Boundary Observers for a Reaction–Diffusion System Under Time-Delayed and Sampled-Data Measurements

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***Abstract*—We construct finite-dimensional observers for a one-dimensional reaction–diffusion system with boundary measurements subject to time-delays and data sampling. The system has a finite number of unstable modes approximated by a Luenberger-type observer. The remaining modes vanish exponentially. For a given reaction coefficient, we show how many modes one should use to achieve a desired rate of convergence. The finite-dimensional part is analyzed using appropriate Lyapunov–Krasovskii functionals that lead to linear matrix inequalitie (LMI)-based convergence conditions feasible for small enough time-delay and sampling period. The LMIs can be used to find appropriate injection gains.**

***Index Terms*—Boundary measurements, data sampling, observers, partial differential equations, time-delays.**

I. Introduction

Time-Delays and data sampling are inevitable in practice due to finite speed of signal processing/transmission and digital nature of most controllers. Since the delay may lead to instability in the reaction–diffusion systems (see the examples in [1] and in Section IV), these phenomena should be carefully studied.

Reaction–diffusion systems with various types of *in-domain* measurements/actuators subject to time-delays and sampling have been considered in [1]–[3]. These papers proposed observers/controllers that work if the delay, sampling period, and the distances between adjacent sensors/actuators are small enough. That is, the system should have enough high-frequency sensors/actuators.

The case of only one *boundary* sensor/actuator is more difficult to study. For diffusion–reaction systems, boundary controllers can be constructed using the backstepping approach [4], [5] or modal decomposition technique [6]–[9]. It has been shown in [10] that both approaches are robust to data sampling. In [11], modal decomposition technique was combined with a predictor to compensate a constant delay in the boundary controller. Robustness to small delays of general linear PDEs was studied in [12]. In this paper, we construct finite-dimensional observers for a one-dimensional (1-D) reaction–diffusion system with boundary measurements subject to time-delays and data sampling. Due to diffusion, there is a finite number of unstable modes, which we approximate by a Luenberger-type observer. The remaining modes vanish exponentially. For a given reaction coefficient, we show how many modes one should use to achieve a desired rate of convergence. Similar constructions have been proposed in [13], where a "lifting" technique and singular perturbation theory were used to obtain qualitative results. To obtain quantitative conditions, we use Lyapunov–Krasovskii functionals that lead to linear matrix inequalities (LMIs), which are feasible for small enough delay and sampling period and allow to find admissible upper bounds of these quantities.

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Lemma 1 (Cauchy–Schwarz inequality): For $f \in L^2(0,1)$

$$\left(\int_0^1 f(x)\,dx\right)^2 \le \int_0^1 (f(x))^2\,dx. \tag{1}$$

Lemma 2 (Wirtinger inequality [14]): If $f \in \mathcal{H}^1(a,b)$ is such that $f(a)=0$ or $f(b)=0$, then

$$\|f\|_{L^2} \le \frac{2(b-a)}{\pi}\|f'\|_{L^2}. \tag{2}$$

II. Time-Delayed Boundary Measurements

Consider the reaction–diffusion system

$$z_t(x,t) = z_{xx}(x,t) + az(x,t) \tag{3a}$$

$$z_x(0,t) = z(1,t) = 0 \tag{3b}$$

$$z(x,0) = z_0(x) \tag{3c}$$

with the state $z : [0,1]\times[0,\infty) \to \mathbb{R}$, reaction coefficient $a \in \mathbb{R}$, and initial function $z_0 : [0,1] \to \mathbb{R}$.

In this section, we construct an observer for the system (3) under the time-delayed boundary measurements

$$y(t) = \begin{cases} z(0, t-\tau(t)), & t-\tau(t) \ge 0 \\ 0, & t-\tau(t) < 0 \end{cases} \tag{4}$$

where $\tau(t) \in [\tau_m, \tau_M] \subset (0,\infty)$ is a known delay such that

$$\exists t_* \in [\tau_m, \tau_M] : \quad \begin{cases} t-\tau(t) \ge 0, & t \ge t_* \\ t-\tau(t) < 0, & t < t_*. \end{cases} \tag{5}$$

The condition $0 < \tau_m \le \tau(t)$ allows to use the step method for the well-posedness analysis (see Lemma 3). We perform robustness analysis with respect to the time delay, that is, the observer will converge to the system state for any $\tau(t) \le \tau_M$ with a small enough τ_M. Following [15], we require (5) to simplify the analysis on the interval where $t-\tau(t) < 0$.

Remark 1: The results of this paper can be extended to a more general system

$$\begin{aligned} &\frac{\partial z}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(p(x)\frac{\partial}{\partial x}z(x,t)\right) + q(x)z(x,t) \\ &a_1 z(0,t) + a_2 z_x(0,t) = 0 \\ &b_1 z(1,t) + b_2 z_x(1,t) = 0 \end{aligned} \tag{6}$$

where $p \in C^1([0,1];(0,\infty))$, $q \in C([0,1];\mathbb{R})$, $a_2 \ne 0$, and $|b_1| + |b_2| \ne 0$. We consider the simplified system (3) to avoid some technical details.

