Systems & Control Letters 136 (2020) 104617

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Network-based control of a semilinear damped beam equation under point and pointlike measurements[☆]

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A R T I C L E I N F O

Article history:
Received 1 May 2019
Received in revised form 25 November 2019
Accepted 22 December 2019
Available online xxxx

Keywords:
Distributed parameter systems
Sampled-data control
Delay
Point measurements
Lyapunov–Krasovskii method

A B S T R A C T

We consider distributed static output-feedback stabilization of a damped semilinear beam equation. Distributed in space measurements are either point or pointlike, where a pointlike measurement is the state value averaged on a small subdomain. Network-based implementation of the control law which enters the PDE through shape functions is studied, where variable sampling intervals and transmission delays are taken into account. Our main objective is to derive and compare the results under both types of measurements in terms of the upper bound on the delays and sampling intervals that preserve the stability for the same (as small as possible) number of sensors/actuators. For locally Lipschitz nonlinearities, regional stabilization is achieved. Numerical results show that the pointlike measurements lead to larger delays and samplings, provided the subdomains, where these measurements are averaged, are not too small.

1. Introduction

The problem of distributed in-domain sensing and actuation for systems governed by the beam equation is related to intelligent materials and structures, in which the mechanical structure is equipped with sensors and actuators in order to achieve a desired performance [1,2]. Recently, intelligent structures have been drawing increasing attention due to their wide range of applications in aerospace, civil structures, medical devices etc. The damped beam equation as a classical Petrovsky type system was studied in [3,4]. Boundary stabilization of the undamped Euler–Bernoulli beam with arbitrary decay rate was addressed in [5].

Data sampling and delays are unavoidable in modern control systems that employ digital technology and in networked control systems (NCSs), where the plant is controlled via communication network. General results on sampled-data control of linear time-invariant partial differential equations (PDEs) were presented in [6]. A model-reduction-based approach to distributed sampled-data control of parabolic systems was suggested in [7,8]. For linear systems of conservation laws, event-triggered boundary control was suggested in [9]. Sampled-data boundary controllers for linear transport and heat equations were introduced in [10] and [11].

Distributed sampled-data and/or delayed control of parabolic PDEs has been studied in [12–15,15–17]. In-domain (point or averaged) sampled-data measurements of the state together with control actions applied through characteristic functions have been considered, and sufficient conditions for the exponential convergence and induced L_2-gain in terms of LMIs have been derived by using the time-delay approach to sampled-data control and appropriate Lyapunov–Krasovskii functionals. All the above results on distributed sampled-data and delayed control were devoted to parabolic systems. Recently, sampled-data observers for the damped semilinear wave equation under the point measurements on the interval were introduced in [18].

For the non-delay case, the continuous-time distributed control of 1D heat equation under the pointlike measurements, i.e. averaged over small subdomains measurements of the state, was suggested in [19] and extended to the sampled-data case in [20]. Distributed sampled-data observers and controllers for 2D semilinear heat equation under the pointlike measurements were proposed in [21]. Although several control methods under various measurements (point, pointlike or averaged over the subdomains that cover all the domain) have been considered for the PDEs, none of them have compared performance under point versus pointlike measurements.

In the present paper, we study, for the first time, stabilization of 1D semilinear damped beam equation under point or pointlike measurements. Network-based implementation of the control law which enters the PDE through the shape functions is studied, where variable sampling intervals and transmission delays are taken into account. Our main objective is to compare the results

☆ This work was supported by Israel Science Foundation (Grant No. 673/19), by C. and H. Manderman Chair at Tel Aviv University, Israel.

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https://doi.org/10.1016/j.sysconle.2019.104617
0167-6911/

under both types of measurements (point and pointlike) in terms of the upper bound on the delays and sampling intervals that preserve the stability for the same (as small as possible) number of sensors/actuators. Under the pointlike measurements, the Linear Matrix Inequalities (LMIs) are derived via the direct Lyapunov–Krasovskii approach (which is applicable also to H_∞ control [12,13,21]), whereas under the point measurements, the Lyapunov–Krasovskii approach is combined with Halanay's inequality (as in [14]).

For locally Lipschitz nonlinearities, regional stabilization is achieved and we find a bound on domain of attraction. Sufficient conditions in terms of LMIs for the exponential stability of the closed-loop system are provided by using the time-delay approach to networked control systems. Numerical results show that the pointlike measurements lead to larger delays and samplings provided the subdomains, where these measurements are averaged, are not too small.

Some preliminary results for network-based control of damped beam equation with globally Lipschitz nonlinearities were presented in [22].

Notation Throughout the paper the notation $P > 0$ with $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of a symmetric matrix will be denoted by $*$. Functions, continuous (continuously differentiable) in all arguments, are referred to as of class $\mathcal{C}$ (of class $\mathcal{C}^1$). $L^2(0,\pi)$ is the Hilbert space of square integrable functions $z(\xi)$, $\xi \in [0,\pi]$ with the corresponding norm $\|z\|_{L^2} = \sqrt{\int_0^\pi z^2(\xi)d\xi}$. $\mathscr{H}^1(0,\pi)$ is the Sobolev space of absolutely continuous scalar functions $z:[0,\pi]\to\mathbb{R}$ with $\frac{dz}{d\xi} \in L^2(0,\pi)$. $\mathscr{H}^2(0,\pi)$ is the Sobolev space of scalar functions $z:[0,\pi]\to\mathbb{R}$ with absolutely continuous $\frac{dz}{d\xi}$ and with $\frac{d^2z}{d\xi^2} \in L^2(0,\pi)$.

2. Problem formulation and preliminaries

Consider the semilinear, damped beam equation

$$z_{tt}(x,t) = -z_{xxxx}(x,t) - \beta z_t(x,t) + \rho\big(z(x,t),x,t\big) + \sum_{j=1}^{N} \chi_j(x) u_{jk}(t), \quad t \geq t_0,\ x \in (0,\pi), \tag{2.1}$$

under the boundary conditions

$$\begin{cases} z(0,t) = z_x(0,t) = 0, \\ z(\pi,t) = z_x(\pi,t) = 0, \end{cases} \tag{2.2}$$

or

$$\begin{cases} z(0,t) = z_{xx}(0,t) = 0, \\ z_x(\pi,t) = z_{xxx}(\pi,t) = 0, \end{cases} \tag{2.3}$$

and the initial conditions

$$\begin{cases} z(x,0) = z_1(x), \\ z_t(x,0) = z_2(x). \end{cases} \tag{2.4}$$

Here $z(x,t) \in \mathbb{R}$ is the state (modeling the beam height position), $u_{jk}(t)$ is the control input, $\beta > 0$ is the damping coefficient (the damping is proportional to an angle of inclination of the center of the beam). It is assumed that ρ is of class $\mathcal{C}^2$ and satisfies $\rho(0,x,t) \equiv 0$.

In Section 3 we consider the case of globally Lipschitz in z nonlinearity ρ. We assume that

$$\phi_m \leq \rho_z(z,x,t) \leq \phi_M \quad \forall z,x,t. \tag{2.5}$$

Then

$$\rho(z,x,t) = \phi(z,x,t)z,\ \phi = \int_0^1 \rho_z(\theta z,x,t)d\theta. \tag{2.6}$$

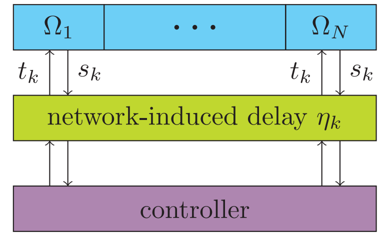

Fig. 1. Networked system.

