On the Design of Sliding-Mode Static-Output-Feedback Controllers for Systems With State Delay

X. R. Han, Emilia Fridman, Sarah K. Spurgeon, and Chris Edwards

***Abstract*—This paper considers the development of sliding-mode-based output-feedback controllers for uncertain systems which are subject to time-varying state delays. A novel method is proposed for design of the switching surface. This method is based on the descriptor approach and leads to a solution in terms of linear matrix inequalities (LMIs). When compared to existing methods (even for systems without delays), the proposed method is efficient and less conservative than other results, giving a feasible solution when the Kimura–Davison conditions are not satisfied. No additional constraints are imposed on the dimensions or structure of the reduced order triple associated with design of the switching surface. The magnitude of the linear gain used to construct the controller is also verified as an appropriate solution to the reachability problem using LMIs. A stability analysis for the full-order time-delay system with discontinuous right-hand side is formulated. This paper facilitates the constructive design of sliding-mode static-output-feedback controllers for a rather general class of time-delay systems. A numerical example from the literature illustrates the efficiency of the proposed method.**

***Index Terms*—Linear matrix inequalities (LMIs), sliding-mode control (SMC), static output feedback (SOF), time delay.**

I. Introduction

SLIDING-MODE control (SMC) [1] is known for its complete robustness to so-called matched uncertainties (which can include time delays that satisfy matching conditions) and disturbances [2]–[4]. The control technique has been applied in many industrial areas [5]–[7]. Many early theoretical developments in SMC assume that all the system states are accessible. In the case where only a subset of states are measurable, which is relevant to a range of practical applications, either output feedback control or the observer-based method are required. Some work has considered implementation of SMC schemes using observers [8]–[10]. In [11], a sliding-mode observer has been shown to give a significant increase in performance in estimation of the unknown variables of a boost converter compared to a traditional current-mode control strategy. A further interesting strand considers the fast output sampling method [12], [13]. Recently, in [14], a fast sampling method is employed for a discrete systems in the presence of time-varying delays where a sliding-mode controller is designed using linear matrix inequalities (LMIs) combined with a delta-operator approach. However, all these methods require additional computation. The most straightforward approach is to consider the study of SMC via static output feedback (SOF).

One problem of interest in the development of SMC via SOF is the design of the switching surface, which is effectively a reduced order SOF problem for a particular subsystem. Two different methods were proposed to design the sliding surface using eigenvalue assignment and eigenvector techniques in [15] and [16]. A canonical form was provided in [18] via which the SOFSMC design problem is routinely converted to an SOF stabilization problem. As stated in [20], all previous-reported methods for the existence problem are, in fact, equivalent to a particular SOF problem. The solution to the general SOF problem, even for linear time-invariant systems, is still open.

LMI methods have been considered within the context of sliding-mode controller design. For example, [21] and [22] presented LMIs methods to design static sliding-mode output-feedback controllers and [23] presented a necessary and sufficient condition to solve the existence problem in terms of LMIs for linear uncertain systems.

It is important to note that all the work described above does not consider an existence problem involving delay and many practical problems include such effects [24], [25]. In [26], the problem of the development of sliding-mode controllers for operation in the presence of single or multiple, constant or time-varying state delays has been solved. This uses the usual regular form method of solution and the uncertainty is assumed to be matched, where matched describes that uncertainty class which is implicit in the range of the input channels and will be rejected by an appropriately designed SMC strategy, although it is important to note that full state availability is assumed. This problem has also been considered in [27] where a class of uncertain time delay systems with multiple fixed delays in the system states is considered. This paper considers unmatched and time-varying parameter uncertainties, together with matched and bounded external disturbances, but again, full state information is assumed to be available to the controller. In [28], Lyapunov functionals were for the first time introduced for the analysis of time-varying delay. In [29], a descriptor

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approach to stability and control of linear systems with time-varying delays, which is based on the Lyapunov–Krasovskii techniques, was combined with results on the SMC of such systems. The systems under consideration were subjected to norm-bounded uncertainties and uncertain bounded delays and the solution given in terms of LMIs. Reference [30] develops a SMC synthesis for a class of uncertain time-delay systems with nonlinear disturbances and unknown delay values whose unperturbed dynamics is linear. The synthesis was based on a new delay-dependent stability criterion. The controller is found to be robust against sufficiently small delay variations and external disturbances.

It is important to emphasize that much of the aforementioned literature on SMC of time-delay systems assumes full-state feedback. Reference [31] considered SMC of systems with time-varying delay. This paper considered a solution via LMIs for the existence problem. The current paper extends this contribution to consider the solution of the existence and reachability problems. Specifically, the selection of parameters for stability of the full-order closed-loop system are obtained via LMIs. In this paper, a compensator-based design problem is considered using the proposed SOF approach. Example from the literature illustrates the efficiency of the method. In Section II, the problem formulation is described. The existence problem is formulated in Section III. In Sections IV and V, stability of the full-order closed-loop system is derived via LMIs, and the reachability problem is presented. Compensator-based design is demonstrated in Section VI.

II. Problem Formulation

Consider an uncertain time-delay system

$$\begin{aligned}\dot{z}(t) &= Az(t) + A_d z\left(t-\tau(t)\right) + B\left(u(t)+\xi(t,z,u)\right)\\ y(t) &= Cz(t)\end{aligned} \tag{1}$$

where $z \in \mathcal{R}^n$, $u \in \mathcal{R}^m$, and $y \in \mathcal{R}^p$ with $m \le p \le n$. The time-varying delay $\tau(t)$ is supposed to be bounded $0 \le \tau(t) \le h$, and it may be either slowly varying (i.e., differentiable delay with $\dot{\tau}(t) \le d < 1$) or fast varying (piecewise continuous delay). Assume that the nominal linear system (A, A_d, B, C) is known and that the input and output matrices B and C are both of full rank. The unknown function $\xi\colon \mathcal{R}_+ \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^n$, which represents the system nonlinearities plus any model uncertainties, is assumed to satisfy the matching condition and

$$\|\xi(t,z,u)\| < k_1\|u\| + a(t,y) \tag{2}$$

for some known function $a : \mathcal{R}_+ \times \mathcal{R}^p \to \mathcal{R}_+$ and positive constant $k_1 < 1$. It can be shown that if $rank(CB) = m$, there exists a coordinate system in which the system (A, A_d, B, C) has the structure

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad A_d = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad C = [0 \quad T] \tag{3}$$

where $B_2 \in \mathcal{R}^{m\times m}$ is nonsingular and $T \in \mathcal{R}^{p\times p}$ is orthogonal. The system can be represented as

$$\begin{aligned}\dot{z}_1(t) &= A_{11}z_1(t) + A_{d11}z_1\left(t-\tau(t)\right)\\ &\quad + A_{12}z_2(t) + A_{d12}z_2\left(t-\tau(t)\right)\\ \dot{z}_2(t) &= \sum_{i=1}^{2}\left(A_{2i}z_i(t) + A_{d2i}z_i\left(t-\tau(t)\right)\right)\\ &\quad + B_2\left(u(t)+\xi(t,z,u)\right)\\ y(t) &= Cz(t).\end{aligned} \tag{4}$$

Consider the following switching function: AQ1

$$\mathcal{S} = \{z(t) \in \mathcal{R}^n : FCz(t) = 0\} \tag{5}$$

for some selected matrix $F \in \mathcal{R}^{m\times p}$ where by design $det(FCB) \neq 0$. Let

$$[F_1 \quad F_2] = FT \tag{6}$$

where $F_1 \in \mathcal{R}^{p-m}$ and $F_2 \in \mathcal{R}^m$. As a result

$$FC = [F_1C_1 \quad F_2] \tag{7}$$

where

$$C_1 = \left[0_{(p-m)\times(n-p)} \quad I_{(p-m)}\right]. \tag{8}$$

Therefore, $FCB = F_2B_2$ and the square matrix F_2 is nonsingular. By assumption, the uncertainty is matched, and therefore the sliding motion is independent of the uncertainty represented by $\xi(\cdot)$. In addition, because the canonical form in (3), where it is necessary that the pair (A_{11}, A_{12}) is controllable and (A_{11}, C_1) is observable, can be viewed as a special case of the regular form normally used in sliding-mode controller design, the switching function can also be expressed as

$$s(t) = z_2(t) + KC_1z_1(t) \tag{9}$$

where $K \in \mathcal{R}^{m\times(p-m)}$ and is defined as $K = F_2^{-1}F_1$.

Then, a simple SMC law, depending on the output information $Fy(t)$ can be defined by

$$u(t) = -\gamma Fy(t) - v(t) \tag{10}$$

where

$$v(t) = \begin{cases} \rho(t,y)\frac{Fy(t)}{\|Fy(t)\|}, & \text{if } Fy(t) \neq 0 \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

where $\rho(t,y)$ is some positive scalar function of the outputs

$$\rho(t,y) = \left(k_1\gamma\,\|Fy(t)\| + \alpha(t,y) + \gamma_2\right)/(1-k_1)$$

where γ and γ_2 are positive design scalars [18]. The closed-loops system (4) and (10) can be described by the following equations:

$$\begin{aligned}\dot{z}_1(t) &= (A_{11} - A_{12}KC_1)z_1(t)\\ &\quad + (A_{d11} - A_{d12}KC_1)z_1\left(t-\tau(t)\right)\end{aligned}$$

$$
\begin{aligned}
\dot{s}(t) =& (A_{21} - \gamma B_2 K C_1) z_1(t) \\
&+ (A_{d21} - \gamma B_2 K C_1) z_1(t - \tau(t)) \\
&+ (A_{22} - \gamma B_2) z_2(t) + (A_{d22} - \gamma B_2) z_2(t - \tau(t)) \\
&+ B_2 (\xi(t, z, u) - v(t)) \\
y(t) =& C z(t). \qquad (12)
\end{aligned}
$$

III. Existence Problem

On the sliding manifold $s(t) = 0$, it is well known [19] that the reduced-order sliding motion is governed by a free motion with system matrix

$$
\begin{aligned}
\dot{z}_1(t) =& (A_{11} - A_{12} K C_1) z_1(t) \\
&+ (A_{d11} - A_{d12} K C_1) z_1(t - \tau(t)). \qquad (13)
\end{aligned}
$$

Consider a Lyapunov–Krasovskii functional

$$
\begin{aligned}
V(t) =& z_1^{\mathrm{T}}(t) P z_1(t) + \int_{t-h}^{t} z_1^{\mathrm{T}}(s) E z_1(s) ds \\
&+ \int_{t-\tau(t)}^{t} z_1^{\mathrm{T}}(s) S z_1(s) ds \\
&+ h \int_{-h}^{0} \int_{t+\theta}^{t} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds d\theta \qquad (14)
\end{aligned}
$$

where the symmetric matrices $P > 0$ and $E, S, R \geq 0$.

