SIAM J. CONTROL OPTIM.
Vol. 48, No. 8, pp. 5028–5052

STABILIZATION OF SECOND ORDER EVOLUTION EQUATIONS WITH UNBOUNDED FEEDBACK WITH TIME-DEPENDENT DELAY*

EMILIA FRIDMAN†, SERGE NICAISE‡, AND JULIE VALEIN§

Abstract. We consider abstract second order evolution equations with unbounded feedback with time-varying delay. Existence results are obtained under some realistic assumptions. We prove the exponential decay under some conditions by introducing an abstract Lyapunov functional. Our abstract framework is applied to the wave, to the beam, and to the plate equations with boundary delays.

Key words. second order evolution equations, wave equations, time-varying delay, stabilization, Lyapunov functional

AMS subject classifications. 93D15, 93D05

DOI. 10.1137/090762105

1. Introduction.

The control of flexible systems, governed by partial differential equations (PDEs), has become a large area of research due to the increasing demand in practical engineering applications (electrical and/or mechanical systems) and in biological applications. Time delay is unavoidable in practice due to measurement lags, analysis times, or computation lags; see, for instance, [1, 8, 31]. Furthermore, in many cases, delay is a source of instability [9]. In the case of distributed parameter systems, even arbitrarily small delays in the feedback may destabilize the system (see, e.g., [6, 21, 29, 22]). Hence the stability issue of systems with delay is of theoretical and practical importance.

There are only a few works on Lyapunov-based technique for PDEs with delay. Most of these works analyze the case of *constant delays*. Thus, stability conditions and exponential bounds were derived for some scalar heat and wave equations with constant delays and with Dirichlet boundary conditions without delay in [32, 33]. Stability and instability conditions for the wave equations with constant delay can be found in [22, 25]. The stability of linear parabolic systems with constant coefficients and internal constant delays has been studied in [11] in the frequency domain. Moreover, we refer to [24] for the stability of second order evolution equation with constant delay in unbounded feedbacks.

Recently the stability of PDEs with *time-varying delays* was analyzed in [4, 7, 26, 27] via the Lyapunov method. In the case of linear systems in a Hilbert space, the conditions of [4, 7, 27] assume that the operator acting on the delayed state is bounded (which means that this condition cannot be applied to boundary delays, for

*Received by the editors June 15, 2009; accepted for publication (in revised form) July 21, 2010; published electronically October 19, 2010.
http://www.siam.org/journals/sicon/48-8/76210.html
†School of Electrical Engineering, Tel Aviv University, Tel Aviv 69978, Israel (emilia@eng.tau.ac.il).
‡Université de Valenciennes et du Hainaut Cambrésis, LAMAV, FR CNRS 2956, Institut des Sciences et Techniques of Valenciennes, F-59313 Valenciennes, Cedex 9, France (Serge.Nicaise@univ-valenciennes.fr).
§Institut Elie Cartan Nancy (IECN), Nancy-Université & INRIA (Project-Team CORIDA), F-54506 Vandoeuvre-lès-Nancy, Cedex, France (Julie.Valein@iecn.u-nancy.fr).

example). The stability of the 1-d (one-dimensional) heat and wave equations with boundary time-varying delays has been studied in [26] via Lyapunov functionals.

The aim of this paper is to consider an abstract setting similar to that in [24] and as large as possible in order to contain a quite large class of problems with time-varying delay feedbacks (the class which contains in particular the results of [26] for the wave equation).

Before going on, let us present our abstract framework. Let H be a real Hilbert space with norm and inner product denoted, respectively, by $\|.\|_H$ and $(.,.)_H$. Let $A : D(A) \to H$ be a self-adjoint operator with a compact inverse in H, which is positive (in the sense that $(Ax, x) > 0$ for all $x \in D(A)$, $x \neq 0$). Let $V := D(A^{1/2})$ be the domain of $A^{1/2}$. We further assume that $D(A)$ is dense in V. Denote by $V' = D(A^{1/2})'$ the dual space of $D(A^{1/2})$ obtained by means of the inner product in H.

Further, for $i = 1, 2$, let U_i be a real Hilbert space (which will be identified to its dual space) with norm and inner product denoted, respectively, by $\|.\|_{U_i}$ and $(.,.)_{U_i}$, and let $B_i \in \mathcal{L}(U_i, D(A^{1/2})')$.

We consider the system

$$\begin{cases} \ddot{\omega}(t) + A\omega(t) + B_1 u_1(t) + B_2 u_2(t - \tau(t)) = 0, \quad t > 0, \\ \omega(0) = \omega_0, \dot{\omega}(0) = \omega_1, \\ u_2(t - \tau(0)) = f^0(t - \tau(0)), \quad 0 < t < \tau(0), \end{cases} \tag{1}$$

where $t \in [0, \infty)$ represents the time, $\tau(t) > 0$ is the time-varying delay, $\omega : [0, \infty) \to H$ is the state of the system, $\dot{\omega}$ is the time derivative of ω, $u_1 \in L^2([0, \infty), U_1)$, $u_2 \in L^2([-\tau(0), \infty), U_2)$ are the input functions, and, finally, $(\omega_0, \omega_1, f^0(\cdot - \tau(0)))$ are the initial data chosen in a suitable space (see below). The time-varying delay $\tau(t)$ satisfies

$$\exists\, d < 1 \quad \forall t > 0, \quad \dot{\tau}(t) \leq d < 1, \tag{2}$$

and

$$\exists\, M > 0 \quad \forall t > 0, \quad 0 < \tau_0 \leq \tau(t) \leq M. \tag{3}$$

Moreover, we assume that

$$\forall\, T > 0, \quad \tau \in W^{2,\infty}([0, T]). \tag{4}$$

Most of the linear equations modeling the vibrations of elastic structures with distributed control with delay can be written in the form (1), where ω stands for the displacement field.

In many problems, in particular, those coming from elasticity, the inputs u_i are given in the feedback form $u_i(t) = B_i^* \dot{\omega}(t)$, which corresponds to collocated actuators and sensors. In this way we obtain the closed loop system

$$\begin{cases} \ddot{\omega}(t) + A\omega(t) + B_1 B_1^* \dot{\omega}(t) + B_2 B_2^* \dot{\omega}(t - \tau(t)) = 0 \quad \text{in } V', \quad t > 0, \\ \omega(0) = \omega_0, \dot{\omega}(0) = \omega_1, \\ B_2^* \dot{\omega}(t - \tau(0)) = f^0(t - \tau(0)), \quad 0 < t < \tau(0). \end{cases} \tag{5}$$

Recall that *without delay*, i.e., if $B_2 = 0$, according to Russell's principle [30], if the decay of the energy is uniformly exponential, then the system is exactly controllable (with controls supported in the set where the feedback mechanism is active). See also the result of [10] for abstract second order evolution equations with bounded feedbacks *without delay*.

The abstract second order evolution equations *without delay* or with *constant delay* of type (5) have been studied in [3] and [24], respectively. In these two papers, the exponential stability (or polynomial stability) is shown, under some conditions, via an observability inequality for the solution of a corresponding conservative system. In our case, for time-varying delay, this method cannot be applied due to the loss of the time translation invariance. Hence we introduce new abstract Lyapunov functionals with exponential terms and an additional term, which take into account the dependence of the delay with respect to time. For the treatment of other problems with Lyapunov technique see [7, 23, 27].

Moreover, contrary to [22, 24], the existence results do not follow from standard semigroup theory because the spatial operator depends on time due to the time-varying delay. Therefore we use the variable norm technique of Kato [14, 15].

Hence the first natural question is the well-posedness of this system. In section 2 we will give a sufficient condition that guarantees that this system (5) is well-posed, where we closely follow the approach developed in [26] for the 1-d heat and wave equations. Second, we may ask if this system is dissipative. We show in section 3 that the condition

$$\exists\, 0 < \alpha < \sqrt{1-d} \quad \forall u \in V, \quad \|B_2^* u\|_{U_2}^2 \leq \alpha \, \|B_1^* u\|_{U_1}^2 \tag{6}$$

guarantees that the energy decays. Note further that if (6) is not satisfied, there exist cases where some instabilities may appear (see [22, 25, 34] for the wave equation with constant delay). Hence this assumption seems realistic.

In a third step, again under the condition (6), we prove the exponential decay of the system (5) by introducing an appropriate Lyapunov functional. Moreover, we give the dependence of the decay rate with respect to the delay; in particular we show that if the delay increases, the decay rate decreases. This is the content of section 4.

Finally, we finish this paper by considering in section 5 different examples where our abstract framework can be applied. To our knowledge, all the examples, with the exception of the first one, are new.

2. Well-posedness of the system.

We aim to show that system (5) is well-posed. For that purpose, we use semigroup theory and an idea from [22]. Let us introduce the auxiliary variable $z(\rho, t) = B_2^* \dot{\omega}(t - \tau(t)\rho)$ for $\rho \in (0, 1)$ and $t > 0$. Note that z satisfies the transport equation

$$\begin{cases} \tau(t)\frac{\partial z}{\partial t} + (1 - \dot{\tau}(t)\rho)\frac{\partial z}{\partial \rho} = 0, \quad 0 < \rho < 1,\, t > 0, \\ z(0, t) = B_2^* \dot{\omega}(t), \\ z(\rho, 0) = B_2^* \dot{\omega}(-\tau(0)\rho) = f^0(-\tau(0)\rho). \end{cases}$$

Therefore, the system (5) is equivalent to

$$\begin{cases} \ddot{\omega}(t) + A\omega(t) + B_1 B_1^* \dot{\omega}(t) + B_2 z(1, t) = 0, \quad t > 0, \\ \tau(t)\frac{\partial z}{\partial t} + (1 - \dot{\tau}(t)\rho)\frac{\partial z}{\partial \rho} = 0, \quad t > 0,\, 0 < \rho < 1, \\ \omega(0) = \omega_0,\, \dot{\omega}(0) = \omega_1,\, z(\rho, 0) = f^0(-\tau(0)\rho), \quad 0 < \rho < 1, \\ z(0, t) = B_2^* \dot{\omega}(t), \quad t > 0. \end{cases} \tag{7}$$

If we introduce

$$U := (\omega,\, \dot\omega,\, z)^T,$$

then U satisfies

$$U' = (\dot\omega,\, \ddot\omega,\, \dot z)^T = \left(\dot\omega,\, -A\omega(t) - B_1B_1^*\dot\omega(t) - B_2 z(1,\, t),\, \frac{\dot\tau(t)\rho - 1}{\tau(t)}\frac{\partial z}{\partial \rho}\right)^T.$$

Consequently the system (5) may be rewritten as the first order evolution equation

$$\left\{\begin{array}{c} U' = \mathcal{A}(t)U, \\ U(0) = (\omega_0,\, \omega_1,\, f^0(-\tau(0).)), \end{array}\right. \tag{8}$$

where the time-dependent operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t)\begin{pmatrix}\omega\\ u\\ z\end{pmatrix} = \begin{pmatrix} u \\ -A\omega - B_1B_1^*u - B_2 z(1), \\ \frac{\dot\tau(t)\rho-1}{\tau(t)}\frac{\partial z}{\partial\rho}\end{pmatrix},$$

with domain

$$\begin{aligned} D(\mathcal{A}(t)) :=& \{(\omega,\, u,\, z) \in V \times V \times H^1((0,\,1),\, U_2);\; z(0) \\ &= B_2^* u,\; A\omega + B_1B_1^*u + B_2 z(1) \in H\}. \end{aligned} \tag{9}$$

We note that the domain of the operator $\mathcal{A}(t)$ is independent of the time t, i.e.,

$$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)) \quad \forall t > 0. \tag{10}$$

Now, we introduce the Hilbert space

$$\mathcal{H} = V \times H \times L^2((0,\,1),\, U_2)$$

equipped with the usual inner product

$$\left\langle \begin{pmatrix}\omega\\ u\\ z\end{pmatrix}, \begin{pmatrix}\tilde\omega\\ \tilde u\\ \tilde z\end{pmatrix}\right\rangle = \left(A^{\frac12}\omega,\, A^{\frac12}\tilde\omega\right)_H + (u,\, \tilde u)_H + \int_0^1 (z(\rho),\, \tilde z(\rho))_{U_2}\, d\rho. \tag{11}$$

A general theory for equations of type (8) has been developed using semigroup theory [14, 15, 28]. The simplest way to prove existence and uniqueness results is to show that the triplet $\{\mathcal{A}, \mathcal{H}, Y\}$, with $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ for some fixed $T > 0$ and $Y = \mathcal{D}(\mathcal{A}(0))$, forms a CD system (or constant domain system; see [14, 15]). More precisely, the following theorem gives some existence and uniqueness results (for its proof see Theorems I and II of [13] or Theorem 1.9 of [14]; see also Theorem 1.2 of [15] or [12], [2]).