A *strong solution* of (3) is a function

$$z \in L^2((0,\infty);\mathcal{H}^2(0,1)) \cap C([0,\infty);\mathcal{H}^1(0,1))$$
$$z_t \in L^2((0,\infty);L^2(0,1)) \quad (7)$$

that satisfies (3c) for $t=0$ and (3a), (3b) for almost all $t>0$. In [16, Th. 7.7], (3) has a unique strong solution for

$$z_0 \in \mathcal{H}^1(0,1) \quad \text{s.t.} \quad z_0(1)=0. \quad (8)$$

To construct a finite-dimensional observer, note that (3) has a finite number of unstable modes, while the remaining modes converge to zero. Namely, the system (3) can be presented as

$$\frac{dz}{dt} + \mathcal{A}z = 0, \quad z(0)=z_0 \quad (9)$$

where $z:[0,\infty)\to L^2(0,1)$ and

$$\mathcal{A}: D(\mathcal{A}) \subset L^2(0,1) \to L^2(0,1)$$
$$\mathcal{A}w = -w'' - aw \quad (10)$$

is a symmetric operator with the domain

$$D(\mathcal{A}) = \{w \in \mathcal{H}^2(0,1) \mid w'(0)=w(1)=0\} \quad (11)$$

dense in $L^2(0,1)$. The eigenfunctions of $\mathcal{A}$, given by

$$\phi_n(x) = \sqrt{2}\cos\left(x\sqrt{\lambda_n + a}\right)$$
$$\lambda_n = \frac{(2n-1)^2\pi^2}{4} - a \qquad n \in \mathbb{N} \quad (12)$$

form an orthonormal basis in $L^2(0,1)$ [16, Corollary 3.26]. Thus, the solution of (3) can be presented as

$$z(\cdot,t) = \sum_{n=1}^{\infty} z_n(t)\phi_n(\cdot) \quad (13)$$

with $z_n(t) = \langle z(\cdot,t), \phi_n\rangle$. Using the symmetry of $\mathcal{A}$

$$\dot{z}_n(t) = \langle z_t(\cdot,t),\phi_n\rangle \overset{(9)}{=} -\langle \mathcal{A}z(\cdot,t),\phi_n\rangle$$
$$= -\langle z(\cdot,t), \mathcal{A}\phi_n\rangle = -\lambda_n\langle z(\cdot,t),\phi_n\rangle = -\lambda_n z_n(t). \quad (14)$$

That is,

$$\dot{z}_n(t) = -\lambda_n z_n(t), \quad n \in \mathbb{N}. \quad (15)$$

Let $\delta>0$ be a desired decay rate of the observer estimation error. Since $\lim_{n\to\infty}\lambda_n = +\infty$, there exists $N\in\mathbb{N}$ such that

$$-\lambda_n \le -\delta, \quad \forall n > N. \quad (16)$$

We will show that (16) implies the exponential convergence of $\sum_{n>N} z_n(t)\phi_n(\cdot)$ with the decay rate δ. Thus, it can be approximated by zero. The term $\sum_{n=1}^{N} z_n(t)\phi_n(\cdot)$ is approximated using the Luenberger-type observer

$$\hat{z}(x,t) = \sum_{n=1}^{N}\hat{z}_n(t)\phi_n(x) \quad (17a)$$

$$\frac{d}{dt}\hat{z}_n(t) = -\lambda_n\hat{z}_n(t) - l_n[\hat{z}(0,t-\tau(t)) - y(t)] \quad (17b)$$

$$\hat{z}_n(t) = 0, \quad t \le 0, \quad n = 1,\ldots,N \quad (17c)$$

with the injection gains $l_1,\ldots,l_N \in \mathbb{R}$.

Remark 2: Our results can be easily extended to arbitrary initial conditions $\hat{z}_n(t) = z_n^0$, $n=1,\ldots,N$. We consider (17c) to avoid some technical details.

Introduce the estimation error

$$e(x,t) = \hat{z}(x,t) - z(x,t). \quad (18)$$

If $e(\cdot,t)\in L^2(0,1)$, it can be presented as

$$e(\cdot,t) = \sum_{n=1}^{\infty} e_n(t)\phi_n(\cdot) \quad (19)$$

where, in view of (13) and (17a)

$$e_n(t) = \hat{z}_n(t) - z_n(t), \quad n \le N \quad (20a)$$
$$e_n(t) = -z_n(t), \quad n > N. \quad (20b)$$

In view of (15) and (17b), relation (20a) implies

$$\dot{e}_n(t) = -\lambda_n e_n(t) - l_n e(0, t-\tau(t)), \quad n \le N \quad (21)$$

which can be presented as

$$\dot{\bar{e}}(t) = A\bar{e}(t) - LC\bar{e}(t-\tau(t)) + L\zeta(t-\tau(t)) \quad (22)$$

with

$$\bar{e} = (e_1,\ldots,e_N)^T$$
$$A = \operatorname{diag}\{-\lambda_1,\ldots,-\lambda_N\}$$
$$L = (l_1,\ldots,l_N)^T$$
$$C = (\phi_1(0),\ldots,\phi_N(0)) = (\sqrt{2},\ldots,\sqrt{2})$$
$$\zeta(t) = \sum_{n=1}^{N} e_n(t)\phi_n(0) - e(0,t). \quad (23)$$

Since $\lambda_1,\ldots,\lambda_N$ are different, the pair (A,C) is observable. Therefore, we can choose $L=(l_1,\ldots,l_N)^T\in\mathbb{R}^N$ such that

$$\exists P > 0: \quad P(A-LC) + (A-LC)^T P < -2\delta P. \quad (24)$$

If $\tau(t)\equiv 0$, then (24) guarantees ISS of (22) with respect to $\zeta(t)$, which decays exponentially (we show this below). Thus, (22) is exponentially stable for $\tau(t)\equiv 0$ and remains so for $\tau(t)\le\tau_M$ with a small enough τ_M. The next theorem allows to find admissible τ_M.