The condition $\dot{V}(t) < 0$ guarantees asymptotic stability of the reduced order system as in [32]. Differentiating $V(t)$ along (13)

$$
\begin{aligned}
\dot{V}(t) =& 2 z_1^{\mathrm{T}}(t) P \dot{z}_1(t) + h^2 \dot{z}_1^{\mathrm{T}}(t) R \dot{z}_1(t) \\
&- h \int_{t-h}^{t} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds + z_1^{\mathrm{T}}(t)(E + S) z_1(t) \\
&- z_1^{\mathrm{T}}(t-h) E z_1(t-h) \\
&- (1 - \dot{\tau}(t)) z_1^{\mathrm{T}}(t - \tau(t)) S z_1(t - \tau(t)). \qquad (15)
\end{aligned}
$$

Further using the identity

$$
\begin{aligned}
-h \int_{t-h}^{t} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds =& -h \int_{t-h}^{t-\tau(t)} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds \\
&- h \int_{t-\tau(t)}^{t} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds \qquad (16)
\end{aligned}
$$

and applying Jensen's inequality

$$
\int_{t-\tau(t)}^{t} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds \geq \frac{1}{h} \int_{t-\tau(t)}^{t} \dot{z}_1^{\mathrm{T}}(s) ds R \int_{t-\tau(t)}^{t} \dot{z}_1(s) ds \qquad (17)
$$

$$
\int_{t-h}^{t-\tau(t)} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds \geq \frac{1}{h} \int_{t-h}^{t-\tau(t)} \dot{z}_1^{\mathrm{T}}(s) ds R \int_{t-h}^{t-\tau(t)} \dot{z}_1(s) ds. \qquad (18)
$$

Then,

$$
\begin{aligned}
\dot{V}(t) \leq& 2 z_1^{\mathrm{T}}(t) P \dot{z}_1^{\mathrm{T}}(t) + h^2 \dot{z}_1^{\mathrm{T}}(t) R \dot{z}_1(t) \\
&- (z_1(t) - z_1(t - \tau(t)))^{\mathrm{T}} R (z_1(t) - z_1(t - \tau(t))) \\
&- (z_1(t - \tau(t)) - z_1(t-h))^{\mathrm{T}} \\
&\times R (z_1(t - \tau(t)) - z_1(t-h)) \\
&+ z_1^{\mathrm{T}}(t)(E + S) z_1(t) - z_1^{\mathrm{T}}(t-h) E z_1(t-h) \\
&- (1 - d) z_1^{\mathrm{T}}(t - \tau(t)) S z_1(t - \tau(t)). \qquad (19)
\end{aligned}
$$

Using the descriptor method as in [33] and the free-weighting matrices technique from [34], the right-hand side of the expression

$$
\begin{aligned}
0 \equiv& 2 \left(z_1^{\mathrm{T}}(t) P_2^{\mathrm{T}} + \dot{z}_1^{\mathrm{T}}(t) P_3^{\mathrm{T}} \right) \\
&\times [-\dot{z}_1(t) + (A_{11} - A_{12} K C_1) z_1(t) \\
&+ (A_{d11} - A_{d12} K C_1) z_1(t - \tau(t))] \qquad (20)
\end{aligned}
$$

with matrix parameters P_2, $P_3 = \epsilon P_2 \in \mathcal{R}^{n-m}$ is added into the right-hand side of (19). Setting $\eta(t) = col\{z_1(t), \dot{z}_1(t), z_1(t-h), z_1(t - \tau(t))\}$, it follows that

$$
\dot{V}(t) \leq \eta^{\mathrm{T}}(t) \Theta \eta(t) \leq 0 \qquad (21)
$$

if the matrix inequality

$$
\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & 0 & \theta_{14} \\ * & \theta_{22} & 0 & \theta_{24} \\ * & * & \theta_{33} & \theta_{34} \\ * & * & * & \theta_{44} \end{bmatrix} < 0 \qquad (22)
$$

is feasible, where

$$
\begin{aligned}
\theta_{11} =& P_2^{\mathrm{T}} (A_{11} - A_{12} K C_1) \\
&+ (A_{11} - A_{12} K C_1)^{\mathrm{T}} P_2 + E + S - R \\
\theta_{12} =& P - P_2^{\mathrm{T}} + \epsilon (A_{11} - A_{12} K C_1)^{\mathrm{T}} P_2 \\
\theta_{14} =& P_2^{\mathrm{T}} (A_{d11} - A_{d12} K C_1) + R \\
\theta_{22} =& -\epsilon P_2 - \epsilon P_2^{\mathrm{T}} + h^2 R \\
\theta_{24} =& \epsilon P_2^{\mathrm{T}} (A_{d11} - A_{d12} K C_1) \\
\theta_{33} =& -(E + R) \\
\theta_{34} =& R \\
\theta_{44} =& -2R - (1 - d) S. \qquad (23)
\end{aligned}
$$

Multiplying matrix Θ from the right and the left by $\mathrm{diag}\{P_2^{-1}, P_2^{-1}, P_2^{-1}, P_2^{-1}\}$ and its transpose, respectively, and denoting

$$
\begin{aligned}
&Q_2 = P_2^{-1} \quad \widehat{P} = Q_2^{\mathrm{T}} P Q_2 \quad \widehat{R} = Q_2^{\mathrm{T}} R Q_2 \\
&\widehat{E} = Q_2^{\mathrm{T}} E Q_2 \quad \widehat{S} = Q_2^{\mathrm{T}} S Q_2
\end{aligned}
$$

it follows $\Theta < 0 \Leftrightarrow \widehat{\Theta} < 0$, where

$$\widehat{\Theta} = \begin{bmatrix} \widehat{\theta}_{11} & \widehat{\theta}_{12} & 0 & \widehat{\theta}_{14} \\ * & \widehat{\theta}_{22} & 0 & \widehat{\theta}_{24} \\ * & * & \widehat{\theta}_{33} & \widehat{\theta}_{34} \\ * & * & * & \widehat{\theta}_{44} \end{bmatrix} < 0 \tag{24}$$

$$\begin{aligned} \widehat{\theta}_{11} &= (A_{11} - A_{12}KC_1)Q_2 \\ &\quad + Q_2^{\mathrm{T}}(A_{11} - A_{12}KC_1)^{\mathrm{T}} + \widehat{E} + \widehat{S} - \widehat{R} \\ \widehat{\theta}_{12} &= \widehat{P} - Q_2 + \epsilon Q_2^{\mathrm{T}}(A_{11} - A_{12}KC_1)^{\mathrm{T}} \\ \widehat{\theta}_{14} &= (A_{d11} - A_{d12}KC_1)Q_2 + \widehat{R} \\ \widehat{\theta}_{22} &= -\epsilon Q_2 - \epsilon Q_2^{\mathrm{T}} + h^2\widehat{R} \\ \widehat{\theta}_{24} &= \epsilon(A_{d11} - A_{d12}KC_1)Q_2 \\ \widehat{\theta}_{33} &= -\widehat{E} - \widehat{R} \\ \widehat{\theta}_{34} &= \widehat{R} \\ \widehat{\theta}_{44} &= -2\widehat{R} - (1-d)\widehat{S}. \end{aligned} \tag{25}$$

Define the variable Q_2 in the following form:

$$Q_2 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{22}\mathcal{M} & \delta Q_{22} \end{bmatrix} \tag{26}$$

where Q_{22} is a $(p-m)\times(p-m)$ matrix, and $\mathcal{M} \in \mathcal{R}^{(p-m)\times(n-p)}$ and $\delta \in \mathcal{R}$ are *a priori* selected tuning parameters. It follows that

$$KC_1Q_2 = [KQ_{22}\mathcal{M} \quad \delta KQ_{22}].$$

Defining

$$Y = KQ_{22}$$

it follows that

$$KC_1Q_2 = [Y\mathcal{M} \quad \delta Y]. \tag{27}$$

To construct K, substitute (27) into (25) to yield

$$\begin{aligned} \widehat{\theta}_{11} &= A_{11}Q_2 - A_{12}[Y \quad \delta Y] + Q_2^{\mathrm{T}}A_{11}^{\mathrm{T}} \\ &\quad - [Y\mathcal{M} \quad \delta Y]^{\mathrm{T}}A_{12}^{\mathrm{T}} + \widehat{E} + \widehat{S} - \widehat{R} \\ \widehat{\theta}_{12} &= \widehat{P} - Q_2 + \epsilon Q_2^{\mathrm{T}}A_{11}^{\mathrm{T}} - \epsilon[Y\mathcal{M} \quad \delta Y]^{\mathrm{T}}A_{12}^{\mathrm{T}} \\ \widehat{\theta}_{14} &= A_{d11}Q_2 - A_{d12}[Y\mathcal{M} \quad \delta Y] + \widehat{R} \\ \widehat{\theta}_{22} &= -\epsilon Q_2 - \epsilon Q_2^{\mathrm{T}} + h^2\widehat{R} \\ \widehat{\theta}_{24} &= \epsilon A_{d11}Q_2 - \epsilon A_{d12}[Y\mathcal{M} \quad \delta Y] \\ \widehat{\theta}_{33} &= -\widehat{E} - \widehat{R} \\ \widehat{\theta}_{34} &= \widehat{R} \\ \widehat{\theta}_{44} &= -2\widehat{R} - (1-d)\widehat{S} \end{aligned} \tag{28}$$

with the tuning parameters δ, ϵ, and $\mathcal{M}$. The following Lemma may now be stated.