THEOREM 2.1. *Assume the following:*

(i) *For all $t \in [0, T]$, $\mathcal{A}(t)$ generates a strongly continuous semigroup on $\mathcal{H}$, and the family $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ is stable with stability constants C and m independent of t (i.e., the semigroup $(S_t(s))_{s\geq 0}$ generated by $\mathcal{A}(t)$ satisfies, for all $0 \leq t_1 \leq \cdots \leq t_k \leq T$, $k \in \mathbb{N}^*$,*

$$\left\| \prod_{j=1}^k S_{t_j}(s_j)u \right\|_{\mathcal{H}} \leq Ce^{m(s_1+\cdots+s_k)}\|u\|_{\mathcal{H}} \quad \forall s_j \geq 0,$$

and for all $u \in \mathcal{H}$).

(ii) *Equation* (10) *holds.*

(iii) $\partial_t \mathcal{A}$ *belongs to* $L^\infty_*([0, T], B(Y, \mathcal{H}))$, *the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0, T]$ *into the set* $B(Y, \mathcal{H})$ *of bounded operators from* Y *into* $\mathcal{H}$.

Then, problem (8) *has a unique mild solution* $U \in C([0, T], \mathcal{H})$ *for any initial data in* $\mathcal{H}$, *and for all* $t \in [0, T]$ *there exists a positive constant* $c(t)$ *such that*

$$\|U(t)\|_{\mathcal{H}} \leq c(t)\|U_0\|_{\mathcal{H}}.$$

If, moreover, the initial data U_0 *belongs to* Y, *then problem* (8) *has a unique strong solution* $U \in C([0, T], Y) \cap C^1([0, T], \mathcal{H})$; *in that last case,* (8) *holds in a strong sense, i.e.,*

$$U'(t) = \mathcal{A}(t)U(t) \quad \forall t > 0, \quad \textit{and} \quad U(0) = U_0.$$

Our goal then is to check the above assumptions for system (8).

Let us suppose that

$$\exists\, 0 < \alpha \leq \sqrt{1-d} \quad \forall u \in V, \quad \|B_2^* u\|^2_{U_2} \leq \alpha \, \|B_1^* u\|^2_{U_1}, \tag{12}$$

where d is given by (2). Note that the choice of α is possible since $d < 1$ by (2).

Now, we will show that the operator $\mathcal{A}(t)$ generates a C_0-semigroup in $\mathcal{H}$ and, by using the variable norm technique of Kato from [14], we will prove that system (8) (and then (5)) has a unique solution.

For this purpose, we introduce the time-dependent inner product on $\mathcal{H}$,

$$\left\langle \begin{pmatrix} \omega \\ u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{\omega} \\ \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle_t = \left(A^{\frac{1}{2}}\omega, \, A^{\frac{1}{2}}\tilde{\omega}\right)_H + (u, \, \tilde{u})_H + q\tau(t) \int_0^1 (z(\rho), \, \tilde{z}(\rho))_{U_2} \, d\rho,$$

where q is a positive constant chosen such that

$$\frac{1}{\sqrt{1-d}} \leq q \leq \frac{2}{\alpha} - \frac{1}{\sqrt{1-d}} \tag{13}$$

with associated norm denoted by $\|.\|_t$. This choice of q is possible since $0 < \alpha \leq \sqrt{1-d}$ by (12). This new inner product is clearly equivalent to the usual inner product (11) on $\mathcal{H}$.

THEOREM 2.2. *Under the assumptions* (2), (3), (4), *and* (12), *for an initial datum* $U_0 \in \mathcal{H}$, *there exists a unique mild solution* $U \in C([0, +\infty), \mathcal{H})$ *to system* (8), *and for all* $t > 0$, *there exists a positive constant* $c(t)$ *such that*

$$\|U(t)\|_{\mathcal{H}} \leq c(t)\|U_0\|_{\mathcal{H}}. \tag{14}$$

If, moreover, the initial datum $U_0 \in D(\mathcal{A}(t))$, *then there exists a unique strong solution*

$$U \in C([0, +\infty), \, D(\mathcal{A}(t))) \cap C^1([0, +\infty), \mathcal{H})$$

to system (8).

Proof. We first notice that

$$\frac{\|\phi\|_t}{\|\phi\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|} \quad \forall t, \quad s \in [0, T], \tag{15}$$

where $\phi = (\omega, u, z)^\top$ and c is a positive constant. Indeed, for all $s, t \in [0, T]$, we have

$$\begin{aligned}\|\phi\|_t^2 - \|\phi\|_s^2 e^{\frac{c}{\tau_0}|t-s|} &= \left(1 - e^{\frac{c}{\tau_0}|t-s|}\right)\left(\left\|A^{\frac12}\omega\right\|_H^2 + \|u\|_H^2\right) \\ &\quad + q\left(\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|}\right)\int_0^1 \|z(\rho)\|_{U_2}^2\, d\rho.\end{aligned}$$

We note that $1 - e^{\frac{c}{\tau_0}|t-s|} \leq 0$. Moreover, $\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|} \leq 0$ for some $c > 0$. Indeed,

$$\tau(t) = \tau(s) + \dot{\tau}(a)(t-s), \quad \text{where } a \in (s, t),$$

and thus,

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{|\dot{\tau}(a)|}{\tau(s)}|t-s|.$$

By (4), $\dot{\tau}$ is bounded, and therefore, there exists $c > 0$ such that

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{c}{\tau_0}|t-s| \leq e^{\frac{c}{\tau_0}|t-s|}$$

by (3), which proves (15).

We now prove that $\mathcal{A}(t)$ is dissipative up to a translation for a fixed $t > 0$. Take $U = (\omega, u, z)^\top \in D(\mathcal{A}(t))$. Then

$$\begin{aligned}\langle \mathcal{A}(t)U, U\rangle_t &= \left\langle \begin{pmatrix} u \\ -A\omega - B_1B_1^*u - B_2z(1) \\ \frac{\dot{\tau}(t)\rho-1}{\tau(t)}\frac{\partial z}{\partial \rho}\end{pmatrix}, \begin{pmatrix}\omega \\ u \\ z\end{pmatrix}\right\rangle_t \\ &= \left(A^{\frac12}u, A^{\frac12}\omega\right)_H - (A\omega + B_1B_1^*u + B_2z(1), u)_H \\ &\quad -q\int_0^1 \left(\frac{\partial z}{\partial\rho}(\rho), z(\rho)\right)_{U_2}(1 - \dot{\tau}(t)\rho)d\rho.\end{aligned}$$

Since $A\omega + B_1B_1^*u + B_2z(1) \in H$, we obtain

$$\begin{aligned}\langle \mathcal{A}(t)U, U\rangle_t &= \left(A^{\frac12}u, A^{\frac12}\omega\right)_H - \langle A\omega, u\rangle_{V',V} - \langle B_1B_1^*u, u\rangle_{V',V} - \langle B_2z(1), u\rangle_{V',V} \\ &\quad - q\int_0^1 \left(\frac{\partial z}{\partial\rho}(\rho), z(\rho)\right)_{U_2}(1 - \dot{\tau}(t)\rho)d\rho \\ &= \langle A\omega, u\rangle_{V',V} - \langle A\omega, u\rangle_{V',V} - \|B_1^*u\|_{U_1}^2 - (z(1), B_2^*u)_{U_2} \\ &\quad - q\int_0^1 \left(\frac{\partial z}{\partial\rho}(\rho), z(\rho)\right)_{U_2}(1 - \dot{\tau}(t)\rho)d\rho\end{aligned}$$

by duality. By integrating by parts in ρ, we obtain

$$\begin{aligned}\int_0^1 \left(\frac{\partial z}{\partial\rho}(\rho), z(\rho)\right)_{U_2}(1 - \dot{\tau}(t)\rho)d\rho &= \int_0^1 \frac12\frac{\partial}{\partial\rho}\left(\|z\|_{U_2}^2\right)(1 - \dot{\tau}(t)\rho)d\rho \\ &= \frac{\dot{\tau}(t)}{2}\int_0^1 \|z\|_{U_2}^2\, d\rho + \frac12\|z(1)\|_{U_2}^2(1 - \dot{\tau}(t)) \\ &\quad -\frac12\|B_2^*u\|_{U_2}^2.\end{aligned}$$

Therefore

$$\langle \mathcal{A}(t)U, U\rangle_t = -\|B_1^*u\|_{U_1}^2 - (z(1), B_2^*u)_{U_2} - \frac{q}{2}\|z(1)\|_{U_2}^2 (1-\dot\tau(t)) + \frac{q}{2}\|B_2^*u\|_{U_2}^2 - \frac{q\dot\tau(t)}{2}\int_0^1 \|z\|_{U_2}^2\, d\rho.$$

By Young's inequality and (12), we find

$$\langle \mathcal{A}(t)U, U\rangle_t \le \left(\frac{\alpha}{2\sqrt{1-d}} + \frac{q\alpha}{2} - 1\right)\|B_1^*u\|_{U_1}^2 + \left(\frac{\sqrt{1-d}}{2} - \frac{q(1-d)}{2}\right)\|z(1)\|_{U_2}^2 + \kappa(t)\langle U, U\rangle_t,$$

where

$$\kappa(t) = \frac{(\dot\tau(t)^2+1)^{1/2}}{2\tau(t)}. \tag{16}$$

Observe that $\frac{\alpha}{2\sqrt{1-d}} + \frac{q\alpha}{2} - 1 \le 0$ and $\frac{\sqrt{1-d}}{2} - \frac{q(1-d)}{2} \le 0$ since q satisfies (13).

This shows that

$$\langle \mathcal{A}(t)U, U\rangle_t - \kappa(t)\langle U, U\rangle_t \le 0, \tag{17}$$

which means that the operator $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$ is dissipative.

