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Finite-dimensional boundary control of the linear Kuramoto-Sivashinsky equation under point measurement with guaranteed L^2-gain

Rami Katz and Emilia Fridman, *Fellow, IEEE*

***Abstract*—Finite-dimensional observer-based controller design for PDEs is a challenging problem. Recently, such controllers were introduced for the 1D heat equation, under the assumption that one of the observation or control operators is bounded. This paper suggests a constructive method for such controllers for 1D parabolic PDEs with both observation and control operators being unbounded. We consider the Kuramoto-Sivashinsky equation (KSE) under either boundary or in-domain point measurement and boundary actuation in the presence of disturbances in the PDE and measurement. We employ a modal decomposition approach via dynamic extension, using eigenfunctions of a Sturm-Liouville operator. The controller dimension is defined by the number of unstable modes, whereas the observer dimension N may be larger. We suggest a direct Lyapunov approach to the full-order closed-loop system, which results in an LMI, for input-to-state stabilization (ISS) and guaranteed L^2-gain, whose elements and dimension depend on N. The value of N and the decay rate are obtained from the LMI. We prove that the LMI is always feasible provided N and the L^2 or ISS gains are large enough, thereby obtaining guarantees for our approach. Moreover, for the case of stabilization, we show that feasibility of the LMI for some N implies its feasibility for $N+1$. Numerical examples demonstrate the efficiency of the method.**

***Index Terms*—Parabolic PDEs, boundary control, observer-based control, modal decomposition, LMI.**

I. Introduction

Parabolic PDEs have many applications in physics and engineering. Among such PDEs, the Kuramoto-Sivashinsky equation (KSE) describes many important processes, including chemical reaction-diffusion, flame propagation and viscous flow (see, e.g, [1]–[4]). Distributed state-feedback and observer-based control of the KSE was suggested in [5] via a modal decomposition approach. A boundary controller for the KSE in case of a small anti-diffusion parameter was designed in [6]. State-feedback stabilization of KSE under boundary or non-local actuation was studied in [7], [8] by using modal decomposition, whereas null controllability of the KSE was studied in [9]. Stability of the linear KSE as well as its stabilization using a distributed control were studied in [10].

Output-feedback controllers are more realistic for implementation. Finite-dimensional static output-feedback controllers were suggested in [11]–[14] via the spatial decomposition method. However, such controllers may require many sensing and actuation devices. Observer-based controllers for parabolic equations have been constructed in [15]–[18], where an observer was designed in the form of a PDE. An advantage of PDE observers is the resulting separation of controller and observer designs. However, they are often difficult for numerical implementation due to high computational complexity.

Finite-dimensional observer-based controllers for parabolic PDEs were suggested in [1], [15], [19], [20], whereas finite-dimensional boundary observers for the heat equation were constructed in [21]. In particular, for bounded control and observation operators, it was shown in [19] that the closed-loop system is stable provided the controller dimension is large enough. A singular perturbation approach that reduces the controller design to a finite-dimensional slow system was suggested in [1], without giving rigorous conditions for finding the dimension of the slow system. A bound (which appeared to be conservative) on the controller dimension was suggested in [20]. Recently an efficient bound on the controller dimension in terms of simple LMIs was suggested for the 1D heat equation in [22], [23] for the case when at least one of the observation or control operators is bounded. The challenging case where both operators are unbounded remained open.

H_∞ control of abstract distributed parameter systems was studied in [24], where the H_∞ control problem was reduced to solvability of operator Riccati equations. LMI-based conditions for H_∞ control of PDEs were derived in [14], [25] and [26]. Recently, input-to-state stability (ISS) of PDEs has regained much interest. ISS of the 1D heat equation with boundary disturbance was studied in [27]. State-feedback with ISS analysis of diagonal boundary control systems was considered in [28]. Non-coercive Lyapunov functionals for ISS of infinite-dimensional system were studied in [29]. A survey of ISS results can be found in [30].

In this paper, for the first time, we provide a constructive method for finite-dimensional observer-based control of a parabolic PDE with the observation and control operators both unbounded. We consider control of the 1D linear KSE under point measurement under either (mixed) Dirichlet or (mixed) Neumann actuation. We use dynamic extension (see e.g. [31], Sect. 3.3) employed for the state-feedback case in [7], [31], [32] and for observer-based control in [23]. This allows to manage with unbounded observation and control operators via modal decomposition. Differently from the existing modal decomposition methods for KSE (see, e.g. [7], [8]), we introduce a method based on a Sturm-Liouville operator with explicit eigenfunctions and eigenvalues. In comparison

R. Katz (rami@benis.co.il) and E. Fridman (emilia@eng.tau.ac.il) are with the School of Electrical Engineering, Tel Aviv University, Israel.

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to [7], [8], where the eigenfunctions and eigenvalues can only be approximated numerically, our novel approach does not require such approximations.

We study internal stabilization with guaranteed L^2-gain and input-to-state stabilization in the presence of disturbances in both the PDE and measurement. Note that stabilization with guaranteed L^2-gain has not been studied yet via modal decomposition for parabolic PDEs. In the design, the controller dimension is defined by the number of unstable modes, whereas the observer's dimension N may be larger than this number. The observer and controller gains are found separately by solving Lyapunov inequalities. We use a direct Lyapunov approach to the full-order closed-loop system to derive LMIs, whose dimension depends on N. These LMIs are used for finding N, the resulting exponential decay rate and the L^2 and ISS gains. We provide feasibility guarantees for the derived LMIs in the cases of L^2 and ISS gains for large enough N and gains. For the case of stabilization we also prove that feasibility for N implies feasibility for $N+1$ (meaning that the decay rate does not deteriorate when the observer dimension increases). Numerical examples demonstrate the efficiency of the presented method.

Preliminary results on stabilization of unperturbed 1D KSE under Dirichlet boundary conditions, were presented in [33].

Notation: $L^2(0,1)$ is the Hilbert space of square integrable functions $f:[0,1]\to\mathbb{R}$ with the inner product $\langle f,g\rangle := \int_0^1 f(x)g(x)dx$ and induced norm $\|f\|^2 := \langle f,f\rangle$. $H^k(0,1)$ is the Sobolev space of functions having k square integrable weak derivatives, with the norm $\|f\|^2_{H^k} := \sum_{j=0}^k \left\|f^{(j)}\right\|^2$. We denote $f\in H_0^1(0,1)$ if $f\in H^1(0,1)$ and $f(0)=f(1)=0$. The Euclidean norm on $\mathbb{R}^n$ is denoted by $|\cdot|$. For $P\in\mathbb{R}^{n\times n}$, $P>0$ means P is symmetric and positive definite. Sub-diagonal elements of a symmetric matrix are denoted by $*$. For $0<U\in\mathbb{R}^{n\times n}$ and $x\in\mathbb{R}^n$ let $|x|_U^2 := x^TUx$. $\mathbb{Z}_+$ denotes the nonnegative integers. $\mathbb{N}$ are the natural numbers.

II. Mathematical preliminaries

Consider the Sturm-Liouville eigenvalue problem

$$\phi''+\lambda\phi=0,\quad x\in(0,1) \tag{1}$$

with one of the following boundary conditions:

$$\text{(D): } \phi(0)=\phi(1)=0,\ \text{(Ne): } \phi'(0)=\phi'(1)=0. \tag{2}$$

These problems induce a sequence of eigenvalues λ_n with corresponding eigenfunctions ϕ_n^D and ϕ_n^{Ne} given by

$$\begin{aligned}&\text{(D): } \lambda_n=n^2\pi^2,\quad \phi_n^D(x)=\sqrt{2}\sin\left(\sqrt{\lambda_n}x\right),\ n\in\mathbb{N},\\ &\text{(Ne): } \lambda_0=0,\qquad \lambda_n=n^2\pi^2,\\ &\qquad \phi_0^{Ne}(x)\equiv 1,\quad \phi_n^{Ne}(x)=\sqrt{2}\cos\left(\sqrt{\lambda_n}x\right),\ n\in\mathbb{N}.\end{aligned} \tag{3}$$

The eigenfunctions from a complete and orthonormal family in $L^2(0,1)$.