Lemma 1: Given *a priori* selected tuning parameters ϵ, δ, and $\mathcal{M} \in R^{(p-m)\times(n-p)}$, then (24) is an LMI in the decision variables $\widehat{P} > 0$, $\widehat{E} \geq 0, \widehat{S} \geq 0$, $\widehat{R} \geq 0$ and matrices $Q_{22} \in \mathcal{R}^{(p-m)\times(p-m)}$, $Q_{11} \in \mathcal{R}^{(n-p)\times(n-p)}$, $Q_{12} \in \mathcal{R}^{(n-p)\times(p-m)}$, $Y \in \mathcal{R}^{m\times(p-m)}$. If a solution to (24) exists, which may be readily obtained from available LMI tools, then the reduced order system (13) is asymptotically stable for all differentiable delays $0 \leq \tau(t) \leq h, \dot{\tau}(t) \leq d < 1$. Moreover, (13) is asymptotically stable for all piecewise-continuous delays $0 \leq \tau(t) \leq h$, if the LMI (24) is feasible with $\widehat{S} = 0$.

Remark: The proposed method is suitable for SOF sliding-mode controller design where Kimura–Davison conditions, written as $n \leq m + p - 1$, are not satisfied as in [20]–[22]. No constraints are imposed on the dimensions of the reduced-order triple A_{11}, A_{12}, C_1. This represents a constructive and efficient approach to output-feedback-based design for a relatively broad class of systems, which is less conservative than existing results [20]–[22]; an example to illustrate the advantages of the method for systems without time-delay is presented in [17].

IV. Stability of the Full-Order Closed-Loop System

It can be shown [18] that there exist a coordinate system in which the system triple $(\bar{A}, \bar{A}_d, \bar{B}, F\bar{C})$ has the property that

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \quad \bar{A}_d = \begin{bmatrix} \bar{A}_{d11} & \bar{A}_{d12} \\ \bar{A}_{d21} & \bar{A}_{d22} \end{bmatrix}$$
$$\bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad F\bar{C} = [0 \quad F_2] \tag{29}$$

where $\bar{A}_{11} = A_{11} - A_{12}KC_1$ and $\bar{A}_{d11} = A_{d11} - A_{d12}KC_1$ and F_2 is a design parameter. Let $\bar{P}$ be a symmetric positive definite matrix partitioned conformably with the matrices in (29) so that

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{bmatrix} \tag{30}$$

then the matrix $\bar{P}$ satisfies the structural constraint

$$\bar{P}\bar{B} = \bar{C}^{\mathrm{T}}F^{\mathrm{T}} \tag{31}$$

if the design matrix $F_2 = B_2^{\mathrm{T}}\bar{P}_2$. The matrix $\bar{P}$ can be shown to be a Lyapunov matrix for

$$\begin{aligned} \bar{A}_0 &= \bar{A} - \gamma\bar{B}F\bar{C} \\ &= \bar{A} - \gamma\bar{B}[0 \quad F_2] \end{aligned} \tag{32}$$

for sufficiently large γ [18]. In the new coordinate system, the uncertain system (1) can be written as

$$\dot{z}(t) = \bar{A}z(t) + \bar{A}_dz(t-\tau) + \bar{B}\left(u(t) + \xi(t,z,u)\right). \tag{33}$$

The closed-loop system will have the form

$$\dot{z}(t) = \bar{A}_0z(t) + \bar{A}_dz(t-\tau) + \bar{B}\left(\xi(t,y_t) - v_y(t)\right). \tag{34}$$

For large enough $\gamma > 0$, these conditions are delay independent with respect to the delay in z_2.} However, for derivation of this condition using Lyapunov–Krasovskii techniques, it is necessary to consider the case where $\dot{\tau} \leq d < 1$. A stability condition for the full-order closed-loop system can be derived using the following Lyapunov–Krasovskii functional:

$$V(t) = z^{\mathrm{T}}(t)\bar{P}z(t) + \int_{t-h}^{t} z^{\mathrm{T}}(s)\bar{E}z(s)ds + \int_{t-\tau(t)}^{t} z^{\mathrm{T}}(s)\bar{S}z(s)ds + h\int_{-h}^{0}\int_{t+\theta}^{t} \dot{z}^{\mathrm{T}}(s)\bar{R}\dot{z}(s)dsd\theta \quad (35)$$

where the matrix $\bar{E}$, $\bar{S} \geq 0$ and $\bar{R} = \begin{bmatrix} \bar{R}_1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$ as it is desired to determine a stability condition for the time delay system which is delay independent with respect to delay in $z_2(t)$. Differentiating $V(t)$ along the closed-loop trajectories

$$\begin{aligned}\dot{V}(t) \leq\, & 2z^{\mathrm{T}}(t)\bar{P}\dot{z}^{\mathrm{T}}(t) + h^2\dot{z}^{\mathrm{T}}(t)\bar{R}\dot{z}(t) \\ & - (z(t) - z(t-\tau(t)))^{\mathrm{T}}\bar{R}(z(t) - (t-\tau(t))) \\ & - (z(t-\tau(t)) - z(t-h))^{\mathrm{T}} \\ & \times \bar{R}(z(t-\tau(t)) - z(t-h)) \\ & + z^{\mathrm{T}}(t)(\bar{E}+\bar{S})z(t) - z^{\mathrm{T}}(t-h)\bar{E}z(t-h) \\ & - (1-d)z^{\mathrm{T}}(t-\tau(t))\bar{S}z(t-\tau(t)). \end{aligned} \quad (36)$$

Substitute the right-hand side of (34) into (36). Setting $\varsigma(t) = col\{z(t), z(t-h), z(t-\tau(t))\}$, it follows that

$$\begin{aligned}\dot{V}(t) \leq\, & \varsigma(t)^{\mathrm{T}}\Phi_h\varsigma(t) + h^2\dot{z}^{\mathrm{T}}(t)\bar{R}\dot{z}(t) \\ & + 2z^{\mathrm{T}}\bar{P}\bar{B}(\xi(t,z,u) - v(t)) < 0 \end{aligned} \quad (37)$$

is satisfied if $\varsigma^{\mathrm{T}}(t)\Phi_h\varsigma(t) + h^2\dot{z}^{\mathrm{T}}(t)\bar{R}\dot{z}(t) < 0$ and $2z^{\mathrm{T}}\bar{P}\bar{B}(\xi(t,z,u) - v(t)) < 0$, where

$$\Phi_h = \begin{bmatrix} \phi_{11} - \bar{R} & 0 & \bar{P}\bar{A}_d + \bar{R} \\ * & -(\bar{E}+\bar{R}) & \bar{R} \\ * & * & -2\bar{R} - (1-d)\bar{S} \end{bmatrix} \quad (38)$$

with

$$\phi_{11} = \bar{A}_0^{\mathrm{T}}\bar{P} + \bar{P}\bar{A}_0 + \bar{S} + \bar{E}. \quad (39)$$

Setting $\varrho(t) = col\{z(t), z(t-h), z(t-\tau), \xi(t,z,u) - v(t)\}$

$$\begin{aligned} & h^2\dot{z}^{\mathrm{T}}(t)\bar{R}\dot{z}(t) \\ & = h^2\Big[z^{\mathrm{T}}(t)\bar{A}_0 + z^{\mathrm{T}}(t-\tau)\bar{A}_d{}^{\mathrm{T}} \\ & \qquad + \left(\xi(t,z,u) - v(t)^{\mathrm{T}}\right)\bar{B}^{\mathrm{T}}\Big]\bar{R} \\ & \quad \times \left[\bar{A}_0 z + \bar{A}_d z(t-\tau) + \bar{B}(\xi(t,z,u) - v(t))\right] \\ & = \varrho^{\mathrm{T}}(t)\begin{bmatrix} \bar{A}_0^{\mathrm{T}} \\ 0_n{}^{\mathrm{T}} \\ \bar{A}_d \\ \bar{B}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} I_{(n-m)} \\ 0 \end{bmatrix} h^2\bar{R}_1 \\ & \quad \times \begin{bmatrix} I_{(n-qm)} \\ 0 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \bar{A}_0^{\mathrm{T}} \\ 0_n{}^{\mathrm{T}} \\ \bar{A}_d \\ \bar{B}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \varrho(t). \end{aligned} \quad (40)$$

Using the Schur complement, $\xi^{\mathrm{T}}(t)\Phi_h\xi(t) + h^2\dot{z}^{\mathrm{T}}(t)\bar{R}\dot{z}(t) < 0$ holds if

$$\begin{bmatrix} \Phi_h & \begin{matrix} h\bar{A}_0^{\mathrm{T}}\begin{bmatrix} I_{(n-m)} \\ 0 \end{bmatrix}\bar{R}_1 \\ 0_{(n,n-m)} \\ h\bar{A}_d{}^{\mathrm{T}}\begin{bmatrix} I_{(n-m)} \\ 0 \end{bmatrix}\bar{R}_1 \end{matrix} \\ *** & -\bar{R}_1 \end{bmatrix} < 0. \quad (41)$$

Inequality (41) is an LMI in the decision variables $\bar{P}_1 > 0$, $\bar{E} \geq 0$, $\bar{S} \geq 0$ and $\bar{R}_1 \geq 0$. Equation (37) is valid if (41) is satisfied and given

$$\begin{aligned} & 2z^{\mathrm{T}}\bar{P}\bar{B}(\xi(t,z,u) - v(t)) \\ & \quad = 2y^{\mathrm{T}}F^{\mathrm{T}}(\xi(t,z,u) - v(t)) \\ & \quad \leq -2y^{\mathrm{T}}F^{\mathrm{T}}v(t) + 2\|Fy(t)\|\|\xi(t,z,u)\| \\ & \quad = -2\rho(t,y)\|Fy(t)\| + 2\|Fy(t)\|\|\xi(t,z,u)\| \\ & \quad < -2\|Fy(t)\|(\rho(t,y) - k_1\|u(t)\| - \alpha(t,y)). \end{aligned} \quad (42)$$

However, by definition

$$\rho(t,y) = (k_1\gamma\|Fy(t)\| + \alpha(t,y) + \gamma_2)/(1-k_1)$$

and so by rearranging

$$\begin{aligned}\rho(t,y) &= k_1\rho(t,y) + k_1\gamma\|Fy(t)\| + \alpha(t,y) + \gamma_2 \\ & \geq k_1(\|v(t)\| + \gamma\|Fy(t)\|) + \alpha(t,y) + \gamma_2 \\ & \geq k_1\|u(t)\| + \alpha(t,y) + \gamma_2. \end{aligned} \quad (43)$$

From (37), if (41) is valid, then from (42) and (43),

$$\dot{V}(t) < -2\gamma_2\|Fy(t)\| < 0 \quad \text{if } z(t) \neq 0 \quad (44)$$

and therefore the system is asymptotically stable.