Let us now prove that $\lambda I - \mathcal{A}(t)$ is surjective for a fixed $t > 0$ and any $\lambda > 0$.

Let $(f, g, h)^T \in \mathcal{H}$. We look for a solution $U = (\omega, u, z)^T \in D(\mathcal{A}(t))$ of

$$(\lambda I - \mathcal{A}(t))\begin{pmatrix}\omega\\ u\\ z\end{pmatrix} = \begin{pmatrix} f\\ g\\ h\end{pmatrix}$$

or, equivalently,

$$\begin{cases} \lambda\omega - u = f,\\ \lambda u + A\omega + B_1B_1^*u + B_2 z(1) = g,\\ \lambda z + \frac{1-\dot\tau(t)\rho}{\tau(t)}\frac{\partial z}{\partial\rho} = h. \end{cases} \tag{18}$$

Suppose that we have found ω with the appropriate regularity. Then, we have

$$u = -f + \lambda\omega \in V.$$

We can then determine z. Indeed, z satisfies the differential equation

$$\lambda z + \frac{1-\dot\tau(t)\rho}{\tau(t)}\frac{\partial z}{\partial\rho} = h$$

and the boundary condition $z(0) = B_2^*u = -B_2^*f + \lambda B_2^*\omega$. Therefore z is explicitly given by

$$z(\rho) = \lambda B_2^*\omega e^{-\lambda\tau(t)\rho} - B_2^* f e^{-\lambda\tau(t)\rho} + \tau(t)e^{-\lambda\tau(t)\rho}\int_0^\rho e^{\lambda\tau(t)\sigma}h(\sigma)d\sigma$$

if $\dot{\tau}(t) = 0$, and

$$z(\rho) = \lambda B_2^* \omega e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\rho)} - B_2^* f e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\rho)} + \tau(t) e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\rho)} \int_0^\rho \frac{h(\sigma)}{1-\dot{\tau}(t)\sigma} e^{-\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\sigma)} d\sigma$$

otherwise. This means that once ω is found with the appropriate properties, we can find z and u. In particular, we have, if $\dot{\tau}(t) = 0$,

$$z(1) = \lambda B_2^* \omega e^{-\lambda\tau(t)} + z^0, \tag{19}$$

where $z^0 = -B_2^* f e^{-\lambda\tau(t)} + \tau(t) e^{-\lambda\tau(t)} \int_0^1 e^{\lambda\tau(t)\sigma} h(\sigma) d\sigma$ is a fixed element of U_2 depending only on f and h, and, otherwise,

$$z(1) = \lambda B_2^* \omega e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} + z^0, \tag{20}$$

where $z^0 = -B_2^* f e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} + \tau(t) e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} \int_0^1 \frac{h(\sigma)}{1-\dot{\tau}(t)\sigma} e^{-\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} d\sigma$ is a fixed element of U_2 depending only on f and h.

It remains to find ω. By (18), ω must satisfy

$$\lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + B_2 z(1) = g + B_1 B_1^* f + \lambda f,$$

and thus by (19),

$$\lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + \lambda e^{-\lambda\tau(t)} B_2 B_2^* \omega = g + B_1 B_1^* f + \lambda f - B_2 z^0 =: q,$$

where $q \in V'$ if $\dot{\tau}(t) = 0$, and by (20),

$$\lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + \lambda e^{\frac{\lambda\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} B_2 B_2^* \omega = g + B_1 B_1^* f + \lambda f - B_2 z^0 =: q,$$

where $q \in V'$ otherwise. Assume $\dot{\tau}(t) = 0$. We then take the duality brackets $\langle ., . \rangle_{V',V}$ with $\phi \in V$:

$$\left\langle \lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + \lambda e^{-\lambda\tau(t)} B_2 B_2^* \omega, \, \phi \right\rangle_{V',V} = \langle q, \, \phi \rangle_{V',V} .$$

Moreover,

$$\begin{aligned} &\left\langle \lambda^2 \omega + A\omega + \lambda B_1 B_1^* \omega + \lambda e^{-\lambda\tau(t)} B_2 B_2^* \omega, \, \phi \right\rangle_{V',V} \\ &= \lambda^2 \langle \omega, \, \phi \rangle_{V',V} + \langle A\omega, \, \phi \rangle_{V',V} + \lambda(\langle B_1 B_1^* \omega, \, \phi \rangle_{V',V} + e^{-\lambda\tau(t)} \langle B_2 B_2^* \omega, \, \phi \rangle_{V',V}) \\ &= \lambda^2 (\omega, \, \phi)_H + \left(A^{\frac{1}{2}} \omega, \, A^{\frac{1}{2}} \phi \right)_H + \lambda((B_1^* \omega, \, B_1^* \phi)_{U_1} + e^{-\lambda\tau(t)} (B_2^* \omega, \, B_2^* \phi)_{U_2}) \end{aligned}$$

because $\omega \in V \subset H$. Consequently, we arrive at the problem

$$\lambda^2 (\omega, \, \phi)_H + \left(A^{\frac{1}{2}} \omega, \, A^{\frac{1}{2}} \phi \right)_H + \lambda((B_1^* \omega, \, B_1^* \phi)_{U_1} + e^{-\lambda\tau(t)} (B_2^* \omega, \, B_2^* \phi)_{U_2}) = \langle q \, \phi \rangle_{V',V} \ \forall \phi \in V. \tag{21}$$

The left-hand side of (21) is continuous and coercive on V. Indeed, we have

$$
\begin{aligned}
&\left|\lambda^2 (\omega, \phi)_H + \left(A^{\frac{1}{2}}\omega, A^{\frac{1}{2}}\phi\right)_H + \lambda((B_1^*\omega, B_1^*\phi)_{U_1} + e^{-\lambda\tau(t)} (B_2^*\omega, B_2^*\phi)_{U_2})\right| \\
&\leq \lambda^2 \|\omega\|_H \|\phi\|_H + \left\|A^{\frac{1}{2}}\omega\right\|_H \left\|A^{\frac{1}{2}}\phi\right\|_H + \lambda(\|B_1^*\omega\|_{U_1} \|B_1^*\phi\|_{U_1} \\
&\qquad\qquad + e^{-\lambda\tau(t)} \|B_2^*\omega\|_{U_2} \|B_2^*\phi\|_{U_2}) \\
&\leq C\lambda^2 \|\omega\|_V \|\phi\|_H + \left\|A^{\frac{1}{2}}\right\|^2 \|\omega\|_V \|\phi\|_V \\
&\qquad + \lambda(\|B_1^*\|^2_{\mathcal{L}(V, U_1)} \|\omega\|_V \|\phi\|_V + e^{-\lambda\tau(t)} \|B_2^*\|^2_{\mathcal{L}(V, U_2)} \|\omega\|_V \|\phi\|_V) \\
&\leq C \|\omega\|_V \|\phi\|_V ,
\end{aligned}
$$

and for $\phi = \omega \in V$,

$$
\begin{aligned}
&\lambda^2 \|\omega\|^2_H + \left(A^{\frac{1}{2}}\omega, A^{\frac{1}{2}}\omega\right)_H + \lambda(\|B_1^*\omega\|^2_{U_1} + e^{-\lambda\tau(t)} \|B_2^*\omega\|^2_{U_2}) \\
&\geq \left\|A^{\frac{1}{2}}\omega\right\|^2_H \geq C \|\omega\|^2_V .
\end{aligned}
$$

Therefore, problem (21) has a unique solution $\omega \in V$ by the Lax–Milgram lemma. We can easily prove the same results in the case where $\dot\tau(t) \neq 0$. Moreover, $A\omega + B_1B_1^*u+B_2z(1) = g+\lambda f-\lambda^2\omega \in H$. In summary, we have found $(\omega, u, z)^T \in D(\mathcal{A}(t))$ satisfying (18). Again as $\kappa(t) > 0$, this proves that

$$
\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))I - \mathcal{A}(t) \text{ is surjective} \tag{22}
$$

for some $\lambda > 0$ and $t > 0$.

According to Theorem I.4.6 of [28], for any $t > 0$, $\tilde{\mathcal{A}}(t)$ is the infinitesimal generator of a C_0-semigroup of contraction on $\mathcal{H}$ (with respect to the inner product $\langle\cdot, \cdot\rangle_t$). Therefore by (15), the family $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}(t) : t \in [0, T]\}$ is a stable family of generators in $\mathcal{H}$ with stability constants independent of t, by Proposition 3.4 from [12] (see also Proposition 1.1 from [14]).

It remains to check the assumption (iii) of Theorem 2.1. As

$$
\dot\kappa(t) = \frac{\ddot\tau(t)\dot\tau(t)}{2\tau(t)(\dot\tau(t)^2 + 1)^{\frac{1}{2}}} - \frac{\dot\tau(t)(\dot\tau(t)^2 + 1)^{\frac{1}{2}}}{2\tau(t)^2}
$$

is bounded on $[0, T]$ for all $T > 0$ (by (3) and (4)), we have

$$
\frac{d}{dt}\mathcal{A}(t)U = \begin{pmatrix} 0 \\ 0 \\ \frac{\ddot\tau(t)\tau(t)\rho-\dot\tau(t)(\dot\tau(t)\rho-1)}{\tau(t)^2} z_\rho \end{pmatrix}
$$

with $\frac{\ddot\tau(t)\tau(t)\rho-\dot\tau(t)(\dot\tau(t)\rho-1)}{\tau(t)^2}$ bounded on $[0, T]$ by (3) and (4). Thus

$$
\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L^\infty_*([0, T], B(D(\mathcal{A}(0)), \mathcal{H})),
$$

the space of equivalence classes of essentially bounded, strongly measurable functions from $[0, T]$ into $B(D(\mathcal{A}(0)), \mathcal{H})$.

In conclusion, the assumptions (i)–(iii) of Theorem 2.1 are verified for all $T > 0$, and thus the problem

$$\begin{cases} \tilde{U}' = \tilde{\mathcal{A}}(t)\tilde{U}, \\ \tilde{U}(0) = U_0 \end{cases}$$

has a unique mild (resp., strong) solution $\tilde{U} \in C([0, +\infty), \mathcal{H})$ (resp., $\tilde{U} \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), \mathcal{H}))$ for $U_0 \in \mathcal{H}$ (resp., $U_0 \in D(\mathcal{A}(0))$). The requested solution of (8) is then given by

$$U(t) = e^{\beta(t)}\tilde{U}(t)$$

with $\beta(t) = \int_0^t \kappa(s)ds$, because, for $U_0 \in D(\mathcal{A}(0))$, one has

$$\begin{aligned} U'(t) &= \kappa(t)e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{U}'(t) \\ &= \kappa(t)e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{\mathcal{A}}(t)\tilde{U}(t) \\ &= e^{\beta(t)}(\kappa(t)\tilde{U}(t) + \tilde{\mathcal{A}}(t)\tilde{U}(t)) \\ &= e^{\beta(t)}\mathcal{A}(t)\tilde{U}(t) = \mathcal{A}(t)e^{\beta(t)}\tilde{U}(t) \\ &= \mathcal{A}(t)U(t), \end{aligned}$$

which concludes the proof. ◻

3. The decay of the energy. We now restrict the hypothesis (12) to obtain the decay of the energy. For that, we suppose that (6) holds, namely,

$$\exists\, 0 < \alpha < \sqrt{1-d} \quad \forall u \in V, \quad \|B_2^* u\|_{U_2}^2 \le \alpha \|B_1^* u\|_{U_1}^2,$$

where d satisfies (2). Note that this is possible since $d < 1$ by (2).

