Extended state observer-based control for systems with locally Lipschitz uncertainties: LMI-based stability conditions

A. Castillo a,*, P. García a, E. Fridman b, P. Albertos a

a Instituto de Automática e Informática Industrial, Universitat Politècnica de València, 46020 Valencia, Spain
b Department of Electrical Engineering and Systems, Tel Aviv University, Tel Aviv 69978, Israel

Abstract

This paper deals with the closed-loop stability of an extended state observer-based control for systems with locally Lipschitz uncertainties. Novel stability conditions are developed, in terms of Linear Matrix Inequalities (LMI), in order to prove its local/global input-to-state or exponential stability, respectively. The stability conditions of this paper do not require neither the uncertainty to satisfy the so-called matched condition, nor the system to be expressed in the canonical integral chain form. LMI-based optimization methodologies are also developed in order to optimize the presented results.

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1. Introduction

It is known that all industrial systems are affected by external disturbances and/or internal uncertainties that bring adverse effects in the controller performance, degrading its nominal behavior, or even causing instability [1–4]. In order to deal with them, different Disturbance/Uncertainty Estimation and Attenuation (DUEA) techniques have been proposed [1], from which the Extended State Observer (ESO)-based control [5,6] has become of notable interest. The ESO was proposed as a methodology to estimate and compensate for unknown uncertainties in real-time. Since it was proposed, it has been subject to theoretical developments [7–9], it has been successfully applied in different scenarios [10–13] and it has become the main core of the Active Disturbance Rejection Control (ADRC) [5,14].

However, there is a lack of numerical methods to guarantee its closed-loop stability when it is used to compensate for state-dependent uncertainties, such as modeling errors or non-linear terms, which widely appear in practice. Initially, the ESO was developed for a system expressed as a chain of n integrators, with the uncertain-term, \( f(\cdot) \), and the control action, \( u(\cdot) \), satisfying the so-called matched condition [6]. In this scenario, its closed-loop stability was firstly guaranteed under the assumption of global boundedness of \( \frac{d}{dt} f(\cdot) \) [15]; a strong assumption that was also taken in the works that followed [16–19], and was not relaxed until 2009 and 2011 in [20] and [21], respectively.

The main problem with this assumption is that, in general, it cannot be strictly guaranteed a priori if \( f(\cdot) \) is dependent on the system state. Some works need to consider that the system is originally stable [21], or that the dependency of \( f(\cdot) \) on the system state is weak-enough [8,9], in order to establish the boundedness of \( \frac{d}{dt} f(\cdot) \). Other results indicate that the stability of an ESO-based controller can be still guaranteed if the partial derivatives of \( f(\cdot) \) are bounded [14,22,23].

On the other hand, to consider that the system is expressed as a chain of n integrators satisfying the matched condition is a strong restriction that cannot be always considered as pointed out in [1,8,9]. By this reason, it was recently developed in [8,9], a Generalized ESO (GESO) for systems, not necessarily expressed as a chain of integrators, with possibly matched uncertainties. But, due to the technical difficulties, its closed-loop stability was proved under the assumption of boundedness of \( \frac{d}{dt} f(\cdot) \).

This paper presents LMI-based stability conditions for the GESO-based control when system is affected by locally Lipschitz uncertainties. In contrast to previous results, the stability conditions of this paper do not rely on the assumption of global boundedness of \( \frac{d}{dt} f(\cdot) \). Instead, simple local requirements over its partial derivatives are taken. LMI-based optimization methodologies, which can be used to get numerical results of the closed-loop response, are also given and tested in some examples. The results of this paper are also be valid for the conventional ESO, or the linear ADRC, since the so-called matched condition, or the plant being expressed in the canonical integral chain form, are particular cases of the problem being considered.

The rest of the paper is structured as follows. Section 1.1 presents the main notation. Section 2 introduces the problem...
being considered. The main results are given in Section 3, where the stability theorems are developed. In Section 4, different LMI-based optimization methodologies are introduced. Finally, Sections 5 and 6 contain numerical examples and the main conclusions, respectively.

1.1. Notation

Through the paper, \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space with vector norm \( \| \cdot \| \). \( \mathbb{R}^{n \times m} \) denotes the set of \( n \times m \) real matrices. The superscript \( 'T \) denotes matrix transposition, while the notation \( P > 0 \) means that \( P \) is positive definite. The symmetric elements of a symmetric matrix are denoted by \( (\cdot) \), while the maximum and minimum eigenvalues of a given matrix, \( P \), are denoted by \( \lambda_{\max}(P) \) and \( \lambda_{\min}(P) \), respectively.

Let \( \xi = [x^T, e_o]^T \in \mathbb{R}^{2n+q} \), with \( x \in \mathbb{R}^n \) and \( e_o \in \mathbb{R}^{n+q} \). A symmetric matrix \( 0 < P_1 \in \mathbb{R}^{(2n+q) \times (2n+q)} \) defines an ellipsoid in \( \mathbb{R}^{2n+q} \) given by

\[
\mathcal{E}_1 \triangleq \left\{ \xi \in \mathbb{R}^{2n+q} \mid \xi^TP_1\xi \leq k_1, \ k_1 > 0 \right\},
\]

whose projection onto \( \mathcal{E}_1 \), is automatically defined by \( P_1^+ \in \mathbb{R}^n \):

\[
\mathcal{E}^T_1 \triangleq \left\{ x \in \mathbb{R}^n \mid x^TP_1^+x \leq k_1, \ k_1 > 0 \right\}.
\]

2. Problem formulation

Let us consider the following class of non-linear systems:

\[
\begin{align*}
\dot{x} &= Ax + Bu + B_f(x, o(t)), \\
y &= Cx
\end{align*}
\tag{1}
\]

where \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) is the system state; \( u \in \mathbb{R}^m \) is the control action; \( y \in \mathbb{R}^p \) is the measurable output; \( A \in \mathbb{R}^{n \times n} \), \( B_0 \in \mathbb{R}^{n \times m} \), \( B_f \in \mathbb{R}^{n \times q} \) and \( C \in \mathbb{R}^{p \times n} \) are the nominal system matrices; \( o(t) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^r \) is a differentiable time-varying function representing the external disturbances; \( f : A \times \mathbb{R}^r \rightarrow \mathbb{R}^n \) is a possibly non-linear function, differentiable in \( A \times \mathbb{R}^r \), for some domain \( A \subseteq \mathbb{R}^n \) containing the origin.