Theorem 1: Consider the system (3) with the measurements (4) subject to (5) and the boundary observer (17) with λ_n, ϕ_n from (12), N satisfying (16) with an arbitrary decay rate $\delta>0$, and $L=(l_1,\ldots,l_N)^T\in\mathbb{R}^N$. Let there exist matrices $P_2,P_3,G\in\mathbb{R}^{N\times N}$ and positive-definite matrices $P,S,R\in\mathbb{R}^{N\times N}$ such that[1]

$$\Phi < 0 \text{ and } \begin{bmatrix} R & G \\ G^T & R \end{bmatrix} \ge 0 \quad (25)$$

where $\Phi = \{\Phi_{ij}\}$ is the symmetric matrix composed from

$$\Phi_{11} = A^TP_2 + P_2^TA + 2\delta P + S - e^{-2\delta\tau_M}R$$
$$\Phi_{12} = P - P_2^T + A^TP_3,\ \Phi_{13} = e^{-2\delta\tau_M}(R-G) - P_2^TLC$$
$$\Phi_{14} = e^{-2\delta\tau_M}G,\ \Phi_{22} = -P_3 - P_3^T + \tau_M^2R$$
$$\Phi_{23} = -P_3^TLC,\ \Phi_{24} = 0,\ \Phi_{33} = -e^{-2\delta\tau_M}(2R - G - G^T)$$
$$\Phi_{34} = e^{-2\delta\tau_M}(R-G),\ \Phi_{44} = -e^{-2\delta\tau_M}(S+R) \quad (26)$$

with A and C from (23). Then, there exists $M>0$, such that

$$\|\hat{z}(\cdot,t) - z(\cdot,t)\|_{L^2} \le Me^{-\delta t}\|z_0\|_{\mathcal{H}^1}, \quad t \ge 0 \quad (27)$$

for any initial function z_0 from (8).

[1] MATLAB codes for solving the LMIs are available at https://github.com/AntonSelivanov/TAC18a

Proof: Since ϕ_n and λ_n defined in (12) are eigenfunctions and eigenvalues of the operator $\mathcal{A}$ defined in (10)

$$\begin{aligned}\hat{z}_t(x,t) &\overset{(17a)}{=} \sum_{n=1}^N \frac{d}{dt}\hat{z}_n(t)\phi_n(x) \overset{(17b)}{=} -\sum_{n=1}^N \lambda_n \hat{z}_n(t)\phi_n(x)\\ &\quad -\sum_{n=1}^N l_n[\hat{z}(0,t-\tau(t))-z(0,t-\tau(t))]\phi_n(x)\\ &= -\sum_{n=1}^N \hat{z}_n(t)\mathcal{A}\phi_n\\ &\quad -\sum_{n=1}^N l_n[\hat{z}(0,t-\tau(t))-z(0,t-\tau(t))]\phi_n(x)\\ &\overset{(10)}{=} \hat{z}_{xx}(x,t)+a\hat{z}(x,t)\\ &\quad - l(x)[\hat{z}(0,t-\tau(t))-z(0,t-\tau(t))] \end{aligned} \tag{28}$$

where $l(x)=\sum_{n=1}^N l_n\phi_n(x)$. The latter, (3), and (18) imply

$$e_t(x,t) = e_{xx}(x,t) + ae(x,t) - l(x)e(0,t-\tau(t)) \tag{29a}$$

$$e_x(0,t) = e(1,t) = 0 \tag{29b}$$

$$e(\cdot,0) = -z_0, \quad e(\cdot,t)=0, \quad t<0. \tag{29c}$$

Lemma 3: There exists a unique strong solution of (29) for any initial function z_0 satisfying (8).

Proof is given in Appendix.

The strong solution $e(\cdot,t)$ of (29) can be presented as the series (19) and, by Parseval's identity

$$\|e(\cdot,t)\|^2_{L^2} = \sum_{n=1}^N e_n^2(t) + \sum_{n>N} e_n^2(t). \tag{30}$$

The second term can be bounded as

$$\begin{aligned}\sum_{n>N} e_n^2(t) &\overset{(20b)}{=} \sum_{n>N} z_n^2(t) \overset{(15)}{=} \sum_{n>N} e^{-2\lambda_n t} z_n^2(0)\\ &\overset{(16)}{\le} e^{-2\delta t}\sum_{n>N} z_n^2(0) \le e^{-2\delta t}\|z(\cdot,0)\|^2_{L^2}\\ &\overset{(29c)}{=} e^{-2\delta t}\|e(\cdot,0)\|^2_{L^2} \overset{\text{Lem.2}}{\le} e^{-2\delta t}\frac{4}{\pi^2}\|e_x(\cdot,0)\|^2_{L^2}. \end{aligned}\tag{31}$$