Lemma 1: [22] Let $h\overset{L^2}{=}\sum_{n=1}^\infty h_n\phi_n^D$. Then $h\in H_0^1(0,1)$ if and only if $\sum_{n=1}^\infty \lambda_n h_n^2<\infty$. Moreover,

$$\|h'\|^2=\sum_{n=1}^\infty \lambda_n h_n^2. \tag{4}$$

Lemma 2: [34] Let $h\overset{L^2}{=}\sum_{n=0}^\infty h_n\phi_n^{Ne}$. Then $h\in H^2(0,1)$ with $h'(0)=h'(1)=0$ if and only if $\sum_{n=1}^\infty \lambda_n^2h_n^2<\infty$. Moreover,

$$\|h''\|^2=\sum_{n=1}^\infty \lambda_n^2h_n^2,\ \|h'\|^2=\sum_{n=1}^\infty \lambda_n h_n^2. \tag{5}$$

Lemma 3: (Sobolev's inequality [13]) Let $h\in H^1(0,1)$. Then, for all $\Gamma>0$:

$$max_{x\in[0,1]}|h(x)|^2\le(1+\Gamma)\|h\|^2+\Gamma^{-1}\|h'\|^2.$$

III. Control with guaranteed L^2 and ISS gains

A. Dirichlet actuation and in-domain point measurement

In this section we consider the perturbed PDE

$$z_t(x,t)=-z_{xxxx}(x,t)-\nu z_{xx}(x,t)+d(x,t), \tag{6}$$

with (mixed) Dirichlet boundary conditions

$$z(0,t)=u(t),\quad z(1,t)=0,\quad z_{xx}(0,t)=z_{xx}(1,t)=0 \tag{7}$$

and in-domain point measurement

$$y(t)=z(x_*,t)+\sigma(t),\ x_*\in(0,1). \tag{8}$$

Here, we consider disturbances satisfying

$$\begin{aligned}&d\in L^2((0,\infty);L^2(0,1))\cap H^1_{\text{loc}}((0,\infty);L^2(0,1)),\\ &\sigma\in L^2(0,\infty)\cap H^1_{\text{loc}}(0,\infty).\end{aligned} \tag{9}$$

Introducting the change of variables

$$w(x,t)=z(x,t)-r(x)u(t),\quad r(x):=1-x \tag{10}$$

we obtain the equivalent ODE-PDE system

$$\begin{aligned}&\dot u(t)=v(t),\\ &w_t(x,t)=-w_{xxxx}(x,t)-\nu w_{xx}(x,t)-r(x)v(t)+d(x,t)\end{aligned} \tag{11}$$

with boundary conditions

$$w(0,t)=w(1,t)=w_{xx}(0,t)=w_{xx}(1,t)=0. \tag{12}$$

and measurement

$$y(t)=w(x_*,t)+r(x_*)u(t)+\sigma(t). \tag{13}$$

Henceforth, we treat $u(t)$ as a state variable and $v(t)$ as the control input, where we choose $u(0)=0$. Given $v(t)$, $u(t)$ can be computed by integrating $\dot u(t)=v(t)$.

We present the solution to (11) as

$$w(x,t)=\textstyle\sum_{n=1}^\infty w_n(t)\phi_n^D(x),\ w_n(t)=\left\langle w(\cdot,t),\phi_n^D\right\rangle \tag{14}$$

where $\left\{\phi_n^D\right\}_{n\in\mathbb{N}}$ are defined in (3). Differentiating under the integral, integrating by parts and using (1) and (2) we have

$$\begin{aligned}&\dot w_n(t)=(-\lambda_n^2+\nu\lambda_n)w_n(t)+b_nv(t)+d_n(t),\\ &w_n(0)=\left\langle w(\cdot,0),\phi_n^D\right\rangle,\ d_n(t)=\left\langle d(\cdot,t),\phi_n^D\right\rangle,\\ &b_n=-\left\langle r,\phi_n^D\right\rangle=-\sqrt{\tfrac{2}{\lambda_n}}\end{aligned} \tag{15}$$

Note that $\{b_n\}_{n=1}^\infty$ satisfy $b_n\neq 0,\ n\ge 1$ and

$$\sum_{n=N+1}^\infty b_n^2\le\frac{2}{\pi^2}\int_N^\infty\frac{dx}{x^2}=\frac{2}{\pi^2N},\ N\ge 1. \tag{16}$$

Let $\delta > 0$ be a desired decay rate. Since $\lim_{n\to\infty} \lambda_n = \infty$, there exists some $N_0 \in \mathbb{N}$ such that

$$-\lambda_n^2 + \nu\lambda_n < -\delta, \quad n > N_0. \tag{17}$$

Let $N \in \mathbb{N}$, $N_0 \le N$. N_0 will define the dimension of the controller, whereas N will be the dimension of the observer. We construct a finite-dimensional observer of the form

$$\hat{w}(x,t) = \sum_{n=1}^{N} \hat{w}_n(t)\phi_n^D(x), \tag{18}$$

where $\hat{w}_n(t)$ satisfy the ODEs

$$\begin{aligned} \dot{\hat{w}}_n(t) &= (-\lambda_n^2 + \nu\lambda_n)\hat{w}_n(t) + b_n v(t) \\ &\quad - l_n\left[\hat{w}(x_*,t) + r(x_*)u(t) - y(t)\right], \\ \hat{w}_n(0) &= 0, \quad 1 \le n \le N, \end{aligned} \tag{19}$$

with $y(t)$ in (13) and scalar observer gains $l_n,\ 1 \le n \le N$.

Denote

$$\begin{aligned} &A_0 = \operatorname{diag}\left\{-\lambda_i^2 + \nu\lambda_i\right\}_{i=1}^{N_0},\ \tilde{A}_0 = \operatorname{diag}\left\{0, A_0\right\}, \\ &C_0 = \left[c_1, \ldots, c_{N_0}\right],\ \tilde{B}_0 = \left[1, b_1, \ldots, b_{N_0}\right]^T. \end{aligned} \tag{20}$$

Assumption 1: The point $x_* \in (0,1)$ satisfies

$$c_n = \phi_n^D(x_*) = \sqrt{2}\sin\left(\sqrt{\lambda_n}x_*\right) \neq 0,\ 1 \le n \le N_0. \tag{21}$$

Assumption 1 is satisfied for $N_0 = 1$ by any $x^* \in (0,1)$, whereas for $N_0 > 1$ the corresponding x^* is subject to the following condition: $x^* \neq k/n < 1,\ k = 1, ..., N_0 - 1,\ n = 2, ..., N_0$. E.g, for $N_0 = 2$ the condition is $x^* \neq \frac{1}{2}$.
Assumption 2: Assume

$$\nu \notin \left\{\pi^2(n^2 + m^2)\ ;\ n, m \ge 0\ n \neq m\right\} \cup \{0\}.$$

Under Assumptions 1 and 2 the pair (A_0, C_0) is observable, by the Hautus lemma. We choose $L_0 = [l_1, \ldots, l_{N_0}]^T$ which satisfies the Lyapunov inequality

$$P_o(A_0 - L_0C_0) + (A_0 - L_0C_0)^T P_o < -2\delta P_o \tag{22}$$

with $0 < P_o \in \mathbb{R}^{N_0 \times N_0}$. Furthermore, let $l_n = 0$ for $n > N_0$. Similarly, it can be verified that $(\tilde{A}_0, \tilde{B}_0)$ is controllable, by the Hautus lemma (see also Lemma 6 in [8], where the Kalman rank condition is used). Let $K_0 \in \mathbb{R}^{1\times(N_0+1)}$ satisfy

$$P_c(\tilde{A}_0 + \tilde{B}_0K_0) + (\tilde{A}_0 + \tilde{B}_0K_0)^T P_c < -2\delta P_c, \tag{23}$$

with $0 < P_c \in \mathbb{R}^{(N_0+1)\times(N_0+1)}$. We propose a (N_0+1)-dimensional controller of the form

$$v(t) = K_0\hat{w}^{N_0}(t),\ \hat{w}^{N_0}(t) = \left[u(t), \hat{w}_1(t), \ldots, \hat{w}_{N_0}(t)\right]^T \tag{24}$$

which is based on the N-dimensional observer (19). Then the closed-loop ODE-PDE system is given by (11), (12), (19) with controller of the form (24).