The function \( f(x, o(t)) \) represents an unknown term that contains the internal uncertainties as well as the external disturbances. The main control purpose is to stabilize system (1), while being actively compensating for \( f(x, o(t)) \). To this purpose, the next control law is considered [8]:

\[
u = K_x \hat{x} + K_f \hat{f},
\tag{2}
\]

where \( K_x \) is a state feedback gain, \( K_f \) is a disturbance feed-forward gain and \( \hat{x}, \hat{f} \) are estimates of \( x \) and \( f(x, o(t)) \), respectively.

To get the estimates \( \hat{x}, \hat{f} \), note that, for all \( x \in A \), system (1) can be equivalently represented in the extended-state form:

\[
\begin{align*}
\dot{\eta} &= \tilde{A}\eta + \tilde{B}_0u + \tilde{B}_ff(x, o(t)), \\
y &= \tilde{C}\eta,
\end{align*}
\tag{3}
\]

where

\[
\begin{align*}
\eta &= [x^T, f^T(x, o(t))]^T, \\
\tilde{A} &= \begin{bmatrix} A & B_f \\ 0 & 0 \end{bmatrix}, \\
\tilde{B}_0 &= \begin{bmatrix} B_0 \\ 0 \end{bmatrix}, \\
\tilde{B}_f &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\tilde{C} &= \begin{bmatrix} C \\ 0 \end{bmatrix}, \\
f(x, o(t)) &= \frac{\partial f}{\partial x}(x, o(t))\hat{x} + \frac{\partial f}{\partial o}(x, o(t))\hat{o}(t).
\end{align*}
\]

This allows to construct the following ESO [8], which provides the desired estimates:

\[
\hat{\eta} = \tilde{A}\hat{\eta} + \tilde{B}_0u + L(y - \tilde{C}\hat{\eta}).
\tag{4}
\]

where \( L \in \mathbb{R}^{(n+q) \times p} \) is the observer gain.

Let us now consider the following assumptions:

**Assumption 1.** The pair \((A, B_a)\) is controllable.

**Assumption 2.** \( \text{rank} \left[ \begin{bmatrix} A & B_f \\ C & 0 \end{bmatrix} \right] = n + q \); and the pair \((A, C)\) is observable.

**Assumption 1** guarantees that \( K_x \) can be found such that \((A + B_0K_x)\) is Hurwitz, while **Assumption 2** guarantees the observability of \((A, C)\) [24].

3. Closed-loop stability

Let us define the observation error as

\[
\begin{align*}
e_o &\triangleq \begin{bmatrix} e_o,x \\ e_o,f \end{bmatrix} \triangleq \begin{bmatrix} x - \hat{x} \\ f(x, o(t)) - \hat{f} \end{bmatrix} = \eta - \hat{\eta},
\end{align*}
\tag{5}
\]

By differentiating (5) and substituting (3) and (4), the observation error dynamics are given by

\[
\begin{align*}
\dot{e}_o &= (\tilde{A} - L\tilde{C})e_o + \tilde{B}_f\hat{f}(x, o(t)).
\end{align*}
\tag{6}
\]

On the other hand, the control action (2) can be rewritten as

\[
u = K_x\hat{x} + K_f\hat{f} = K_x\hat{x} + K_f\hat{f}(x, o(t)) - K_f e_o,x + K_f e_o,f
\nonumber
\]

\[
= K_x\hat{x} + K_f\hat{f}(x, o(t)) + E_0e_o
\]

where \( E_0 \triangleq -[K_x, K_f] \).

Therefore, by substituting (7) into (1) and incorporating (6), the following closed-loop is obtained:

\[
\dot{\xi} = \Phi\xi + \Gamma_f f(x, o(t)) + \Gamma_o\hat{f}(x, o(t)),
\tag{8}
\]

where

\[
\xi = [x^T, e_o^T]^T, \quad \Phi \triangleq \begin{bmatrix} A + B_0K_x & B_0E_0 \\ 0 & L \end{bmatrix}, \\
\Gamma_f \triangleq \begin{bmatrix} B_fK_x + B_f \\ 0 \end{bmatrix}, \quad \Gamma_o \triangleq \begin{bmatrix} 0 \end{bmatrix}.
\]

In the next sections, the local/global input-to-state and exponential stability of the resulting closed-loop (8) is analyzed. To this purpose, let us define a ball \( B_r \triangleq \{ x \in \mathbb{R}^n \mid \|x\| \leq r \} \subseteq A, r > 0 \), and let us state the following assumption, being needed for well-posedness problem formulation:

**Assumption 3.** Under **Assumptions 1–2** and the control law (2), (4), and in the absence of external disturbances (i.e. \( o(t) = 0, \dot{o}(t) = 0 \)), the state \( x^* = 0 \) is the unique equilibrium point of (8) in \( B_r \).

