w(k)\|^2$ in (34).

By applying Schur complements to the terms $\|y(k)\|^2+\|z(k)\|^2$ and multiplying the resulting matrix by diag$\{I_n, \sqrt{\mu_1}X_1, \sqrt{\mu_2}X_2, I_{4n}, \rho I_n, \gamma I_q, I_{2n}, \rho I_n, I_p\}$ and its transpose, from the right and from the left, respectively, we find that (34) is satisfied if

$$\left[\begin{array}{c|cccc} \Gamma_1 & \begin{matrix} 0 \\ \mu_1 R_{1a} \\ 0 \\ 0 \\ 0_{2n\times n} \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ (\mu_{2-}+\mu_{2+})R_{2a} \\ 0 \\ 0 \\ 0_{2n\times n} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} rE_0^{\mathrm{T}} \\ 0 \\ rE_1^{\mathrm{T}} \\ rE_2^{\mathrm{T}} \\ 0_{2n\times n} \\ r\mu_1 E_1^{\mathrm{T}} \\ r\mu_2 E_2^{\mathrm{T}} \\ 0 \\ rE_3^{\mathrm{T}} \end{matrix} & \begin{matrix} C_0^{\mathrm{T}} \\ 0 \\ C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \\ 0_{2n\times n} \\ \mu_1 C_1^{\mathrm{T}} \\ \mu_2 C_2^{\mathrm{T}} \\ 0 \\ 0 \end{matrix} \\ \hline * & -\mu_1 R_{1a} & 0 & 0 & 0 \\ * & * & -(\mu_{2-}+\mu_{2+})R_{2a} & 0 & 0 \\ * & * & * & -rI_n & 0 \\ * & * & * & * & -I_p \end{array}\right] < 0, \tag{45}$$

where $R_{ia}=X_i^{\mathrm{T}}X_i$, $r=\rho^2$ and

$$\Gamma_1 = \begin{bmatrix} \Gamma_n & \mu_1 P^{\mathrm{T}}\begin{bmatrix} 0_{n\times n} \\ A_1 \\ 0_{4n\times n} \end{bmatrix} & \mu_2 P^{\mathrm{T}}\begin{bmatrix} 0_{n\times n} \\ A_2 \\ 0_{4n\times n} \end{bmatrix} & P^{\mathrm{T}}\begin{bmatrix} 0_{n\times n} \\ H \\ 0_{4n\times n} \end{bmatrix} & P^{\mathrm{T}}\begin{bmatrix} 0_{n\times n} \\ B_1 \\ 0_{4n\times n} \end{bmatrix} \\ * & -\mu_1 R_{1a} & 0 & 0 & 0 \\ * & * & -\mu_2 R_{2a} & 0 & 0 \\ * & * & * & -rI_n & 0 \\ * & * & * & * & -\gamma^2 I_q \end{bmatrix}, \tag{46}$$

We thus obtained the following:

Theorem 3.1. *Given* $\gamma>0$, (21) *is internally stable and has* L_2*-gain less than* γ *for all delays satisfying* (23), *if there exist* $n\times n$ *matrices* $0<P_1,\ P_2,\ P_3,\ S_i>0, Y_{i1},\ Y_{i2}, R_i, T_i, R_{ia},\ i=1,2$ *and a scalar* $r>0$ *such that LMI* (45) *is feasible.*

3.2. Continuous-time results

By combining the descriptor model transformation and the corresponding LKF [7]

$$V_n = x^{\mathrm{T}}(t)P_1x(t) + \sum_{i=1}^{2}\left[\int_{-h_i}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_i\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta + \int_{t-h_i}^{t}x^{\mathrm{T}}(s)S_ix(s)\,\mathrm{d}s\right], \quad R_i>0, \quad S_i>0, \tag{47}$$

with the technique of [19] and the above arguments we obtain:

Theorem 3.2. *Given* $\gamma>0$, (12) *is internally stable and has* L_2*-gain less than* γ *for all delays satisfying* (23), *if there exist* $n\times n$ *matrices* $0<P_1,\ P_2,\ P_3,\ S_i>0, Y_{i1},\ Y_{i2}, R_i, T_i, R_{ia},\ i=1,2$ *and a scalar* $\rho>0$ *such that LMI* (45) *is feasible, where* $\mu_{2-}+\mu_{2+}$ *should be changed by* $2\mu_2$ *and* Ψ_n *should be substituted by* Ψ_{nc} *given by*

$$\Psi_{nc} = P^{\mathrm{T}}\begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix} + \begin{bmatrix} 0 & A_0^{\mathrm{T}} \\ I & -I \end{bmatrix}P + \begin{bmatrix} \sum_{i=1}^{2}S_i & 0 \\ 0 & \sum_{i=1}^{2}h_iR_i \end{bmatrix} + \sum\nolimits_{i=1}^{2}\begin{bmatrix} Y_i \\ 0 \end{bmatrix} + \sum\nolimits_{i=1}^{2}\begin{bmatrix} Y_i \\ 0 \end{bmatrix}^{\mathrm{T}}. \tag{48}$$

Remark 3.2. Assumptions A1 and A1d are automatically satisfied if the time domain criteria of Theorems 3.2 and 3.1 are satisfied. This is different from the frequency domain results, where these assumptions should be verified.