The later implies

$$\phi_m \leq \phi \leq \phi_M \tag{2.7}$$

for all z,x,t, where ϕ_m and ϕ_M are known bounds. In Section 4 we consider the case of locally Lipschitz ρ, where (2.7) holds for $|z| \leq D$, where D is some constant.

Boundary conditions (2.2) correspond to a beam with both ends clamped, whereas boundary conditions (2.3) correspond to a pinned (hinged) end at $x=0$ and a guided (sliding) end at $x=\pi$.

Let the points $0 = x_0 < x_1 < \cdots < x_N = \pi$ divide the domain $\Omega = [0,\pi]$ into N sampling domains $[x_{j-1},x_j) = \Omega_j$, $j = 1,\ldots,N$, with the characteristic functions

$$\chi_j(x) = \begin{cases} 1, & x \in \Omega_j, \\ 0, & x \notin \Omega_j, \end{cases} \quad j = 1,\ldots,N. \tag{2.8}$$

The size of the domains may be variable, but bounded

$$x_j - x_{j-1} \triangleq \Delta_j \leq \Delta.$$

A sensor and an actuator are placed in each domain Ω_j.

We consider a network-based control, where measurements from sensors to controller and control signals from the controller to actuators are transmitted through communication network (see Fig. 1). Let

$$0 = s_0 < s_1 < \cdots < s_k < \ldots, \quad \lim s_k = \infty,$$

be the sampling time instants on the side of sensors. The sampling intervals in time may be variable, but have a known bound $0 \leq s_{k+1} - s_k \leq MATI$, $k = 0,1,\ldots$, where MATI is Maximum Allowable Transmission Interval. The total round-trip transmission time-varying delays from sensors to actuators is $\eta_k \leq MAD$, where MAD is the Maximum Allowable Delay. The updating time of the actuators is $t_k = s_k + \eta_k$ and we assume that $t_k < t_{k+1}$, $k = 0,1,\ldots$.

We consider two types of measurements for all $j = 1,\ldots,N$, $k = 0,1,2,\ldots$: *pointlike* measurements

$$y_j = \int_{\Omega_j} c_j(\xi) z(\xi, s_k) d\xi, \tag{2.9}$$

$$c_j(x) = \begin{cases} \varepsilon_j^{-1}, & x \in \Omega_j^y \\ 0, & elsewhere \end{cases} \tag{2.10}$$

where Ω_j^y is subinterval of Ω_j of the length ε_j (see Fig. 2) and $\varepsilon_j \leq \varepsilon$ with some $\varepsilon \in (0,\Delta)$, or *point* measurements in the middle of Ω_j

$$y_j = z(\hat{x}_j, s_k), \quad \hat{x}_j = \frac{x_{j-1}+x_j}{2}. \tag{2.11}$$

For less conservative results and adequate comparison, both measurements are centered with respect to Ω_j.

Remark 2.1. Note that in many cases a sensor cannot measure exactly in one point: the measuring device relies on some physical phenomenon and, in fact, the sensor measures an average over a certain region occupied by the measuring device.

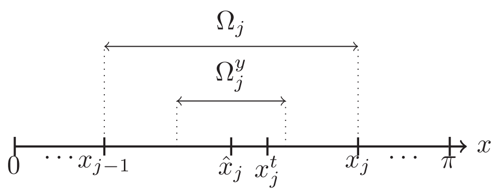

Fig. 2. Subdomains Ω_j^y of pointlike measurements.

It is known that the open-loop system (2.1) under the boundary conditions (2.2) or (2.3) is stable due to the damping term, and becomes unstable for large enough ϕ_M. Our aim is to design a controller that exponentially stabilizes the system, and compare the performance under the point and pointlike measurements. We suggest a static output-feedback controller of the form

$$u_j(t) = -Ky_j(t_k - \eta_k), \quad t \in [t_k, t_{k+1}) \tag{2.12}$$

based on the measurements y_j given by (2.9) or (2.11). This controller can be implemented by zero-order hold devices. By using the time-delay approach to networked control systems [23,24], the resulting control input can be modeled as a delayed one:

$$u_j(t) = -Ky_j(t - \tau(t)), \quad t \geq t_0,$$

where $\tau(t) = t - t_k + \eta_k,\ t \in [t_k, t_{k+1})$ and

$$\tau(t) \leq t_{k+1} - t_k + \eta_k \leq MATI + MAD \triangleq \tau_M.$$

For brevity, later the time argument of τ will be omitted.

2.1. Well-posedness of the closed-loop system

We prove the well-posedness of the closed-loop system (2.1), (2.12) under the boundary conditions (2.2) (for boundary conditions (2.3), the well-posedness can be established similarly). We use the step method. For $t \in [t_0, t_1)$, the closed-loop system can be represented as an abstract differential equation by defining the state $\zeta(t) = [\zeta_0(t)\ \zeta_1(t)]^T = [z(t)\ z_t(t)]^T$ and the operators

$$\mathscr{A} = \begin{bmatrix} 0 & I \\ -\frac{\partial^4}{\partial x^4} & -\beta I \end{bmatrix}, \quad F(\zeta, t) = \begin{bmatrix} 0 \\ F_1(\zeta_0, t) \end{bmatrix}. \tag{2.13}$$

Here $F_1 : \mathscr{H}^2(0, \pi) \times [t_0, \infty) \to L^2(0, \pi)$ is defined as

$$F_1(\zeta_0, t) = \rho(\zeta_0, \cdot, t) - K \sum_{j=1}^{N} \chi_j(x) y_j(t_0 - \eta_0)$$

so that it is continuous in t for each $\zeta_0 \in \mathscr{H}^2(0, \pi)$. The resulting differential equation

$$\dot{\zeta}(t) = \mathscr{A}\zeta(t) + F(\zeta(t), t), \quad t \geq t_0 \tag{2.14}$$

is considered in the Hilbert space $\mathscr{H} = \mathscr{H}_0^2 \times L^2(0, \pi)$, where

$$\mathscr{H}_0^2 = \left\{ \zeta_0 \in \mathscr{H}^2(0, \pi) \middle| \zeta_0(0) = \zeta_{0x}(0) = \zeta_0(\pi) = \zeta_{0x}(\pi) = 0 \right\},$$

and the induced norm $\|\zeta\|_{\mathscr{H}}^2 = \|\zeta_{0xx}\|_{L^2}^2 + \|\zeta_1\|_{L^2}^2$. The operator $\mathscr{A}$ with the dense domain

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{bmatrix} \zeta_0 \\ \zeta_1 \end{bmatrix} \in \mathscr{H}^4(0, \pi) \bigcap \mathscr{H}_0^2 \times \mathscr{H}_0^2 \right\}$$

generates an exponentially stable semigroup [25].

Consider first the case of uniformly bounded ρ_z, where F is continuous in t and globally Lipschitz in ζ

$$\|F_1(\zeta_0, t) - F_1(\bar{\zeta}_0, t)\|_{L^2} \leq L \|\zeta_0 - \bar{\zeta}_0\|_{L^2} \tag{2.15}$$

with some constant $L > 0$ for $\zeta_0, \bar{\zeta}_0 \in \mathscr{H}^2(0, \pi), t \in [t_0, t_1]$. Then by Theorem 6.1.2 of [26], there exists a unique mild solution $\zeta \in \mathcal{C}([t_0, t_1); \mathscr{H})$ of (2.14) initialized by

$$\zeta_0(t_0) = z_0 \in \mathscr{H}_L^2, \quad \zeta_1(t_0) = z_1 \in L^2(0, \pi). \tag{2.16}$$

We note that $F : \mathscr{H} \times [t_0, \infty) \to \mathscr{H}$ is continuously differentiable. If $\zeta(t_0) \in \mathscr{D}(\mathscr{A})$, then this mild solution is in $\mathcal{C}^1([t_0, t_1); \mathscr{H})$ and it is a classical solution of (2.1), (2.3) with $\zeta(t) \in \mathscr{D}(\mathscr{A})$ (see Theorem 6.1.5 of [26]). By using the same argument step by step on $[t_1, t_2], [t_2, t_3], \ldots$, we obtain the well-posedness of the closed-loop system for all $t \geq t_0$.