Well-posedness of the closed-loop system (11), (19) with $y(t)$ defined in (13) and controller (24), under the assumption (9) on the disturbances $d(x,t)$ and $\sigma(t)$ follows from Theorem 6.3.3 in [35]. In partiucalr, let

$$\mathcal{G} = \left\{h \in H^4(0,1) | h(0) = h(1) = h''(0) = h''(1) = 0\right\}. \tag{25}$$

Then, if $z(\cdot,0) = w(\cdot,0) \in \mathcal{G}$ there exists a unique classical solution satisfying

$$\begin{aligned} &\xi \in C([0,\infty); \mathcal{G}) \cap C^1([0,\infty); L^2(0,1)), \\ &\xi(t) = \operatorname{col}\left\{w(\cdot,t), u(t), \hat{w}_1(t), \ldots, \hat{w}_N(t)\right\} \end{aligned} \tag{26}$$

such that $\xi(t) \in \mathcal{G} \times \mathbb{R}^{N+1},\ t > 0$. The details are omitted due to space constraints (see [34] and [33]).

Let $\gamma > 0$ and $\rho_w, \rho_u \ge 0$ be scalars. We introduce the performance index

$$\begin{aligned} J_{(\rho_w,\rho_u,\gamma)}(\infty) = \textstyle\int_0^\infty &\Big[\rho_w^2 \left\|w(\cdot,t)\right\|_{L^2}^2 + \rho_u^2 u^2(t) \\ &-\gamma^2\left(\left\|d(\cdot,t)\right\|_{L^2}^2 + \sigma^2(t)\right)\Big]\,dt. \end{aligned} \tag{27}$$

The closed-loop ODE-PDE system (11), (12), (19), (24) has L^2-gain less or equal to γ if $J_{(\rho_w,\rho_u,\gamma)}(\infty) \le 0$ for all disturbances $d(x,t)$ and σ(t) satisfying (9) along the solutions of the closed-loop system starting from $w(\cdot,0) \equiv 0$. We will find conditions that guarantee that the following inequality holds along the closed-loop system:

$$\begin{aligned} &\dot{V} + 2\delta V + W \le 0, \\ &W = \rho_w^2\left\|w(\cdot,t)\right\|_{L^2}^2 + \rho_u^2 u^2(t) - \gamma^2\left(\left\|d(\cdot,t)\right\|_{L^2}^2 + \sigma^2(t)\right) \end{aligned} \tag{28}$$

with $V(t)$ given by

$$V(t) = \left|X_N(t)\right|_P^2 + \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t) \tag{29}$$

and $\delta = 0$. Indeed, integration of (28) in t from 0 to ∞ leads to $J_{(\rho_w,\rho_u,\gamma)}(\infty) \le 0$ for $w(\cdot,0) \equiv 0$. For the case $\delta > 0$ and $\rho_w = \rho_u = 0$, (28) and the comparison principle imply ISS of the closed-loop system for all $T > 0$:

$$V(T) \le e^{-2\delta T}V(0) + \tfrac{\gamma^2}{2\delta}\sup_{0\le t\le T}\left[\left\|d(\cdot,t)\right\|_{L^2}^2 + \sigma^2(t)\right]. \tag{30}$$

Note that by Lemma 1, Wirtinger's inequality (see [36], Sec. 3.10) and Parseval's equality, the following holds for $t \ge 0$

$$\begin{aligned} V(t) &\ge \sigma_{min}(P)\left|u(t)\right|^2 \\ &\quad + \tfrac{\pi^2}{4+\pi^2}\min\left(\tfrac{\sigma_{min}(P)}{2\lambda_N}, 1\right)\left\|w(\cdot,t)\right\|_{H^1}^2, \\ V(0) &\le M_0\left\|w_x(\cdot,0)\right\|_{L^2}^2 \le M_0\left\|w(\cdot,0)\right\|_{H^1}^2 \end{aligned} \tag{31}$$

for some $M_0 > 0$. Thus, (30) yields for some $\overline{M} > \underline{M} > 0$:

$$\begin{aligned} \underline{M}\left[|u(t)|^2 + \left\|w(\cdot,t)\right\|_{H^1}^2\right] &\le \overline{M}e^{-2\delta T}\left\|w(\cdot,0)\right\|_{H^1}^2 \\ + \tfrac{\gamma^2}{2\delta}\sup_{0\le t\le T}&\left[\left\|d(\cdot,t)\right\|_{L^2}^2 + \sigma^2(t)\right] \quad \forall T > 0, \end{aligned} \tag{32}$$

leading to the upper bound $\frac{\gamma}{\sqrt{2\delta}}$ on the ISS gain.

Remark 1: The performance index (27), expressed in terms of $w(x,t)$ and $u(t)$, is considered for simplicity. Note that for a performance index

$$\begin{aligned} \bar{J}_{(\bar{\rho}_z,\bar{\rho}_u,\gamma)}(\infty) = \textstyle\int_0^\infty &\Big[\bar{\rho}_z^2\left\|z(\cdot,t)\right\|_{L^2}^2 + \bar{\rho}_u^2 u^2(t) \\ &-\gamma^2\left(\left\|d(\cdot,t)\right\|_{L^2}^2 + \sigma^2(t)\right)\Big]\,dt, \end{aligned} \tag{33}$$

where $\gamma > 0$ and $\bar{\rho}_z, \bar{\rho}_u \ge 0$, the triangle and Young inequalities imply $\bar{J}_{(\bar{\rho}_z,\bar{\rho}_u,\gamma)}(\infty) \le J_{\left(\sqrt{2}\bar{\rho}_z, \sqrt{\frac{2}{3}\bar{\rho}_z^2 + \bar{\rho}_u^2}, \gamma\right)}(\infty)$.

Let

$$e_n(t) = w_n(t) - \hat{w}_n(t),\ 1 \le n \le N \tag{34}$$

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be the estimation error. By (14) and (18), the innovation term $\hat{w}(x_*,t)+r(x_*)u(t)-y(t)$ in (19) can be presented as

$$\begin{array}{l}\hat{y}(t)-y(t)=-\sum_{n=1}^{N}c_ne_n(t)-\zeta_N(t)-\sigma(t),\\ \zeta_N(t)=w(x_*,t)-\sum_{n=1}^{N}w_n(t)\phi_n^D(x_*)\\ =\int_0^{x_*}\left[w_x(x,t)-\sum_{n=1}^{N}w_n(t)\frac{d}{dx}\phi_n^D(x)\right]dx.\end{array} \tag{35}$$

Note that the Young inequality implies

$$\begin{array}{l}\zeta_N^2(t)\le\left(\int_0^{x_*}\left|w_x(x,t)-\sum_{n=1}^{N}w_n(t)\frac{d}{dx}\phi_n^D(x)\right|dx\right)^2\\ \le\left\|w_x(\cdot,t)-\sum_{n=1}^{N}w_n(t)\frac{d}{dx}\phi_n^D(\cdot)\right\|^2\overset{(4)}{=}\sum_{n=N+1}^{\infty}\lambda_nw_n^2(t).\end{array} \tag{36}$$

Then the error equations have the form

$$\begin{array}{l}\dot{e}_n(t)=(-\lambda_n^2+\nu\lambda_n)e_n(t)+d_n(t)\\ \quad -l_n\left(\sum_{n=1}^{N}c_ne_n(t)+\zeta_N(t)+\sigma(t)\right),\quad 1\le n\le N_0,\\ \dot{e}_n(t)=(-\lambda_n^2+\nu\lambda_n)e_n(t)+d_n(t),\quad N_0+1\le n\le N.\end{array} \tag{37}$$