3.1. Input-to-state stability

In order to prove ISS, let us consider that

**Assumption 4.** There exist scalars, \( \beta_f \geq 0, \beta_o \geq 0, \beta_{dx} \geq 0, \beta_{do} \geq 0 \), such that,

\[
\begin{align*}
\|f(x, o(t))\| &\leq \beta_f, \\
\|\dot{o}(t)\| &\leq \beta_o, \\
\left|\frac{\partial f}{\partial x}(x, o(t))\right| &\leq \beta_{dx}, \quad \left|\frac{\partial f}{\partial o}(x, o(t))\right| \leq \beta_{do},
\end{align*}
\]

for all \( x \in B_r, t \geq 0 \).
Assumption 4 states that \( f(x, \omega(t)) \) and its partial derivatives are bounded in \( B_r \times \mathbb{R}_0^+ \) (not necessarily globally bounded). This also implies that \( f(x, \omega(t)) \) is Lipschitz in \( B_r \times \mathbb{R}_0^+ \) and, if the control action is chosen such that \( x(t) \) does not leave \( B_r \), it also ensures the existence and uniqueness of the solution of (1) for all \( r \geq 0 \). Note that, in contrast to previous works \([8],[21]\) whose stability results rely on different assumptions that imply the boundedness of \( f(x, \omega(t)) \), Assumption 4 just guarantees the following worst-case upper bound:

\[
\|f(x, \omega(t))\| \leq \beta_{dx} \beta_{\omega} + \beta_{dx} \dot{x}(t),
\]

for all \( x \in B_r \), and \( t \geq 0 \).

Now, let us recall the next well-known result that is needed for the subsequent analysis:

**Lemma 1 (ISS).** Define \( V(\xi(t)) = \xi(t)^T P \xi(t), \) with \( P > 0 \). Let \( V(\xi(t)) \triangleq V(\xi(t)) \) be absolutely continuous and let \( g_1(\xi(t), t), g_2(\xi(t), t) \) be essentially bounded functions, i.e., \( \|g_1(\xi(t), t)\| \leq \alpha_1, \|g_2(\xi(t), t)\| \leq \alpha_2, \) for all \( t \geq 0 \), with \( \alpha_1 \geq 0, \alpha_2 > 0 \). If there exist \( \delta > 0, \gamma_1 \geq 0, \gamma_2 > 0 \) such that

\[
\dot{V}(\xi(t)) + \delta V(\xi(t)) - \gamma_1 \|g_1(\xi(t), t)\|^2 - \gamma_2 \|g_2(\xi(t), t)\|^2 \leq 0, \quad \forall t \geq 0
\]

then, the ellipsoid \( \mathcal{E} = \left\{ \xi \in \mathbb{R}^{2n+q} \mid \xi^T P \xi \leq \frac{\gamma_1^2 + \gamma_2^2}{\delta} \right\} \), is a passivity and exponentially attractive set, with decay rate \( \delta/2 \), for \( \xi(t) \).

**Proof.** The proof is similar to the one presented in Lemma 4.1 of \([26]\), where the term \( b\|\omega(t)\|^2 \) is substituted by \( \gamma_1 \|g_1(\xi(t), t)\|^2 + \gamma_2 \|g_2(\xi(t), t)\|^2 \).

The above lemma is employed in the next theorem representing conditions for the local ISS of (1) controlled by (2), (4).

**Theorem 1 (Local ISS).** Let \( i \equiv [0, \infty) \). Under Assumptions 1-4, given any \( \delta_i \), let there exist positive definite \( P_i \in \mathbb{R}^{(2n+q) \times (2n+q)} \) and scalars \( \tau_i > 0, \gamma_i \geq 0, \gamma_2 > 0 \), that satisfy the following LMIs:

\[
\begin{bmatrix}
\psi_{ls}^i \quad \psi_{ls}^i
\end{bmatrix}
\begin{bmatrix}
P_i F_i + \tau_i P_i + \Delta r_m & \tau_i P_i + \Delta F_m
-\tau_i F_i + \tau_i P_i & 0 & 0
\end{bmatrix}
\begin{bmatrix}
F_i \quad F_m \quad P_i F_i
\end{bmatrix} \leq 0,
\]

being \( \psi_{ls}^i \triangleq P_i \Phi_i + \Phi_i^T P_i + \delta_i P_i + \tau_i \beta_{dx} \Delta_F^r \Delta_F, \Delta_r \triangleq (B_u K_r) \), and \( \Delta_F \triangleq \{A + B_u K_r, B_u E_0\} \).

Assume additionally that

\[
\tau_i \triangleq \sqrt{\frac{\gamma_1 r_m^2 + \gamma_2 (\beta_{dx} \beta_{\omega})^2}{\lambda(P_i)^2 \delta_i}} < r.
\]

Then, for all states \( \xi \) starting from the initial ellipsoid

\[
\mathcal{E}_0 \triangleq \left\{ \xi \in \mathbb{R}^{2n+q} \mid \xi^T P \xi \leq \frac{\tau_i^2 (\lambda(P_i)^2)}{\delta_i} \right\},
\]

the solution \( x(t) \) of the closed-loop system (1)-(2), (4), does not leave the ball \( B_r \), and it exponentially approaches, with a decay rate \( \delta_i \), to the attractive ellipsoid

\[
\mathcal{E}_\infty \triangleq \left\{ x \in \mathbb{R}^n \mid x^T P \xi \leq \frac{\gamma_1 r_m^2 + \gamma_2 (\beta_{dx} \beta_{\omega})^2}{\delta_\infty} \right\}.
\]

**Proof.** See Appendix B. □

Fig. 1 represents an illustration of the sets that are considered in this problem. The domain \( \mathcal{A} \) is the region of \( \mathbb{R}^q \) in which \( f(x, \omega(t)) \) is continuously differentiable and, therefore, the set in which the extended state representation (3) is equivalent to the original system (1). \( B_r \) is the ball where Assumptions 3-4 hold. Theorem 1 states that, if (9)-(10) hold, then for any \( \xi(0) \triangleq x^T(0), e(0)^T \in \mathcal{E}_0 \), the state \( x(t) \) does not leave \( B_r \) and it approaches to \( \mathcal{E}_\infty \). If \( \xi(0) \notin \mathcal{E}_0 \), convergence is not guaranteed as the state could leave \( B_r \).