In the case of locally Lipschitz ρ with locally Lipschitz condition (2.15), well-posedness follows from Theorems 6.1.4 and 6.1.5 of [26]). Note that if the solution admits a priori estimate, then the solution exists on the entire interval $[t_k, t_{k+1}]$ (see Theorems 6.1.4 and 6.1.5 of [26] and [27]). The a priori estimates on the solutions starting from the domain of attraction will be guaranteed by the regional stability conditions of Theorem 4.1.

3. Stabilization of damped beam equations: globally Lipschitz nonlinearities

Throughout this section we assume that ρ is globally Lipschitz in z, i.e. that (2.7) holds for all z, x, t.

3.1. Continuous-time exponential stabilization

In the sequel, we will present some preliminary results in the non-delayed continuous-time case. Namely, we will design exponentially stabilizing controllers for (2.1), based on the pointlike measurements (2.9), where s_k is changed by t,

$$u_j(t) = -K \int_{\Omega_j} c_j(\xi) z(\xi, t) d\xi \tag{3.1}$$

or under the point measurements

$$u_j(t) = -Kz(\hat{x}_j, t) \tag{3.2}$$

with some controller gain $K > 0$.

By employing the mean-value theorem as suggested in [19], we present the controller (3.1) under the pointlike measurements in the form

$$u_j(t) = -K \int_{\Omega_j} c_j(\xi) z(\xi, t) d\xi = -Kz(x_j^t, t), \tag{3.3}$$

where $x_j^t \in \Omega_j^y$ is some point (see Fig. 2). Then, both controllers (3.1) and (3.2) can be represented as a state feedback, and disturbance given by

$$u_j = -K[z(x, t) - f_j], \tag{3.4}$$

where

$$f_j = z(x, t) - z(x_j^t, t) \tag{3.5}$$

for pointlike-based controller (3.1), and

$$f_j = z(x, t) - z(\hat{x}_j, t) \tag{3.6}$$

for point-based controller (3.2).

Then the closed-loop system under both controllers has the form

$$z_{tt} = -z_{xxxx} - \beta z_t + [\phi(z, x, t) - K]z + K \sum_{j=1}^{N} \chi_j f_j, \ t \geq t_0, \tag{3.7}$$

where f_j are given by (3.5) or (3.6), under the boundary conditions (2.2) or (2.3). By applying arguments of the previous section, we find that the closed-loop system has a unique mild (classical) solution initialized by $[z(\cdot,t_0),z_t(\cdot,t_0)]^T\in\mathscr{H}$ ($[z(\cdot,t_0),z_t(\cdot,t_0)]^T\in\mathscr{D}(\mathscr{A})$).

For the stability analysis of the damped beam equation (3.7) we employ the following Lyapunov function:

$$V_0(t)=p_3\int_0^\pi z_{xx}^2dx+\int_0^\pi[z\ z_t]P[z\ z_t]^Tdx, \tag{3.8}$$

where $P=\begin{bmatrix}p_1 & p_2\\ * & p_3\end{bmatrix}$ and $p_3>0$. To guarantee the positivity of V_0, we apply Wirtinger's inequality (A.1) with $\sigma=1$ for boundary conditions (2.2) and $\sigma=\frac{1}{4}$ for boundary conditions (2.3). Then the positivity of V_0 is guaranteed if

$$P_0\triangleq\begin{bmatrix}p_1+\sigma^2p_3 & p_2\\ * & p_3\end{bmatrix}>0, \tag{3.9}$$

since

$$V_0(t)\geq\int_0^\pi[z\ z_t]P_0[z\ z_t]^Tdx.$$

Proposition 3.1. *Consider the closed-loop system (3.7) under the boundary conditions (2.2) or (2.3) with the bounds ϕ_m,ϕ_M, Δ and $\varepsilon>0$ for pointlike and $\varepsilon=0$ for point measurements. Let $\sigma=1$ and $\sigma=\frac{1}{4}$ for boundary conditions (2.2) and (2.3) respectively. Given $K>\phi_M-\sigma^2$ and a positive scalar δ, let there exist scalars p_1,p_2,p_3 and $\lambda_1\geq0$ that satisfy the LMIs: (3.9),*

$$\delta p_3-p_2\leq0, \tag{3.10}$$

and

$$\Xi|_{\phi=\phi_m,\phi_M}\leq0, \tag{3.11}$$

$$\Xi\triangleq\begin{bmatrix}2p_2(\phi-K)+2\delta p_1-\sigma\lambda_1 & \psi_{12} & Kp_2\\ * & 2p_2+2p_3(\delta-\beta) & Kp_3\\ * & * & \psi_{33}\end{bmatrix}\leq0,$$

$$\psi_{12}=p_1+p_2(2\delta-\beta)+p_3(\phi-K),$$
$$\psi_{33}=-[2c(p_2-\delta p_3)-\lambda_1]\pi^2/(\Delta+\varepsilon)^2.$$

Then the closed-loop system is exponentially stable with a decay rate δ, meaning that the following inequality holds:

$$V_0(t)\leq\exp(-2\delta t)V_0(t_0),\quad t\geq t_0.$$

The stability conditions under the pointlike measurements for $\varepsilon\to0$ coincide with the conditions under the point measurements. In the linear case with $\phi\equiv\phi_M$, the gain

$$K=\phi_M-\sigma^2+\beta^2/4 \tag{3.12}$$

leads to the maximal achievable decay rate $\delta=0.5\beta$ for $\Delta\to0$.

Proof. Differentiating (3.8) along (3.7) we obtain

$$\begin{aligned}\dot V_0(t)&=2\int_0^\pi\Big[p_3z_{xx}z_{xxt}+p_1zz_t+p_2z_t^2+(p_2z+p_3z_t)z_{tt}\Big]dx\\&=2\int_0^\pi\Big[p_3z_{xx}z_{xxt}+p_1zz_t+p_2z_t^2\\&\quad+(p_2z+p_3z_t)\big((\phi-K)z-z_{xxxx}-\beta z_t\big)\Big]dx\\&\quad+2K\sum_{j=1}^N\int_{\Omega_j}f_j(p_2z+p_3z_t)dx\end{aligned} \tag{3.13}$$

Integrating by parts twice, and substituting boundary conditions, we have

$$-p_3\int_0^\pi z_tz_{xxxx}dx=-p_3\int_0^\pi z_{xxt}z_{xx}dx,$$
$$-p_2\int_0^\pi zz_{xxxx}dx=-p_2\int_0^\pi z_{xx}^2dx.$$

Then

$$\dot V_0+2\delta V_0\leq2(\delta p_3-p_2)\int_0^\pi z_{xx}^2dx+\int_0^\pi[z\ z_t]C[z\ z_t]^Tdx$$
$$+\ 2K\sum_{j=1}^N\int_{\Omega_j}f_j(p_2z+p_3z_t)dx \tag{3.14}$$

where

$$C=\begin{bmatrix}2p_2(\phi-K)+2\delta p_1 & p_1+p_2(2\delta-\beta)+p_3(\phi-K)\\ * & 2p_2+2p_3(\delta-\beta)\end{bmatrix}.$$