Denote

$$\begin{array}{l}X_N(t)=\operatorname{col}\left\{\hat{w}^{N_0}(t),e^{N_0}(t),\hat{w}^{N-N_0}(t),e^{N-N_0}(t)\right\},\\ e^{N_0}(t)=\operatorname{col}\{e_i(t)\}_{i=1}^{N_0},e^{N-N_0}(t)=\operatorname{col}\{e_i(t)\}_{i=N_0+1}^{N},\\ \hat{w}^{N-N_0}(t)=\operatorname{col}\{\hat{w}_i(t)\}_{i=N_0+1}^{N},\ \tilde{K}_0=\left[K_0,0_{1\times(2N-N_0)}\right],\\ F=\begin{bmatrix}\tilde{A}_0+\tilde{B}_0K_0 & \tilde{L}_0C_0 & 0 & \tilde{L}_0C_1\\ 0 & A_0-L_0C_0 & 0 & -L_0C_1\\ B_1K_0 & 0 & A_1 & 0\\ 0 & 0 & 0 & A_1\end{bmatrix},\\ \tilde{L}_0=\operatorname{col}\{0,L_0\}\in\mathbb{R}^{N_0+1},\ \mathcal{L}=\operatorname{col}\left\{\tilde{L}_0,-L_0,0\right\}\end{array} \tag{38}$$

Using (15), (19), (24), (37) and (38), we present the closed-loop system as

$$\begin{array}{l}\dot{X}_N(t)=FX_N(t)+\mathcal{L}\zeta_N(t)+\mathcal{L}\sigma(t)+d^N(t),\quad t\ge 0,\\ \dot{w}_n(t)=(-\lambda_n^2+\nu\lambda_n)w_n(t)+b_n\tilde{K}_0X_N(t)+d_n(t),\ n>N.\end{array} \tag{39}$$

Here

$$\begin{array}{l}d^N(t)=\operatorname{col}\left\{0,d^{N_0}(t),0,d^{N-N_0}(t)\right\},\\ d^{N_0}(t)=\operatorname{col}\{d_i(t)\}_{i=1}^{N_0},\ d^{N-N_0}(t)=\operatorname{col}\{d_i(t)\}_{i=N_0+1}^{N}.\end{array}$$

Recall that we are interested in determining conditions which guarantee (28), with $V(t)$ given in (29). By Parseval's equality W can be presented as

$$\begin{array}{l}W=|X_N(t)|_{\Xi}^2+\rho_w^2\sum_{n=N+1}^{\infty}w_n^2(t)\\ \quad -\gamma^2\left|d^N(t)\right|^2-\gamma^2\sum_{n=N+1}^{\infty}d_n^2(t)-\gamma^2\sigma^2(t),\\ \Xi_1=\begin{bmatrix}\rho_u & 0 & 0 & 0 & 0\\ 0 & \rho_wI_{N0} & \rho_wI_{N0} & 0 & 0\\ 0 & 0 & 0 & \rho_wI_{N-N_0} & \rho_wI_{N-N_0}\end{bmatrix},\ \Xi=\Xi_1^T\Xi_1.\end{array} \tag{40}$$

Differentiating $V(t)$ along the solution to (39) we have

$$\begin{array}{l}\dot{V}+2\delta V=X_N^T(t)\left[PF+F^TP+2\delta P\right]X_N(t)\\ +2X_N^T(t)P\mathcal{L}\left[\zeta_N(t)+\sigma(t)\right]+2X_N^T(t)Pd^N(t)\\ +2\sum_{n=N+1}^{\infty}(-\lambda_n^3+\nu\lambda_n^2+\delta\lambda_n)w_n^2(t)\\ +2\sum_{n=N+1}^{\infty}\lambda_nw_n(t)\left[b_n\tilde{K}_0X_N(t)+d_n(t)\right].\end{array} \tag{41}$$

Furthermore, (16) and the Young inequality imply

$$\begin{array}{l}\sum_{n=N+1}^{\infty}2\lambda_nw_n(t)\left[b_n\tilde{K}_0X_N(t)+d_n(t)\right]\\ \overset{(16)}{\le}\frac{2\alpha}{\pi^2N}\left|\tilde{K}_0X_N(t)\right|^2+\frac{\alpha+\alpha_1}{\alpha\alpha_1}\sum_{n=N+1}^{\infty}\lambda_n^2w_n^2(t)\\ +\alpha_1\sum_{n=N+1}^{\infty}d_n^2(t).\end{array} \tag{42}$$

where $\alpha,\alpha_1>0$. By using (40), (41) and (42) we find

$$\begin{array}{l}\dot{V}+2\delta V+W\le X_N^T(t)\left[PF+F^TP+2\delta P+\frac{2\alpha}{\pi^2N}\tilde{K}_0^T\tilde{K}_0\right.\\ \left.+\Xi\right]X_N(t)+2X_N^T(t)P\mathcal{L}\left[\zeta_N(t)+\sigma(t)\right]+2X_N^T(t)Pd^N(t)\\ -\gamma^2\left[\sigma^2(t)+\left|d^N(t)\right|^2\right]+\left(\alpha_1-\gamma^2\right)\sum_{n=N+1}^{\infty}d_n^2(t)\\ +2\sum_{n=N+1}^{\infty}\left(-\theta_n^{(1)}+\frac{\lambda_n}{2\alpha}+\frac{\lambda_n}{2\alpha_1}\right)\lambda_nw_n^2(t),\end{array} \tag{43}$$

where

$$\theta_n^{(1)}=\lambda_n^2-\nu\lambda_n-\delta-\frac{\rho_w^2}{2\lambda_n},\quad n>N. \tag{44}$$

By monotonicity of $\{\lambda_n\}_{n=1}^{\infty}$, we have

$$-\theta_n^{(1)}+\frac{\lambda_n}{2\alpha}+\frac{\lambda_n}{2\alpha_1}\le-\theta_{N+1}^{(1)}+\frac{\lambda_{N+1}}{2\alpha}+\frac{\lambda_{N+1}}{2\alpha_1}\le 0,$$

implying due to (36)

$$\begin{array}{l}2\sum_{n=N+1}^{\infty}(-\theta_n^{(1)}+\frac{\lambda_n}{2\alpha}+\frac{\lambda_n}{2\alpha_1})\lambda_nw_n^2(t)\\ \quad\le-2\left(\theta_{N+1}^{(1)}-\frac{\lambda_{N+1}}{2\alpha}-\frac{\lambda_{N+1}}{2\alpha_1}\right)\zeta_N^2(t).\end{array} \tag{45}$$

Let $\eta(t)=\operatorname{col}\left\{X_N(t),\zeta_N(t),d^N(t),\sigma(t)\right\}$ and $\alpha_1=\gamma^2$. Then, (43) and (45) imply

$$\dot{V}+2\delta V+W\le\eta^T(t)\Psi_N^{(1)}\eta(t)\le 0$$

provided

$$\begin{array}{l}\Psi_N^{(1)}=\left[\begin{array}{cc|cc}\Phi_N^{(1)}+\Xi & P\mathcal{L} & P & P\mathcal{L}\\ * & -2\left(\theta_{N+1}^{(1)}-\frac{\lambda_{N+1}}{2\alpha}-\frac{\lambda_{N+1}}{2\gamma^2}\right) & 0 & 0\\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c}{-\gamma^2I}\end{array}\right]<0,\\ \Phi_N^{(1)}=PF+F^TP+2\delta P+\frac{2\alpha}{\pi^2N}\tilde{K}_0^T\tilde{K}_0.\end{array} \tag{46}$$

Applying Schur complement, we find that (46) holds iff

$$\left[\begin{array}{cc|cc|cc|c}\Phi_N^{(1)} & P\mathcal{L} & 0 & 0 & P & P\mathcal{L} & \Xi_1^T\\ * & -2\theta_{N+1}^{(1)} & 1 & 1 & 0 & 0 & 0\\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{-\operatorname{diag}\left(\frac{\alpha}{\lambda_{N+1}},\frac{\gamma^2}{\lambda_{N+1}}\right)} & \multicolumn{2}{c|}{0} & 0\\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{-\gamma^2I} & 0\\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{*} & -I\end{array}\right]<0. \tag{47}$$

Note that if (47) holds for $\delta=0$, then we obtain internal exponential stability of the closed-loop system with a small enough decay rate $\delta_0>0$. Summarizing, we have:

Theorem 1: Consider the system (11) with boundary conditions (7), perturbed in-domain measurement (13) and control law (24), (19). Here, $d(x,t)$ and $\sigma(t)$ are disturbances satisfying (9). Let $\delta=0$, $N_0\in\mathbb{N}$ satisfy (17) and $N\in\mathbb{N}$ satisfy $N_0\le N$. Let L_0 and K_0 be obtained using (22) and (23), respectively. Given $\gamma>0$, let there exist $0<P\in\mathbb{R}^{(2N+1)\times(2N+1)}$ and scalar $\alpha>0$ such that (47) holds with $\theta_n^{(1)}$ and Ξ_1 given in (40). Then the above system is internally exponentially stable and satisfies $J_{(\rho_w,\rho_u,\gamma)}(\infty)\le 0$ for $w(\cdot,0)\equiv 0$. Given $\rho_w,\rho_u>0$, (47) is feasible for N and γ large enough.