To bound the first summand of (30), i.e., the state of (22), we first show that $\zeta(t)$ exponentially converges to zero. Since $\phi_n(1) = e(1,t) = 0$ and $\|\phi_n'\|^2_{L^2} = \lambda_n + a$, we have

$$\begin{aligned}\zeta^2(t) &= \left(\sum_{n=1}^N e_n(t)\phi_n(0) - e(0,t)\right)^2\\ &= \left(\int_0^1\left(\sum_{n=1}^N e_n(t)\phi_n'(x) - e_x(x,t)\right)dx\right)^2\\ &\overset{\text{Lem.1}}{\le} \left\|\sum_{n=1}^N e_n(t)\phi_n'(\cdot) - e_x(\cdot,t)\right\|^2_{L^2}\\ &= \left\|\sum_{n>N} e_n(t)\phi_n'\right\|^2_{L^2} = \sum_{n>N}(\lambda_n+a)e_n^2(t)\\ &\le e^{-2\delta t}\sum_{n=1}^\infty(\lambda_n+a)e_n^2(0) = e^{-2\delta t}\|e_x(\cdot,0)\|^2_{L^2}.\end{aligned}\tag{32}$$

The last inequality is obtained in a manner similar to (31). Consequently,

$$\zeta^2(t-\tau(t)) \le e^{-2\delta(t-\tau(t))}\|e_x(\cdot,0)\|^2_{L^2} \le e^{2\delta\tau_M}e^{-2\delta t}\|e_x(\cdot,0)\|^2_{L^2}. \tag{33}$$

Consider the functional $V_\tau = V_0 + V_S + V_R$ with

$$\begin{aligned}V_0 &= \bar{e}^T(t)P\bar{e}(t)\\ V_S &= \int_{t-\tau_M}^t e^{-2\delta(t-s)}\bar{e}^T(s)S\bar{e}(s)\,ds\\ V_R &= \tau_M\int_{-\tau_M}^0\int_{t+\theta}^t e^{-2\delta(t-s)}\dot{\bar{e}}^T(s)R\dot{\bar{e}}(s)\,ds\,d\theta.\end{aligned}\tag{34}$$

We consider $V_\tau(t)$ on $[t_*,\infty)$ with t_* from (5). On this interval, (22) does not depend on $\bar{e}(t)$ with $t<0$. Thus, we formally set $\bar{e}(t)=\bar{e}(0)$ for $t<0$ to define V_τ on $[t_*,\tau_M)$ (see [15]). We have

$$\begin{aligned}\dot{V}_0 + 2\delta V_0 &= 2\bar{e}^TP\dot{\bar{e}} + 2\delta\bar{e}^TP\bar{e}\\ \dot{V}_S + 2\delta V_S &= \bar{e}^TS\bar{e} - e^{-2\delta\tau_M}\bar{e}^T(t-\tau_M)S\bar{e}(t-\tau_M)\\ \dot{V}_R + 2\delta V_R &= \tau_M^2\dot{\bar{e}}^TR\dot{\bar{e}} - \tau_M\int_{t-\tau_M}^t e^{-2\delta(t-s)}\dot{\bar{e}}^T(s)R\dot{\bar{e}}(s)\,ds.\end{aligned}\tag{35}$$

Using Jensen's inequality [17, Proposition B.8] and reciprocally convex approach [18, Th. 1], we have

$$\begin{aligned}&-\tau_M\int_{t-\tau_M}^t e^{-2\delta(t-s)}\dot{\bar{e}}^T(s)R\dot{\bar{e}}(s)\,ds \le -\tau_M e^{-2\delta\tau_M}\\ &\quad\times\left[\int_{t-\tau(t)}^t \dot{\bar{e}}^T(s)R\dot{\bar{e}}(s)\,ds + \int_{t-\tau_M}^{t-\tau(t)}\dot{\bar{e}}^T(s)R\dot{\bar{e}}(s)\,ds\right]\\ &\le -e^{-2\delta\tau_M}\frac{\tau_M}{\tau(t)}\left[\int_{t-\tau(t)}^t\dot{\bar{e}}(s)\,ds\right]^TR\left[\int_{t-\tau(t)}^t\dot{\bar{e}}(s)\,ds\right]\\ &\quad - e^{-2\delta\tau_M}\frac{\tau_M}{\tau_M-\tau(t)}\left[\int_{t-\tau_M}^{t-\tau(t)}\dot{\bar{e}}(s)\,ds\right]^TR\left[\int_{t-\tau_M}^{t-\tau(t)}\dot{\bar{e}}(s)\,ds\right]\\ &\le -e^{-2\delta\tau_M}\begin{bmatrix}\bar{e}(t)-\bar{e}(t-\tau(t))\\ \bar{e}(t-\tau(t))-\bar{e}(t-\tau_M)\end{bmatrix}^T\begin{bmatrix}R & G\\ G^T & R\end{bmatrix}\\ &\qquad\times\begin{bmatrix}\bar{e}(t)-\bar{e}(t-\tau(t))\\ \bar{e}(t-\tau(t))-\bar{e}(t-\tau_M)\end{bmatrix}.\end{aligned}\tag{36}$$

Similarly to [19], we use the descriptor representation of (22)

$$0 = 2[\bar{e}^TP_2^T + \dot{\bar{e}}^TP_3^T][-\dot{\bar{e}} + A\bar{e} - LC\bar{e}(t-\tau(t)) + L\zeta(t-\tau(t))]. \tag{37}$$