Taking into account (3.10), due to Wirtinger's inequality

$$2(\delta p_3-p_2)\int_0^\pi z_{xx}^2dx\leq2\sigma(\delta p_3-p_2)\int_0^\pi z_x^2dx. \tag{3.15}$$

For the pointlike measurements, Wirtinger's inequality (A.1) with $(b-a)=(\Delta_j+\varepsilon_j)/2$ and $\sigma=1/4$ is employed

$$\begin{aligned}\int_0^\pi z_x^2dx&=\sum_{j=1}^N\Big[\int_{x_{j-1}}^{x_j^t}z_x^2dx+\int_{x_j^t}^{x_j}z_x^2dx\Big]\\&\geq\tfrac{\pi^2}{(\Delta_j+\varepsilon_j)^2}\sum_{j=1}^N\Big[\int_{x_j}^{x_j^t}[z(x,t)-z(x_j^t,t)]^2dx\\&\quad+\int_{x_j^t}^{x_j}[z(x,t)-z(x_j^t,t)]^2dx\Big]\\&\geq\tfrac{\pi^2}{(\Delta+\varepsilon)^2}\sum_{j=1}^N\int_{\Omega_j}[z(x,t)-z(x_j^t,t)]^2dx\\&=\tfrac{\pi^2}{(\Delta+\varepsilon)^2}\sum_{j=1}^N\int_{\Omega_j}(f_j)^2dx.\end{aligned} \tag{3.16}$$

For the point measurements, inequality (3.16) holds with x_j^t replaced by $\hat x_j$ and $\varepsilon=0$.

By S-procedure, we add the non-negative term (due to Wirtinger's inequality (A.1))

$$\lambda_1\int_0^\pi\left(z_x^2-\sigma z^2\right)dx\geq0,\ \ \lambda_1\geq0 \tag{3.17}$$

to $\dot V_0+2\delta V_0$. Denote $\eta_\Xi=[z\quad z_t\quad f_j]^T$. Then, under (3.11)

$$\begin{aligned}\dot V_0+2\delta V_0&\leq-[2\sigma(p_2-\delta p_3)-\lambda_1]\frac{\pi^2}{(\Delta+\varepsilon)^2}\sum_{j=1}^N\int_{\Omega_j}f_j^2dx\\&+\int_0^\pi[z\ z_t]C[z\ z_t]^Tdx+2K\int_{\Omega_j}(p_2z+p_3z_t)f_jdx\\&-\lambda_1\sigma\int_0^\pi z^2dx\leq\sum_{j=1}^N\int_{\Omega_j}\eta_\Xi^T\Xi\eta_\Xi dx\leq0,\end{aligned} \tag{3.18}$$

where Ξ is given by (3.11). Note that Ξ is affine in ϕ. Thus, it is sufficient to verify (3.11) in the vertices ϕ_m,ϕ_M.

In the sequel, we find the gain K that leads to a larger decay rate. For $\Delta\to0$ the inequality $\dot V_0+2\delta V_0\leq0$ holds if $p_2\geq\delta p_3$ and (3.9), and $\Xi\leq0$ with $\lambda_1=2\sigma(p_2-\delta p_3)$ are feasible, i.e. if

$$\Xi_0=\begin{bmatrix}\xi_{11} & \xi_{12}\\ * & 2p_2+2p_3(\delta-\beta)\end{bmatrix}\leq0, \tag{3.19}$$

$$\xi_{11}=2\delta p_1+2p_2(\phi-K-\sigma^2)+2\sigma^2\delta p_3,$$
$$\xi_{12}=p_1+p_2(2\delta-\beta)+p_3(\phi-K).$$

The inequalities $\Xi_0\leq0$ and $P_0>0$ coincide with the Lyapunov inequality for the exponential stability with a decay rate δ of the second-order ODE

$$\dot\zeta(t)=\begin{bmatrix}0 & 1\\ \phi-K-\sigma^2 & -\beta\end{bmatrix}\zeta(t),\quad\zeta\in\mathbb{R}^2. \tag{3.20}$$

In the linear case with $\phi\equiv\phi_M$, (3.20) is the first mode in the modal decomposition of $z_{tt}=-z_{xxxx}-\beta z_t+(\phi-K)z$ under the corresponding boundary conditions, and thus, the choice of K given by (3.12) leads to the maximal achievable decay rate $\delta=0.5\beta$ for $\Delta\to0$. □

Remark 3.1. The proposed controllers (3.1) and (3.2) are based on a proportional action. To stabilize the undamped beam equation or to improve the performance, it is desirable to add a derivative action. However, we cannot benefit from the spatially sampled derivative actions (terms like (3.1) and (3.2) with z changed by z_x). Indeed, our method for spatial sampling is based on presentation (3.4), where the approximation errors f_j are compensated in $\dot{V}_0$ by the negative term $2(\delta p_3 - p_2)\int z_{xx}^2 dx$ via Wirtinger's inequality (cf. (3.14), (3.15) and (3.16)). In order to manage with $z_x(\hat{x}_j)$ or $\int_{\Omega_j} c_j(x) z_{xx}(x)dx$ we need to obtain $c\int z_{tx}^2 dx$ with some $c < 0$ in $\dot{V}_0$, but we have no such term in $\dot{V}_0$. Moreover, in the delayed case, for the wave equation, arbitrary small delay in the damping term may destabilize the system [28]. We suppose that the same may be true for the beam equation. For sure, our Lyapunov–Krasovskii method cannot cope with such delay. Thus, we do not introduce the derivative action in the controller.

3.2. Network-based exponential stabilization

Differently from the continuous-time control, for the network-based control, the stability analysis of the closed-loop system under the point and pointlike measurements is provided not in the same way: the direct Lyapunov–Krasovskii method is applied under the pointlike measurements (like under the averaged measurements in [12,13]), whereas the analysis under the point measurements requires additional application of Halanay's inequality (as introduced in [14]).

Similar to (3.4), the delayed control input can be represented as:

$$u_j(t-\tau) = -K[z(x,t) - f_j - \vartheta_j], \quad t \ge t_0. \tag{3.21}$$

Here

$$f_j = z(x,t) - z(x_j^t, t),\ \ \vartheta_j = \int_{\Omega_j} c_j(x)[z(x,t) - z(x,t-\tau)]dx \tag{3.22}$$

for the *pointlike* measurements (2.9), and

$$f_j = z(x,t-\tau) - z(\hat{x}_j, t-\tau),\ \ \vartheta_j = z(x,t) - z(x,t-\tau) \tag{3.23}$$

for the *point* measurements (2.11). Then the closed-loop system has the form

$$\begin{aligned} z_{tt} &= -z_{xxxx} - \beta z_t + [\phi(z,x,t) - K]z \\ &+K\sum_{j=1}^{N} \chi_j\left[f_j + \vartheta_j\right], \quad t \ge t_0. \end{aligned} \tag{3.24}$$

Theorem 3.1. *Consider the closed-loop system (3.24) under the boundary conditions (2.2) or (2.3) with the bounds $\phi_m, \phi_M, \Delta, \tau_M$ (and $\varepsilon > 0$ for the pointlike measurements). Let the controller gain $K > \phi_M - \sigma^2$ be given. Define $\sigma = 1$ and $\sigma = \frac{1}{4}$ for boundary conditions (2.2) and (2.3) respectively. Then the following holds:*