Proof: We show that (47) is always feasible for large enough N and $\gamma>0$. Assume, without loss of generality, $\gamma\ge 1$. First, consider $\Xi=\Xi_1^T\Xi_1$ with Ξ_1 given in (40). Since Ξ is symmetric, the equality $|\Xi|=\max_{|g|\le 1}\left|g^T\Xi g\right|=\max_{|g|\le 1}\left|\Xi_1g\right|^2$ implies $|\Xi|=|\Xi_1|^2\le\max\left(\rho_u^2,2\rho_w^2\right)$ is independent of N. Thus, there exists $0<\mu\in\mathbb{R}$ such that

$$-\mu I+\Xi<0,\ \forall N\in\mathbb{N}. \tag{48}$$

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Next, note that (16) and (21) imply $\{c_n\}_{n=1}^{\infty} \in l^{\infty}(\mathbb{N})$ and $\{b_n\}_{n=1}^{\infty} \in l^2(\mathbb{N})$, respectively. By arguments of Theorem 3.2 in [22], there exist some $\kappa > 0$ and $\Lambda > 0$, independent of N, such that $\left|e^{(F+\delta I)t}\right| \leq \Lambda \cdot \sqrt{N}\left(1+t+t^2\right)e^{-\kappa t}$ for all $t > 0$. Therefore, $P \in \mathbb{R}^{(2N+1)\times(2N+1)}$ which solves

$$P(F+\delta I) + (F+\delta I)^T = -\mu I \tag{49}$$

satisfies $|P| \leq \Lambda_1 \cdot N$, where $0 < \Lambda_1 \in \mathbb{R}$ is independent of N. Substituting (49), $\lambda_{N+1} = \pi^2\left(N+1\right)^2$ and $\alpha = 1$ into the top left block of (46) we first show

$$\begin{bmatrix} -\mu I + \Xi + \frac{2}{\pi^2 N}\tilde{K}_0^T\tilde{K}_0 & P\mathcal{L} \\ * & -2\left(\theta_{N+1}^{(1)} - \frac{\lambda_{N+1}}{2} - \frac{\lambda_{N+1}}{2\gamma^2}\right)\end{bmatrix} < 0 \tag{50}$$

holds for large enough N. From (48), $\gamma \geq 1$ and $\lambda_{N+1} \approx \left(N+1\right)^2$, the diagonal blocks are negative provided N is large enough. Applying Schur complement (50) holds iff

$$\begin{aligned} -\mu I + \Xi + \tfrac{2}{\pi^2 N}\tilde{K}_0^T\tilde{K}_0 & \\ + \frac{1}{2\left(\theta_{N+1}^{(1)} - \frac{\lambda_{N+1}}{2} - \frac{\lambda_{N+1}}{2\gamma^2}\right)} & P\mathcal{L}\mathcal{L}^T P < 0. \end{aligned} \tag{51}$$

Note that $\theta_n^{(1)} \approx n^4$ for large n, whereas $\left|\tilde{K}_0\right|$ and $|\mathcal{L}|$ are independent of N. Taking into account $|P| \leq \Lambda_1 \cdot N$ and increasing N, (50) holds for large enough N. Finally, consider (46) with N large enough for (50) to hold. Applying Schur complement and choosing γ large enough, (46) holds. ■

Remark 2: Let (47) hold with $\delta > 0$ and $\rho_w = \rho_u = 0$ (i.e $\Xi_1 = 0$). Then the closed-loop system (11), (7), (24), (19) is ISS and its solutions satisfy (30) and (32).

Next, we consider the case of ISS with $d(x,t) \equiv 0$ (i.e., only the measurement disturbance $\sigma(t)$ is present in (8)) and show that feasibility of LMI (47) with some N implies the feasibility of LMI (47) with $N+1$.

Proposition 1: Consider ISS with $d(x,t) \equiv 0$. Let $\delta > 0$, $N_0 \in \mathbb{N}$ satisfy (17) and $N \in \mathbb{N}$ satisfy $N_0 \leq N$. Let the gains L_0 and K_0 be obtained using (22) and (23). Assume that for some $0 < P \in \mathbb{R}^{(2N+1)\times(2N+1)}$ and scalars $\gamma, \alpha, \alpha_1 > 0$, the LMI (47) holds with $\theta_n^{(1)}$ and Ξ_1 given in (40) respectively. Then, there exists some $0 < P_1 \in \mathbb{R}^{(2N+3)\times(2N+3)}$ such that (47) holds *with* N *and* P *replaced by* $N+1$ *and* P_1, respectively, and the same $\gamma, \alpha, \alpha_1 > 0$.

Proof: Let $d(x,t) \equiv 0$. Recall that for ISS we have $\rho_w = \rho_u = 0$, which implies $\Xi_1 = 0$ in (40). Recall $\hat{w}^{N_0}(t)$, $e^{N_0}(t)$, $\hat{w}^{N-N_0}(t)$, $e^{N-N_0}(t)$ and $X_N(t)$ defined in (24) and (38). For $N+1$, we rewrite $X_{N+1}(t)$ as $X_N(t)$ with the remaining $e_{N+1}(t), \hat{w}_{N+1}(t)$ written in the end: $\begin{bmatrix} X_N(t) \\ e_{N+1}(t) \\ \hat{w}_{N+1}(t)\end{bmatrix} = Q_1 X_{N+1}(t)$, where Q_1 is a permutation matrices. Let $P_1 = Q_1^T \operatorname{diag}\{P, q_1, q_2\} Q_1$, where $q_1, q_2 > 0$ are scalars. Substitution of P_1, $\alpha > 0$, $\gamma > 0$ and $\rho_u = \rho_w = 0$ into (46) results in the following equivalent LMI for $\eta(t) = \operatorname{col}\{X_N(t), \zeta_{N+1}(t), \sigma(t), e_{N+1}(t), \hat{w}_{N+1}(t)\}$:

$$\left[\begin{array}{cc|c|cc} \Phi_{N+1}^{(1)} & P\mathcal{L} & P\mathcal{L} & 0 & q_2\tilde{K}_0^T \\ * & -2\left(\theta_{N+2}^{(1)} - \frac{\lambda_{N+2}}{2\alpha}\right) & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{*} & -\gamma^2 I & 0 & 0 \\ \hline \multicolumn{2}{c|}{*} & * & \multicolumn{2}{c}{\Pi_1} \end{array}\right] < 0, \tag{52}$$

$$\begin{aligned} &\Pi_1 = -2\theta_{N+1}^{(1)} \operatorname{diag}\{q_1, q_2\}, \\ &\Phi_{N+1}^{(1)} = PF + F^T P + 2\delta P + \tfrac{2\alpha}{\pi^2(N+1)}\tilde{K}_0^T\tilde{K}_0. \end{aligned}$$

In particular, note that $\Pi_1 < 0$, since $\theta_{N+1}^{(1)} < 0$ by assumption. Applying Schur complement and taking $q_2 \to 0^+$ small and $q_1 \to \infty$ large, we have that feasibility of LMI (47) with some N implies the feasibility of LMI (47) with $N+1$. ■

Remark 3: For the case of ISS and L^2-gain with nonzero $d(x,t)$, there is a coupling of $e_{N+1}(t)$ and $d_{N+1}(t)$ in the ODEs (37). Therefore, $e_{N+1}(t)$ is no longer exponentially decaying. Furthermore, coupling of $X_N(t)$ and $e_{N+1}(t)$ appears in the innovation term (35). In these cases, the proof that feasibility for N implies feasibility for $N+1$ remains unclear.