Lemma 2: Given large enough γ, let there exist $n \times n$ matrices $\bar{P}_1 > 0$, $\bar{E} \geq 0$, $\bar{S} \geq 0$, $\bar{R}_1 \geq 0$ from the LMI solver such that LMI (41) holds. Given that the design parameters k_1, $\alpha(t,y)$, γ_2, and $\bar{P}_2$ have been selected so that (44) holds, the closed-loop system (33) is asymptotically stable for all differentiable delays $0 \leq \tau(t) \leq h$, $\dot{\tau}(t) \leq d \leq 1$.

V. Finite-Time Reachability to the Sliding Manifold

Corollary: An ideal sliding motion takes place on the surface S if

$$\left\|B_2^{-1}\bar{A}_0^L z(t)\right\| + \left\|B_2^{-1}\bar{A}_d^L z(t-\tau)\right\| < \gamma_2 - \eta \quad (45)$$

where the matrices $\bar{A}_0^L$ and $\bar{A}_d^L$ represent the last m rows of $\bar{A}_0$ and $\bar{A}_d$, respectively, and η is a small scalar satisfying $0 < \eta < \gamma_2$.

Proof:

$$\dot{s}(t) = F\bar{C}\bar{A}_0\bar{z}(t) + F\bar{C}\bar{A}_d z(t-\tau) + F_2B_2(\xi(t,z,u) - v(t)). \quad (46)$$

Let $V_c : \mathcal{R}^m \rightarrow \mathcal{R}$ be defined by

$$V_c(s) = s^{\mathrm{T}}(t)\left(F_2^{-1}\right)^{\mathrm{T}} \bar{P}_2 F_2^{-1} s(t). \quad (47)$$

Then, using the fact that $F_2^{\mathrm{T}} = \bar{P}_2 B_2$ it follows that

$$\left(F_2^{-1}\right)^{\mathrm{T}} \bar{P}_2 F_2^{-1} F \bar{C} \bar{A}_0 = B_2^{-1} \bar{A}_0^L$$
$$\left(F_2^{-1}\right)^{\mathrm{T}} \bar{P}_2 F_2^{-1} F \bar{C} \bar{A}_d = B_2^{-1} \bar{A}_d^L. \quad (48)$$

Then, it can be verified that

$$\begin{aligned}\dot{V}_c =& 2s^{\mathrm{T}}(t) B_2^{-1} \bar{A}_0^L z(t) + 2s^{\mathrm{T}}(t) B_2^{-1} \bar{A}_d^L z(t-\tau)\\ &+ 2s^{\mathrm{T}}(t)\left(\xi(t,z,u) - v(t)\right)\\ \leq& 2\|s(t)\| \left\|B_2^{-1}\bar{A}_0^L z(t)\right\| + 2\|s(t)\|\\ &\times \left\|B_2^{-1}\bar{A}_d^L z(t-\tau)\right\| - 2\gamma_2\|s(t)\|\\ <& -2\eta\|s(t)\| \end{aligned} \quad (49)$$

if $z(t)$ and $z(t-\tau) \in \Omega$. It follows that there exists a t_0 such that $z(t)$ and $z(t-\tau) \in \Omega$ for all $t > t_0$. Consequently, (49) holds for all $t > t_0$. A sliding motion will thus be attained in finite time.

Example 1: The following model of a liquid monopropellant rocket motor has been considered in [35]. It is assumed that the variable $\kappa = 0.8$ in this case, where $A_d(1,1) = -\kappa$ and $A(1,1) = \kappa - 1$. The outputs have been chosen to be the second and fourth states so that in (1)

$$A = \begin{bmatrix} -0.2 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & -1 & 1\\ 0 & 1 & 1 & 0 \end{bmatrix} \quad A_d = \begin{bmatrix} -0.8 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (50)$$

Clearly, the Kimura–Davison conditions are not met. Here, the rate at which the delay varies with time has been examined with $d = 0$, a slow varying delay. The gain from the LMI tool solver with $\delta = 50$, $\epsilon = 0.5$, $h = 0.45s$, $\bar{P}_2 = 1$, $\gamma = 2.9$, and $M = [5\ 0.2]$, yields $F = [-1\ \ -2.0754]$. The poles of the sliding-mode dynamics are $\{-2.67,\ -0.4,\ -0.2\}$. A simulation was performed with the initial state values $[1\ 1\ 1\ 1]$. As shown, the LMI solver gave a feasible result for stability for $h \leq 0.45s$, but in simulation, the closed-loop system only became unstable for $h \geq 1s$ (Fig. 1); this is due to the conservativeness of the method. The LMI solver gave feasible closed-loop stability results for controller gain $\gamma \geq 2.9$ while a choice of $\gamma \geq 1$ in simulation was able to stabilize the system with the compensation of longer settling time. The switching function for $h = 0.45s$, $\gamma = 2.9$ is shown in Fig. 2.

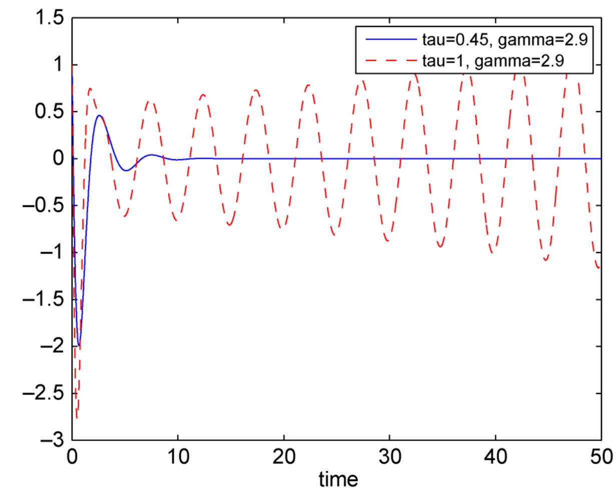

Fig. 1. Output against time.

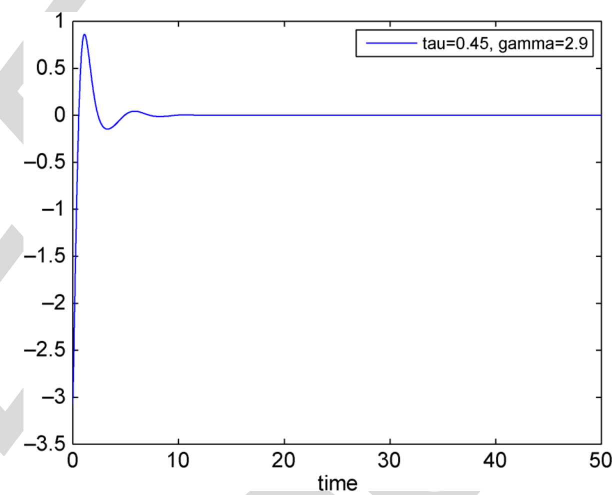

Fig. 2. Switching function.

VI. Compensator-Based Existence Problem

For certain system triples (A_{11}, A_{12}, C_1), LMI (24) is known to be infeasible. In this case, consider a dynamic compensator similar to that of El-Khazali and DeCarlo [36]

$$\dot{z}_c(t) = H z_c(t) + D y(t) \quad (51)$$

where the matrices $H \in \mathcal{R}^{q\times q}$ and $D \in \mathcal{R}^{q\times p}$ are to be determined. Define a new hyperplane in the augmented state space, formed from the plant and compensator state spaces, as

$$S_c = \left\{(z(t), z_c(t)) \in \mathcal{R}^{n+q}\ :\ F_c z_c(t) + F C z(t) = 0\right\} \quad (52)$$

where $F_c \in \mathcal{R}^{m\times q}$ and $F \in \mathcal{R}^{m\times p}$. Define $D_1 \in \mathcal{R}^{q\times(p-m)}$ and $D_2 \in \mathcal{R}^{q\times m}$ as

$$[D_1 \quad D_2] = DT \quad (53)$$

then the compensator can be written as

$$\dot{z}_c(t) = H z_c(t) + D_1 C_1 z_1(t) + D_2 z_2(t) \quad (54)$$

where C_1 is defined in (8). The sliding motion, obtained by eliminating the coordinates $z_2(t)$, can be written as

$$\begin{aligned}\dot{z}_1(t) &= (A_{11} - A_{12}KC_1)z_1(t) - A_{12}K_c z_c(t)\\ &\quad + (A_{d11} - A_{d12}KC_1)z_1(t-\tau) - A_{d12}K_c z_c(t-\tau)\\ \dot{z}_c(t) &= (D_1 - D_2K)C_1 z_1(t) + (H - D_2K_c)z_c(t) \qquad (55)\end{aligned}$$

where $K = F_2^{-1}F_1$ and $K_c = F_2^{-1}F_c$, then similar to [37], the design problem becomes one of selecting a compensator, represented by the matrices D_1, D_2, and H, and a hyperplane, represented by the matrices K and K_c, so that the system

$$\begin{aligned}\begin{bmatrix}\dot{z}_1(t)\\ \dot{z}_c(t)\end{bmatrix} &= \underbrace{\begin{bmatrix}A_{11} - A_{12}KC_1 & -A_{12}K_c\\ (D_1 - D_2K)C_1 & H - D_2K_c\end{bmatrix}}_{A_c}\begin{bmatrix}z_1(t)\\ z_c(t)\end{bmatrix}\\ &\quad + \underbrace{\begin{bmatrix}A_{d11} - A_{d12}KC_1 & -A_{d12}K_c\\ 0 & 0\end{bmatrix}}_{A_{cd}}\begin{bmatrix}z_1(t-\tau)\\ z_c(t-\tau)\end{bmatrix} \qquad (56)\end{aligned}$$

is stable. To obtain the compensator gains this problem can be shown to be a new output-feedback problem with

$$\begin{aligned}A_c &= \underbrace{\begin{bmatrix}A_{11} & 0\\ 0 & 0\end{bmatrix}}_{A_q} - \underbrace{\begin{bmatrix}A_{12} & 0\\ D_2 & -I_q\end{bmatrix}}_{B_q}\underbrace{\begin{bmatrix}K & K_c\\ D_1 & H\end{bmatrix}}_{K_q}\underbrace{\begin{bmatrix}C_1 & 0\\ 0 & I_q\end{bmatrix}}_{C_q}\\ A_{cd} &= \underbrace{\begin{bmatrix}A_{d11} & 0\\ 0 & 0\end{bmatrix}}_{A_{qd}} - \underbrace{\begin{bmatrix}A_{d12} & 0\\ 0 & 0\end{bmatrix}}_{B_{qd}}\underbrace{\begin{bmatrix}K & K_c\\ D_1 & H\end{bmatrix}}_{K_q}\underbrace{\begin{bmatrix}C_1 & 0\\ 0 & I_q\end{bmatrix}}_{C_q}. \qquad (57)\end{aligned}$$

The existence problem represented by system (56), where A_c and A_{cd} are partitioned as in (57) and D_2 is a tuning parameter, can be solved as for the noncompensated case (13). Similarly to (27),

$$\begin{aligned}K_qC_qQ_2 &= K_q\begin{bmatrix}0_{(p-m+q)\times(n-p)} & I_{p-m+q}\end{bmatrix}\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{22}\mathcal{M} & \delta Q_{22}\end{bmatrix}\\ &= \begin{bmatrix}K_qQ_{22}\mathcal{M} & \delta K_qQ_{22}\end{bmatrix}\\ &= \begin{bmatrix}Y\mathcal{M} & \delta Y\end{bmatrix} \qquad (58)\end{aligned}$$

where $Y = K_qQ_{22}$, $\mathcal{M} \in \mathcal{R}^{(p-m+q)\times(n-p)}$ is a tuning matrix.