(i) The closed-loop system under the point measurements (with notations (3.23)) is exponentially stable if given $\delta_0 > \delta_1 > 0$, there exist scalars p_1, p_2, p_3, q_{12} and nonnegative scalars $\lambda_1, \lambda_2, r, s$ that satisfy LMIs (3.9), (3.10),

$$R = \begin{bmatrix} r & q_{12} \\ * & r \end{bmatrix} \ge 0, \tag{3.25}$$

and

$$\Phi|_{\phi=\phi_m,\phi_M} \le 0, \tag{3.26}$$

$$\Phi \triangleq \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & se^{-2\delta_0\tau_M} & Kp_2 & -2\delta_1 p_2 \\ * & \varphi_{22} & Kp_3 & 0 & Kp_3 & 0 \\ * & * & \varphi_{33} & \varphi_{34} & 0 & 2\delta_1 p_2 \\ * & * & * & \varphi_{44} & 0 & 0 \\ * & * & * & * & \varphi_{55} & 0 \\ * & * & * & * & * & -2\delta_1 p_3 \end{bmatrix}, \tag{3.27}$$

where

$$\begin{aligned} \varphi_{11} &= 2p_2(\phi - K) + 2p_1(\delta_0 - \delta_1) \\ &\quad + 2\sigma^2(\delta_0 p_3 - p_2) + s(1 - e^{-2\delta_0\tau_M}), \\ \varphi_{12} &= p_1 + p_2(2\delta_0 - \beta) + p_3(\phi - K), \\ \varphi_{13} &= Kp_2 + 2\delta_1 p_1 + se^{-2\delta_0\tau_M},\ \ \varphi_{22} = 2p_2 + 2p_3(\delta_0 - \beta) + \tau_M^2 r, \\ \varphi_{33} &= -(r+s)e^{-2\delta_0\tau_M} - 2\delta_1 p_1,\ \ \varphi_{34} = -(s + q_{12})e^{-2\delta_0\tau_M}, \\ \varphi_{44} &= -(r+s)e^{-2\delta_0\tau_M},\ \ \varphi_{55} = -2\delta_1 p_3 \sigma\frac{\pi^2}{\Delta^2}. \end{aligned} \tag{3.28}$$

The resulting decay rate δ is a unique positive solution of $\delta = \delta_0 - \delta_1 \exp(2\delta\tau_M)$.

(ii) The closed-loop system under the pointlike measurements (with notations (3.22)) is exponentially stable with a decay rate $\delta > 0$ if given δ, there exist scalars p_1, p_2, p_3, q_{12} and nonnegative scalars $\lambda_1, \lambda_2, r, s$ that satisfy LMIs (3.9), (3.10), (3.25) and

$$\Psi|_{\phi=\phi_m,\phi_M} \le 0, \tag{3.29}$$

$$\Psi \triangleq \begin{bmatrix} \omega_{11} & \omega_{12} & Kp_2 & 0 & Kp_2 & 0 \\ * & \omega_{22} & Kp_3 & 0 & Kp_3 & 0 \\ * & * & \omega_{33} & \omega_{34} & 0 & se^{-2\delta\tau_M} \\ * & * & * & \omega_{44} & 0 & se^{-2\delta\tau_M} \\ * & * & * & * & \omega_{55} & 0 \\ * & * & * & * & * & \omega_{66} \end{bmatrix}, \tag{3.30}$$

where

$$\begin{aligned} \omega_{11} &= 2p_2(\phi - K) + 2\delta p_1 - \sigma\lambda_1 + \lambda_2\Delta\varepsilon^{-1}, \\ \omega_{12} &= p_1 + p_2(2\delta - \beta) + p_3(\phi - K), \\ \omega_{22} &= 2p_2 + 2p_3(\delta - \beta) + \tau_M^2 r\Delta\varepsilon^{-1}, \\ \omega_{33} &= \omega_{44} = -(r+s)e^{-2\delta\tau_M},\ \ \omega_{34} = -(q_{12} + s)e^{-2\delta\tau_M}, \\ \omega_{55} &= -[2\sigma(p_2 - \delta p_3) - \lambda_1]\frac{\pi^2}{(\Delta + \varepsilon)^2}, \\ \omega_{66} &= -\lambda_2 - s(e^{-2\delta\tau_M} - 1). \end{aligned} \tag{3.31}$$

Proof. In order to derive stability conditions for (3.24) we employ Lyapunov–Krasovskii functional of the form

$$V(t) = V_0(t) + V_s(t) + V_r(t),\ \ t \in [t_k, t_{k+1}), \tag{3.32}$$

where $V_0(t)$ is given by (3.8) and

$$V_s(t) = s\sum_{j=1}^{N}\int_{\Omega_j}\int_{t-\tau_M}^{t} e^{2\delta_0(s-t)}\kappa^2(x,s)ds\,dx, \tag{3.33}$$

$$V_r(t) = r\tau_M\sum_{j=1}^{N}\int_{\Omega_j}\int_{-\tau_M}^{0}\int_{t+\theta}^{t} e^{2\delta_0(s-t)}\kappa_s^2(x,s)ds\,d\theta\,dx,$$

with some scalars $s, r \ge 0$, and κ given by

$$\kappa(x,s) = \begin{cases} \int_{\Omega_j} c_j(\xi) z(\xi,s)d\xi, & \textit{pointlike measurements}, \\ z(x,s), & \textit{point measurements}. \end{cases}$$

Here, V_s and V_r treat time-delay terms as introduced in [14] for the point and in [21] for the pointlike measurements. By differentiating V_s and V_r we have

$$\dot{V}_s + 2\delta_0 V_s = s\sum_{j=1}^{N}\int_{\Omega_j}\left(\kappa^2(x,t) - e^{-2\delta_0\tau_M}\kappa^2(x,t-\tau_M)\right)dx, \tag{3.34}$$

and

$$\dot{V}_r + 2\delta_0 V_r \le \tau_M r\sum_{j=1}^{N}\int_{\Omega_j}\left[\tau_M\kappa_t^2 - e^{-2\delta_0\tau_M}\int_{t-\tau_M}^{t}\kappa_s^2(x,s)ds\right]dx. \tag{3.35}$$

Note that $\vartheta_j = \kappa(x,t) - \kappa(x,t-\tau)$. Denote

$\nu = \kappa(x,t-\tau) - \kappa(x,t-\tau_M)$.

Then, under (3.25) by Lemma 3.4 of [29] we find

$$-\tau_M r \sum_{j=1}^{N} \int_{\Omega_j} \int_{t-\tau_M}^{t} \kappa^2 ds\, dx \leq -\sum_{j=1}^{N} \int_{\Omega_j} [\vartheta_j\ \nu] R [\vartheta_j\ \nu]^T dx.$$

(i) For the closed-loop system (3.24) under the point measurements (with notations (3.23)) we have in (3.34) $\kappa^2(x,t-\tau_M) = (z - \vartheta_j - \nu)^2$. Then, differentiating (3.32) along (3.24) with (3.23) we have (cf. (3.18))

$$\dot V + 2\delta_0 V \leq -2(p_2 - \delta_0 p_3) \int_0^\pi z_{xx}^2 dx + \int_0^\pi [z\ z_t] C [z\ z_t]^T dx$$
$$+ 2K \sum_{j=1}^{N} \int_{\Omega_j} (p_2 z + p_3 z_t)\left(f_j + \vartheta_j\right) dx$$
$$+ s \sum_{j=1}^{N} \int_{\Omega_j} \left(z^2 - e^{-2\delta_0 \tau_M}(z - \vartheta_j - \nu)^2\right) dx$$
$$+ \tau_M^2 r \int_0^\pi z_t^2 dx - e^{-2\delta_0\tau_M} \sum_{j=1}^{N} \int_{\Omega_j} [\vartheta_j\ \nu] R [\vartheta_j\ \nu]^T dx,$$