Let us choose the energy

$$E(t) := \frac{1}{2}\left(\left\|A^{\frac{1}{2}}\omega\right\|_H^2 + \|\dot{\omega}\|_H^2 + q\tau(t)\int_0^1 \|B_2^*\dot{\omega}(t-\tau(t)\rho)\|_{U_2}^2\, d\rho\right), \tag{23}$$

where q is a positive constant satisfying

$$\frac{1}{\sqrt{1-d}} < q < \frac{2}{\alpha} - \frac{1}{\sqrt{1-d}}, \tag{24}$$

which exists by (6). Note that this energy corresponds to the time-dependent inner product on $\mathcal{H}$ defined before.

PROPOSITION 3.1. *If* (2), (3), (4), *and* (6) *hold, then for all* $(\omega_0, \omega_1, f^0(-\tau.))^T \in D(\mathcal{A}(0))$, *the energy of the corresponding regular solution of* (5) *is nonincreasing and there exists a positive constant* C *depending only on* α, d, *and* q *such that*

$$E'(t) \le -C\left(\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau(t))\|_{U_2}^2\right). \tag{25}$$

Proof. Deriving (23), we obtain

$$\begin{aligned} E'(t) = {} & \left(A^{\frac{1}{2}}\omega, A^{\frac{1}{2}}\dot{\omega}\right)_H + (\dot{\omega}, \ddot{\omega})_H + \frac{q\dot{\tau}(t)}{2}\int_0^1 \|B_2^*\dot{\omega}(t-\tau(t)\rho)\|_{U_2}^2\, d\rho \\ & + q\tau(t)\int_0^1 (B_2^*\dot{\omega}(t-\tau(t)\rho), B_2^*\ddot{\omega}(t-\tau(t)\rho))_{U_2}(1-\dot{\tau}(t)\rho)d\rho. \end{aligned}$$

Since $\ddot{\omega} = -(A\omega + B_1B_1^*\dot{\omega} + B_2B_2^*\dot{\omega}(t-\tau(t))) \in H$,

$$\begin{aligned} E'(t) &= \langle A\omega, \dot{\omega}\rangle_{V',V} - \langle \dot{\omega}, A\omega + B_1B_1^*\dot{\omega} + B_2B_2^*\dot{\omega}(t-\tau(t))\rangle_{V,V'} \\ &\quad + \frac{q\dot{\tau}(t)}{2}\int_0^1 \|B_2^*\dot{\omega}(t-\tau(t)\rho)\|_{U_2}^2\, d\rho \\ &\quad + q\tau(t)\int_0^1 (B_2^*\dot{\omega}(t-\tau\rho),\, B_2^*\ddot{\omega}(t-\tau(t)\rho))_{U_2}\,(1-\dot{\tau}(t)\rho)d\rho. \end{aligned}$$

Then

$$\begin{aligned} E'(t) &= \langle A\omega, \dot{\omega}\rangle_{V',V} - \langle \dot{\omega}, A\omega\rangle_{V,V'} - \langle \dot{\omega}, B_1B_1^*\dot{\omega}\rangle_{V,V'} - \langle \dot{\omega}, B_2B_2^*\dot{\omega}(t-\tau(t))\rangle_{V,V'} \\ &\quad + \frac{q\dot{\tau}(t)}{2}\int_0^1 \|B_2^*\dot{\omega}(t-\tau(t)\rho)\|_{U_2}^2\, d\rho \\ &\quad + q\tau(t)\int_0^1 (B_2^*\dot{\omega}(t-\tau(t)\rho),\, B_2^*\ddot{\omega}(t-\tau(t)\rho))_{U_2}\,(1-\dot{\tau}(t)\rho)d\rho \\ &= -\|B_1^*\dot{\omega}\|_{U_1}^2 - (B_2^*\dot{\omega},\, B_2^*\dot{\omega}(t-\tau(t)))_{U_2} + \frac{q\dot{\tau}(t)}{2}\int_0^1 \|B_2^*\dot{\omega}(t-\tau(t)\rho)\|_{U_2}^2\, d\rho \\ &\quad + q\tau(t)\int_0^1 (B_2^*\dot{\omega}(t-\tau(t)\rho),\, B_2^*\ddot{\omega}(t-\tau(t)\rho))_{U_2}\,(1-\dot{\tau}(t)\rho)d\rho. \end{aligned}$$

Moreover, recalling that $z(\rho, t) = B_2^*\dot{\omega}(t-\tau(t)\rho)$, and thus $z_\rho(\rho,t) = -\tau(t)B_2^*\ddot{\omega}(t-\tau(t)\rho)$, we see that

$$\begin{aligned} &\int_0^1 (B_2^*\dot{\omega}(t-\tau(t)\rho),\, B_2^*\ddot{\omega}(t-\tau(t)\rho))_{U_2}\,(1-\dot{\tau}(t)\rho)d\rho \\ &\quad = -\frac{1}{\tau(t)}\int_0^1 \left(z(\rho,t),\, \frac{\partial z}{\partial\rho}(\rho,t)\right)_{U_2}(1-\dot{\tau}(t)\rho)d\rho \\ &\quad = -\frac{1}{2\tau(t)}\int_0^1 \frac{\partial}{\partial\rho}\left(\|z(\rho,t)\|_{U_2}^2\right)(1-\dot{\tau}(t)\rho)d\rho \\ &\quad = -\frac{\dot{\tau}(t)}{2\tau(t)}\int_0^1 \|z(\rho,t)\|_{U_2}^2\, d\rho - \frac{1-\dot{\tau}(t)}{2\tau(t)}\|z(1,t)\|_{U_2}^2 + \frac{1}{2\tau(t)}\|z(0,t)\|_{U_2}^2 \\ &\quad = -\frac{\dot{\tau}(t)}{2\tau(t)}\int_0^1 \|B_2^*\dot{\omega}(t-\tau(t)\rho)\|_{U_2}^2\, d\rho - \frac{1-\dot{\tau}(t)}{2\tau(t)}\|B_2^*\dot{\omega}(t-\tau(t))\|_{U_2}^2 \\ &\qquad + \frac{1}{2\tau(t)}\|B_2^*\dot{\omega}(t)\|_{U_2}^2. \end{aligned}$$

Consequently,

$$\begin{aligned} E'(t) &= -\|B_1^*\dot{\omega}\|_{U_1}^2 - (B_2^*\dot{\omega},\, B_2^*\dot{\omega}(t-\tau(t)))_{U_2} \\ &\quad - \frac{q(1-\dot{\tau}(t))}{2}\|B_2^*\dot{\omega}(t-\tau(t))\|_{U_2}^2 + \frac{q}{2}\|B_2^*\dot{\omega}(t)\|_{U_2}^2. \end{aligned}$$

Young's inequality, (2), and (6) yield

$$E'(t) \le \left(\frac{\alpha}{2\sqrt{1-d}} + \frac{q\alpha}{2} - 1\right)\|B_1^*\dot{\omega}\|_{U_1}^2 + \left(\frac{\sqrt{1-d}}{2} - \frac{q(1-d)}{2}\right)\|B_2^*\dot{\omega}(t-\tau(t))\|_{U_2}^2.$$

Therefore, this estimate leads to

$$E'(t) \leq -C\left(\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau(t))\|_{U_2}^2\right)$$

with

$$C = \min\left\{\left(1 - \frac{q\alpha}{2} - \frac{\alpha}{2\sqrt{1-d}}\right), \left(\frac{q(1-d)}{2} - \frac{\sqrt{1-d}}{2}\right)\right\},$$

which is positive according to the assumption (24). ∎

Remark 3.2. The choice to apply Young's inequality with a factor $\sqrt{1-d}$ in the proof of the above proposition is made in order to give the stability result under the best assumption between α and d.

Remark 3.3. In the case where the delay is constant in time (and thus $d = 0$), we recover some results from [24].

Remark 3.4. If (6) is not satisfied, there exist cases where instabilities may appear (see [22, 25, 34]) for the wave equation with constant (in time) delay. Hence this condition appears to be quite realistic.

COROLLARY 3.5. *If* (2), (3), (4), *and* (6) *hold, then for all* $(\omega_0, \omega_1, f^0(-\tau.))^T \in \mathcal{H}$, *the energy of the corresponding mild solution of* (5) *is nonincreasing.*

Proof. The proof is a direct consequence of the estimate (14) from Theorem 2.2 and the density of $D(\mathcal{A}(0))$ into $\mathcal{H}$. ∎

4. Exponential stability. In this section, we prove, under some assumptions, the exponential stability of (5) by using an appropriate abstract Lyapunov functional, defined by

$$\mathcal{E}(t) = E(t) + \gamma\left(\mathcal{E}_2(t) + (\mathcal{M}\omega(t), \dot{\omega}(t))_H\right), \tag{26}$$

where γ is a positive small constant that will be chosen later on, E is the standard energy defined by (23) with q verifying (24), and $\mathcal{E}_2$ is defined by

$$\mathcal{E}_2(t) := q\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho}\,\|B_2^*\dot{\omega}(t-\tau(t)\rho)\|_{U_2}^2\,d\rho, \tag{27}$$

where δ is a fixed positive real number. Moreover, the operator $\mathcal{M} : V \to H$ satisfies the assumptions

$$\exists C_0, C_1, C_2 > 0,\ \frac{d}{dt}(\mathcal{M}\omega(t), \dot{\omega}(t))_H \leq -C_0E_0(t) + C_1\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + C_2\|B_2^*\dot{\omega}(t-\tau(t))\|_{U_2}^2 \tag{28}$$

for all solutions ω of (5) with initial data in $D(\mathcal{A}(0))$ and where E_0 is the natural energy for the problem without delay,

$$E_0(t) := \frac{1}{2}\left(\left\|A^{\frac{1}{2}}\omega(t)\right\|_H^2 + \|\dot{\omega}(t)\|_H^2\right),$$

and

$$\exists C > 0 \quad \forall t > 0, \quad |(\mathcal{M}\omega(t), \dot{\omega}(t))_H| \leq CE_0(t). \tag{29}$$

First, we note that the energies E and $\mathcal{E}$ are equivalent under (29).

LEMMA 4.1. *Assume* (2), (3), (4), (6), *and* (29). *For* γ *small enough, there exists a positive constant* $C_3(\gamma)$ *such that*

$$(1 - C\gamma)E(t) \leq \mathcal{E}(t) \leq C_3(\gamma)E(t), \text{ where } 1 - C\gamma > 0. \tag{30}$$

Proof. It is easy to see that

$$\mathcal{E}(t) \leq C_3(\gamma)E(t),$$

with $C_3(\gamma) = \max(1 + \gamma C, 1 + 2\gamma)$ by (29), since $e^{-2\delta\tau(t)\rho} \leq 1$.

For the second inequality of (30), we note that, since $\gamma\mathcal{E}_2(t) \geq 0$ and by (29),

$$\begin{aligned} \mathcal{E}(t) &\geq E(t) - C\gamma E_0(t) \\ &\geq (1 - C\gamma)E(t), \end{aligned}$$

and thus we obtain (30) with $1 - C\gamma > 0$ for γ small enough ($\gamma < 1/C$). ∎

To prove the exponential decay of (5), we need the following lemma.