Remark 3.3. Conditions equivalent to those of Theorems 3.2 and 3.3 may be derived, for the case with delays of the type of τ_2 by a direct application of the Lyapunov–Krasovskii method to the original system (as introduced in [3] and extended to the discrete-time case in [9]).

4. Examples

In order to verify the conditions of Theorems 2.1 and 2.2 for the continuous case, or the one of Theorem 2.3 for discrete-time, a constant non-singular matrix D of a specific diagonal block structure is sought that satisfies, for say G of Remark 2.2, the following inequality:

$$D^{-T}G^{\mathrm{T}}(-j\omega)D^{\mathrm{T}}DG(j\omega)D^{-1} < I, \quad \forall\omega \in [0\ \infty), \quad D = \mathrm{diag}\{X_1,\ X_2\}.$$

Denoting $Q = D^{\mathrm{T}}D = \mathrm{diag}\{R_{1a}, R_{2a}\}$, the latter inequality becomes:

$$G^{\mathrm{T}}(\mathrm{j}\omega)QG(\mathrm{j}\omega) < Q, \quad Q > 0, \ \forall\omega \in [0\ \infty) \tag{49a,b}$$

Since $G(j\omega)$ is a complex matrix, the left side of (49a) is a complex Hermitian matrix. Denoting

$$G_r(\omega) = \mathrm{Re}(G(\mathrm{j}\omega)) \quad \text{and} \quad G_i(\omega) = \mathrm{Im}\ (G(\mathrm{j}\omega))$$

(49a,b) become

$$\begin{bmatrix} G_r^{\mathrm{T}}(\omega) & G_i^{\mathrm{T}}(\omega) \\ -G_i^{\mathrm{T}}(\omega) & G_r^{\mathrm{T}}(\omega) \end{bmatrix}\begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} G_r(\omega) & -G_i(\omega) \\ G_i(\omega) & G_r(\omega) \end{bmatrix} - \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} < 0, \quad Q > 0, \ \ \forall\ \omega \in [0\ \infty).$$

Discretizing the range of ω by selecting N properly spread points, $\omega_k,\ k = 1, \ldots, N$, in $(0\ \infty)$, the latter inequality is solved by seeking $Q > 0$ that satisfies the following LMIs:

$$\begin{bmatrix} G_r^{\mathrm{T}}(\omega_k)QG_r(\omega_k) + G_i^{\mathrm{T}}(\omega_k)QG_i(\omega_k) - \lambda Q & -G_r^{\mathrm{T}}(\omega_k)QG_i(\omega_k) + G_i^{\mathrm{T}}(\omega_k)QG_r(\omega_k) \\ * & G_r^{\mathrm{T}}(\omega_k)QG_r(\omega_k) + G_i^{\mathrm{T}}(\omega_k)QG_i(\omega_k) - \lambda Q \end{bmatrix} < 0, \quad k = 1, \ldots, N, \tag{50}$$

where $0 < \lambda < 1$ is a scalar, close to 1, that introduces some margin for the LMIs of (50) to be satisfied also for ω in between the selected points ω_k.

Example 1 (*Kharitonov and Niculescu [14]*). *Continuous-time system*. Consider (1) with

$$A_0 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ -0.4 & 0 \end{bmatrix}, \quad A_2 = 0. \tag{51}$$

The nominal non-delayed system (i.e. (51) with $\tau_1 = 0$) is not asymptotically stable and thus the descriptor nominal LKF is not applicable. For the case of *constant* delay $\tau_1 = 4 + \eta_1$ the following stability interval was found by the frequency domain analysis [14]: $-0.6209 < \eta_1 < 0.7963$. For $h_1 = 4$ the following interval of fast-varying delay $\tau_1(t)$ was found in [4] by using complete LKF : [3.989, 4.011].

Using the procedure of (50) for $\dot{\tau}_1 \leqslant 1$, stability is guaranteed for $\tau_1(t) \in [3.958, 4.042]$. This result is obtained by solving (50) for $N = 200$ frequency points that are logspaced between 10^{-2} and 10^2, taking $\lambda = 0.9$ and checking the resulting Q at 10 000 logspaced frequency points. The corresponding interval for the fast-varying delay (for the same parameters of N and λ) is: [3.971, 4.029].