B. Neumann actuation and collocated measurement

In this section we consider the perturbed PDE (6), with disturbances $d(x,t)$ and $\sigma(t)$ satisfying (9), (mixed) Neumann boundary conditions

$$z_x(0,t) = u(t),\ z_x(1,t) = z_{xxx}(0,t) = z_{xxx}(1,t) = 0. \tag{53}$$

and collocated boundary measurement

$$y(t) = z(0,t) + \sigma(t). \tag{54}$$

By change of variables

$$w(x,t) = z(x,t) - r(x)u(t), \quad r(x) := x - \frac{x^2}{2} \tag{55}$$

we obtain the ODE-PDE system

$$\begin{aligned} &\dot{u}(t) = v(t), \\ &w_t(x,t) = -w_{xxxx}(x,t) - \nu w_{xx}(x,t) \\ &\qquad\qquad + \nu u(t) - r(x)v(t) + d(x,t) \end{aligned} \tag{56}$$

with boundary conditions

$$w_x(0,t) = w_x(1,t) = w_{xxx}(0,t) = w_{xxx}(1,t) = 0. \tag{57}$$

and measurement

$$y(t) = w(0,t) + \sigma(t). \tag{58}$$

We present the solution to (56) as

$$w(x,t) = \textstyle\sum_{n=0}^{\infty} w_n(t)\phi_n^{Ne}(x),\ w_n(t) = \left\langle w(\cdot,t), \phi_n^{Ne}\right\rangle, \tag{59}$$

with $\left\{\phi_n^{Ne}\right\}_{n=0}^{\infty}$ given in (3). Differentiating under the integral sign, integrating by parts and using (1) and (2) we have

$$\begin{aligned} &\dot{w}_0(t) = \nu u(t) + b_0 v(t) + d_0(t), \\ &\dot{w}_n(t) = (-\lambda_n^2 + \nu\lambda_n)w_n(t) + b_n v(t) + d_n(t),\ n \in \mathbb{N} \\ &w_n(0) = \left\langle w(\cdot,0), \phi_n^{Ne}\right\rangle,\ b_n = \tfrac{\sqrt{2}}{\lambda_n},\ n \in \mathbb{Z}_+. \end{aligned} \tag{60}$$

Note that in this case $\{b_n\}_{n=0}^{\infty}$ satisfy

$$\sum_{n=N+1}^{\infty} b_n^2 \leq \frac{2}{\pi^4}\int_N^{\infty} \frac{1}{x^4}dx \leq \frac{2}{3\pi^4 N^3},\ N \geq 1. \tag{61}$$

Let $\delta \geq 0$, $N_0 \in \mathbb{Z}_+$ satisfy (17) and $N \in \mathbb{Z}_+,\ N_0 \leq N$. Let scalars $\gamma > 0$ and $\rho_w, \rho_u \geq 0$. Recall the performance index given by (27). We want to find a control law $v(t)$ which guarantees (28), where $V(t)$ is given by (29).

We construct a finite-dimensional observer of the form (18),

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with summation starting from $n=0$. Here $\hat{w}_n(t)$ satisfy

$$\begin{array}{l}\dot{\hat{w}}_0(t)=\nu u(t)+b_0 v(t)-l_0\left[\hat{w}(0,t)-y(t)\right],\\ \dot{\hat{w}}_n(t)=(-\lambda_n^2+\nu\lambda_n)\hat{w}_n(t)+b_n v(t)\\ \qquad -l_n\left[\hat{w}(0,t)-y(t)\right],\ n\in\mathbb{N},\\ \hat{w}_n(0)=0,\quad 0\le n\le N.\end{array} \tag{62}$$

with $y(t)$ in (58) and scalar observer gains $l_n,\ 0\le n\le N$.

Recall A_0 and $\tilde{A}_0$ defined in (20) and denote

$$\begin{array}{l}\tilde{A}_0^{(1)}=\mathrm{diag}\left(\begin{bmatrix}0&0\\ \nu&0\end{bmatrix},A_0\right)\in\mathbb{R}^{(N_0+2)\times(N_0+2)},\\ L_0^{(1)}=[l_0,\ldots,l_{N_0}]^T,\ \tilde{L}_0^{(1)}=\mathrm{col}\left\{0,L_0^{(1)}\right\}\in\mathbb{R}^{N_0+2},\\ C_0^{(1)}=[c_0,\ldots,c_{N_0}],\ \tilde{B}_0^{(1)}=[1,b_0,\ldots,b_{N_0}]^T,\\ c_0=1,\quad c_n=\phi_n^{Ne}(0)=\sqrt{2},\ n\ge 1.\end{array} \tag{63}$$

Let Assumptions 1 and 2 hold. Then, the observer and controller gains $L_0^{(1)}$ and K_0 can be chosen to satisfy

$$\begin{array}{l}P_o(\tilde{A}_0-L_0^{(1)}C_0^{(1)})+(\tilde{A}_0-L_0^{(1)}C_0^{(1)})^TP_o<-2\delta P_o,\\ P_c(\tilde{A}_0^{(1)}+\tilde{B}_0^{(1)}K_0)+(\tilde{A}_0^{(1)}+\tilde{B}_0^{(1)}K_0)^TP_c<-2\delta P_c,\end{array} \tag{64}$$

with $0<P_o\in\mathbb{R}^{(N_0+1)\times(N_0+1)}$ and $0<P_c\in\mathbb{R}^{(N_0+2)\times(N_0+2)}$. Let $l_n=0,\ n>N_0$. We propose a (N_0+2)-dimensional controller of the form

$$v(t)=K_0\hat{w}^{N_0}(t),\ \hat{w}^{N_0}(t)=\left[u(t),\hat{w}_0(t),\ldots,\hat{w}_{N_0}(t)\right]^T, \tag{65}$$

which is based on the $N+1$-dimensional observer (62).

Using the estimation error $e_n(t)=w_n(t)-\hat{w}_n(t),\ 0\le n\le N$, (59) and (18), the innovation term $\hat{w}(0,t)-y(t)$ in (62) can be presented as (35) (with summation starting at $n=0$), where $\zeta_N(t)$ is now given by

$$\zeta_N(t)=w(0,t)-\textstyle\sum_{n=0}^N w_n(t)\phi_n^{Ne}(0). \tag{66}$$

To bound $\zeta_N(t)$, let $g(x,t):=w(x,t)-\sum_{n=0}^N w_n(t)\phi_n^{Ne}(x)$ and $\Gamma>0$. By Sobolev's inequality

$$\zeta_N^2(t)\le(1+\Gamma)\left\|g(\cdot,t)\right\|^2+\Gamma^{-1}\left\|g_x(\cdot,t)\right\|^2. \tag{67}$$

By (1), (2) and (5)

$$\zeta_N^2(t)\le\sum_{n=N+1}^{\infty}\mu_n w_n^2(t),\quad \mu_n=1+\Gamma+\frac{1}{\Gamma}\lambda_n. \tag{68}$$

Then the error equations have the form

$$\begin{array}{l}\dot{e}_0(t)=-l_0\left(\sum_{n=0}^N c_n e_n(t)+\zeta_N(t)+\sigma(t)\right)+d_0(t),\\ \dot{e}_n(t)=(-\lambda_n^2+\nu\lambda_n)e_n(t)+d_n(t)\\ \qquad -l_n\left(\sum_{n=0}^N c_n e_n(t)+\zeta_N(t)+\sigma(t)\right),\ 1\le n\le N_0,\\ \dot{e}_n(t)=(-\lambda_n^2+\nu\lambda_n)e_n(t)+d_n(t),\quad N_0+1\le n\le N.\end{array} \tag{69}$$