LEMMA 4.2. *Assume* (2), (3), (4), *and* (6). *Then*

$$\frac{d}{dt}\mathcal{E}_2(t) \leq -2\delta\mathcal{E}_2(t) + q\,\|B_2^*\dot{\omega}(t)\|_{U_2}^2 . \tag{31}$$

Proof. Direct calculations show that

$$\frac{d}{dt}\mathcal{E}_2(t) = \frac{\dot{\tau}(t)}{\tau(t)}\mathcal{E}_2(t) + q\tau(t)\int_0^1 (-2\delta\dot{\tau}(t)\rho)e^{-2\delta\tau(t)\rho}\,\|B_2^*\dot{\omega}(t - \tau(t)\rho)\|_{U_2}^2\,d\rho + J,$$

where J is equal to

$$J := 2q\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho}\,(B_2^*\dot{\omega}(t - \tau(t)\rho), B_2^*\ddot{\omega}(t - \tau(t)\rho))_{U_2}\,(1 - \dot{\tau}(t)\rho)d\rho.$$

Recalling that $z(\rho, t) = B_2^*\dot{\omega}(t - \tau(t)\rho)$ and then $z_\rho(\rho, t) = -\tau(t)B_2^*\ddot{\omega}(t - \tau(t)\rho)$, we see that

$$J = -2q\int_0^1 e^{-2\delta\tau(t)\rho}\left(z(\rho, t), \frac{\partial z}{\partial\rho}(\rho, t)\right)_{U_2}(1 - \dot{\tau}(t)\rho)d\rho.$$

By integrating by parts in ρ, we obtain

$$\begin{aligned} J &= -J + 2q\int_0^1 e^{-2\delta\tau(t)\rho}\,\|z(\rho, t)\|_{U_2}^2\,(-2\delta\tau(t)(1 - \dot{\tau}(t)\rho) - \dot{\tau}(t))d\rho \\ &\quad - 2qe^{-2\delta\tau(t)}\,\|z(1, t)\|_{U_2}^2\,(1 - \dot{\tau}(t)) + 2q\,\|z(0, t)\|_{U_2}^2 , \end{aligned}$$

which yields

$$\begin{aligned} J &= q\int_0^1 e^{-2\delta\tau(t)\rho}\,\|B_2^*\dot{\omega}(t - \tau(t)\rho)\|_{U_2}^2\,(-2\delta\tau(t)(1 - \dot{\tau}(t)\rho) - \dot{\tau}(t))d\rho \\ &\quad -qe^{-2\delta\tau(t)}\,\|B_2^*\dot{\omega}(t - \tau(t))\|_{U_2}^2\,(1 - \dot{\tau}(t)) + q\,\|B_2^*\dot{\omega}(t)\|_{U_2}^2 . \end{aligned}$$

Consequently,

$$\frac{d}{dt}\mathcal{E}_2(t) = -2\delta\mathcal{E}_2(t) - q(1 - \dot{\tau}(t))e^{-2\delta\tau(t)}\,\|B_2^*\dot{\omega}(t - \tau(t))\|_{U_2}^2 + q\,\|B_2^*\dot{\omega}(t)\|_{U_2}^2 .$$

We thus get (31) by (2). ∎

Now, we are able to state the main result of this paper.

THEOREM 4.3. *Assume that* (2), (3), (4), (6), (28), *and* (29) *hold. Then there exist positive constants* ν *and* K *such that*

$$E(t) \leq Ke^{-\nu t}E(0) \quad \forall t > 0,$$

for all solutions of (5) *with initial data in* $D(\mathcal{A}(0))$.

Proof. We have, by the definition (26) of $\mathcal{E}$,

$$\frac{d}{dt}\mathcal{E}(t) = \frac{d}{dt}E(t) + \gamma\frac{d}{dt}\mathcal{E}_2(t) + \gamma\frac{d}{dt}\left(\mathcal{M}\omega(t), \dot{\omega}(t)\right)_H.$$

By (25), (28), and (29),

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) \quad &\leq \quad \left(\frac{\alpha}{2\sqrt{1-d}} + \frac{q\alpha}{2} - 1\right)\|B_1^*\dot{\omega}(t)\|_{U_1}^2 \\ &+ \left(\frac{\sqrt{1-d}}{2} - \frac{q(1-d)}{2}\right)\|B_2^*\dot{\omega}(t-\tau(t))\|_{U_2}^2 \\ &- 2\delta\gamma\mathcal{E}_2(t) + \gamma q\|B_2^*\dot{\omega}(t)\|_{U_2}^2 - \gamma C_0E_0(t) + \gamma C_1\|B_1^*\dot{\omega}(t)\|_{U_1}^2 \\ &+ \gamma C_2\|B_2^*\dot{\omega}(t-\tau(t))\|_{U_2}^2.\end{aligned}$$

Using (6), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) \leq &\left(\frac{\alpha}{2\sqrt{1-d}} + \frac{q\alpha}{2} - 1 + \gamma(q\alpha + C_1)\right)\|B_1^*\dot{\omega}(t)\|_{U_1}^2 \\ &+ \left(\frac{\sqrt{1-d}}{2} - \frac{q(1-d)}{2} + \gamma C_2\right)\|B_2^*\dot{\omega}(t-\tau(t))\|_{U_2}^2 - 2\delta\gamma\mathcal{E}_2(t) - \gamma C_0E_0(t).\end{aligned}$$

Now we take γ small enough; more precisely, we take $\gamma > 0$ such that

$$\gamma \leq \min\left(\frac{1 - \frac{\alpha}{2\sqrt{1-d}} - \frac{q\alpha}{2}}{q\alpha + C_1}, \frac{\frac{q(1-d)}{2} - \frac{\sqrt{1-d}}{2}}{C_2}\right).$$

Note that $(1 - \frac{\alpha}{2\sqrt{1-d}} - \frac{q\alpha}{2})/(q\alpha + C_1)$ and $(\frac{q(1-d)}{2} - \frac{\sqrt{1-d}}{2})/C_2$ are positive by the choice of q from (24). Then

$$\frac{d}{dt}\mathcal{E}(t) \leq -\gamma(2\delta\mathcal{E}_2(t) + C_0E_0(t)).$$

As $\tau(t) \leq M$ (by (3)), we have

$$\frac{d}{dt}\mathcal{E}(t) \leq -\gamma\left(C_0E_0(t) + 2\delta e^{-2\delta M}q\tau(t)\int_0^1\|B_2^*\dot{\omega}(t-\tau(t)\rho)\|_{U_2}^2\,d\rho\right),$$

and then, in view of the definition of E, there exists a constant $\gamma' > 0$ (depending on γ and δ: $\gamma' \leq \gamma\min(C_0, 4\delta e^{-2\delta M})$) such that

$$\frac{d}{dt}\mathcal{E}(t) \leq -\gamma' E(t).$$

By applying Lemma 4.1, we arrive at

$$\frac{d}{dt}\mathcal{E}(t) \leq -\frac{\gamma'}{C_3(\gamma)}\mathcal{E}(t).$$

Therefore

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{-\frac{\gamma'}{C_3(\gamma)}t} \quad \forall t > 0,$$

and Lemma 4.1 allows us to conclude the proof:

$$E(t) \leq \frac{1}{1-C\gamma}\mathcal{E}(t) \leq \frac{1}{1-C\gamma}\mathcal{E}(0)e^{-\frac{\gamma'}{C_3(\gamma)}t} \leq \frac{C_3(\gamma)}{1-C\gamma}E(0)e^{-\frac{\gamma'}{C_3(\gamma)}t}. \qquad \square$$

Remark 4.4. In the proof of Theorem 4.3, we note that we can explicitly calculate the decay rate ν of the energy, given by

$$\nu = \frac{\gamma}{C_3(\gamma)} \min\left(C_0,\, 4\delta e^{-2\delta M}\right),$$

with $C_3(\gamma) = \max(1+\gamma C, 1+2\gamma)$,

$$\gamma < \frac{1}{C}, \quad \gamma \leq \frac{1 - \frac{\alpha}{2\sqrt{1-d}} - \frac{q\alpha}{2}}{q\alpha + C_1}, \quad \text{and} \quad \gamma \leq \frac{\frac{q(1-d)}{2} - \frac{\sqrt{1-d}}{2}}{C_2}$$

(by Lemma 4.1 and Theorem 4.3), where C, C_0, C_1, C_2 are given by (28) and (29), α is defined by (6), q by (24), and δ is a positive real number. Recalling that M is the upper bound of τ, if the delay τ becomes larger, the decay rate is slower. Moreover, we can choose δ such that the decay of the energy is as quick as possible for given parameters. For that purpose, we note that the function $\delta \to 4\delta e^{-2\delta M}$ admits a maximum at $\delta = \frac{1}{2M}$ and that this maximum is $\frac{2}{Me}$. Thus the larger decay rate of the energy is given by

$$\nu_{max} = \frac{\gamma}{C_3(\gamma)} \min\left(C_0,\, \frac{2}{Me}\right).$$

COROLLARY 4.5. *Under the assumptions of Theorem* 4.3*, we have*

$$E(t) \leq Ke^{-\nu t}E(0) \quad \forall t > 0,$$

for all solutions of (5) *with initial data in* $\mathcal{H}$.

Proof. The proof is a simple consequence of the density of $D(\mathcal{A}(0))$ in $\mathcal{H}$ and of the estimate

$$E(t) \leq E(0),$$

for the solutions of (5) with initial data in $\mathcal{H}$ (see Corollary 3.5). $\square$

5. Examples. We wrap up this paper by considering different examples for which our abstract framework can be applied. To our knowledge, all the examples, with the exception of the first, are new. In all examples, we assume that the delay function τ satisfies the assumptions (2)–(4).