The above results refer to intervals of τ_1 for which the LMIs in (50) possess a positive solution Q. It is noted, however, that although (50) possesses a marginally infeasible solution for the fast-varying $\tau_1(t) \in [3.967, 4.033]$, the resulting Q still satisfies (49a,b) for the tested 10 000 frequency points.

Consider next the BRL for the system (12) with (51), and with $C_0 = [0\ 1]$, $C_1 = [0.2\ 0]$, $B_1^{\mathrm{T}} = [1\ 0]$, $E_i = 0,\ i = 0, \ldots, 3$. The existing methods in the case of time-varying delays are not applicable, because the non-delayed nominal system is not asymptotically stable. By using Theorem 2.2 and the procedure of (50) for $h = 4$, $\mu_1 = 0.01$ the resulting values of $\gamma = 10.4$ (for $\dot{\tau}_1 \leqslant 1$) and $\gamma = 10.9$ (for fast-varying delays) are guaranteed. This result is obtained by solving (50) for $N = 200$ taking $\lambda = 0.95$ and checking the resulting Q at 10 000 frequency points.

Example 2 (*Wu et al. [19]*). *Continuous-time system*. Consider (1) with

$$A_0 = \begin{bmatrix} -0.5 & -2 \\ 1 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.5 & -1 \\ 0 & 0.6 \end{bmatrix}, \quad A_2 = E_2 = 0, \ H = I, \quad E_i = 0.2I, \ \ i = 0, 1. \tag{52}$$

In this example for $\dot{\tau}_1 \leqslant 0.9$ the following stability interval was obtained in [19]: $\tau_1(t) \in [0, 0.242]$. The LMIs of Theorem 3.2 are feasible for all fast-varying delays, where $h_1 = \mu_1 = 0.146$. The LMIs of Theorem 3.2, with $T_i = 0, \ i = 1, 2$, are feasible for smaller values: $h_1 = \mu_1 = 0.133$. Hence, the system is stable for all fast-varying delays in a larger interval: $\tau_1(t) \in [0, 0.292]$. For delays $\dot{\tau}_1 \leqslant 1$ the corresponding interval is $[0, 0.388]$. By applying the frequency domain result of Theorem 2.2 it is found that the system is asymptotically stable for delays in a slightly wider intervals: [0 0.298] in the fast-varying case and [0 0.4] for $\dot{\tau}_1 \leqslant 1$.

In the case of constant delay $\tau_1 = h_1$ we find, by Theorem 3.2 for $T_i = 0, \ i = 1, 2$, that $h_1 \leqslant 0.68$. Non-zero $T_i, \ i = 1, 2$, improve the result and achieve $h_1 \leqslant 0.84$. In the case of known system matrices ($H = 0$), the T_i do not improve the result. Thus, in the fast-varying case, we have $h_1 = \mu_1 = 0.34$.

Example 3 (*Fridman and Shaked [9]*). *Discrete-time system*. We consider (21) where

$$A_0 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.97 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \quad A_1 = 0, \quad B_1 = 0 \quad \text{and} \quad H = 0. \tag{53}$$

For the constant $\tau_2 = h_2$, the maximum value of h_2 for which the asymptotic stability of the system is guaranteed via the descriptor approach is $h_2 = 16$ [2,9]. Using augmentation it is found that the system considered is asymptotically stable for all $h_2 \leqslant 18$. For time-varying τ_2 the stability is guaranteed by [9] for all $\tau_2(k)$ from the following segments : [0, 8], [3, 10], [5, 11], [8, 12] and [10, 13]. Note that the conditions of [20] are not feasible even for $0 \leqslant \tau_2(k) \leqslant 1$.

Treating next the case of uncertain system matrices with $H = \text{diag}\{0.1, 0.02\}$, $E_0 = I_2$ and $E_1 = 0.5I_2$, by [9] the stability is guaranteed for all $\tau_2(k)$ from [0, 3], [1, 4], [3, 5] and [5, 6].

By Theorem 3.1 all the results are equivalent to [9] and the free weighting matrices do not lead to improvement. The same results are obtained also in the frequency domain by Theorem 2.3.