By applying Wirtinger's inequality (A.1) we obtain

$$-2(p_2 - \delta_0 p_3) \int_0^\pi z_{xx}^2 dx \leq -2(p_2 - \delta_0 p_3)\sigma^2 \int_0^\pi z^2 dx.$$

Next, Halanay's inequality (A.3) is applied. For some $\delta_1 < \delta_0$ we have

$$W \triangleq \dot V(t) + 2\delta_0 V(t) - 2\delta_1 \sup_{-h \leq \theta \leq 0} V(t+\theta)$$
$$\leq \dot V(t) + 2\delta_0 V(t) - 2\delta_1 p_3 \int_0^\pi z_{xx}^2(x,t-\tau) dx$$
$$- 2\delta_1 \int_0^\pi \begin{bmatrix} z(x,t-\tau) \\ z_t(x,t-\tau) \end{bmatrix}^T P \begin{bmatrix} z(x,t-\tau) \\ z_t(x,t-\tau) \end{bmatrix} dx.$$

By applying Wirtinger's inequality (A.1) and further using (3.16) with $\varepsilon = 0$ we obtain

$$- 2\delta_1 p_3 \int_0^\pi z_{xx}^2(x,t-\tau) dx \leq -2\delta_1 p_3 \sigma \int_0^\pi z_x^2(x,t-\tau) dx$$
$$\leq -2\delta_1 p_3 \sigma \frac{\pi^2}{\Delta^2} \sum_{j=1}^{N} \int_{\Omega_j} \left(f_j\right)^2 dx.$$

Denote $\eta_\Phi = [z \quad z_t \quad \vartheta_j \quad \nu \quad f_j \quad z_t(x,t-\tau)]^T$. Then,

$$W \leq -2(p_2 - \delta_0 p_3)\sigma^2 \int_0^\pi z^2 dx + \int_0^\pi [z\ z_t] C [z\ z_t]^T dx$$
$$+ 2K \sum_{j=1}^{N} \int_{\Omega_j} (p_2 z + p_3 z_t)\left(f_j + \vartheta_j\right) dx$$
$$+ s \int_0^\pi \left(z^2 - e^{-2\delta_0\tau_M}(z - \vartheta_j - \nu)^2\right) dx$$
$$+ \tau_M^2 r \int_0^\pi z_t^2 dx - e^{-2\delta_0\tau_M} \int_0^\pi [\vartheta_j\ \nu] R [\vartheta_j\ \nu]^T dx$$
$$- 2\delta_1 \int_0^\pi [z - \vartheta_j\ z_t(x,t-\tau)] P [z - \vartheta_j\ z_t(x,t-\tau)]^T dx$$
$$- 2\delta_1 p_3 \sigma^2 \frac{\pi^2}{(\Delta+\varepsilon)^2} \sum_{j=1}^{N} \int_{\Omega_j} \left(f_j\right)^2 dx \leq \sum_{j=1}^{N} \int_{\Omega_j} \eta_\Phi{}^T \Phi \eta_\Phi dx \leq 0$$

if $\Phi \leq 0$, where Φ is given by (3.26). Note that Φ is affine in ϕ. Thus it is sufficient to verify (3.26) in the vertices ϕ_m and ϕ_M.

(ii) Consider now the closed-loop system (3.24) under the pointlike measurements (with notations (3.22)). We have in (3.34) $\kappa^2(x,t-\tau_M) = (\kappa - \vartheta_j - \nu)^2$. Differentiating (3.32) with $\delta_0 = \delta$ along (3.24) with notations (3.22) and taking into account (3.18) we have

$$\dot V + 2\delta V \leq -[2\sigma(p_2 - \delta p_3) - \lambda_1] \frac{\pi^2}{(\Delta+\varepsilon)^2} \sum_{j=1}^{N} \int_{\Omega_j} \left(f_j\right)^2 dx$$
$$- \sigma\lambda_1 \int_0^\pi z^2 dx + \int_0^\pi [z\ z_t] C [z\ z_t]^T dx$$
$$+ 2K \sum_{j=1}^{N} \int_{\Omega_j} (p_2 z + p_3 z_t)\left(f_j + \vartheta_j\right) dx$$
$$+ s \sum_{j=1}^{N} \int_{\Omega_j} \left[\kappa^2(x,t) - e^{-2\delta\tau_M}(\kappa - \vartheta_j - \nu)^2\right] dx$$
$$+ \tau_M^2 r \sum_{j=1}^{N} \int_{\Omega_j} \kappa_t^2 dx - e^{-2\delta\tau_M} \sum_{j=1}^{N} \int_{\Omega_j} [\vartheta_j\ \nu] R [\vartheta_j\ \nu]^T dx.$$

Due to Jensen's inequality

$$\int_{\Omega_j} \kappa_t^2 dx = \Delta_j \left(\int_{\Omega_j} c_j(\xi) z_t(\xi,t) d\xi \right)^2$$
$$\leq \Delta_j \int_{\Omega_j} c_j(\xi) d\xi \int_{\Omega_j} c_j(\xi) z_t^2(\xi,t) d\xi$$
$$\leq \frac{\Delta_j}{\varepsilon_j} \int_{\Omega_j} z_t^2(x,t) dx \leq \frac{\Delta}{\varepsilon} \int_{\Omega_j} z_t^2(x,t) dx.$$

Similarly

$$\int_{\Omega_j} \kappa^2 dx = \Delta_j \left(\int_{\Omega_j} c_j(\xi) z(\xi,t) d\xi \right)^2 \leq \frac{\Delta}{\varepsilon} \int_{\Omega_j} z^2(x,t) dx. \tag{3.36}$$

Denote $\eta_\Psi = [z \quad z_t \quad \vartheta_j \quad \nu \quad f_j \quad \kappa(x,t)]^T$. By adding to $\dot V + 2\delta V$ the left-hand side of

$$\lambda_2 \left(\frac{\Delta}{\varepsilon} \int_{\Omega_j} z^2(x,t) dx - \int_{\Omega_j} \kappa^2 dx \right) \geq 0$$

we arrive at

$$\dot V + 2\delta V \leq \textstyle\sum_{j=1}^{N} \int_{\Omega_j} \eta_\Psi{}^T \Psi \eta_\Psi dx \leq 0$$

if $\Psi \leq 0$, where Ψ is given by (3.29). Note that Ψ is affine in ϕ. Thus it is sufficient to verify (3.29) in the vertices ϕ_m, ϕ_M. $\square$

3.3. Numerical example

Consider the damped beam equation (2.1) under the boundary conditions (2.2), with the parameters

$$\beta = 4, \quad \phi_M = -\phi_m = 6.2. \tag{3.37}$$

The open-loop system with $\psi = 6.2z$ is unstable. Indeed, simulation of the solution of the open-loop system with $z(x,t_0) = x^2(x-\pi)^2$, $z_t(x,t_0) = 0$, $t_0 = 0$ (by employing the finite-difference method) shows instability (see Fig. 3).

We choose the controller gain $K = \phi_M - \sigma^2 + \frac{\beta^2}{4} = 9.2$ and compare the performance of the resulting closed-loop system in terms of maximal τ_M that preserves stability for both types of measurements (2.9) and (2.11) under the same (as small as possible) number of sensors. For simplicity, we assume that the intervals Ω_j have the same length $\Delta = \frac{\pi}{N}$ and formulate results in terms of the number of sensors N (that coincides with the number of actuators).