Conditions (9)-(10) have a simple meaning. Eq. (9) guarantees that the obtained ellipsoids, \( \mathcal{E}_0, \mathcal{E}_\infty \), are positively invariant and exponentially attractive, i.e. any trajectory \( \xi(t) \) starting inside the ellipsoids is kept inside them for all \( t > 0 \), while trajectories starting outside approach to them. Eq. (10) guarantees that \( \mathcal{E}_0^\perp \) and \( \mathcal{E}_\infty^\perp \) are strictly inside \( B_r \). It is clear that, if (9)-(10) are satisfied, then the stability result of Theorem 1 hold.

Also, Theorem 1 defines two \( L \)-independent sets of parameters, i.e., \( s_i \triangleq \{p_i, \delta_i, \tau_i, \gamma_1, \gamma_2 \} \) with \( i = 0, \infty \), that may satisfy (9)-(10). The parameters in \( s_0 \) define the set of allowable initial states \( \mathcal{E}_0 \), while the parameters in \( s_\infty \) define the terminal ellipsoid \( \mathcal{E}_\infty \). This provides an additional degree of freedom that, once \( \mathcal{E}_0 \) is chosen as large as possible and \( \mathcal{E}_\infty \) as small as possible (as depicted in Fig. 1), in Section 4, different optimization methodologies to address this issue are introduced.

Finally, the next corollaries can be established from Theorem 1. Corollary 1 shows that local ISS is guaranteed for weak-enough uncertainties if \( A + B_u K_r, A - LC \) are Hurwitz. Corollaries 2 and 3 represent simplified stability conditions for the cases of matched uncertainties and \( A = B_r, \mathcal{A} = \mathbb{R}^q \), respectively.

**Corollary 1 (Local ISS for Weak-Enough Uncertainties).** Consider that Assumption 4 is satisfied with a small-enough \( \beta_{dx} \) and \( \beta_{dx} \). Then, if \( A + B_u K_r, A - LC \) are Hurwitz, the solution \( x(t) \) of the closed-loop system (1)-(2), (4) is locally ISS for any \( \xi(0) \) sufficiently close to the origin.

**Proof.** Since \( A + B_u K_r \) and \( A - LC \) are Hurwitz, given any \( \delta_i \), there exist \( P_i \) such that \( P_i \Phi_i + \Phi_i^T P_i + \delta_i P_i < 0 \). Then, by Schur complements, \( \Psi_{ls}^i < 0 \) for large enough \( \gamma_1, \gamma_2, \tau_i \) and small enough \( \beta_{dx} \). On the other hand, for a given \( P_i, \gamma_1, \gamma_2, \tau_i, \) condition (10) is satisfied if \( \beta_{dx} \) and \( \beta_{dx} \) are sufficiently small. Therefore, if the initial state is chosen sufficiently close to the origin, then \( \xi(0) \in \mathcal{E}_0 \) and Theorem 1 is verified. □

**Corollary 2 (Local ISS for Matched Uncertainties).** Consider that \( B_r = B_u, \gamma_1 = -1 \). Given any \( \delta_i \), let there exist positive definite \( P_i \in \mathbb{R}^{(2n+q) \times (2n+q)} \) and scalars \( \tau_i > 0, \gamma_2 \geq 0 \), that satisfy:

\[
\begin{bmatrix}
\psi_{ls}^i \quad \psi_{ls}^i
\end{bmatrix}
\begin{bmatrix}
P_i F_i & P_i F_m
-\tau_i & 0
\end{bmatrix} \leq 0, \quad \tau_i \triangleq \sqrt{\frac{\gamma_1 r_m^2 + \gamma_2 (\beta_{dx} \beta_{\omega})^2}{\lambda(P_i)^2 \delta_i}} < r \quad (11)
\]

Then, for any arbitrarily large \( \beta_{dx} \), and for all states \( \xi \) starting from \( \mathcal{E}_0 = \left\{ \xi \in \mathbb{R}^{2n+q} \mid \xi^T P \xi \leq \frac{\gamma_1 r_m^2 + \gamma_2 (\beta_{dx} \beta_{\omega})^2}{\delta_0} \right\} \), the solution \( x(t) \) of
the closed-loop system (1)–(2), (4), does not leave the ball \( \mathcal{B}_r \) and it exponentially approaches, with a decay rate \( \delta_\infty /2 \), to the attractive ellipsoid \( \mathcal{E}_\infty = \left\{ x \in \mathbb{R}^n \mid x^T P_x x \leq \frac{2\delta_\infty}{\delta_\infty} \frac{P_0 (x_0^0)^2}{\delta_\infty} \right\} \).

**Proof.** If \( B_{r_0} = B_r \) and \( K_f = -I \), then \( \Gamma_f = 0 \) and \( \Delta_f = 0 \). In this case the LMI in (9) is reduced to (11), subject to \( y_{\gamma_1} \geq 0 \). As \( y_{\gamma_1} \) is a free parameter, it can be set \( y_{\gamma_1} = 0 \), which completes the proof. □

**Corollary 3 (Global ISS).** Consider that \( A \equiv B_r \equiv \mathbb{R}^n \), i.e. \( r \to \infty \). Given any \( \delta_\infty \), let there exist a positive definite \( P_\infty \in \mathbb{R}^{(2n+q) \times (2n+q)} \) and scalars \( \tau_0 > 0, y_{\gamma_1} \geq 0, y_{\gamma_\infty} \geq 0 \), that satisfy \( \psi_{\infty,0} \leq 0 \). Then, for any initial state, the solution \( x(t) \) of the closed-loop system (1)–(2), (4), exponentially approaches, with a decay rate \( \delta_\infty /2 \), to the attractive ellipsoid \( \mathcal{E}_\infty \).