Example 2: Consider the delay system

$$A = \begin{bmatrix}0 & 25 & -1\\ 1 & 0 & 0\\ -5 & 0 & 1\end{bmatrix} \quad A_d = \begin{bmatrix}0.1 & 0.1 & 0\\ 0 & 0.3 & -0.1\\ 0 & 0.2 & 0\end{bmatrix}$$

$$B = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \quad C = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad (59)$$

from which

$$A_{11} = \begin{bmatrix}0 & 25\\ 1 & 0\end{bmatrix} \quad A_{12} = \begin{bmatrix}-1\\ 0\end{bmatrix} \quad C_1 = \begin{bmatrix}1 & 0\end{bmatrix}.$$

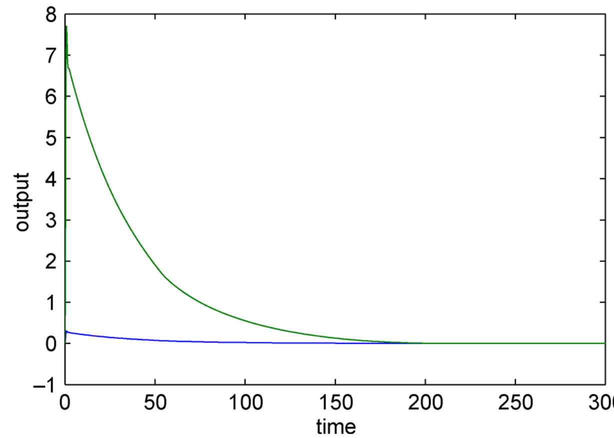

Fig. 3. Compensator-based controller design $h = 2.5s$.

It follows that

$$\lambda(A_{11} - A_{12}KC_1) = \pm\sqrt{(25 + K^2)}$$

and so (24) is infeasible. Now, consider designing a first-order compensator. Choosing $D_2 = 1$, it follows that

$$A_q = \begin{bmatrix}0 & 25 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad A_{qd} = \begin{bmatrix}0.1 & 0.1 & 0\\ 0 & 0.3 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$B_q = \begin{bmatrix}-1 & 0\\ 0 & 0\\ 1 & -1\end{bmatrix} \quad B_{qd} = \begin{bmatrix}-0.1 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

$$C_q = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Choosing $\delta = 50$, $\epsilon = 5$, $\mathcal{M} = [10 \quad 4]'$, and $d = 0$ (slowly varying delay) with the maximum allowable delay $h = 0.25s$, the LMITOOL solver returns

$$K_q = \begin{bmatrix}-25.38 & -0.37\\ -25.32 & -5.03\end{bmatrix}.$$

The augmented system with compensator given by

$$A_a = \begin{bmatrix}H & DC\\ 0 & A\end{bmatrix} \quad A_{da} = \begin{bmatrix}0 & 0\\ 0 & A_d\end{bmatrix}$$

$$B_a = \begin{bmatrix}0\\ B\end{bmatrix} \quad C_a = \begin{bmatrix}I_q & 0\\ 0 & C\end{bmatrix}$$

is asymptotically stablized by the controller

$$[F_c \quad F] = [-0.369 \quad -25.38 \quad 1].$$

Taking the controller in the form of (10) and (11) where $\gamma = 10$, $\rho = 10$. Simulation results with the switching gain and initial conditions $[0.5,\ 0,\ 0]$, as shown in Figs. 3 and 4.

VII. Conclusion

A descriptor Lyapunov–Krasovskii functional method has been introduced for SOF switching function design for systems

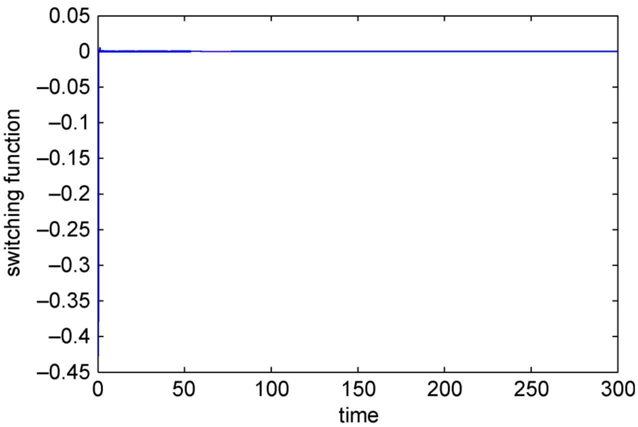

Fig. 4. Compensator-based controller design $h = 2.5s$.

with state time-varying delays. The delay is assumed bounded with a known upper bound, either slowly or fast varying. In addition, a novel stability analysis of the full-order closed-loop discontinuous time-delay system has been performed via the Krasovskii method, which is delay independent in $z_2(t)$ (and thus the delay is restricted to be slowly varying) and delay dependent in $z_1(t)$, i.e., in the state of the reduced-order system. The proposed SOF design approach also applies to compensator-based design. Examples show the effectiveness of the method. For future work, the results can be extended to the interval delay case, where the lower bound on the delay is taken into account. The Razumikin approach can be employed for the stability analysis of the full-order closed-loop system with fast varying delay.

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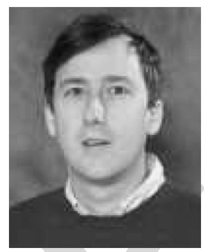

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AQ1 = “Assume that” was deleted. Please check if correct.
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AQ6 = Please provide issue number and month of publication in Ref. [16].
AQ7 = Please provide page range in Ref. [17].
AQ8 = Please provide page range in Ref. [31].
AQ9 = Please specify the degree received by Mr. Han from Fushun Mining College, Fushun, China, in 2002.

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On the Design of Sliding-Mode Static-Output-Feedback Controllers for Systems With State Delay

X. R. Han, Emilia Fridman, Sarah K. Spurgeon, and Chris Edwards

***Abstract*—This paper considers the development of sliding-mode-based output-feedback controllers for uncertain systems which are subject to time-varying state delays. A novel method is proposed for design of the switching surface. This method is based on the descriptor approach and leads to a solution in terms of linear matrix inequalities (LMIs). When compared to existing methods (even for systems without delays), the proposed method is efficient and less conservative than other results, giving a feasible solution when the Kimura–Davison conditions are not satisfied. No additional constraints are imposed on the dimensions or structure of the reduced order triple associated with design of the switching surface. The magnitude of the linear gain used to construct the controller is also verified as an appropriate solution to the reachability problem using LMIs. A stability analysis for the full-order time-delay system with discontinuous right-hand side is formulated. This paper facilitates the constructive design of sliding-mode static-output-feedback controllers for a rather general class of time-delay systems. A numerical example from the literature illustrates the efficiency of the proposed method.**

***Index Terms*—Linear matrix inequalities (LMIs), sliding-mode control (SMC), static output feedback (SOF), time delay.**

I. INTRODUCTION

SLIDING-MODE control (SMC) [1] is known for its complete robustness to so-called matched uncertainties (which can include time delays that satisfy matching conditions) and disturbances [2]–[4]. The control technique has been applied in many industrial areas [5]–[7]. Many early theoretical developments in SMC assume that all the system states are accessible. In the case where only a subset of states are measurable, which is relevant to a range of practical applications, either output feedback control or the observer-based method are required. Some work has considered implementation of SMC schemes using observers [8]–[10]. In [11], a sliding-mode observer has been shown to give a significant increase in performance in estimation of the unknown variables of a boost converter compared to a traditional current-mode control strategy. A further interesting strand considers the fast output sampling method [12], [13]. Recently, in [14], a fast sampling method is employed for a discrete systems in the presence of time-varying delays where a sliding-mode controller is designed using linear matrix inequalities (LMIs) combined with a delta-operator approach. However, all these methods require additional computation. The most straightforward approach is to consider the study of SMC via static output feedback (SOF).

One problem of interest in the development of SMC via SOF is the design of the switching surface, which is effectively a reduced order SOF problem for a particular subsystem. Two different methods were proposed to design the sliding surface using eigenvalue assignment and eigenvector techniques in [15] and [16]. A canonical form was provided in [18] via which the SOFSMC design problem is routinely converted to an SOF stabilization problem. As stated in [20], all previous-reported methods for the existence problem are, in fact, equivalent to a particular SOF problem. The solution to the general SOF problem, even for linear time-invariant systems, is still open.

LMI methods have been considered within the context of sliding-mode controller design. For example, [21] and [22] presented LMIs methods to design static sliding-mode output-feedback controllers and [23] presented a necessary and sufficient condition to solve the existence problem in terms of LMIs for linear uncertain systems.