5.1. The wave equation.

5.1.1. The 1-d wave equation. In this subsection, we show that our abstract framework applies to the 1-d wave equation

$$
\left\{
\begin{array}{ll}
\frac{\partial^2 u}{\partial t^2}(x,\, t) - a\frac{\partial^2 u}{\partial x^2}(x,\, t) = 0, & 0 < x < \pi,\, t > 0, \\
u(0,\, t) = 0, & t > 0, \\
a\frac{\partial u}{\partial x}(\pi,\, t) = -\alpha_1 \frac{\partial u}{\partial t}(\pi,\, t) - \alpha_2 \frac{\partial u}{\partial t}(\pi,\, t - \tau(t)), & t > 0, \\
u(x,\, 0) = u^0(x),\ \frac{\partial u}{\partial t}(x,\, 0) = u^1(x), & 0 < x < \pi, \\
\frac{\partial u}{\partial t}(\pi,\, t - \tau(0)) = f^0(t - \tau(0)), & 0 < t < \tau(0),
\end{array}
\right. \tag{32}
$$

where $\alpha_1,\, \alpha_2 > 0,\ a > 0$. This system have been studied in [26]; we also refer the reader to [34] for a constant delay. First, we rewrite this system in the form (5). For that purpose, we introduce $H = L^2(0,\, \pi)$ and the operator $A : D(A) \to H$ defined by

$$
A\varphi = -a\frac{d^2}{dx^2}\varphi,
$$

where $D(A) = \{\varphi \in H^2(0,\, \pi) \cap V\, ;\ \frac{\partial \varphi}{\partial x}(\pi) = 0\}$ and $V = \{H^1(0,\, \pi)\, ;\ \varphi(0) = 0\}$. The operator A is self-adjoint and positive with a compact inverse in H. We now define $U = U_1 = U_2 = \mathbb{R}$ and the operators $B_i : U \to D(A^{\frac{1}{2}})'$ given by

$$
B_i k = \sqrt{\alpha_i}\, k\, \delta_\pi, \quad i = 1, 2.
$$

It is easy to verify that $B_i^*(\varphi) = \sqrt{\alpha_i}\, \varphi(\pi)$ for $\varphi \in D(A^{1/2})$, and thus $B_i B_i^*(\varphi) = \alpha_i\, \varphi(\pi)\delta_\pi$ for $\varphi \in D(A^{1/2})$ and $i = 1,\, 2$. Then the system (32) can be rewritten in the form (5). We notice that (12) is equivalent to

$$
\exists\, 0 < \alpha \leq \sqrt{1-d}, \quad \alpha_2 \leq \alpha\, \alpha_1. \tag{33}
$$

Taking $\alpha = \alpha_2/\alpha_1$, (33) is equivalent to

$$
\alpha_2^2 \leq (1-d)\alpha_1^2, \tag{34}
$$

which is the condition (10) from [26]. Consequently, under the condition (34), by Theorem 2.2, this system is well-posed, and by Proposition 3.1 the energy decays for $\alpha_2^2 < (1-d)\alpha_1^2$.

To prove the exponential stability of (32), we introduce the Lyapunov functional (26) with the operator $\mathcal{M} : V \to H$ defined by

$$
\mathcal{M}u = 2x\,\frac{\partial u}{\partial x}. \tag{35}
$$

Then (28) holds with $C_0 = 2$, $C_1 = \pi(1 + 2a\alpha_1^2)$, and $C_2 = 2a\pi\alpha_2^2$ (see (48) from [26]) and (29) holds with $C = 2\pi \max(1, 1/a)$. Therefore, our abstract framework applies here, and system (32) is exponentially stable under the previous hypotheses. We then recover the results from [26].

5.1.2. The multidimensional wave equation. In this subsection, we study the stability of the wave equation with a time-varying delay in the boundary condition. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be an open bounded set with a boundary Γ of class C^2. We assume that Γ is divided into two parts Γ_D and Γ_N, i.e., $\Gamma = \Gamma_D \cup \Gamma_N$, with $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$ and $\Gamma_D \neq \emptyset$. Moreover, we assume that

$$\Gamma_N^2 \subseteq \Gamma_N^1 = \Gamma_N.$$

In this domain Ω, we consider the initial boundary value problem

$$(36)\quad \begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) = 0 & \text{in} \quad \Omega \times (0,+\infty), \\ u(x,t) = 0 & \text{on} \quad \Gamma_D \times (0,+\infty), \\ \frac{\partial u}{\partial \nu}(x,t) = -\alpha_1 \frac{\partial u}{\partial t}(x,t)\chi_{\Gamma_N^1} - \alpha_2 \frac{\partial u}{\partial t}(x,t-\tau(t))\chi_{\Gamma_N^2} & \text{on} \quad \Gamma_N \times (0,+\infty), \\ u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial t}(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in} \quad \Gamma_N^2 \times (0,\tau(0)), \end{cases}$$

where $\nu(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$, and $\partial u/\partial \nu$ is the normal derivative. Note that system (36) has been studied, for instance, in [5, 16, 17, 18, 19, 20] without delay and in [22] with a constant delay.

Let us denote by $v \cdot w$ the Euclidean inner product between two vectors $v, w \in \mathbb{R}^n$. We assume that there exists $x_0 \in \mathbb{R}^n$ such that denoting by m the standard multiplier

$$m(x) := x - x_0,$$

we have

$$(37)\quad m(x) \cdot \nu(x) \leq 0 \quad \text{on} \quad \Gamma_D$$

and, for some positive constant δ,

$$(38)\quad m(x) \cdot \nu(x) \geq \delta > 0 \quad \text{on} \quad \Gamma_N.$$

In the particular case where $\Omega = O_1 \setminus O_2$, with O_1 and O_2 convex sets such that $O_2 \subset O_1$, assumptions (37), (38) hold with $\Gamma_N = \partial O_1$ and $\Gamma_D = \partial O_2$ for any $x_0 \in O_2$.

First, we rewrite this system in the form (5). For this purpose, we introduce $H = L^2(\Omega)$ and the operator $A : D(A) \to H$ defined by

$$A\varphi = -\Delta\varphi,$$

where $D(A) = \{\varphi \in H^2(\Omega) \cap V : \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \Gamma_N\}$, where, as usual,

$$V = H^1_{\Gamma_D}(\Omega) = \{ \varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_D \}.$$

The operator A is self-adjoint and positive with a compact inverse in H. We now define $U_1 = L^2(\Gamma_N^1)$, $U_2 = L^2(\Gamma_N^2)$, and the operators $B_i^* : V \to U_i$ as

$$(39)\quad B_i^*\varphi = \sqrt{\alpha_i}\, \varphi_{|\Gamma_N^i}, \qquad i = 1, 2,$$

where $\varphi_{|\Gamma_N^i}$ is the trace operator for φ. The operator $B_i : U_i \to V'$ is then defined by duality:

$$(40)\quad \langle B_i u, v\rangle_{V',V} = \sqrt{\alpha_i} \int_{\Gamma_N^i} uv \, d\Gamma.$$

Thus the system (36) can be rewritten in the form (5). We notice that (12) is equivalent to (33), and then, as previously, to (34).

Consequently, under the condition (34), system (36) is well-posed by Theorem 2.2, and the energy decays by Proposition 3.1 for $\alpha_2^2 < (1-d)\alpha_1^2$.

To prove the exponential stability of (36), we introduce the Lyapunov functional (26) with the operator $\mathcal{M} : V \to H$ defined by

$$\mathcal{M}u = 2m \cdot \nabla u + (n-1)u. \tag{41}$$

Then we can easily prove that (29) holds by Poincaré's inequality. Moreover, we have the following.

LEMMA 5.1. *Condition* (28) *holds.*

Proof. Let $u \in H^2(\Omega)$. Then the standard multiplier identity gives

$$\begin{aligned} \frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u + (n-1)u]u_t dx\right\} &= -\int_\Omega \{u_t^2 + |\nabla u|^2\}dx \\ &+ \int_{\Gamma_N} (m\cdot\nu)(u_t^2 - |\nabla u|^2)d\Gamma + \int_{\Gamma_N} [2m\cdot\nabla u + (n-1)u]\frac{\partial u}{\partial \nu}d\Gamma \\ &+ \int_{\Gamma_D} (m\cdot\nu)\left|\frac{\partial u}{\partial\nu}\right|^2 d\Gamma. \end{aligned} \tag{42}$$

From (42) and Young's inequality, and recalling (37) and that, by (38) $m\cdot\nu \geq \delta$ on Γ_N, we have

$$\begin{aligned} \frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u + (n-1)u]u_t dx\right\} &\leq -\int_\Omega \{u_t^2 + |\nabla u|^2\}dx \\ &+ \int_{\Gamma_N} (m\cdot\nu)u_t^2 d\Gamma - \delta\int_{\Gamma_N} |\nabla u|^2 d\Gamma + \frac{c}{\varepsilon}\int_{\Gamma_N}\left(\frac{\partial u}{\partial\nu}\right)^2 d\Gamma \\ &+ \varepsilon\int_{\Gamma_N} (|\nabla u|^2 + u^2)d\Gamma \end{aligned} \tag{43}$$

for some positive constants ε, c. Using the trace inequality and then Poincaré's theorem, we have, for some c', $c'' > 0$,

$$\int_{\Gamma_N} u^2 d\Gamma \leq c'\|u\|^2_{H^1(\Omega)} \leq c''\int_\Omega |\nabla u|^2 dx.$$

This estimate in (43) yields, for ε small enough ($\varepsilon < \min(\delta,\, 1/(2c''))$),

$$\begin{aligned} \frac{d}{dt}\left\{\int_\Omega [2m\cdot\nabla u + (n-1)u]u_t dx\right\} &\leq -C_0E_0(t) \\ &+ C\int_{\Gamma_N} u_t^2 d\Gamma + C\int_{\Gamma_N}\left(\frac{\partial u}{\partial\nu}\right)^2 d\Gamma \end{aligned} \tag{44}$$

for suitable positive constants C_0, C. Therefore, using the boundary condition (36) and the Cauchy–Schwarz inequality in (44), we obtain (28). ◻

Therefore, our abstract framework still applies, and system (36) is exponentially stable under the above assumptions.

5.2. The beam equation. In this subsection, we show that our abstract framework can be applied to the 1-d beam equation

$$
(45) \qquad \begin{cases} \frac{\partial^2\omega}{\partial t^2}(x,\,t) + \frac{\partial^4\omega}{\partial x^4}(x,\,t) = 0, & 0 < x < 1,\, t > 0, \\ \omega(0,\,t) = \frac{\partial\omega}{\partial x}(0,\,t) = 0, & t > 0, \\ \frac{\partial^2\omega}{\partial x^2}(1,\,t) = 0, & t > 0, \\ \frac{\partial^3\omega}{\partial x^3}(1,\,t) = \alpha_1\frac{\partial\omega}{\partial t}(1,\,t) + \alpha_2\frac{\partial\omega}{\partial t}(1,\,t-\tau(t)), & t > 0, \\ \omega(x,\,0) = \omega^0(x),\ \frac{\partial\omega}{\partial t}(x,\,0) = \omega^1(x), & 0 < x < 1, \\ \frac{\partial\omega}{\partial t}(1,\,t-\tau(0)) = f^0(t-\tau(0)), & 0 < t < \tau(0), \end{cases}
$$

where $\alpha_1,\,\alpha_2 > 0$. First, we rewrite this system in the form (5). For this purpose, we introduce $H = L^2(0,\,1)$ and the operator $A : D(A) \to H$ defined by

$$
A\varphi = \frac{d^4}{dx^4}\varphi,
$$

where $D(A) = \{\varphi \in H^4(0,\,1) \cap V\,;\, \frac{\partial^2\varphi}{\partial x^2}(1) = \frac{\partial^3\varphi}{\partial x^3}(1) = 0\}$ and $V = \{\varphi \in H^2(0,\,1)\,;\, \varphi(0) = \frac{\partial\varphi}{\partial x}(0) = 0\}$, which is a self-adjoint and positive operator with a compact inverse in H. We now define $U = U_1 = U_2 = \mathbb{R}$ and the operators $B_i : U \to D(A^{\frac{1}{2}})'$ given by

$$
B_i k = \sqrt{\alpha_i}\, k\, \delta_1, \qquad i = 1, 2.
$$

It is easy to verify that $B_i^*(\varphi) = \sqrt{\alpha_i}\,\varphi(1)$ for $\varphi \in D(A^{1/2})$, and thus $B_iB_i^*(\varphi) = \alpha_i\,\varphi(1)\delta_1$ for $\varphi \in D(A^{1/2})$ and $i = 1,\,2$. Then the system (45) can be rewritten in the form (5). We notice that (12) is equivalent to (33), and by taking $\alpha = \alpha_2/\alpha_1$, (33) is equivalent to (34).