Let

$$\begin{array}{l}e^{N_0}(t)=\left[e_0(t),\ldots,e_{N_0}(t)\right]^T,\ \tilde{K}_0=\left[K_0,0\right]\in\mathbb{R}^{1\times 2N+3}\\ \mathcal{L}^{(1)}=\mathrm{col}\left\{\tilde{L}_0^{(1)},-L_0^{(1)},0\right\}\in\mathbb{R}^{2N+3},\\ F^{(1)}=\begin{bmatrix}\tilde{A}_0^{(1)}+\tilde{B}_0^{(1)}K_0 & \tilde{L}_0^{(1)}C_0^{(1)} & 0 & \tilde{L}_0^{(1)}C_1\\ 0 & \tilde{A}_0-L_0^{(1)}C_0^{(1)} & 0 & -L_0^{(1)}C_1\\ B_1K_0 & 0 & A_1 & 0\\ 0 & 0 & 0 & A_1\end{bmatrix}.\end{array} \tag{70}$$

Using (38), (60), (62), (65), (69) and (70) we arrive at the closed-loop system

$$\begin{array}{l}\dot{X}_N(t)=F^{(1)}X_N(t)+\mathcal{L}^{(1)}\zeta_N(t)+\mathcal{L}^{(1)}\sigma(t)+d^N(t),\\ \dot{w}_n(t)=(-\lambda_n^2+\nu\lambda_n)w_n(t)+b_n\tilde{K}_0X_N(t)+d_n(t),\ n>N.\end{array} \tag{71}$$

We derive conditions which guarantee (28), with $V(t)$ in (29). Differentiation of $V(t)$ along the solution to (71) gives

$$\begin{array}{l}\dot{V}+2\delta V=X_N^T(t)\left[PF^{(1)}+\left(F^{(1)}\right)^TP+2\delta P\right]X_N(t)\\ +2X_N^T(t)P\mathcal{L}^{(1)}\left[\zeta_N(t)+\sigma(t)\right]+2X_N^T(t)Pd^N(t)\\ +2\sum_{n=N+1}^{\infty}(-\lambda_n^3+\nu\lambda_n^2+\delta\lambda_n)w_n^2(t)\\ +2\sum_{n=N+1}^{\infty}\lambda_n w_n(t)\left[b_n\tilde{K}_0X_N(t)+d_n(t)\right].\end{array} \tag{72}$$

By the Young inequality and b_n given in (60), we have that

$$\begin{array}{l}\sum_{n=N+1}^{\infty}2\lambda_n w_n(t)\left[b_n\tilde{K}_0X_N(t)+d_n(t)\right]\\ \overset{(16)}{\le}\frac{2\alpha}{\pi^2N}\left|\tilde{K}_0X_N(t)\right|^2+\sum_{n=N+1}^{\infty}\left(\frac{\lambda_n}{\alpha}+\frac{\lambda_n^2}{\alpha_1}\right)w_n^2(t)\\ +\alpha_1\sum_{n=N+1}^{\infty}d_n^2(t).\end{array} \tag{73}$$

holds with $\alpha,\alpha_1>0$. By (40), (72) and (73) we find

$$\begin{array}{l}\dot{V}+2\delta V+W\le X_N^T(t)\left[PF^{(1)}+\left(F^{(1)}\right)^TP+2\delta P+\Xi\right.\\ \left.+\frac{2\alpha}{\pi^2N}\tilde{K}_0^T\tilde{K}_0\right]X_N(t)+2X_N^T(t)P\left[\mathcal{L}^{(1)}\left(\zeta_N(t)+\sigma(t)\right)+\right.\\ \left.d^N(t)\right]-\gamma^2\left[\sigma^2(t)+\left|d^N(t)\right|^2\right]+\left(\alpha_1-\gamma^2\right)\sum_{n=N+1}^{\infty}d_n^2(t)\\ +2\sum_{n=N+1}^{\infty}\left(-\theta_n^{(2)}+\frac{\lambda_n}{2\alpha\mu_n}+\frac{\lambda_n^2}{2\alpha_1\mu_n}\right)\mu_n w_n^2(t)\end{array} \tag{74}$$

where $\mu_n,\ n>N$ is defined in (68) and

$$\theta_n^{(2)}=\frac{\lambda_n^3-\nu\lambda_n^2-\delta\lambda_n-0.5\rho_w^2}{\mu_n},\quad n>N. \tag{75}$$

By monotonicity of $\lambda_n,\ n\ge 0$ we have

$$\begin{array}{l}-\theta_n^{(2)}+\frac{\lambda_n}{2\alpha\mu_n}+\frac{\lambda_n^2}{2\alpha_1\mu_n}\\ \le-\theta_{N+1}^{(2)}+\frac{\lambda_{N+1}}{2\alpha\mu_{N+1}}+\frac{\lambda_{N+1}^2}{2\alpha_1\mu_{N+1}}\le 0\quad\forall n>N.\end{array}$$

Then, due to (68) we obtain

$$\begin{array}{l}2\sum_{n=N+1}^{\infty}\left(-\theta_n^{(2)}+\frac{\lambda_n}{2\alpha\mu_n}+\frac{\lambda_n^2}{2\alpha_1\mu_n}\right)\mu_n w_n^2(t)\\ \quad\le 2\left(-\theta_{N+1}^{(2)}+\frac{\lambda_{N+1}}{2\alpha\mu_{N+1}}+\frac{\lambda_{N+1}^2}{2\alpha_1\mu_{N+1}}\right)\zeta_N^2(t).\end{array} \tag{76}$$

Let $\eta(t)=\mathrm{col}\left(X_N(t),\zeta_N(t),d^N(t),\sigma(t)\right)$ and $\alpha_1=\gamma^2$. Then, (74) and (76) imply

$$\dot{V}+2\delta V+W\le\eta^T(t)\Psi_N^{(2)}\eta(t)\le 0$$

provided

$$\begin{array}{l}\Psi_N^{(2)}=\left[\begin{array}{cc|cc}\Phi_N^{(2)}+\Xi & P\mathcal{L}^{(1)} & P & P\mathcal{L}^{(1)}\\ * & -2\theta_{N+1}^{(2)}+\frac{\lambda_{N+1}\left(\gamma^2+\alpha\lambda_{N+1}\right)}{\alpha\gamma^2\mu_{N+1}} & 0 & 0\\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c}{-\gamma^2I_2}\end{array}\right]<0,\\ \Phi_N^{(2)}=PF^{(1)}+\left(F^{(1)}\right)^TP+2\delta P+\frac{2\alpha}{\pi^2N}\tilde{K}_0^T\tilde{K}_0,\\ \mu_{N+1}=1+\Gamma+\frac{1}{\Gamma}\lambda_{N+1},\end{array} \tag{77}$$

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By Schur complement (77) holds if and only if

$$\left[\begin{array}{cc|cc|cc|c} \Phi_N^{(2)} & P\mathcal{L}^{(1)} & 0 & 0 & P & P\mathcal{L}^{(1)} & \Xi_1^T \\ * & -2\theta_{N+1}^{(2)} & 1 & 1 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{-\operatorname{diag}\left(\frac{\alpha\mu_{N+1}}{\lambda_{N+1}}, \frac{\gamma^2\mu_{N+1}}{\lambda_{N+1}^2}\right)} & \multicolumn{2}{c|}{0} & 0 \\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{-\gamma^2 I} & 0 \\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{*} & -I \end{array}\right] < 0. \tag{78}$$

Note that if (78) holds for $\delta = 0$ then we obtain internal exponential stability of the closed-loop system with a small enough decay rate $\delta_0 > 0$. Summarizing, we have:

Theorem 2: Consider the system (56) with boundary conditions (57), boundary measurement (58) and control law (65). Here, $d(x,t)$ and $\sigma(t)$ are disturbances satisfying (9). Let $\delta = 0$, $N_0 \in \mathbb{N}$ satisfy (17) and $N \in \mathbb{N}$ satisfy $N_0 \leq N$. Let L_0 and K_0 be obtained using (64). Given $\gamma > 0$ and $\Gamma > 0$, let there exist $0 < P \in \mathbb{R}^{(2N+2)\times(2N+2)}$ and a scalar $\alpha > 0$ satisfying (78) with $\theta_n^{(2)}$ given by (75). Then (56) is internally exponentially stable and satisfies $J_{(\rho_w,\rho_u,\gamma)}(\infty) \leq 0$ for $w(\cdot,0) \equiv 0$. Furthermore, given $\rho_w, \rho_u > 0$, the LMI (47) is always feasible for N and $\gamma > 0$ large enough.