**Proof.** Since \( r \to \infty \), condition (10) holds in any case. Also, if there exist \( \delta_\infty, P_\infty, \tau_0, y_{\gamma_1}, y_{\gamma_\infty} \) such that \( \psi_{\infty,0} \leq 0 \), it can be always set \( \delta_\infty \leq \delta_\infty, P_\infty = P_\infty, \tau_0 = \tau_0, y_{\gamma_1} = y_{\gamma_1} \) and \( y_{\gamma_\infty} = y_{\gamma_\infty} \), which satisfy \( \psi_{\infty,0} \leq 0 \). By decreasing \( \delta_\infty \), the set of allowable initial states, \( \mathcal{E}_\infty \), can be made arbitrarily large. □

### 3.2. Exponential stability

In order to prove ES, let us consider that

**Assumption 5.** There exist scalars, \( \beta_{\delta_0} \geq 0, \beta_{\delta_\infty} \geq 0 \), and matrices, \( P_1 \in \mathbb{R}^{n \times n}, P_2 \in \mathbb{R}^{n \times n} \), such that

\[
\| f(x, \omega(t)) \| \leq \| P_1 x \|, \quad \left\| \frac{df}{dx} (x, \omega(t)) \right\| \leq \beta_{\delta_0},
\]

\[
\| \dot{w}(t) \| \leq \beta_{\delta_\infty}, \quad \left\| \frac{df}{d\omega} (x, \omega(t)) \dot{w}(t) \right\| \leq \beta_{\delta_\infty} \| P_2 x \|
\]

for all \( x \in \mathcal{B}_r, t \geq 0 \). △

**Assumption 5** is stronger than 4 as it further restricts the class of uncertainties to those whose terms \( f(x, \omega(t)) \) and \( \frac{df}{d\omega} (x, \omega(t)) \) vanish when the state goes to zero.

In the same way, let us recall the next result being needed for the subsequent analysis:

**Lemma 2 (ES).** Define \( W(\xi(t)) = \xi(t)^T P \xi(t) \), with \( P > 0 \). Let \( \bar{V}(t) \equiv V(\xi(t)) \) be absolutely continuous. If there exists \( \delta > 0 \) such that

\[
\bar{V}(t) + \delta \bar{V}(t) \leq 0, \quad \forall t \geq 0
\]

then \( \bar{V}(t) \) is exponentially stable with decay-rate \( \delta /2 \).

**Proof.** The proof follows from Lemma 1. □

The above lemma is employed to establish the following theorem representing conditions for the local ES.

**Theorem 2 (Local ES).** Let \( \delta \in [0, \infty) \). Under **Assumptions 1–3 and 5**, given any \( \delta_0 \), let there exist positive definite \( P_1 \in \mathbb{R}^{(2n+q) \times (2n+q)} \) and scalars \( \tau_0 \geq 0, \tau_3 \geq 0, \tau_4 \geq 0 \), that satisfy the following LMIs:

\[
\psi_{\infty,0} \triangleq \left[ \begin{array}{c}
\psi_{\infty,0}^{(1)} \\
\psi_{\infty,0}^{(2)} \\
\psi_{\infty,0}^{(3)} \\
\psi_{\infty,0}^{(4)}
\end{array} \right] \leq 0,
\]

being

\[
\psi_{\infty,0}^{(1)} \triangleq P_1 \Gamma_1 + \tau_3 \beta_{\delta_0}^2 \Delta_f \Delta_f^T P_1 P_2, \quad \psi_{\infty,0}^{(2)} \triangleq \psi_{\infty,0}^{(3)} \leq 0,
\]

\[
\psi_{\infty,0}^{(4)} \triangleq P_1 \Gamma_1 + \tau_4 \beta_{\delta_0}^2 \Delta_f \Delta_f^T P_1 P_2 + \tau_4 \beta_{\delta_0}^2 \Delta_f \Delta_f^T P_2
\]

Then, for all states \( \xi \) starting from the initial ellipsoid \( \mathcal{E}_0 \triangleq \left\{ \xi \in \mathbb{R}^{2n+q+1} \mid \xi^T P_0 \xi \leq \lambda(P_0)^2 \right\} \), the solution \( x(t) \) of the closed-loop system (1)–(2), (4), does not leave the ball \( \mathcal{B}_r \) and it is exponentially stable with a decay rate \( \delta_\infty /2 \).

**Proof.** See Appendix B. □

**Theorem 2** also defines two \( i \)-independent sets of parameters, i.e. \( \delta_i \equiv \{ P_i, \delta_i, \tau_i, t_i, \gamma_i \} \) with \( i = 0, \infty \), which may satisfy (13). The set \( \delta_0 \) should be optimized so that the initial ellipsoid is obtained as large as possible, while the set \( \delta_\infty \) should be optimized so that the higher exponential decay rate is obtained. These optimization issues are discussed in Section 4.

**Remark 1.** The same arguments employed in Corollaries 1–3 could be reproduced for the case of ES.

### 4. Optimization issues

In this section, different optimization problems are introduced in order to check, and optimize, the stability conditions presented in Theorems 1 and 2.