Accordingly, under the condition (34), this system is well-posed by Theorem 2.2, and the energy decays by Proposition 3.1 for $\alpha_2^2 < (1-d)\alpha_1^2$.

To prove the exponential stability of (45), we introduce the Lyapunov functional (26) with the operator $\mathcal{M} : V \to H$ defined by (35).

The following lemma shows that (28) and (29) hold.

LEMMA 5.2. *The conditions* (28) *and* (29) *hold.*

Proof. Condition (29) follows directly from Young's inequality:

$$
\begin{aligned}
|(\mathcal{M}\omega, \dot{\omega})_H| &= \left|2\int_0^1 x\frac{\partial\omega}{\partial x}(x,t)\frac{\partial\omega}{\partial t}(x,t)dx\right| \\
&\leq \int_0^1 \left(\left(\frac{\partial\omega}{\partial x}(x,t)\right)^2 + \left(\frac{\partial\omega}{\partial t}(x,t)\right)^2\right) dx.
\end{aligned}
$$

For the other assertion, we note that

$$
\frac{d}{dt}\,(\mathcal{M}\omega, \dot{\omega})_H = \int_0^1 \left(2x\,\frac{\partial^2\omega}{\partial x\partial t}(x,t)\frac{\partial\omega}{\partial t}(x,t) - 2x\,\frac{\partial\omega}{\partial x}(x,t)\frac{\partial^4\omega}{\partial x^4}(x,t)\right) dx.
$$

But, by integrating by parts, we obtain

$$
2\int_0^1 x\,\frac{\partial^2\omega}{\partial x\partial t}(x,t)\frac{\partial\omega}{\partial t}(x,t)dx = -\int_0^1 \left(\frac{\partial\omega}{\partial t}(x,t)\right)^2 dx + \left(\frac{\partial\omega}{\partial t}(1,t)\right)^2.
$$

Moreover, again integrating by parts yields

$$\int_0^1 x\,\frac{\partial\omega}{\partial x}(x,t)\frac{\partial^4\omega}{\partial x^4}(x,t)dx = -\int_0^1 \frac{\partial\omega}{\partial x}(x,t)\frac{\partial^3\omega}{\partial x^3}(x,t)dx - \int_0^1 x\,\frac{\partial^2\omega}{\partial x^2}(x,t)\frac{\partial^3\omega}{\partial x^3}(x,t)dx + \frac{\partial\omega}{\partial x}(1,t)\frac{\partial^3\omega}{\partial x^3}(1,t),$$

with

$$\int_0^1 x\,\frac{\partial^2\omega}{\partial x^2}(x,t)\frac{\partial^3\omega}{\partial x^3}(x,t)dx = -\frac{1}{2}\int_0^1 \left(\frac{\partial^2\omega}{\partial x^2}(x,t)\right)^2 dx + \frac{1}{2}\left(\frac{\partial^2\omega}{\partial x^2}(1,t)\right)^2$$

and

$$\int_0^1 \frac{\partial\omega}{\partial x}(x,t)\frac{\partial^3\omega}{\partial x^3}(x,t)dx = -\int_0^1 \left(\frac{\partial^2\omega}{\partial x^2}(x,t)\right)^2 dx + \frac{\partial\omega}{\partial x}(1,t)\frac{\partial^2\omega}{\partial x^2}(1,t) - \frac{\partial\omega}{\partial x}(0,t)\frac{\partial^2\omega}{\partial x^2}(0,t).$$

Consequently,

$$\int_0^1 x\,\frac{\partial\omega}{\partial x}(x,t)\frac{\partial^4\omega}{\partial x^4}(x,t)dx = \frac{3}{2}\int_0^1 \left(\frac{\partial^2\omega}{\partial x^2}(x,t)\right)^2 dx - \frac{\partial\omega}{\partial x}(1,t)\frac{\partial^2\omega}{\partial x^2}(1,t) + \frac{\partial\omega}{\partial x}(0,t)\frac{\partial^2\omega}{\partial x^2}(0,t) - \frac{1}{2}\left(\frac{\partial^2\omega}{\partial x^2}(1,t)\right)^2 + \frac{\partial\omega}{\partial x}(1,t)\frac{\partial^3\omega}{\partial x^3}(1,t).$$

Therefore, the boundary conditions satisfied by ω lead to

$$\frac{d}{dt}\left(\mathcal{M}\omega,\dot{\omega}\right)_H = -\int_0^1 \left(\frac{\partial\omega}{\partial t}(x,t)\right)^2 dx + \left(\frac{\partial\omega}{\partial t}(1,t)\right)^2 - 3\int_0^1 \left(\frac{\partial^2\omega}{\partial x^2}(x,t)\right)^2 dx - 2\frac{\partial\omega}{\partial x}(1,t)\frac{\partial^3\omega}{\partial x^3}(1,t).$$

By Young's inequality, we have

$$\left|-2\frac{\partial\omega}{\partial x}(1,t)\frac{\partial^3\omega}{\partial x^3}(1,t)\right| \le \epsilon\left(\frac{\partial\omega}{\partial x}(1,t)\right)^2 + \frac{1}{\epsilon}\left(\frac{\partial^3\omega}{\partial x^3}(1,t)\right)^2 \quad \forall\epsilon > 0.$$

Moreover, by the trace inequality and Poincaré's inequality, there exists a constant $C > 0$ such that

$$\left(\frac{\partial\omega}{\partial x}(1,t)\right)^2 \le C\int_0^1 \left(\frac{\partial^2\omega}{\partial x^2}(x,t)\right)^2 dx.$$

Thus, by the dissipation condition at 1 of (45),

$$\left|-2\frac{\partial\omega}{\partial x}(1,t)\frac{\partial^3\omega}{\partial x^3}(1,t)\right| \le C\epsilon\int_0^1 \left(\frac{\partial^2\omega}{\partial x^2}(x,t)\right)^2 dx + \frac{2\alpha_1^2}{\epsilon}\left(\frac{\partial\omega}{\partial t}(1,t)\right)^2 + \frac{2\alpha_2^2}{\epsilon}\left(\frac{\partial\omega}{\partial t}(1,t-\tau(t))\right)^2.$$

Therefore, it holds that

$$\frac{d}{dt}(\mathcal{M}\omega, \dot{\omega})_H \leq -\int_0^1 \left(\frac{\partial \omega}{\partial t}(x,t)\right)^2 dx - (3 - C\epsilon)\int_0^1 \left(\frac{\partial^2 \omega}{\partial x^2}(x,t)\right)^2 dx$$
$$+\left(1 + \frac{2\alpha_1^2}{\epsilon}\right)\left(\frac{\partial \omega}{\partial t}(1,t)\right)^2 + \frac{2\alpha_2^2}{\epsilon}\left(\frac{\partial \omega}{\partial t}(1, t-\tau(t))\right)^2 \ \forall \epsilon > 0.$$

It suffices to take $\epsilon \leq 2/C$ to obtain

$$\frac{d}{dt}(\mathcal{M}\omega, \dot{\omega})_H \leq -\int_0^1 \left(\left(\frac{\partial \omega}{\partial t}(x,t)\right)^2 + \left(\frac{\partial^2 \omega}{\partial x^2}(x,t)\right)^2\right) dx + C_1\alpha_1\left(\frac{\partial \omega}{\partial t}(1,t)\right)^2$$
$$+ C_2\alpha_2\left(\frac{\partial \omega}{\partial t}(1, t-\tau(t))\right)^2,$$

with C_1, $C_2 > 0$, which corresponds to (28). □

Therefore, by our abstract framework the system (45) is exponentially stable under the above assumptions.

5.3. The plate equation. In this subsection, we study the stability of the plate equation with boundary time-varying delay. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be an open bounded set with a boundary Γ of class C^4. We assume that Γ is divided into two parts Γ_D and Γ_N, i.e., $\Gamma = \Gamma_D \cup \Gamma_N$, with $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$ and $\Gamma_D \neq \emptyset$. Moreover, we assume that

$$\Gamma_N^2 \subseteq \Gamma_N^1 = \Gamma_N.$$

In this domain Ω, we consider the initial boundary value problem

$$(46)\quad \begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) + \Delta^2 u(x,t) = 0 & \text{in} \quad \Omega \times (0,+\infty), \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0 & \text{on} \quad \Gamma_D \times (0,+\infty), \\ \Delta u(x,t) = 0 & \text{on} \quad \Gamma_N \times (0,+\infty), \\ \frac{\partial \Delta u}{\partial \nu}(x,t) = \alpha_1 \frac{\partial u}{\partial t}(x,t)\chi_{\Gamma_N} + \alpha_2 \frac{\partial u}{\partial t}(x,t-\tau(t))\chi_{\Gamma_N^2} & \text{on} \quad \Gamma_N \times (0,+\infty), \\ u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial t}(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in} \quad \Gamma_N^2 \times (0,\tau(0)). \end{cases}$$

We assume that (37) holds, with the standard multiplier $m(x) := x - x_0$, for some $x_0 \in \mathbb{R}^n$. Note that the hypothesis (38) is not necessary.

To rewrite this system in the form (5), we introduce $H = L^2(\Omega)$ and the operator $A : D(A) \to H$ given by

$$A\varphi = \Delta^2 \varphi,$$

where $D(A) = \{\varphi \in H^4(\Omega) \cap V : \Delta\varphi = \frac{\partial \Delta\varphi}{\partial \nu} = 0 \text{ on } \Gamma_N\}$ and $V = \{\varphi \in H^2(\Omega) : \varphi = \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \Gamma_D\}$. The operator A is self-adjoint and positive with a compact inverse in H. The operators B_1^* and B_2^* are given here by (39), and B_1, B_2 are given by (40) with $U_1 = L^2(\Gamma_N^1)$, $U_2 = L^2(\Gamma_N^2)$.

Thus the system (46) can be rewritten in the form (5). We notice that (12) is equivalent to (33) and then, as previously, to (34).

Again, under the hypothesis (34), this system is well-posed by Theorem 2.2, and the energy decays by Proposition 3.1 for $\alpha_2^2 < (1-d)\alpha_1^2$.

To prove the exponential stability of (46), we introduce the Lyapunov functional (26) with the operator $\mathcal{M} : V \to H$ defined by (41). Then we can easily prove that (29) holds by Poincaré's theorem. Moreover, we have the following.