It is important to note that all the work described above does not consider an existence problem involving delay and many practical problems include such effects [24], [25]. In [26], the problem of the development of sliding-mode controllers for operation in the presence of single or multiple, constant or time-varying state delays has been solved. This uses the usual regular form method of solution and the uncertainty is assumed to be matched, where matched describes that uncertainty class which is implicit in the range of the input channels and will be rejected by an appropriately designed SMC strategy, although it is important to note that full state availability is assumed. This problem has also been considered in [27] where a class of uncertain time delay systems with multiple fixed delays in the system states is considered. This paper considers unmatched and time-varying parameter uncertainties, together with matched and bounded external disturbances, but again, full state information is assumed to be available to the controller. In [28], Lyapunov functionals were for the first time introduced for the analysis of time-varying delay. In [29], a descriptor

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approach to stability and control of linear systems with time-varying delays, which is based on the Lyapunov–Krasovskii techniques, was combined with results on the SMC of such systems. The systems under consideration were subjected to norm-bounded uncertainties and uncertain bounded delays and the solution given in terms of LMIs. Reference [30] develops a SMC synthesis for a class of uncertain time-delay systems with nonlinear disturbances and unknown delay values whose unperturbed dynamics is linear. The synthesis was based on a new delay-dependent stability criterion. The controller is found to be robust against sufficiently small delay variations and external disturbances.

It is important to emphasize that much of the aforementioned literature on SMC of time-delay systems assumes full-state feedback. Reference [31] considered SMC of systems with time-varying delay. This paper considered a solution via LMIs for the existence problem. The current paper extends this contribution to consider the solution of the existence and reachability problems. Specifically, the selection of parameters for stability of the full-order closed-loop system are obtained via LMIs. In this paper, a compensator-based design problem is considered using the proposed SOF approach. Example from the literature illustrates the efficiency of the method. In Section II, the problem formulation is described. The existence problem is formulated in Section III. In Sections IV and V, stability of the full-order closed-loop system is derived via LMIs, and the reachability problem is presented. Compensator-based design is demonstrated in Section VI.

II. Problem Formulation

Consider an uncertain time-delay system

$$\begin{aligned}\dot{z}(t) &= Az(t) + A_d z\left(t-\tau(t)\right) + B\left(u(t)+\xi(t,z,u)\right)\\ y(t) &= Cz(t)\end{aligned} \tag{1}$$

where $z \in \mathcal{R}^n$, $u \in \mathcal{R}^m$, and $y \in \mathcal{R}^p$ with $m \leq p \leq n$. The time-varying delay $\tau(t)$ is supposed to be bounded $0 \leq \tau(t) \leq h$, and it may be either slowly varying (i.e., differentiable delay with $\dot{\tau}(t) \leq d < 1$) or fast varying (piecewise continuous delay). Assume that the nominal linear system (A, A_d, B, C) is known and that the input and output matrices B and C are both of full rank. The unknown function $\xi\colon \mathcal{R}_+ \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^n$, which represents the system nonlinearities plus any model uncertainties, is assumed to satisfy the matching condition and

$$\|\xi(t,z,u)\| < k_1\|u\| + a(t,y) \tag{2}$$

for some known function $a: \mathcal{R}_+ \times \mathcal{R}^p \to \mathcal{R}_+$ and positive constant $k_1 < 1$. It can be shown that if $rank(CB) = m$, there exists a coordinate system in which the system (A, A_d, B, C) has the structure

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad A_d = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad C = [0 \quad T] \tag{3}$$

where $B_2 \in \mathcal{R}^{m\times m}$ is nonsingular and $T \in \mathcal{R}^{p\times p}$ is orthogonal. The system can be represented as

$$\begin{aligned}\dot{z}_1(t) &= A_{11}z_1(t) + A_{d11}z_1\left(t-\tau(t)\right)\\ &\quad + A_{12}z_2(t) + A_{d12}z_2\left(t-\tau(t)\right)\\ \dot{z}_2(t) &= \sum_{i=1}^{2}\left(A_{2i}z_i(t) + A_{d2i}z_i\left(t-\tau(t)\right)\right)\\ &\quad + B_2\left(u(t)+\xi(t,z,u)\right)\\ y(t) &= Cz(t).\end{aligned} \tag{4}$$

Consider the following switching function: AQ1

$$\mathcal{S} = \{z(t) \in \mathcal{R}^n : FCz(t) = 0\} \tag{5}$$

for some selected matrix $F \in \mathcal{R}^{m\times p}$ where by design $det(FCB) \neq 0$. Let

$$[F_1 \quad F_2] = FT \tag{6}$$

where $F_1 \in \mathcal{R}^{p-m}$ and $F_2 \in \mathcal{R}^m$. As a result

$$FC = [F_1C_1 \quad F_2] \tag{7}$$

where

$$C_1 = \left[0_{(p-m)\times(n-p)} \quad I_{(p-m)}\right]. \tag{8}$$

Therefore, $FCB = F_2B_2$ and the square matrix F_2 is nonsingular. By assumption, the uncertainty is matched, and therefore the sliding motion is independent of the uncertainty represented by $\xi(\cdot)$. In addition, because the canonical form in (3), where it is necessary that the pair (A_{11}, A_{12}) is controllable and (A_{11}, C_1) is observable, can be viewed as a special case of the regular form normally used in sliding-mode controller design, the switching function can also be expressed as

$$s(t) = z_2(t) + KC_1z_1(t) \tag{9}$$

where $K \in \mathcal{R}^{m\times(p-m)}$ and is defined as $K = F_2^{-1}F_1$.

Then, a simple SMC law, depending on the output information $Fy(t)$ can be defined by

$$u(t) = -\gamma Fy(t) - v(t) \tag{10}$$

where

$$v(t) = \begin{cases} \rho(t,y)\frac{Fy(t)}{\|Fy(t)\|}, & \text{if } Fy(t) \neq 0 \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

where $\rho(t,y)$ is some positive scalar function of the outputs

$$\rho(t,y) = \left(k_1\gamma\,\|Fy(t)\| + \alpha(t,y) + \gamma_2\right)/(1-k_1)$$

where γ and γ_2 are positive design scalars [18]. The closed-loops system (4) and (10) can be described by the following equations:

$$\begin{aligned}\dot{z}_1(t) &= (A_{11} - A_{12}KC_1)z_1(t)\\ &\quad + (A_{d11} - A_{d12}KC_1)z_1\left(t-\tau(t)\right)\end{aligned}$$

$$\begin{aligned}\dot{s}(t) =& (A_{21} - \gamma B_2 K C_1) z_1(t)\\ &+ (A_{d21} - \gamma B_2 K C_1) z_1(t - \tau(t))\\ &+ (A_{22} - \gamma B_2) z_2(t) + (A_{d22} - \gamma B_2) z_2(t - \tau(t))\\ &+ B_2(\xi(t, z, u) - v(t))\\ y(t) =& C z(t). \end{aligned} \tag{12}$$

III. Existence Problem

On the sliding manifold $s(t) = 0$, it is well known [19] that the reduced-order sliding motion is governed by a free motion with system matrix

$$\begin{aligned}\dot{z}_1(t) = (A_{11} - A_{12} K C_1) z_1(t)\\ +(A_{d11} - A_{d12} K C_1) z_1(t - \tau(t)). \end{aligned} \tag{13}$$

Consider a Lyapunov–Krasovskii functional

$$\begin{aligned}V(t) =& z_1^{\mathrm{T}}(t) P z_1(t) + \int_{t-h}^{t} z_1^{\mathrm{T}}(s) E z_1(s) ds\\ &+ \int_{t-\tau(t)}^{t} z_1^{\mathrm{T}}(s) S z_1(s) ds\\ &+ h \int_{-h}^{0} \int_{t+\theta}^{t} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds d\theta \end{aligned} \tag{14}$$

where the symmetric matrices $P > 0$ and $E, S, R \geq 0$.

The condition $\dot{V}(t) < 0$ guarantees asymptotic stability of the reduced order system as in [32]. Differentiating $V(t)$ along (13)

$$\begin{aligned}\dot{V}(t) =& 2 z_1^{\mathrm{T}}(t) P \dot{z}_1(t) + h^2 \dot{z}_1^{\mathrm{T}}(t) R \dot{z}_1(t)\\ &- h \int_{t-h}^{t} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds + z_1^{\mathrm{T}}(t)(E + S) z_1(t)\\ &- z_1^{\mathrm{T}}(t-h) E z_1(t-h)\\ &- (1 - \dot{\tau}(t)) z_1^{\mathrm{T}}(t - \tau(t)) S z_1(t - \tau(t)). \end{aligned} \tag{15}$$

Further using the identity

$$\begin{aligned}-h \int_{t-h}^{t} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds = -h \int_{t-h}^{t-\tau(t)} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds\\ -h \int_{t-\tau(t)}^{t} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds \end{aligned} \tag{16}$$

and applying Jensen's inequality

$$\int_{t-\tau(t)}^{t} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds \geq \frac{1}{h} \int_{t-\tau(t)}^{t} \dot{z}_1^{\mathrm{T}}(s) ds R \int_{t-\tau(t)}^{t} \dot{z}_1(s) ds \tag{17}$$

$$\int_{t-h}^{t-\tau(t)} \dot{z}_1^{\mathrm{T}}(s) R \dot{z}_1(s) ds \geq \frac{1}{h} \int_{t-h}^{t-\tau(t)} \dot{z}_1^{\mathrm{T}}(s) ds R \int_{t-h}^{t-\tau(t)} \dot{z}_1(s) ds. \tag{18}$$

Then,

$$\begin{aligned}\dot{V}(t) \leq& 2 z_1^{\mathrm{T}}(t) P \dot{z}_1^{\mathrm{T}}(t) + h^2 \dot{z}_1^{\mathrm{T}}(t) R \dot{z}_1(t)\\ &- (z_1(t) - z_1(t - \tau(t)))^{\mathrm{T}} R (z_1(t) - z_1(t - \tau(t)))\\ &- (z_1(t - \tau(t)) - z_1(t-h))^{\mathrm{T}}\\ &\times R (z_1(t - \tau(t)) - z_1(t-h))\\ &+ z_1^{\mathrm{T}}(t)(E + S) z_1(t) - z_1^{\mathrm{T}}(t-h) E z_1(t-h)\\ &- (1 - d) z_1^{\mathrm{T}}(t - \tau(t)) S z_1(t - \tau(t)). \end{aligned} \tag{19}$$