#### 4.1. Numerical optimization of Theorem 1

Let \( \delta_i \equiv \{ P_i, \delta_i, \tau_i, \gamma_i \} \), with \( i = 0, \infty \), be the sets of parameters in Theorem 1 that should be optimized. The set \( \delta_0 \) is optimized such that \( \mathcal{E}_\infty \) is minimized [27]. This can be performed by solving

\[
\max_{\gamma_0, \delta_0, \tau_0} \alpha \quad \text{s.t.} \quad \psi_{\infty,0}^{(1)}, \psi_{\infty,0}^{(2)} \leq 0,
\]

\[
\gamma_0 \beta_{\gamma_0}^2 + \gamma_2 \beta_{\gamma_2}^2 \geq \delta_\infty,
\]

\[
\alpha > 0, \quad \delta_\infty > 0, \quad \tau_0 > 0, \quad \gamma_0 > 0, \quad \gamma_2 > 0.
\]

The first constraint assures that condition (9) of Theorem 1 is satisfied. The second constraint assures that \( \delta \geq \alpha \). The third constraint forces that \( \tau_0 \) in (10) takes the form \( \tau_0 = \sqrt{1 - \sqrt{\frac{1}{\alpha}} \cdot \beta_{\gamma_0}^2 + \beta_{\gamma_2}^2} \).

Finally, note that the feasibility of (14) guarantees (9) but not (10), which should be checked with the obtained values in \( \delta_\infty \).

The set \( \delta_0 \) is optimized such that \( \mathcal{E}_0 \) is maximized. This can be done by minimizing the condition number of \( P_0 \) (so that \( \mathcal{E}_0 \) is as similar as possible to a sphere), while forcing condition (10) to be strictly satisfied, i.e. as an equality. In this way, the largest ellipsoid such that its projection strictly fits inside \( \mathcal{B}_r \) is obtained. This can be performed by solving the next optimization problem for different fixed (and decreasing) values of \( \alpha \), until the following equality holds \( \tau_0 = r \).

\[
\min_{\{\gamma_0, \delta_0, \tau_0\}} \gamma_0 \quad \text{s.t.} \quad \psi_{\infty,0}^{(1)}, \psi_{\infty,0}^{(2)} \leq 0,
\]

\[
\gamma_0 \beta_{\gamma_0}^2 + \gamma_2 \beta_{\gamma_2}^2 \geq \delta_0,
\]

\[
\gamma_0 \geq 1, \quad \alpha > 0, \quad \delta_0 > 0, \quad \tau_0 > 0, \quad \gamma_0 > 0, \quad \gamma_2 > 0.
\]
4.2. Numerical optimization of Theorem 2

Let $\bar{s}_i \triangleq \{p_i, \delta_i, \tau_1, \tau_2, \tau_3\}$, with $i = 0, \infty$, be the set of parameters in Theorem 2 that should be optimized. The set $s_\infty$ is optimized such that $\delta_\infty$ is maximized. This can be performed by solving:

$$\max_{\delta_\infty} \delta_\infty, \quad \text{s.t. } \Psi_{\delta_{\infty}}^\infty \leq 0,$$

$$P_{\infty} \geq 0, \quad \delta_\infty > 0, \quad \tau_{1,\infty} > 0, \quad \tau_{2,\infty} > 0, \quad \tau_{3,\infty} > 0. \quad (16)$$

The set $s_0$ is optimized in order to obtain the largest $\tilde{\delta}_0$. This can be done by minimizing the condition number of $P_0$ by solving:

$$\min_{\gamma, \gamma'} \gamma, \quad \text{s.t. } \Psi_{\gamma, \gamma'}^0 \leq 0, \quad P_0 > 0, \quad \gamma I_{2n+q} \geq P_0 \geq I_{2n+q}, \quad \delta_0 > 0, \quad \tau_{10} > 0, \quad \tau_{20} > 0, \quad \tau_{30} > 0. \quad (17)$$

5. Numerical examples

5.1. Example 1

Let us consider the following system:

$$\begin{align*}
\dot{x}(t) &= x(t) + u(t) + \beta x(t) + \omega(t), \quad \forall |x| < R, \\
y &= x,
\end{align*} \quad (18)$$

where $\beta \geq 0$ is an unknown parameter and $\omega(t) = \sin(t)$ represents the external disturbance.

The control law (2), (4) is applied with $L = [41, 400]^T$, $K_x = -2$ and $K_f = -1$. Let us consider $r = 1$. For all $x \in B_r$ and $t \geq 0$, Assumption 4 is satisfied with

$$\|f(x, \omega(t))\| \leq \beta + 1, \quad \|\dot{\omega}(t)\| \leq 1,$$

$$\left| \frac{\partial f}{\partial x}(x, \omega(t)) \right| \leq \beta, \quad \left| \frac{\partial f}{\partial \omega}(x, \omega(t)) \right| \leq 1.$$

Theorem 1 is applied to check the closed-loop ISS. Consider an upper bound of $\beta \leq 2$. Theorem 1 is optimized according to (14)-(15). The resulting $s_0, s_\infty$ are presented in Table 1. Since both $s_0, s_\infty$ satisfy (9)-(10) (concretely (10)), the results of Theorem 1 are valid.

A simulation result is presented in Fig. 2, where system (18), with $\beta = 2$, is controlled under (2), (4). The initial state is set to $x(0) = \tilde{x}(0) = x_0$ and $\tilde{f}(0) = 0$. It is verified that $\xi(0) \in \epsilon_0$ for all $x_0 \leq 0.625$, so the simulation is performed with $x_0 = 0.625$. It can be seen that the simulation results match with the results given by Theorem 1.

On the other hand, Theorem 1 can be also employed to get robustness properties of the closed-loop response against the uncertain parameter $\beta$. It is verified that the closed-loop becomes unstable for $\beta \geq 10.17$. By evaluation of Theorem 1 it is found that none set, $s_i$, satisfying (9), can be found if $\beta > 10.04$, which is remarkable.

5.2. Example 2

Let us consider the system presented in [8], which, in order to satisfy Assumption 3, is conveniently rewritten after translating its equilibrium point up to the origin:

$$\begin{align*}
\dot{x}_1 &= x_2 + f(x), \\
\dot{x}_2 &= -2x_1 - x_2 + u(t), \\
y &= x_1,
\end{align*}$$

being $f(x) = e^{x_1} - 1$, and $\beta > 0$ a constant unknown parameter.