Summing up (35) and (37), for $\gamma>0$, we obtain

$$\dot{V}_\tau(t) + 2\delta V_\tau(t) - \gamma\zeta^2(t-\tau(t)) \le \psi^T(t)\Psi\psi(t) \tag{38}$$

where $\psi = \mathrm{col}\{\bar{e}(t),\dot{\bar{e}}(t),\bar{e}(t-\tau(t)),\bar{e}(t-\tau_M),\zeta(t-\tau(t))\}$

$$\Psi = \left[\begin{array}{c|c} \Phi & \begin{matrix}P_2^TL\\ P_3^TL\\ 0_{2N\times 1}\end{matrix}\\ \hline \begin{matrix}L^TP_2 & L^TP_3 & 0_{1\times 2N}\end{matrix} & -\gamma\end{array}\right]. \tag{39}$$

Since $\Phi<0$, the inequality $\Psi<0$ holds for a large enough $\gamma\in\mathbb{R}$. Moreover, $\Phi<0$ holds with δ replaced by $\delta+\epsilon$ if $\epsilon>0$ is small enough. Thus,

$$\begin{aligned}\dot{V}_\tau(t) &\le -2(\delta+\epsilon)V_\tau(t) + \gamma\zeta^2(t-\tau(t))\\ &\overset{(33)}{\le} -2(\delta+\epsilon)V_\tau(t) + \gamma e^{2\delta\tau_M}e^{-2\delta t}\|e_x(\cdot,0)\|^2_{L^2}.\end{aligned}\tag{40}$$

The comparison principle implies that

$$V_\tau(t) \le e^{-2\delta(t-t_*)}V_\tau(t_*) + \frac{\gamma e^{2\delta\tau_M}}{2\epsilon}e^{-2\delta t}\|e_x(\cdot,0)\|^2_{L^2}. \tag{41}$$

Due to (5), $\dot{\bar{e}}(t) = A\bar{e}(t)$ for $t\in[0,t_*)$, thus, $|\bar{e}(t)| \le e^{\kappa t}|\bar{e}(0)|$ for $t\in[0,t_*)$ with some $\kappa > 0$. Therefore, for some $C > 0$

$$\begin{aligned} V_\tau(t_*) &\le C \max_{t\in[t_*-\tau_M,t_*]} |\bar{e}(t)|^2 \\ &\le Ce^{2\kappa t_*}|\bar{e}(0)|^2 \le Ce^{2\kappa t_*}\sum_{n=1}^{\infty} e_n^2(0) \\ &= Ce^{2\kappa t_*}\|e(\cdot,0)\|^2_{L^2} \overset{\text{Lem.2}}{\le} Ce^{2\kappa t_*}\frac{4}{\pi^2}\|e_x(\cdot,0)\|^2_{L^2}. \end{aligned} \tag{42}$$

The latter and (41) imply

$$\sum_{n=1}^{N} e_n^2(t) \le \lambda^{-1}_{\min}(P)V_\tau(t) \le M_1 e^{-2\delta t}\|e_x(\cdot,0)\|^2_{L^2} \tag{43}$$

with some $M_1 > 0$. Finally, we have

$$\begin{aligned} &\|\hat{z}(\cdot,t) - z(\cdot,t)\|^2_{L^2} = \|e(\cdot,t)\|^2_{L^2} \\ &= \sum_{n=1}^{N} e_n^2(t) + \sum_{n=N+1}^{\infty} e_n^2(t) \overset{(43),(31)}{\le} M^2 e^{-2\delta t}\|e_x(\cdot,0)\|^2_{L^2} \end{aligned} \tag{44}$$

with some $M > 0$. Thus, (27) is true. ■

Remark 3: We have to use the $\mathcal{H}^1$-norm in the right-hand side of (27), since the L^2-norm does not take into account the point values that we use as measurements (4). Namely, we cannot bound ζ without using the space derivative as in (33).

Corollary 1: The observer (17) with $L = (l_1,\dots,l_N)^T$ satisfying (24) converges to (3) with the decay rate δ in the sense of (27) if the delay bound τ_M is small enough.

Proof: Take P from (24), $P_2 = P$, $P_3 = \varepsilon I > 0$, $R = \mu^{-1}I > 0$, $G = S = 0$, and $\tau_M = 0$. Then,

$$\Phi \overset{(26)}{=} \left[\begin{array}{c|c} M_1 & M_2 \\ \hline M_2^T & M_3 \end{array}\right]$$

with

$$M_1 = \begin{bmatrix} A^TP + PA + 2\delta P - \mu^{-1}I & \varepsilon A^T \\ * & -2\varepsilon I \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \mu^{-1}I - PLC & 0 \\ -\varepsilon LC & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} -2\mu^{-1}I & \mu^{-1}I \\ * & -\mu^{-1}I \end{bmatrix}.$$

Clearly,

$$M_3 < 0 \quad \text{and} \quad M_3^{-1} = -\mu\begin{bmatrix} I & I \\ I & 2I \end{bmatrix}.$$