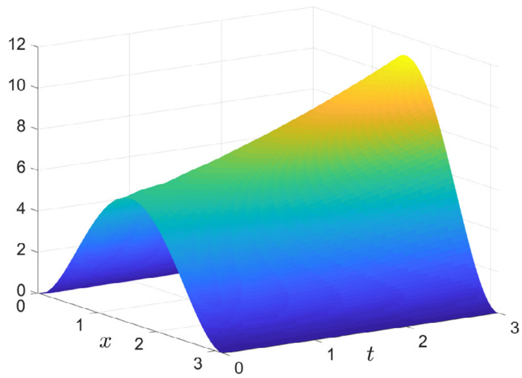

Fig. 3. The unstable open-loop system.

Table 1
Maximal values of τ_M for $\phi \equiv \phi_M$ and $\delta \to 0$.

N	Point	Point like	
	τ_M $\delta_0 \approx \delta_1 = 0.4$	τ_M	ε/Δ
4	0.019	0.015	0.0006
		0.019	0.001
		0.061	0.1
7	0.24	0.16	0.05
		0.24	0.1
		0.31	0.2
1000	0.33	0.27	0.1
		0.33	0.15
		0.38	0.2

Table 2
Maximal values of τ_M for $|\phi| \le \phi_M$ and $\delta \to 0$.

N	Point	Point like	
	τ_M	τ_M	ε/Δ
7	0.011 $\delta_0 \approx \delta_1 = 0.4$	0.007	0.0001
		0.011	0.0003
		0.022	0.001
10	0.07 $\delta_0 \approx \delta_1 = 0.4$	0.03	0.001
		0.07	0.009
		0.1	0.02
1000	0.16 $\delta_0 \approx \delta_1 = 0.2$	0.08	0.01
		0.16	0.04
		0.28	0.1

We start with the continuous-time controller. Consider the case of either linear $\rho = \phi_M z$, where we verify the LMIs of Proposition 3.1 in one vertex $\phi = \phi_M$, or the general case with $\phi \in [\phi_m, \phi_M]$, where the LMIs are verified in both vertices. For small $\delta > 0$, the conditions of Proposition 3.1 hold for a minimal $N = 3$ in the linear case, and a minimal $N = 4$ in the general case. By taking $N \to \infty$ we approach the maximal achievable decay rate $\delta \to \beta/2$ for both measurements.

Consider further the network-based control. By verifying the LMI conditions of Theorem 3.1, we find maximal values of τ_M that preserve the exponential stability of the closed-loop system with a small decay rate. Tables 1 and 2 show these maximal values of τ_M under the point measurements (2.11) and under the pointlike measurements (2.9) (in the latter case we show also the corresponding values of the ratio ε/Δ). It is seen that for larger values of ε/Δ, the controller under the pointlike measurements preserves the stability for larger values of τ_M than the controller under the point measurements.

4. Regional stabilization: locally Lipschitz nonlinearities

In this section, we consider network-based stabilization of system (2.1) with a locally Lipschitz nonlinearity under boundary conditions (2.2) of (2.3) and pointlike (2.9) or point (2.11) measurements. Assume that ρ is *locally Lipschitz* in the first argument, and satisfies

$$\phi_m \le \rho_z \le \phi_M \quad \forall |z| \le D,\ x \in [0,\pi],\ t \ge t_0 \tag{4.1}$$

for some $D > 0$. Consider the initial conditions (2.4). For locally Lipschitz ρ, we are looking for a bound on the set of initial functions $[z_1, z_2]^T \in \mathscr{H}$, starting from which the solutions of the system exist for all $t \ge t_0$ and are exponentially converging to zero with a decay rate $\delta > 0$.

Consider a region of initial conditions defined by

$$\mathcal{X}_{D_0} = \left\{ [z_1, z_2]^T \in \mathscr{H} \middle| \int_0^\pi (z_{1xx}^2 + z_2^2) dx \le D_0^2 \right\}, \tag{4.2}$$

where $D_0 > 0$ is some constant. We will derive conditions that guarantee the following: all the solutions of (2.1) under (2.2) or (2.3) starting from $\mathcal{X}_{D_0}$ are exponentially converging with a decay rate $\delta > 0$ and satisfy $|z| \le D \quad \forall x \in [0,\pi],\ t \ge t_0$, meaning that the following implication holds:

$$[z_1, z_2]^T \in \mathcal{X}_{D_0} \Rightarrow |z| < D \quad \forall x \in [0,\pi],\ t \ge t_0. \tag{4.3}$$

We aim to find a region of initial conditions $\mathcal{X}_{D_0}$ (a bound on the domain of attraction) that leads to exponentially converging solutions with as large as possible D_0. For simplicity, we assume $\eta_0 \equiv 0$. Note that if $\eta_0 > 0$, additional conditions for the solution bound on the first time interval are needed, where the system is in the open-loop (see [30]).

Theorem 4.1. *Consider the closed-loop system (3.24) subject to the locally Lipschitz nonlinearity that satisfies (4.1) under the boundary conditions (2.2) or (2.3) and the initial conditions (2.4). Let the controller gain K be given by (3.12). Define $\sigma = 1$ and $\sigma = \frac{1}{4}$ for boundary conditions (2.2)* and *(2.3) respectively. Assume that LMIs of Theorem 3.1, where (3.9) is changed by a stronger condition $P > 0$, are feasible. Let*

$$D_0^2 < \frac{\sigma p_3 D^2}{\pi[(\lambda_{max}(P) + \tau_M \cdot s \cdot s_0)\sigma^{-2} + p_3]}, \tag{4.4}$$

$$s_0 \triangleq \begin{cases} 1 & \text{for point measurements,} \\ \frac{\Delta}{\varepsilon} & \text{for pointlike measurements,} \end{cases}$$

where $\lambda_{max}(P)$ denotes the maximal eigenvalue of P. Then the system is regionally exponentially stable with a decay rate δ for all initial conditions from $\mathcal{X}_{D_0}$.

Proof. We first assume that implication (4.3) holds, then (4.1) is satisfied and LMIs of Theorem 3.1 yield exponential convergence of $V(t)$ given by (3.32), and thus $V(t) \le V(t_0)$ for all $t \ge t_0$. Then by Sobolev's and Wirtinger's inequality

$$\begin{aligned} \max_{x\in[0,\pi]} z^2 &\le \pi \int_0^\pi z_x^2 dx \le \frac{\pi}{\sigma} \int_0^\pi z_{xx}^2 dx \\ &\le \frac{\pi}{\sigma p_3} V(t) \le \frac{\pi}{\sigma p_3} V(t_0) \quad \forall t \ge t_0. \end{aligned} \tag{4.5}$$

To upper-bound $V(t_0)$, we follow [30]. Since $\eta_0 = 0$, the solution to the closed-loop system does not depend on the values of $z(x,t)$ for $t < t_0$, and we may define the initial conditions to be constant: $z(x,t) = z_1(x),\ z_t(x,t) = z_2(x)\ \forall t \le t_0$. Then (cf. (3.33) and (3.36))

$$V(t_0) \le \int_0^\pi [\lambda_{max}(P)(z_1^2 + z_2^2) + \tau_M \cdot s \cdot s_0 \cdot z_1^2 + p_3 z_{1xx}^2] dx.$$