As proposed in [8], the observer gain is set to $L = [41, -65, 125]^T$, the state-feedback gain is set to $K_x = [-4, -4]$ and the disturbance feed-forward gain is set to $K_f = -5$. Let us fix $B_r$ of radius $r = 1$. For all $x \in B_r$ and $t \geq 0$, Assumption 5 is satisfied with

$$\|f(x, \omega(t))\| \leq \beta e^2 + 1, \quad \|\dot{\omega}(t)\| \leq 1,$$

$$\left| \frac{\partial f}{\partial x}(x, \omega(t)) \right| \leq \beta e^{\beta^2}, \quad \left| \frac{\partial f}{\partial \omega}(x, \omega(t)) \right| \leq 1.$$

Theorem 2 is optimized by solving (16)-(17) in order to get robustness properties of the closed-loop against the uncertain parameter $\beta$. It is found that none set of parameters, $\bar{s}_i$, satisfying (13), can be found if $\beta > 0.54$.

Simulation results are depicted in Figs. 3–4. The initial state is set to $x_1(0) = x_0, \dot{x}_1(0) = x_0, x_2(0) = 0, \dot{x}_2(0) = 0$ and $f(0) = 0$. It can be checked that, for $x_0 \leq 0.21$, this initial state belongs to the allowable set of initial states for all $\beta < 0.54$; so it is set $x_0 = 0.21$. Figs. 3–4 depict simulation results for $\beta = 0.54$, $\beta = 0.54$, and $\beta = 1.59$, respectively. The trajectories leave $B_r$ for $\beta > 1.59$ and the closed-loop becomes unstable for $\beta > 2.04$.

6. Conclusions

In this paper, different LMI-based stability conditions for a generalized extended state observer-based control have been developed. The provided stability conditions do not rely on the assumption of global boundedness of the total disturbance derivative. Furthermore, they can be easily optimized by LMI solvers to get numerical properties of the closed-loop behavior. The results presented in this paper are also valid for the conventional Extended State Observer or the Active Disturbance Rejection Control, since both techniques are particular cases of the problem being considered.
for all $r \geq 0$, then, the ellipsoid
\[ \mathcal{E}_i \triangleq \left\{ \xi \in \mathbb{R}^{2n+q} \mid \xi^T P_i \xi \leq \frac{\gamma_i}{\delta_i} \right\} \]
(A.2)

is positively invariant and exponentially attractive.

Hence, the proof is reduced to show how (9)–(10) imply (A.1). The derivative of $V_i$ with $\xi$ substituted by (8) is
\[ \dot{V}_i = \xi^T (P_i \Phi_i + \Phi_i^T P_i) \xi + 2\xi^T P_i \Gamma_i f + 2\xi^T P_i \Gamma_2 d \xi + 2\xi^T P_i \Gamma_3 d f, \]
(A.3)

where, for simplicity, $f \triangleq f(x, \omega(t))$, $d \xi \triangleq \frac{df}{d\xi}(x, \omega(t)) \xi$, and $d \xi \triangleq \frac{df}{d\xi}(x, \omega(t)) \xi(t)$.

Substituting (A.3) into (A.1) leads to
\[ \xi^T (P_i \Phi_i + \Phi_i^T P_i) \xi + 2\xi^T P_i \Gamma_i f + 2\xi^T P_i \Gamma_2 d \xi + 2\xi^T P_i \Gamma_3 d f - \gamma_i \xi^T f - \gamma_i d \xi^T f - d \xi^T d \xi \leq 0, \]

being expressed as $\phi^T (\psi^i_{\xi_0,0}) \phi \leq 0$, $\phi \triangleq [\xi^T, f, d \xi, d f, d f]^T$.

\[ \psi^i_{\xi_0,0} \triangleq \begin{bmatrix} P_i \Phi_i + \Phi_i^T P_i & \beta P_i \Gamma_1 & P_i \Gamma_2 & P_i \Gamma_3 & \end{bmatrix} \begin{bmatrix} -\gamma_i & 0 & 0 & 0 \\ 0 & -\gamma_i & 0 & 0 \\ 0 & 0 & -\gamma_i & 0 \\ 0 & 0 & 0 & -\gamma_i \end{bmatrix} \begin{bmatrix} \sigma \xi \xi_0 & \xi \xi_0 & f \xi \xi_0 & f \xi \xi_0 \end{bmatrix} \]

So, if $\psi^i_{\xi_0,0} \preceq 0$, then (A.1) is satisfied. The next step follows by the application of the $S$-procedure in the term $d \xi$. It is known that
\[ d \xi = \frac{df}{d\xi}(x, \omega(t)) [A \xi + B_j u + B_j f] \]
\[ = \frac{df}{d\xi}(x, \omega(t)) [A \xi + B_j K_0 (x - c_{\omega_0}) + K_0 (f - c_{\omega_0})] + B_j f \]
\[ = \frac{df}{d\xi}(x, \omega(t)) [A \xi + \Delta \xi + \Delta f], \]

with $\Delta \xi \triangleq [(A + B_j K_0), B_j e_{\omega_1}, \Delta \xi \triangleq (B_j K_0 + B_j) \dot{f}, \Delta f \triangleq (B_j K_0 + B_j) f$.