By Schur's complement lemma, $\Phi < 0$ is equivalent to

$$\begin{aligned} &M_1 - M_2M_3^{-1}M_2^T \\ &= \begin{bmatrix} P(A-LC) + (A-LC)^TP + 2\delta P & \varepsilon(A-LC)^T \\ \varepsilon(A-LC) & -2\varepsilon I \end{bmatrix} \\ &\qquad + \mu\begin{bmatrix} PLC \\ \varepsilon LC \end{bmatrix}\begin{bmatrix} PLC \\ \varepsilon LC \end{bmatrix}^T < 0. \end{aligned} \tag{45}$$

In view of (24), the later holds for small $\varepsilon > 0$ and $\mu > 0$. Thus, $\Phi < 0$ is feasible for $\tau_M = 0$. By continuity, it remains so for a small $\tau_M > 0$. Then, Theorem 1 implies (27). ■

The well-posedness of (8), (29) with $\tau(t) \equiv 0$ can be proved using [20, Th. 6.3.1]. Then, Theorem 1 and Corollary 1 imply the following result.

Corollary 2: For $\tau(t) \equiv 0$, the observer (17) with $L = (l_1,\dots,l_N)^T$ satisfying (24) exponentially converges to (3) with the decay rate δ in the sense of (27).

Remark 4: The LMIs of Theorem 1 allow to find appropriate injection gain $L = (l_1,\dots,l_N)^T$. Following [21, Sec. 5.2], one can take $P_3 = \varepsilon P_2$, where ε is a tuning parameter, and use $Y = P_2^TL$ as a new decision variable. After solving the resulting LMIs, the injection gain can be found as $L = (P_2^T)^{-1}Y$.

III. Sampled-Data Boundary Measurements

In this section, we construct an observer for the system (3) under the sampled in time boundary measurements

$$y(t) = z(0,t_k), \quad t\in[t_k,t_{k+1}), \quad k\in\mathbb{N} \tag{46}$$

where $0 = t_1 < t_2 < t_3 < \cdots$ are sampling instants satisfying

$$0 < t_{k+1} - t_k \le h, \quad \lim_{k\to\infty} t_k = \infty. \tag{47}$$

Remark 5: The output (46) can be presented as (4) with

$$\tau(t) = t - t_k, \quad t\in[t_k,t_{k+1}), \quad k\in\mathbb{N} \tag{48}$$

such that $0 \le \tau(t) \le \tau_M = h$ and (5) is satisfied with $t_* = 0$. The condition $0 < \tau_m \le \tau(t)$ was imposed only to establish the well-posedness of (29) (see Lemma 3), and we will show that it is not required for the measurements (46). Therefore, the results of Theorem 1 can be applied. However, we will perform a more subtle analysis using the ideas of [22], which take into account the saw-tooth shape of $\tau(t)$ and lead to simpler convergence conditions.

Similarly to (17), the boundary observer is constructed as

$$\begin{aligned} \hat{z}(x,t) &= \sum_{n=1}^{N} \hat{z}_n(t)\phi_n(x) \\ \frac{d}{dt}\hat{z}_n(t) &= -\lambda_n\hat{z}_n(t) - l_n[\hat{z}(0,t_k) - y(t)] \\ &\qquad t\in[t_k,t_{k+1}), \quad k\in\mathbb{N} \\ \hat{z}_n(0) &= 0, \quad n = 1,\dots,N. \end{aligned} \tag{49}$$

Theorem 2: Consider the system (3) with the measurements (46) subject to (47) and the boundary observer (49) with λ_n, ϕ_n from (12), N satisfying (16) with an arbitrary decay rate $\delta > 0$, and $L = (l_1,\dots,l_N)^T \in \mathbb{R}^N$. Let there exist matrices $P_2, P_3 \in \mathbb{R}^{N\times N}$ and positive-definite matrices $P, W \in \mathbb{R}^{N\times N}$ such that[2] $\Upsilon < 0$, where $\Upsilon = \{\Upsilon_{ij}\}$ is the symmetric matrix composed from

$$\begin{aligned} \Upsilon_{11} &= (A-LC)^TP_2 + P_2^T(A-LC) + 2\delta P \\ \Upsilon_{12} &= P - P_2^T + (A-LC)^TP_3, \quad \Upsilon_{13} = -P_2^TLC \\ \Upsilon_{22} &= -P_3 - P_3^T + h^2e^{2\delta h}W, \quad \Upsilon_{23} = -P_3^TLC \\ \Upsilon_{33} &= -\frac{\pi^2}{4}W \end{aligned} \tag{50}$$

with A and C from (23). Then, there exists $M > 0$ such that (27) holds for any initial function z_0 from (8).