Remark 4: By arguments similar to Proposition 1, it can be shown that either for $d(x,t) \equiv 0$ (ISS) or $d(x,t) \equiv 0,\ \sigma(t) \equiv 0$ (H^1-stabilization), feasibility of (78) with some $N \geq N_0$ implies feasibility of the LMI with $N+1$.

Remark 5: For the unperturbed system (6) with boundary conditions (53) and measurement (54), where $d(x,t) \equiv \sigma(t) \equiv 0$, our approach can be used to prove H^2-exponential stability of the closed-loop system by considering the Lyapunov functional (29) with $\{\lambda_n\}_{n=N+1}^{\infty}$ replaced by $\{\lambda_n^2\}_{n=N+1}^{\infty}$ [34].

IV. Examples

Consider KSE (6) with $\nu = 10$. This choice corresponds to an unstable open-loop system for both Dirichlet (one unstable mode) and Neumann (two unstable modes) actuations. Feasibility of LMIs was verified using the Matlab LMI toolbox.

A. Dirichlet actuation and in-domain measurement

Consider the perturbed KSE (6) under boundary conditions (7) and preturbed measurement (8) with $x_* = \pi^{-1}$. Here, the disturbances $d(x,t)$ and $\sigma(t)$ satisfy (9). For the case of input-to-state stabilization we choose K_0 and L_0 given by

$$K_0 = [7.1415, 26.0901]\,, \quad L_0 = 2.3419. \tag{79}$$

For the corresponding L^2-gain problem we consider $\rho_w = 0.1$, $\rho_u = 0.2$ and $\delta = 0$. Similarly to (79), the gains K_0 and L_0 were found by solving (22) and (23) with strong inequality replaced by equality and $\delta = \delta_0 = 1.5$. The resulting gains are given by

$$K_0 = [3.0672, 15.911]\,, \quad L_0 = 1.501. \tag{80}$$

The LMI (47) (with $\delta = 0$ and gains (80) for L^2-gain analysis and with $\delta = 1$ and gains (79) for ISS) is verified for $N \in \{4,6,8,10,12\}$. For each choice of N, we find the smallest γ which guarantees the feasibility of the LMI. The results are presented in Table I. Note that for ISS γ decreases as N grows, while for L^2-gain the resulting γ does not grow for larger N.

N	4	6	8	10	12
γ (ISS)	0.8	0.5	0.3	0.3	0.2
γ (L^2-gain)	15	15	15	15	15

TABLE I
FEASIBILITY OF LMIS - DIRICHLET. N VS MINIMAL γ.

Next, we carry out two simulations of the closed-loop system for the unperturbed (i.e. $d(x,t) \equiv \sigma(t) \equiv 0$) and the perturbed case with

$$d(x,t) = 0.25\sin(10x+t), \ \sigma(t) = 0.25\cos(30t). \tag{81}$$

In both simulations we have $N = 4$ and gains given by (79). We choose initial conditions

$$u(0) = 0, \quad z(x,0) = w(x,0) = 25(x-x^2)^3, \ x \in [0,1]. \tag{82}$$

Note that $w(\cdot,0) \in \mathcal{G}$, where $\mathcal{G}$ is defined in (25). The H^1 norm of $w(\cdot,t)$ is approximated by truncating (4) after 60 coefficients. Then, (15) with $1 \leq n \leq 60$ and (19) are simulated using MATLAB with $v(t) = K_0\hat{w}^{N_0}(t)$ and $\hat{w}^{N_0}(t)$ in (24). The value of $\zeta_N(t)$ in (35) is approximated using

$$\zeta_N(t) \approx \sum_{n=5}^{60} w_n(t)\phi_n^D(x_*). \tag{83}$$

The simulation results are presented in Figure 1. From the simulations of exponential stability, we obtain a decay rate 1.17, which is slightly larger than the theoretical decay rate $\delta = 1$ found from the LMIs.

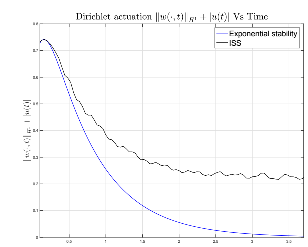

Fig. 1. Dirichlet actuation: $\|w(\cdot,t)\|_{H^1} + |u(t)|$ vs. t

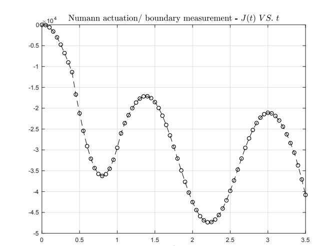

Fig. 2. Neumann actuation: $J(t)$ v.s t

B. Neumann actuation and collocated measurement

Consider the perturbed KSE (6), boundary conditions (53) and perturbed measurement (54). The disturbances again satisfy (9). For the case of ISS, let K_0 and L_0 given by

$$K_0 = [477.83, 32.61, -3315.44]\,, \ L_0 = [-6.147, 8.101]^T. \tag{84}$$

For L^2-gain analysis we consider $\rho_w = 0.1$, $\rho_u = 0.2$ and $\delta = 0$. The gains K_0 and L_0 were found by solving (64) with strong inequality replaced by equality and $\delta = \delta_0 = 1$. The corresponding gains are

$$K_0 = [291.602, 13.311, -2043.3]\,, \ L_0 = [-1.967, 3.741]^T. \tag{85}$$

Let $\Gamma = 1$. The LMI (78) (with $\delta = 0$ and gains (85) for L^2-gain analysis and with $\delta = 1$ and gains (84) for ISS) was verified for $N \in \{5,7,9,11,13\}$. For each choice of N, we find the smallest γ which guarantees the feasibility of the LMI.

N	5	7	9	11	13
γ (ISS)	3.6	1.7	1	0.6	0.5
γ (L^2-gain)	31	31	31	31	31

TABLE II
FEASIBILITY OF LMIS - NEUMANN. N VS MINIMAL γ.

The results are presented in Table II. Also in this case, for ISS γ decreases as N grows, whereas for L^2-gain the resulting γ does not grow for larger N.

Next, we perform a simulation for the corresponding L^2-gain with $\gamma = 31$ and $N = 5$. The observer and controller gains are given by (85). The chosen disturbances are given by (81). We choose zero initial conditions. For $t \in [0, 3.5]$ we simulate the ODEs (60), $0 \leq n \leq 60$ and (62) with $v(t)$ defined in (65). The value of $\zeta_N(t)$ in (66) is approximated similarly to (83). By truncating Parseval's equality at $n = 60$ we approximate the value of $J_{(\rho_w,\rho_u,\gamma)}(t)$ (see (27)). The results appear in Figure 2, confirming the theoretical analysis. We also carry out simulations with γ less than 31 (obtained in LMIs). Simulations show that it is possible to reduce γ to approximately 18, while maintaining $J(t) \leq 0$ for $t \in [0, 3.5]$. The latter may indicate the conservatism of the LMIs.

V. CONCLUSIONS

This paper introduced finite-dimensional observer-based boundary controllers for linear parabolic PDEs under point measurement via modal decomposition. The results were presented for stabilization with guaranteed L^2-gain and ISS gain. The presented method allows for challenging finite-dimensional observer-based control of various PDEs, and for design in the case of delayed inputs and outputs.

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