By employing Wirtinger's inequality (A.1) twice, we have

$$\sigma^2 \int_0^\pi z_1^2 dx \le \int_0^\pi z_{1xx}^2 dx,$$

implying

$$\begin{aligned} V(t_0) \le \int_0^\pi & \Big[[(\lambda_{max}(P) + \tau_M \cdot s \cdot s_0)\sigma^{-2} + p_3] z_{1xx}^2 \\ & + \lambda_{max}(P) z_2^2 \Big] dx \\ & \le [(\lambda_{max}(P) + \tau_M \cdot s \cdot s_0)\sigma^{-2} + p_3] D_0^2. \end{aligned}$$

Taking into account (4.4), the latter inequality together with (4.5) yield for $t \ge t_0$

$$\max_{x\in[0,\pi]} z^2 \le \frac{\pi}{\sigma p_3} V(t_0) < D^2. \tag{4.6}$$

We prove next that under the LMIs of the theorem implication (4.3) holds. By contradiction, assume that given some $[z_1, z_2]^T \in \mathcal{X}_{D_0}$ there exists $t^* > t_0$ such that

$$\max_{x\in[0,\pi]} z^2(x,t) < D^2 \;\; t \in [t_0, t^*) \text{ and } \max_{x\in[0,\pi]} z^2(x,t^*) = D^2. \tag{4.7}$$

Then due to continuity of $V(t)$, we have $V(t) \le V(t_0)$ for all $t \in [t_0, t^*]$, implying (4.6) for $t \in [t_0, t^*]$, i.e. $\max_{x\in[0,\pi]} z^2(x,t^*) < D^2$. The latter contradicts (4.7) and completes the proof. □

Remark 4.1. Note that in the LMIs of Theorem 4.1 (for the stability analysis), we can always choose $p_3 = 1$ and have equivalent conditions. Then in order to enlarge the bound D_0, in the LMIs of Theorem 4.1 we choose $p_3 = 1$ and add the LMI $P < \lambda I$, where we minimize λ.

4.1. Numerical example — regional stabilization

Consider the damped beam equation (2.1) with $\rho = 0.1z|z|^q$ ($q \ge 1$) under boundary conditions (2.2) or (2.3). Here $|\rho_z| \le 0.1(1+q)|z|^q \le \phi_M = -\phi_m$ if $|z| \le \left(\frac{10\phi_M}{1+q}\right)^{1/q}$. We choose $q = 1.5$ and $\phi_M = 6.2$, meaning that (4.1) holds with $D = 8.5042$. As in Section 3.3, we choose $\beta = 4$ and $K = 9.2$. We use the results of Table 2 for the minimal number of sensors $N = 7$ with $\tau_M = 0.011$, where in the case of pointlike measurements $\frac{\varepsilon}{\Delta} = 0.0003$. To maximize the domain of attraction $\mathcal{X}_{D_0}$, we employ Remark 4.1. Under both, point and pointlike measurements, we obtain

$$D_0^2 \le 0.0379 D^2 = 1.6556^2,$$

where the LMIs are feasible with $\lambda = 7.3926$, $p_1 = 7.235$, $p_2 = 1.0039$, $s = 4.6352 \cdot 10^{-11}$.

For simulations we consider the initial conditions of the form $z_1 = \gamma x^2(x-\pi)^2$, $z_2 = 0$, $x \in [0,\pi]$. Therefore, the initial conditions that guarantee stability satisfy the following inequality:

$$\int_0^\pi (z_{1xx}^2 + z_2^2) dx = \gamma^2 \frac{4\pi^5}{5} \le D_0^2,$$

leading to $|\gamma| \le 0.1058$. Simulations of solutions of the networked system under $N = 7$, $\tau_M = 0.011$, $MAD = 0.0007$ confirm the theoretical results. Fig. 4 shows the energy $E(t) \triangleq \int_0^\pi (z_{xx}^2 + z_t^2) dx$ of the closed-loop system for $\gamma = 0.1058$ and $\gamma = 5.3394$ under the point measurements. Note that simulations under the pointlike measurements look similar. From simulations we see that stability is preserved for much larger $|\gamma| \le 5.3$ that may illustrate the conservativeness of the results.

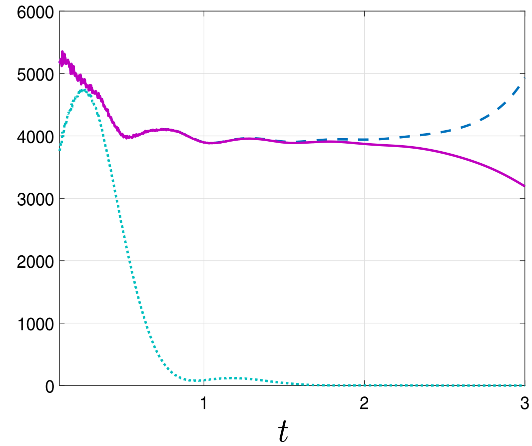

Fig. 4. $E(t)$ of the closed-loop system: $\gamma = 0.1058$ — solid (scaled by $1.5 \cdot 10^3$), $\gamma = 5.3395$ — dotted.

5. Conclusions

In this paper we designed a network-based distributed controller for the damped semilinear beam equation, based on point or pointlike measurements. Quantitative LMI-based conditions were provided for the minimal number of sensors/actuators and the maximal values of delays and sampling intervals that preserve the exponential stability of the closed-loop system. For beam equations with locally Lipschitz nonlinearities, regional network-based stabilization was studied.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Maria Terushkin: Writing - original draft, Visualization, Investigation. **Emilia Fridman:** Conceptualization, Supervision.

Appendix. Useful lemmas

The following inequalities will be useful:

Lemma A.1. *Let $z \in \mathcal{H}^1[a,b]$ be a scalar function, with the boundary values stated below. Then the Wirtinger inequality holds [31]:*

$$\sigma \int_a^b z^2(\xi) d\xi \le \frac{(b-a)^2}{\pi^2} \int_a^b \left[\frac{dz(\xi)}{d\xi}\right]^2 d\xi \tag{A.1}$$

where

$$\sigma = \begin{cases} 1, & \text{if} \quad z(a) = z(b) = 0; \\ \frac{1}{4}, & \text{if} \quad z(a) = 0 \text{ or } z(b) = 0. \end{cases}$$

Moreover, the Sobolev inequality is satisfied:

$$\max_{z\in(a,b)} z^2(x) \le (b-a) \int_a^b z_x^2(x) dx. \tag{A.2}$$

Lemma A.2 (*Halanay's Inequality [32] & p. 138 of [29]*)**.** *Let $0 < \delta_1 < \delta_0$ and let $V : [t_0 - h, \infty) \longrightarrow [0,\infty)$ be an absolutely continuous function that satisfies*

$$\dot V(t) + 2\delta_0 V(t) - 2\delta_1 \sup_{-h\le\theta\le 0} V(t+\theta) \le 0, \;\; t \ge t_0.$$

Then

$$V(t) \leq \exp\big(-2\delta(t-t_0)\big) \sup_{-h\leq\theta\leq 0} V(t_0+\theta),\ t \geq t_0, \tag{A.3}$$

where $\delta > 0$ *is a unique positive solution of* $\delta = \delta_0 - \delta_1 \exp(2\delta h)$*.*

Lemma A.3 (*Jensen's Inequality [33]*)**.** *Let* $c : [a, b] \to [0, \infty)$ *and* $z : [a, b] \to \mathbb{R}$ *be such that the integration concerned is well defined. Then*

$$\left[\int_a^b c(\xi)z(\xi)d\xi\right]^2 \leq \int_a^b c(\xi)d\xi \int_a^b c(\xi)z^2(\xi)d\xi. \tag{A.4}$$

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