Hence, by Assumption 4 and for all $x \in \mathcal{B}_i$, the following upper bound can be established:
\[ d \xi^T d \xi \leq \beta_{\Delta \xi} [\xi^T A \xi \xi + 2\xi^T A \xi \Delta \xi + f^T A \xi \Delta \xi + f^T A \xi \Delta f], \]
(A.4)

which can be written as $\phi^T (\psi^i_{\xi_0,1}) \phi \preceq 0$, with
\[ \psi^i_{\xi_0,1} \triangleq \begin{bmatrix} -\beta_{\Delta \xi} A \xi & -\beta_{\Delta \xi} A \xi & 0 & 0 \\ 0 & -\beta_{\Delta \xi} A \xi & 0 & 0 \\ 0 & 0 & -\beta_{\Delta \xi} A \xi & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma \xi \xi_0 & \xi \xi_0 & f \xi \xi_0 & f \xi \xi_0 \end{bmatrix} \]

The knowledge of $\psi^i_{\xi_0,1} \preceq 0$ implies that $\psi^i_{\xi_0,0} \preceq 0$ if there exist $\tau > 0$ such that $\phi^T (\psi^i_{\xi_0,\tau}) \phi \preceq 0$. This holds if $\phi^T (\psi^i_{\xi_0,0} - \tau_{\psi^i_{\xi_0,\tau}}) \phi \preceq 0$, leading to (9).

So, if (9) holds, then the ellipsoid (A.2) is attractive and positively invariant. Condition (10) follows from a short analysis of the sets that are being considered in this problem. Lemma 1 considers that $g_1(x, t) = f(x, \omega(t))$ and $g_2(x, t) = \frac{df}{d\omega}(x, \omega(t)) \omega(t)$ are bounded. However, by Assumption 4, this bound can be only established in $\mathcal{B}_i$. So it must be required that $\omega(t)$ lies inside $\mathcal{B}_i$ for all $t \geq 0$.

As mentioned in Section 1.1, the ellipsoid (A.2) has a projection onto $\mathbb{R}^n$ given by
\[ \xi^T \triangleq \left\{ x \in \mathbb{R}^n \mid x^T P_i x \leq \frac{\gamma_i}{\delta_i} \right\}. \]

Therefore, it must be imposed that $\xi^T \subseteq \mathcal{B}_i$, which is satisfied if (10) holds. Finally, the theorem follows by defining two independent solutions, given by $i \in [0, \infty)$, such that both satisfy (9)–(10). In this case, two attractive ellipsoids, i.e, $\mathcal{E}_0, \mathcal{E}_\infty$, are obtained. If the initial state is restricted to be inside $\mathcal{E}_0$, then $\omega(t)$ approaches $\mathcal{E}_\infty$ with an exponential rate $\delta_{\infty}/2$. 

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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**Appendix A. Proof of Theorem 1**

By Lemma 1, consider the Lyapunov function $V_i(\xi) = \xi^T P_i \xi$, $P_i > 0$. Let us set $g_1(x, t) = f(x, \omega(t))$, $g_2(x, t) = \frac{df}{d\omega}(x, \omega(t)) \omega(t)$ with $\alpha_1 = \beta_f$ and $\alpha_2 = \beta_{\Delta \omega} \Delta \omega$. If there exist $\gamma_i > 0$, $\gamma_i > 0$, $\Delta \xi > 0$, then
\[ \dot{V}_i + \dot{V}_i + \gamma_i \|f(x, \omega(t))\|^2 - \gamma_i \|\frac{df}{d\omega}(x, \omega(t)) \omega(t)\|^2 \leq 0, \]
(A.1)
Appendix B. Proof of Theorem 2

By Lemma 2, let us consider the Lyapunov function $V_l = \xi^T P_l \xi$, where the derivative $\dot{V}_l$, which is given by (A.3), is substituted into (12) and expressed as $\dot{V}_l = \phi^T (\psi_{\theta,0}) \phi \leq 0$, with $\phi = [\xi^T, f, d_f, df]$. Then

$$\psi_{\theta,0} = \begin{bmatrix}
P \phi_c + \phi_c^T \delta P \phi_c & P F_1 & P F_2 & P F_2^T \\
(\phi) & 0 & 0 & 0 \\
(\phi) & (\phi) & 0 & 0 \\
(\phi) & (\phi) & (\phi) & 0 \\
\end{bmatrix}$$

The term $df$ satisfies the inequality (A.4), which leads to $\phi^T (\psi_{\theta,1}) \phi \leq 0$, being $\psi_{\theta,1}$ defined in (A.5). The terms $f$ and $df$ satisfy the inequalities in Assumption 5 leading to $\phi^T (\psi_{\theta,2}) \phi \leq 0$, with

$$\psi_{\theta,1} = \begin{bmatrix}
-H^T \Pi_1^T \Pi_1 H & 0 & 0 & 0 \\
(\phi) & 1 & 0 & 0 \\
(\phi) & (\phi) & 0 & 0 \\
(\phi) & (\phi) & (\phi) & 0 \\
\end{bmatrix}$$

$$\psi_{\theta,2} = \begin{bmatrix}
-\beta_2^2 H^T \Pi_2^T \Pi_2 H & 0 & 0 & 0 \\
(\phi) & 0 & 0 & 0 \\
(\phi) & (\phi) & 0 & 0 \\
(\phi) & (\phi) & (\phi) & 1 \\
\end{bmatrix}$$

where $\Pi = [\Pi_1, \Pi_2, \Pi_3]$ is defined so that $x = H \xi$.

By the $\mathcal{L}_2$-procedure, the knowledge of $\psi_{\theta,0} \leq 0$, $\psi_{\theta,1} \leq 0$, $\psi_{\theta,2} \leq 0$, implies that $\psi_{\theta,0} \leq 0$ if there exist $\tau_1 > 0$, $\tau_2 > 0$ and $\tau_3 > 0$ such that

$$\dot{\phi}^T \psi_{\theta,0} \phi \leq \tau_1 \phi^T \psi_{\theta,1} \phi + \tau_2 \phi^T \psi_{\theta,1} \phi + \tau_3 \phi^T \psi_{\theta,2} \phi \leq 0,$$

leading to (13).

Finally, similarly to Theorem 1, the proof follows by defining two independent solutions that satisfy (13); while the set of allowable initial states is defined so that it strictly fits inside $\mathcal{B}_k$.

References