Proof: Similarly to (29), the estimation error $e(x,t) = \hat{z}(x,t) - z(x,t)$ satisfies

$$\begin{aligned} e_t(x,t) &= e_{xx}(x,t) + ae(x,t) - l(x)e(0,t_k) \\ &\qquad t\in[t_k,t_{k+1}), \quad k\in\mathbb{N} \\ e_x(0,t) &= e(1,t) = 0 \\ e(\cdot,0) &= -z_0 \end{aligned} \tag{51}$$

[2]MATLAB codes for solving the LMIs are available at https://github.com/AntonSelivanov/TAC18a

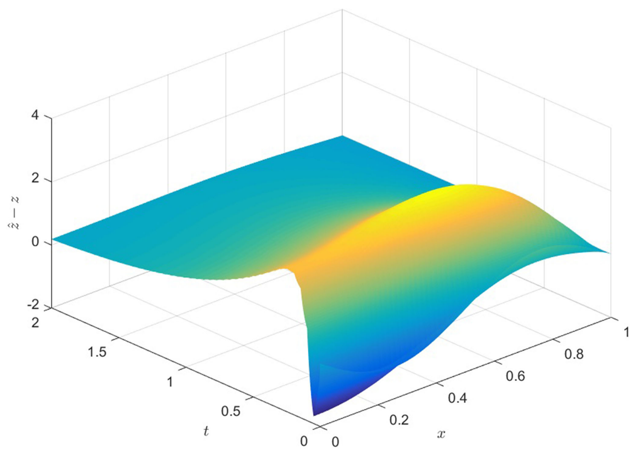

Fig. 1. Estimation error $\hat{z}(x,t)-z(x,t)$ of the observer (49) under the sampled-data measurements (46).

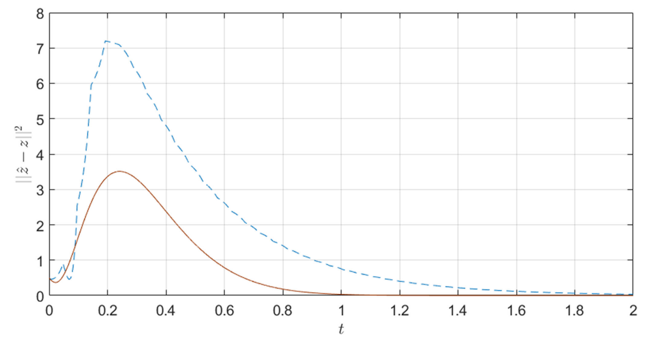

Fig. 2. Evolution of $\|\hat{z}(\cdot,t)-z(\cdot,t)\|^2_{L^2}$ for sampled-data (dashed blue line) and continuous-time (solid red line) measurements.

where $l(x)=\sum_{n=1}^N l_n\phi_n(x)$. Similarly to Lemma 3, the well-posedness of (8) and (51) is established considering $f(x,t)=-l(x)e(0,t_k)$ as constant inhomogeneities on every step $[t_k,t_{k+1})$, $k\in\mathbb{N}$. Presenting e as (19), we obtain [cf. (22)]

$$\dot{\bar{e}}(t)=(A-LC)\bar{e}(t)-LCv(t)+L\zeta(t_k),\quad t\in[t_k,t_{k+1})\tag{52}$$

where $v(t)=\bar{e}(t_k)-\bar{e}(t)$ for $t\in[t_k,t_{k+1})$ and the other notations are from (23). Consider the functional $V_h=V_0+V_W$ with $V_0=\bar{e}^T(t)P\bar{e}(t)$ and

$$\begin{aligned}V_W&=h^2e^{2\delta h}\int_{t_k}^t e^{-2\delta(t-s)}\dot{\bar{e}}^T(s)W\dot{\bar{e}}(s)\,ds\\&\quad-\frac{\pi^2}{4}\int_{t_k}^t e^{-2\delta(t-s)}v^T(s)Wv(s)\,ds,\quad t\in[t_k,t_{k+1}).\end{aligned}\tag{53}$$

Note that $V_W\geq 0$ due to the exponential Wirtinger inequality [23, Lemma 1]. Moreover, V_h does not increase in the jumps at t_k and is continuous elsewhere. We have

$$\begin{aligned}\dot{V}_0+2\delta V_0&=2\bar{e}^TP\dot{\bar{e}}+2\delta\bar{e}^TP\bar{e}\\\dot{V}_W+2\delta V_W&=h^2e^{2\delta h}\dot{\bar{e}}^T(t)W\dot{\bar{e}}(t)-\frac{\pi^2}{4}v^T(t)Wv(t)\\0&=2[\bar{e}^TP_2^T+\dot{\bar{e}}^TP_3^T]\\&\quad\times[-\dot{\bar{e}}+(A-LC)\bar{e}(t)-LCv(t)+L\zeta(t_k)],\\&\qquad t\in[t_k,t_{k+1}).\end{aligned}\tag{54}$$

Summing up, we obtain

$$\dot{V}_h+2\delta V_h-\gamma\zeta^2(t_k)=\xi^T\Xi\xi\tag{55}$$

where $\xi=\operatorname{col}\{\bar{e},\dot{\bar{e}},v,\zeta(t_k)\}$ and

$$\Xi=\left[\begin{array}{c|c}\Upsilon&\begin{matrix}P_2^TL\\P_3^TL\\0_{N\times 1}\end{matrix}\\\hline L^TP_2\;\;L^TP_3\;\;0_{1\times N}&-\gamma\end{array}\right].\tag{56}$$

The rest of the proof is similar to that of Theorem 1. ■

Corollary 3: The observer (49) with $L=(l_1,\ldots,l_N)^T$ satisfying (24) converges to (3) with the decay rate δ in the sense of (27) if the sampling period h is small enough.