Using the descriptor method as in [33] and the free-weighting matrices technique from [34], the right-hand side of the expression

$$\begin{aligned}0 \equiv& 2\left(z_1^{\mathrm{T}}(t) P_2^{\mathrm{T}} + \dot{z}_1^{\mathrm{T}}(t) P_3^{\mathrm{T}}\right)\\ &\times [-\dot{z}_1(t) + (A_{11} - A_{12} K C_1) z_1(t)\\ &+(A_{d11} - A_{d12} K C_1) z_1(t - \tau(t))] \end{aligned} \tag{20}$$

with matrix parameters P_2, $P_3 = \epsilon P_2 \in \mathcal{R}^{n-m}$ is added into the right-hand side of (19). Setting $\eta(t) = col\{z_1(t), \dot{z}_1(t), z_1(t-h), z_1(t - \tau(t))\}$, it follows that

$$\dot{V}(t) \leq \eta^{\mathrm{T}}(t) \Theta \eta(t) \leq 0 \tag{21}$$

if the matrix inequality

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & 0 & \theta_{14}\\ * & \theta_{22} & 0 & \theta_{24}\\ * & * & \theta_{33} & \theta_{34}\\ * & * & * & \theta_{44} \end{bmatrix} < 0 \tag{22}$$

is feasible, where

$$\begin{aligned}\theta_{11} =& P_2^{\mathrm{T}}(A_{11} - A_{12} K C_1)\\ &+ (A_{11} - A_{12} K C_1)^{\mathrm{T}} P_2 + E + S - R\\ \theta_{12} =& P - P_2^{\mathrm{T}} + \epsilon (A_{11} - A_{12} K C_1)^{\mathrm{T}} P_2\\ \theta_{14} =& P_2^{\mathrm{T}}(A_{d11} - A_{d12} K C_1) + R\\ \theta_{22} =& -\epsilon P_2 - \epsilon P_2^{\mathrm{T}} + h^2 R\\ \theta_{24} =& \epsilon P_2^{\mathrm{T}}(A_{d11} - A_{d12} K C_1)\\ \theta_{33} =& -(E + R)\\ \theta_{34} =& R\\ \theta_{44} =& -2R - (1 - d) S. \end{aligned} \tag{23}$$

Multiplying matrix Θ from the right and the left by $\mathrm{diag}\{P_2^{-1}, P_2^{-1}, P_2^{-1}, P_2^{-1}\}$ and its transpose, respectively, and denoting

$$\begin{aligned} Q_2 = P_2^{-1} \quad \widehat{P} = Q_2^{\mathrm{T}} P Q_2 \quad \widehat{R} = Q_2^{\mathrm{T}} R Q_2\\ \widehat{E} = Q_2^{\mathrm{T}} E Q_2 \quad \widehat{S} = Q_2^{\mathrm{T}} S Q_2 \end{aligned}$$

it follows $\Theta < 0 \Leftrightarrow \widehat{\Theta} < 0$, where

$$\widehat{\Theta} = \begin{bmatrix} \widehat{\theta}_{11} & \widehat{\theta}_{12} & 0 & \widehat{\theta}_{14} \\ * & \widehat{\theta}_{22} & 0 & \widehat{\theta}_{24} \\ * & * & \widehat{\theta}_{33} & \widehat{\theta}_{34} \\ * & * & * & \widehat{\theta}_{44} \end{bmatrix} < 0 \tag{24}$$

$$\begin{aligned} \widehat{\theta}_{11} &= (A_{11} - A_{12}KC_1)Q_2 \\ &\quad + Q_2^{\mathrm{T}}(A_{11} - A_{12}KC_1)^{\mathrm{T}} + \widehat{E} + \widehat{S} - \widehat{R} \\ \widehat{\theta}_{12} &= \widehat{P} - Q_2 + \epsilon Q_2^{\mathrm{T}}(A_{11} - A_{12}KC_1)^{\mathrm{T}} \\ \widehat{\theta}_{14} &= (A_{d11} - A_{d12}KC_1)Q_2 + \widehat{R} \\ \widehat{\theta}_{22} &= -\epsilon Q_2 - \epsilon Q_2^{\mathrm{T}} + h^2\widehat{R} \\ \widehat{\theta}_{24} &= \epsilon(A_{d11} - A_{d12}KC_1)Q_2 \\ \widehat{\theta}_{33} &= -\widehat{E} - \widehat{R} \\ \widehat{\theta}_{34} &= \widehat{R} \\ \widehat{\theta}_{44} &= -2\widehat{R} - (1-d)\widehat{S}. \end{aligned} \tag{25}$$

Define the variable Q_2 in the following form:

$$Q_2 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{22}\mathcal{M} & \delta Q_{22} \end{bmatrix} \tag{26}$$

where Q_{22} is a $(p-m) \times (p-m)$ matrix, and $\mathcal{M} \in \mathcal{R}^{(p-m)\times(n-p)}$ and $\delta \in \mathcal{R}$ are *a priori* selected tuning parameters. It follows that

$$KC_1Q_2 = [KQ_{22}\mathcal{M} \quad \delta KQ_{22}].$$

Defining

$$Y = KQ_{22}$$

it follows that

$$KC_1Q_2 = [Y\mathcal{M} \quad \delta Y]. \tag{27}$$

To construct K, substitute (27) into (25) to yield

$$\begin{aligned} \widehat{\theta}_{11} &= A_{11}Q_2 - A_{12}[Y \quad \delta Y] + Q_2^{\mathrm{T}}A_{11}^{\mathrm{T}} \\ &\quad - [Y\mathcal{M} \quad \delta Y]^{\mathrm{T}}A_{12}^{\mathrm{T}} + \widehat{E} + \widehat{S} - \widehat{R} \\ \widehat{\theta}_{12} &= \widehat{P} - Q_2 + \epsilon Q_2^{\mathrm{T}}A_{11}^{\mathrm{T}} - \epsilon[Y\mathcal{M} \quad \delta Y]^{\mathrm{T}}A_{12}^{\mathrm{T}} \\ \widehat{\theta}_{14} &= A_{d11}Q_2 - A_{d12}[Y\mathcal{M} \quad \delta Y] + \widehat{R} \\ \widehat{\theta}_{22} &= -\epsilon Q_2 - \epsilon Q_2^{\mathrm{T}} + h^2\widehat{R} \\ \widehat{\theta}_{24} &= \epsilon A_{d11}Q_2 - \epsilon A_{d12}[Y\mathcal{M} \quad \delta Y] \\ \widehat{\theta}_{33} &= -\widehat{E} - \widehat{R} \\ \widehat{\theta}_{34} &= \widehat{R} \\ \widehat{\theta}_{44} &= -2\widehat{R} - (1-d)\widehat{S} \end{aligned} \tag{28}$$

with the tuning parameters δ, ϵ, and $\mathcal{M}$. The following Lemma may now be stated.

Lemma 1: Given *a priori* selected tuning parameters ϵ, δ, and $\mathcal{M} \in R^{(p-m)\times(n-p)}$, then (24) is an LMI in the decision variables $\widehat{P} > 0$, $\widehat{E} \geq 0, \widehat{S} \geq 0$, $\widehat{R} \geq 0$ and matrices $Q_{22} \in \mathcal{R}^{(p-m)\times(p-m)}$, $Q_{11} \in \mathcal{R}^{(n-p)\times(n-p)}$, $Q_{12} \in \mathcal{R}^{(n-p)\times(p-m)}$, $Y \in \mathcal{R}^{m\times(p-m)}$. If a solution to (24) exists, which may be readily obtained from available LMI tools, then the reduced order system (13) is asymptotically stable for all differentiable delays $0 \leq \tau(t) \leq h, \dot{\tau}(t) \leq d < 1$. Moreover, (13) is asymptotically stable for all piecewise-continuous delays $0 \leq \tau(t) \leq h$, if the LMI (24) is feasible with $\widehat{S} = 0$.

Remark: The proposed method is suitable for SOF sliding-mode controller design where Kimura–Davison conditions, written as $n \leq m + p - 1$, are not satisfied as in [20]–[22]. No constraints are imposed on the dimensions of the reduced-order triple A_{11}, A_{12}, C_1. This represents a constructive and efficient approach to output-feedback-based design for a relatively broad class of systems, which is less conservative than existing results [20]–[22]; an example to illustrate the advantages of the method for systems without time-delay is presented in [17].

IV. Stability of the Full-Order Closed-Loop System

It can be shown [18] that there exist a coordinate system in which the system triple $(\bar{A}, \bar{A}_d, \bar{B}, F\bar{C})$ has the property that

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \quad \bar{A}_d = \begin{bmatrix} \bar{A}_{d11} & \bar{A}_{d12} \\ \bar{A}_{d21} & \bar{A}_{d22} \end{bmatrix}$$
$$\bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad F\bar{C} = [0 \quad F_2] \tag{29}$$

where $\bar{A}_{11} = A_{11} - A_{12}KC_1$ and $\bar{A}_{d11} = A_{d11} - A_{d12}KC_1$ and F_2 is a design parameter. Let $\bar{P}$ be a symmetric positive definite matrix partitioned conformably with the matrices in (29) so that

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{bmatrix} \tag{30}$$

then the matrix $\bar{P}$ satisfies the structural constraint

$$\bar{P}\bar{B} = \bar{C}^{\mathrm{T}}F^{\mathrm{T}} \tag{31}$$

if the design matrix $F_2 = B_2^{\mathrm{T}}\bar{P}_2$. The matrix $\bar{P}$ can be shown to be a Lyapunov matrix for

$$\begin{aligned} \bar{A}_0 &= \bar{A} - \gamma\bar{B}F\bar{C} \\ &= \bar{A} - \gamma\bar{B}[0 \quad F_2] \end{aligned} \tag{32}$$

for sufficiently large γ [18]. In the new coordinate system, the uncertain system (1) can be written as

$$\dot{z}(t) = \bar{A}z(t) + \bar{A}_d z(t-\tau) + \bar{B}\left(u(t) + \xi(t,z,u)\right). \tag{33}$$

The closed-loop system will have the form

$$\dot{z}(t) = \bar{A}_0 z(t) + \bar{A}_d z(t-\tau) + \bar{B}\left(\xi(t,y_t) - v_y(t)\right). \tag{34}$$