5. Conclusions

Stability and L_2 (l_2)-gain analysis of uncertain linear continuous-time and discrete-time systems with uncertain bounded time-varying delays is studied under the assumption that the nominal values of delays are not equal to zero. A new type of a *moderately varying* delay ($\dot{\tau}(t) \leqslant 1$ for almost all $t \geqslant 0$, or $\tau(k+1) - \tau(k) < 1$ for all $k \geqslant 0$), is revealed. This was treated in the past as a *fast-varying* delay (without constraints on $\dot{\tau}(t)$ or on $\tau(k+1) - \tau(k)$). Input–output approach is applied to stability and is extended to BRL. New BRLs are derived in the case of delayed state and *objective* vector, which allows the solution of the delayed state-feedback H_∞ control problem. Both, frequency domain and time domain criteria are derived. In the time domain, a descriptor type LKF is chosen, which is combined with the free weighting matrices technique of [19]. Note that equivalent LMI conditions may be derived in the fast-varying delay case directly in the time domain by the appropriate construction of the LKF [3,9], where to the same nominal LKF new terms are added by additional terms. However, when the delay is moderately varying, the Lyapunov-based results are significantly improved.

For the first time (frequency domain) BRLs are derived for systems with time-varying delays in the case, where the non-delayed system is not asymptotically stable (but the system becomes asymptotically stable for positive values of the delay). In the time-domain, this case cannot be treated via (simple) descriptor type nominal LKF. In the continuous-time, the discretized Lyapunov functional method [11] can be applied to derive LMI conditions. This subject is currently under study. In the discrete-time, the system can always be augmented in such a way that the non-delayed system becomes asymptotically stable [9].

References

[1] P.-A. Bliman, Extension of Popov absolute stability criterion to non-autonomous systems with delays, Internat. J. Control 73 (2000) 1349–1361.
[2] W.-H. Chen, Z.-H. Guan, X. Lu, Delay-dependent guaranteed cost control for uncertain discrete-time systems with delay, IEE Proc. Control Theory Appl. 150 (2003) 412–416.
[3] E. Fridman, Stability of linear functional differential equations: A new Lyapunov technique, in: Proceedings of MTNS, Leuven, July 2004.
[4] E. Fridman, Stability of systems with uncertain non-small delay: a new 'complete' Lyapunov–Krasovskii, IEEE Trans. Automat. Control, 51(5) (2006) 885–890.
[5] E. Fridman, M.I. Gil', A direct frequency domain approach to stability of linear systems with time-varying delays, in: Proceedings of 13th Mediterranian Conference on Control and Automation, Cyprus, June 2005.
[6] E. Fridman, A. Seuret, J.-P. Richard, Robust sampled-data stabilization of linear systems: an input delay approach, Automatica 40 (2004) 1441–1446.
[7] E. Fridman, U. Shaked, A descriptor system approach to H_∞ control of time-delay systems, IEEE Trans. Automat. Control 47 (2002) 253–270.
[8] E. Fridman, U. Shaked, An improved stabilization method for linear time-delay systems, IEEE Trans. Automat. Control 47 (11) (2002) 1931–1937.
[9] E. Fridman, U. Shaked, Stability and guaranteed cost control of uncertain discrete delay systems, Internat. J. Control 78 (4) (2005) 235–246.
[10] J.K. Hale, S.M.V. Lunel, Introduction to Functional Differential Equations, Springer, New-York, 1993.
[11] K. Gu, V. Kharitonov, J. Chen, Stability of Time-Delay Systems, Birkhauser, Boston, 2003.
[12] Y.-P. Huang, K. Zhou, Robust stability of uncertain time-delay systems, IEEE Trans. Automat. Control 45 (2000) 2169–2173.
[13] C.-Y. Kao, B. Lincoln, Simple stability criteria for systems with time-varying delays, Automatica 40 (2004) 1429–1434.

[14] V. Kharitonov, S. Niculescu, On the stability of linear systems with uncertain delay, IEEE Trans. Automat. Control 48 (2002) 127–132.
[15] V. Kolmanovskii, J.-P. Richard, Stability of some linear systems with delays, IEEE Trans. Automat. Control 44 (1999) 984–989.
[16] X. Li, C. de Souza, Criteria for robust stability and stabilization of uncertain linear systems with state delay, Automatica 33 (1997) 1657–1662.
[17] Y.S. Moon, P. Park, W.H. Kwon, Y.S. Lee, Delay-dependent robust stabilization of uncertain state-delayed systems, Internat. J. Control 74 (2001) 1447–1455.
[18] S.I. Niculescu, Delay Effects on Stability: A Robust Control Approach, Lecture Notes in Control and Information Sciences, vol. 269, Springer, London, 2001.
[19] M. Wu, Y. He, J.-H. She, G.-P. Liu, Delay-dependent criteria for robust stability of time-varying delay systems, Automatica 40 (2004) 1435–1439.
[20] S. Xu, T. Chen, Robust H_∞ control for uncertain discrete-time systems with time-varying delays via exponential output feedback controllers, Systems Control Lett. 51 (2004) 171–183.
[21] J. Zhang, C. Knopse, P. Tsiotras, Stability of time-delay systems: equivalence between Lyapunov and small-gain conditions, IEEE Trans. Automat. Control 46 (2001) 482–486.