LEMMA 5.3. *Condition* (28) *holds.*

Proof. Direct calculation gives

(47)
$$\begin{aligned}\frac{d}{dt}\int_\Omega (2m\cdot\nabla u + (n-1)u)\, u_t dx \quad &= \quad \int_\Omega 2m\cdot\nabla u_t u_t dx + (n-1)\int_\Omega u_t^2 dx \\ &\quad - \int_\Omega (2m\cdot\nabla u)\,\Delta^2 u dx - (n-1)\int_\Omega u\Delta^2 u dx.\end{aligned}$$

By Green's formula, we find

$$\int_\Omega 2m\cdot\nabla u_t u_t dx = -n\int_\Omega u_t^2 dx + \int_\Gamma (m\cdot\nu) u_t^2 d\Gamma.$$

Moreover, again two applications of Green's formula lead to

$$\begin{aligned}\int_\Omega (2m\cdot\nabla u)\,\Delta^2 u dx = 2\int_\Omega &\Delta(m\cdot\nabla u)\Delta u dx \\ &- 2\int_\Gamma \frac{\partial}{\partial\nu}(m\cdot\nabla u)\Delta u d\Gamma + 2\int_\Gamma \frac{\partial \Delta u}{\partial\nu}(m\cdot\nabla u)d\Gamma,\end{aligned}$$

with

$$\Delta(m\cdot\nabla u)\Delta u = 2(\Delta u)^2 + m\cdot\nabla(\Delta u)\Delta u = 2(\Delta u)^2 + \frac{1}{2}m\cdot\nabla((\Delta u)^2).$$

Then

$$\begin{aligned}\int_\Omega (2m\cdot\nabla u)\,\Delta^2 u dx &= 4\int_\Omega (\Delta u)^2 dx + \int_\Omega m\cdot\nabla((\Delta u)^2)dx - 2\int_\Gamma \frac{\partial}{\partial\nu}(m\cdot\nabla u)\Delta u d\Gamma \\ &\quad +2\int_\Gamma \frac{\partial\Delta u}{\partial\nu}(m\cdot\nabla u)d\Gamma \\ &= 4\int_\Omega (\Delta u)^2 dx - n\int_\Omega (\Delta u)^2 dx + \int_\Gamma (m\cdot\nu)(\Delta u)^2 d\Gamma \\ &\quad -2\int_\Gamma \frac{\partial}{\partial\nu}(m\cdot\nabla u)\Delta u d\Gamma + 2\int_\Gamma \frac{\partial\Delta u}{\partial\nu}(m\cdot\nabla u)d\Gamma\end{aligned}$$

by Green's formula. For the last term of (47), we again use Green's formula twice:

$$\int_\Omega u\Delta^2 u dx = \int_\Omega (\Delta u)^2 dx - \int_\Gamma \frac{\partial u}{\partial\nu}\Delta u d\Gamma + \int_\Gamma \frac{\partial\Delta u}{\partial\nu} u d\Gamma.$$

Consequently, (47) becomes

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (2m\cdot\nabla u+(n-1)u)\,u_t dx\\
&\quad= -\int_\Omega \left(u_t^2+3(\Delta u)^2\right)dx+\int_\Gamma (m\cdot\nu)\left(u_t^2-(\Delta u)^2\right)d\Gamma\\
&\qquad+\int_\Gamma \left(2\frac{\partial}{\partial\nu}(m\cdot\nabla u)d\Gamma+(n-1)\frac{\partial u}{\partial\nu}\right)\Delta u d\Gamma\\
&\qquad-\int_\Gamma \frac{\partial\Delta u}{\partial\nu}\left(2(m\cdot\nabla u)+(n-1)u\right)d\Gamma.
\end{aligned}$$

As $u=\partial u/\partial\nu=0$ on Γ_D, $\nabla u=0$ on Γ_D, and

$$\frac{\partial}{\partial\nu}(m\cdot\nabla u)=m\cdot\nu\frac{\partial^2 u}{\partial\nu^2}=(m\cdot\nu)\Delta u \quad \text{on } \Gamma_D.$$

Therefore the boundary conditions of (46) implies

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (2m\cdot\nabla u+(n-1)u)\,u_t dx \;=\; & -\int_\Omega \left(u_t^2+3(\Delta u)^2\right)dx-\int_{\Gamma_D}(m\cdot\nu)(\Delta u)^2 d\Gamma\\
&+\int_{\Gamma_N}(m\cdot\nu)u_t^2 d\Gamma+2\int_{\Gamma_D}(m\cdot\nu)(\Delta u)^2 d\Gamma\\
&-\int_{\Gamma_N}\frac{\partial\Delta u}{\partial\nu}\left(2(m\cdot\nabla u)+(n-1)u\right)d\Gamma.
\end{aligned}$$

By (37), we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (2m\cdot\nabla u+(n-1)u)\,u_t dx \;\le\; & -\int_\Omega \left(u_t^2+3(\Delta u)^2\right)dx+\int_{\Gamma_N}(m\cdot\nu)u_t^2 d\Gamma\\
&-\int_{\Gamma_N}\frac{\partial\Delta u}{\partial\nu}\left(2(m\cdot\nabla u)+(n-1)u\right)d\Gamma.
\end{aligned}$$

From Young's inequality, we deduce that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (2m\cdot\nabla u+(n-1)u)\,u_t dx \le & -\int_\Omega \left(u_t^2+3(\Delta u)^2\right)dx+c\int_{\Gamma_N}u_t^2 d\Gamma\\
&+\frac{C}{\epsilon}\int_{\Gamma_N}\left(\frac{\partial\Delta u}{\partial\nu}\right)^2 d\Gamma+\epsilon\int_{\Gamma_N}\left((\nabla u)^2+u^2\right)d\Gamma,
\end{aligned}$$

with $C,c>0$. We conclude the proof of this lemma by using a trace inequality, Poincaré's inequality, and the boundary condition of (46). ∎

In conclusion, our abstract framework applies again, and system (46) is exponentially stable under the previous hypotheses.

REFERENCES

[1] C. Abdallah, P. Dorato, J. Benitez-Read, and R. Byrne, *Delayed positive feedback can stabilize oscillatory systems*, in Proceedings of the IEEE American Control Conference (ACC) 1993, San Francisco, 1993, pp. 3106–3107.

[2] F. Ali Mehmeti, *Nonlinear Waves in Networks*, Math. Res. 80, Akademie-Verlag, Berlin, 1994.
[3] K. Ammari and M. Tucsnak, *Stabilization of second order evolution equations by a class of unbounded feedbacks*, ESAIM Control Optim. Calc. Var., 6 (2001), pp. 361–386.
[4] T. Caraballo, J. Real, and L. Shaikhet, *Method of Lyapunov functionals construction in stability of delay evolution equations*, J. Math. Anal. Appl., 334 (2007), pp. 1130–1145.
[5] G. Chen, *Control and stabilization for the wave equation in a bounded domain*, SIAM J. Control Optim., 17 (1979), pp. 66–81.
[6] R. Datko, *Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks*, SIAM J. Control Optim., 26 (1988), pp. 697–713.
[7] E. Fridman and Y. Orlov, *Exponential stability of linear distributed parameter systems with time-varying delays*, Automatica, 45 (2009), pp. 194–201.
[8] K. P. Hadeler, *Delay equations in biology*, in Functional Differential Equations and Approximation of Fixed Points (Proc. Summer School and Conf., Univ. Bonn, Bonn, 1978), Lecture Notes in Math. 730, Springer, Berlin, 1979, pp. 136–156.
[9] J. Hale and S. Verduyn Lunel, *Introduction to Functional Differential Equations*, Appl. Math. Sci. 99, Springer-Verlag, New York, 1993.
[10] A. Haraux, *Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps*, Portugal. Math., 46 (1989), pp. 245–258.
[11] C. Huang and S. Vandewalle, *An analysis of delay-dependent stability for ordinary and partial differential equations with fixed and distributed delays*, SIAM J. Sci. Comput., 25 (2004), pp. 1608–1632.
[12] T. Kato, *Linear evolution equations of "hyperbolic" type*, J. Fac. Sci. Univ. Tokyo Sect. I, 17 (1970), pp. 241–258.
[13] T. Kato, *Linear evolution equations of "hyperbolic" type.* II, J. Math. Soc. Japan, 25 (1973), pp. 648–666.
[14] T. Kato, *Linear and quasilinear equations of evolution of hyperbolic type*, in Hyperbolicity: Lectures given at a Summer School of the Centro Internazionale Matematico Estivo (C.I.M.E.), Cortona (Arezzo), Italy, Volume 2, G. da Prato and G. Geymonat, eds., Liguori Editore, 1977, pp. 125–191.
[15] T. Kato, *Abstract Differential Equations and Nonlinear Mixed Problems*, Lezioni Fermiane [Fermi Lectures], Scuola Normale Superiore, Pisa, 1985.
[16] V. Komornik, *Rapid boundary stabilization of the wave equation*, SIAM J. Control Optim., 29 (1991), pp. 197–208.
[17] V. Komornik, *Exact Controllability and Stabilization: The Multiplier Method*, RAM: Res. Appl. Math., Masson, Paris, 1994.
[18] V. Komornik and E. Zuazua, *A direct method for the boundary stabilization of the wave equation*, J. Math. Pures Appl. (9), 69 (1990), pp. 33–54.
[19] J. E. Lagnese, *Note on boundary stabilization of wave equations*, SIAM J. Control Optim., 26 (1988), pp. 1250–1256.
[20] I. Lasiecka and R. Triggiani, *Uniform exponential energy decay of wave equations in a bounded region with $L_2(0,\infty;L_2(\Gamma))$-feedback control in the Dirichlet boundary conditions*, J. Differential Equations, 66 (1987), pp. 340–390.
[21] H. Logemann, R. Rebarber, and G. Weiss, *Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop*, SIAM J. Control Optim., 34 (1996), pp. 572–600.
[22] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim., 45 (2006), pp. 1561–1585.
[23] S. Nicaise and C. Pignotti, *Stabilization of the wave equation with variable coefficients and boundary condition of memory type*, Asymptot. Anal., 50 (2006), pp. 31–67.
[24] S. Nicaise and J. Valein, *Stabilization of second order evolution equations with unbounded feedback with delay*, ESAIM Control Optim. Calc. Var., 16 (2010), pp. 420–456.
[25] S. Nicaise and J. Valein, *Stabilization of the wave equation on 1-D networks with a delay term in the nodal feedbacks*, Netw. Heterog. Media, 2 (2007), pp. 425–479.
[26] S. Nicaise, J. Valein, and E. Fridman, *Stability of the heat and of the wave equations with boundary time-varying delays*, Discrete Contin. Dyn. Syst. Ser. S, 2 (2009), pp. 559–581.
[27] Y. Orlov and E. Fridman, *On stability of linear retarded distributed parameter systems of parabolic type*, in Topics in Time-Delay Systems, J.-J. Loiseau, W. Michiels, S. I. Niculescu, and R. Sipahi, eds., Springer, Berlin, 2009, pp. 49–59.
[28] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. 44, Springer-Verlag, New York, 1983.

[29] R. Rebarber and S. Townley, *Robustness with respect to delays for exponential stability of distributed parameter systems*, SIAM J. Control Optim., 37 (1998), pp. 230–244.
[30] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions*, SIAM Rev., 20 (1978), pp. 639–739.
[31] I. H. Suh and Z. Bien, *Use of time delay action in the controller design*, IEEE Trans. Automat. Control, 25 (1980), pp. 600–603.
[32] T. X. Wang, *Stability in abstract functional-differential equations.* II. *Applications*, J. Math. Anal. Appl., 186 (1994), pp. 835–861.
[33] T. X. Wang, *Exponential stability and inequalities of solutions of abstract functional differential equations*, J. Math. Anal. Appl., 324 (2006), pp. 982–991.
[34] G. Q. Xu, S. P. Yung, and L. K. Li, *Stabilization of wave systems with input delay in the boundary control*, ESAIM Control Optim. Calc. Var., 12 (2006), pp. 770–785.