Proof: Take P from (24), $P_2=P$, $P_3=\varepsilon I>0$, $W=\mu^{-1}I>0$, and $h=0$. Calculating the Schur complement, we find that $\Upsilon<0$ is equivalent to (45), which, in view of (24), holds for small $\varepsilon>0$ and $\mu>0$. Thus, $\Upsilon<0$ is feasible for $h=0$ and, by continuity, remains so for a small $\tau_M>0$. Then, Theorem 2 implies (27). ■

Remark 6: The LMIs of Theorem 2 can be transformed to solve the design problem in a manner similar to Remark 4.

Remark 7: If the sampling is uniform, i.e., $t_k=kh$, the system (52) can be studied using the discretization [21, Sec. 7.1.1]. Combining it with the modal decomposition technique, one will obtain necessary and sufficient conditions for (3), (46), (49) to satisfy (27). The advantage of the Lyapunov–Krasovskii approach developed here is that it leads to simple conditions under variable sampling (47).

IV. Example

Consider the system (3) with $a=25$ and sampled in time boundary measurements (46) subject to (47). We consider an unstable plant since otherwise $\hat{z}(x,t)=0$ is an exponentially converging estimate. Let $\delta=1$ be the desired rate of convergence of the observation error. Since (16) holds with $N=2$, the observer (49) with appropriate injection gains l_1,l_2 provides exponentially converging state estimate for a small enough sampling period h. To find l_1, l_2, and h, we take small h and increase it while the design LMIs with $\varepsilon=0.5$ (see Remarks 4 and 6) remain feasible. This gives

$$h=0.048,\quad L=\begin{bmatrix}l_1\\l_2\end{bmatrix}\approx\begin{bmatrix}23.2\\-1.1\end{bmatrix}.\tag{57}$$

The analytical bound for the uniform sampling is $h\approx 0.081$, which we found using the method described in Remark 7. Note that we used the Lyapunov functional with the Wirtinger-based term (53) that leads to simple LMIs on the account of some conservatism. Less conservative conditions may be derived using other types of Lyapunov functionals (see, e.g., [24]).

The results of numerical simulations for the initial function

$$z_0(x)=\sin(2\pi x),\quad x\in[0,1]\tag{58}$$

are given in Figs. 1 and 2. For comparison, Fig. 2 also shows the error under the continuous measurements $y(t)=z(0,t)$.

The observer (49) coincides with (17) for $\tau(t)$ defined in (48). Thus, it can be studied using Theorem 1 and Remark 4. In the considered example, these conditions lead to a smaller sampling period $h=0.031$ with approximately the same injection gains l_1,l_2.

V. Conclusion

We have designed finite-dimensional observers for a 1-D reaction–diffusion system under delayed and sampled in time boundary measurements. We showed how to choose the observer injection gains and proved that it provides exponentially converging estimate if the time-delay or sampling period are small enough. The obtained LMIs allow to find admissible bounds on the delay and sampling period. The proposed observers can be used to design network-based controllers for parabolic systems. This may be a subject of the future research.

Appendix
Proof of Lemma 3

The proof is based on [16, Th. 7.7] and the step method. Since $t - \tau(t) \le 0$ for $t \in [0, \tau_m]$

$$f(x,t) = -l(x)e(0, t-\tau(t)), \quad t \in [0,\tau_m] \tag{59}$$

can be viewed as inhomogeneity $f : [0,\tau_m] \to L^2(0,1)$ and

$$\int_0^{\tau_m} \|f(s)\|^2_{L^2}\, ds \overset{(29c)}{\le} \int_0^{\tau_m} \|l(\cdot)z_0(0)\|^2_{L^2}\, ds = \tau_m z_0^2(0)\|l\|^2_{L^2} < \infty. \tag{60}$$

Therefore, $f \in L^2((0,\tau_m); L^2(0,1))$ and [16, Th. 7.7] guarantees the existence of a unique strong solution $e \in C([0,\tau_m]; \mathcal{H}^1)$.

Since $t - \tau(t) \le \tau_m$ for $t \in [\tau_m, 2\tau_m]$

$$f(x,t) = -l(x)e(0, t-\tau(t)), \quad t \in [\tau_m, 2\tau_m] \tag{61}$$

can be viewed as inhomogeneity $f : [\tau_m, 2\tau_m] \to L^2(0,1)$. Since $e(\cdot,t)$ is continuous on $[0,\tau_m]$ in $\mathcal{H}^1$, $e(0,t)$ is also continuous on $[0,\tau_m]$

$$|e(0,t_1) - e(0,t_2)| = \left| \int_0^1 (e_x(y,t_1) - e_x(y,t_2))\, dy \right| \le \|e_x(\cdot,t_1) - e_x(\cdot,t_2)\|_{L^2}. \tag{62}$$

Thus, there exists $M_e \in \mathbb{R}$ such that $\sup_{t\le\tau_m} |e(0,t)| \le M_e$. Clearly,

$$\int_{\tau_m}^{2\tau_m} \|f(s)\|^2_{L^2}\, ds \le \tau_m M_e^2 \|l\|^2_{L^2} < \infty. \tag{63}$$

Therefore, $f \in L^2((\tau_m, 2\tau_m); L^2(0,1))$ and [16, Th. 7.7] guarantees the existence of a unique strong solution $e \in C([\tau_m, 2\tau_m]; \mathcal{H}^1)$. Repeating the same reasoning consequently on every interval $[j\tau_m, (j+1)\tau_m]$ with $j = 2, 3, \ldots$, we obtain the existence of a unique strong solution on $[0,\infty)$.

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