For large enough $\gamma > 0$, these conditions are delay independent with respect to the delay in z_2.} However, for derivation of this condition using Lyapunov–Krasovskii techniques, it is necessary to consider the case where $\dot{\tau} \leq d < 1$. A stability condition for the full-order closed-loop system can be derived using the following Lyapunov–Krasovskii functional:

$$V(t) = z^{\mathrm{T}}(t)\bar{P}z(t) + \int_{t-h}^{t} z^{\mathrm{T}}(s)\bar{E}z(s)ds + \int_{t-\tau(t)}^{t} z^{\mathrm{T}}(s)\bar{S}z(s)ds + h\int_{-h}^{0}\int_{t+\theta}^{t} \dot{z}^{\mathrm{T}}(s)\bar{R}\dot{z}(s)dsd\theta \quad (35)$$

where the matrix $\bar{E}$, $\bar{S} \geq 0$ and $\bar{R} = \begin{bmatrix} \bar{R}_1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$ as it is desired to determine a stability condition for the time delay system which is delay independent with respect to delay in $z_2(t)$. Differentiating $V(t)$ along the closed-loop trajectories

$$\begin{aligned}\dot{V}(t) \leq\, & 2z^{\mathrm{T}}(t)\bar{P}\dot{z}^{\mathrm{T}}(t) + h^2\dot{z}^{\mathrm{T}}(t)\bar{R}\dot{z}(t) \\ & - \left(z(t) - z\left(t-\tau(t)\right)\right)^{\mathrm{T}}\bar{R}\left(z(t) - \left(t-\tau(t)\right)\right) \\ & - \left(z\left(t-\tau(t)\right) - z(t-h)\right)^{\mathrm{T}} \\ & \times \bar{R}\left(z\left(t-\tau(t)\right) - z(t-h)\right) \\ & + z^{\mathrm{T}}(t)(\bar{E}+\bar{S})z(t) - z^{\mathrm{T}}(t-h)\bar{E}z(t-h) \\ & - (1-d)z^{\mathrm{T}}\left(t-\tau(t)\right)\bar{S}z\left(t-\tau(t)\right). \end{aligned} \quad (36)$$

Substitute the right-hand side of (34) into (36). Setting $\varsigma(t) = col\{z(t), z(t-h), z(t-\tau(t))\}$, it follows that

$$\begin{aligned}\dot{V}(t) \leq\, & \varsigma(t)^{\mathrm{T}}\Phi_h\varsigma(t) + h^2\dot{z}^{\mathrm{T}}(t)\bar{R}\dot{z}(t) \\ & + 2z^{\mathrm{T}}\bar{P}\bar{B}\left(\xi(t,z,u) - v(t)\right) < 0 \end{aligned} \quad (37)$$

is satisfied if $\varsigma^{\mathrm{T}}(t)\Phi_h\varsigma(t) + h^2\dot{z}^{\mathrm{T}}(t)\bar{R}\dot{z}(t) < 0$ and $2z^{\mathrm{T}}\bar{P}\bar{B}(\xi(t,z,u) - v(t)) < 0$, where

$$\Phi_h = \begin{bmatrix} \phi_{11} - \bar{R} & 0 & \bar{P}\bar{A}_d + \bar{R} \\ * & -(\bar{E}+\bar{R}) & \bar{R} \\ * & * & -2\bar{R} - (1-d)\bar{S} \end{bmatrix} \quad (38)$$

with

$$\phi_{11} = \bar{A}_0^{\mathrm{T}}\bar{P} + \bar{P}\bar{A}_0 + \bar{S} + \bar{E}. \quad (39)$$

Setting $\varrho(t) = col\{z(t), z(t-h), z(t-\tau), \xi(t,z,u) - v(t)\}$

$$\begin{aligned} & h^2\dot{z}^{\mathrm{T}}(t)\bar{R}\dot{z}(t) \\ &= h^2\Big[z^{\mathrm{T}}(t)\bar{A}_0 + z^{\mathrm{T}}(t-\tau)\bar{A}_d{}^{\mathrm{T}} \\ &\qquad + \left(\xi(t,z,u) - v(t)^{\mathrm{T}}\right)\bar{B}^{\mathrm{T}}\Big]\bar{R} \\ &\quad \times \left[\bar{A}_0 z + \bar{A}_d z(t-\tau) + \bar{B}\left(\xi(t,z,u) - v(t)\right)\right] \\ &= \varrho^{\mathrm{T}}(t)\begin{bmatrix} \bar{A}_0^{\mathrm{T}} \\ 0_n{}^{\mathrm{T}} \\ \bar{A}_d \\ \bar{B}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} I_{(n-m)} \\ 0 \end{bmatrix} h^2\bar{R}_1 \\ &\quad \times \begin{bmatrix} I_{(n-qm)} \\ 0 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \bar{A}_0^{\mathrm{T}} \\ 0_n{}^{\mathrm{T}} \\ \bar{A}_d \\ \bar{B}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}\varrho(t). \end{aligned} \quad (40)$$

Using the Schur complement, $\xi^{\mathrm{T}}(t)\Phi_h\xi(t) + h^2\dot{z}^{\mathrm{T}}(t)\bar{R}\dot{z}(t) < 0$ holds if

$$\begin{bmatrix} \Phi_h & \begin{matrix} h\bar{A}_0^{\mathrm{T}}\begin{bmatrix} I_{(n-m)} \\ 0 \end{bmatrix}\bar{R}_1 \\ 0_{(n,n-m)} \\ h\bar{A}_d{}^{\mathrm{T}}\begin{bmatrix} I_{(n-m)} \\ 0 \end{bmatrix}\bar{R}_1 \end{matrix} \\ *** & -\bar{R}_1 \end{bmatrix} < 0. \quad (41)$$

Inequality (41) is an LMI in the decision variables $\bar{P}_1 > 0$, $\bar{E} \geq 0$, $\bar{S} \geq 0$ and $\bar{R}_1 \geq 0$. Equation (37) is valid if (41) is satisfied and given

$$\begin{aligned} & 2z^{\mathrm{T}}\bar{P}\bar{B}\left(\xi(t,z,u) - v(t)\right) \\ &\quad = 2y^{\mathrm{T}}F^{\mathrm{T}}\left(\xi(t,z,u) - v(t)\right) \\ &\quad \leq -2y^{\mathrm{T}}F^{\mathrm{T}}v(t) + 2\left\|Fy(t)\right\|\left\|\xi(t,z,u)\right\| \\ &\quad = -2\rho(t,y)\left\|Fy(t)\right\| + 2\left\|Fy(t)\right\|\left\|\xi(t,z,u)\right\| \\ &\quad < -2\left\|Fy(t)\right\|\left(\rho(t,y) - k_1\left\|u(t)\right\| - \alpha(t,y)\right). \end{aligned} \quad (42)$$

However, by definition

$$\rho(t,y) = \left(k_1\gamma\left\|Fy(t)\right\| + \alpha(t,y) + \gamma_2\right)/(1-k_1)$$

and so by rearranging

$$\begin{aligned}\rho(t,y) &= k_1\rho(t,y) + k_1\gamma\left\|Fy(t)\right\| + \alpha(t,y) + \gamma_2 \\ &\geq k_1\left(\left\|v(t)\right\| + \gamma\left\|Fy(t)\right\|\right) + \alpha(t,y) + \gamma_2 \\ &\geq k_1\left\|u(t)\right\| + \alpha(t,y) + \gamma_2. \end{aligned} \quad (43)$$

From (37), if (41) is valid, then from (42) and (43),

$$\dot{V}(t) < -2\gamma_2\left\|Fy(t)\right\| < 0 \quad \text{if } z(t) \neq 0 \quad (44)$$

and therefore the system is asymptotically stable.

Lemma 2: Given large enough γ, let there exist $n \times n$ matrices $\bar{P}_1 > 0$, $\bar{E} \geq 0$, $\bar{S} \geq 0$, $\bar{R}_1 \geq 0$ from the LMI solver such that LMI (41) holds. Given that the design parameters k_1, $\alpha(t,y)$, γ_2, and $\bar{P}_2$ have been selected so that (44) holds, the closed-loop system (33) is asymptotically stable for all differentiable delays $0 \leq \tau(t) \leq h$, $\dot{\tau}(t) \leq d \leq 1$.

V. Finite-Time Reachability to the Sliding Manifold

Corollary: An ideal sliding motion takes place on the surface S if

$$\left\|B_2^{-1}\bar{A}_0^L z(t)\right\| + \left\|B_2^{-1}\bar{A}_d^L z(t-\tau)\right\| < \gamma_2 - \eta \quad (45)$$

where the matrices $\bar{A}_0^L$ and $\bar{A}_d^L$ represent the last m rows of $\bar{A}_0$ and $\bar{A}_d$, respectively, and η is a small scalar satisfying $0 < \eta < \gamma_2$.

Proof:

$$\dot{s}(t) = F\bar{C}\bar{A}_0\bar{z}(t) + F\bar{C}\bar{A}_d z(t-\tau) + F_2B_2\left(\xi(t,z,u) - v(t)\right). \quad (46)$$

Let $V_c : \mathcal{R}^m \rightarrow \mathcal{R}$ be defined by

$$V_c(s) = s^{\mathrm{T}}(t)\left(F_2^{-1}\right)^{\mathrm{T}} \bar{P}_2 F_2^{-1} s(t). \tag{47}$$

Then, using the fact that $F_2^{\mathrm{T}} = \bar{P}_2 B_2$ it follows that

$$\begin{aligned}\left(F_2^{-1}\right)^{\mathrm{T}} \bar{P}_2 F_2^{-1} F \bar{C} \bar{A}_0 &= B_2^{-1} \bar{A}_0^L \\ \left(F_2^{-1}\right)^{\mathrm{T}} \bar{P}_2 F_2^{-1} F \bar{C} \bar{A}_d &= B_2^{-1} \bar{A}_d^L.\end{aligned} \tag{48}$$

Then, it can be verified that

$$\begin{aligned}\dot{V}_c =& 2s^{\mathrm{T}}(t) B_2^{-1} \bar{A}_0^L z(t) + 2s^{\mathrm{T}}(t) B_2^{-1} \bar{A}_d^L z(t-\tau) \\ &+ 2s^{\mathrm{T}}(t)\left(\xi(t,z,u) - v(t)\right) \\ \leq& 2\|s(t)\| \left\|B_2^{-1} \bar{A}_0^L z(t)\right\| + 2\|s(t)\| \\ &\times \left\|B_2^{-1} \bar{A}_d^L z(t-\tau)\right\| - 2\gamma_2 \|s(t)\| \\ <& -2\eta \|s(t)\|\end{aligned} \tag{49}$$

if $z(t)$ and $z(t-\tau) \in \Omega$. It follows that there exists a t_0 such that $z(t)$ and $z(t-\tau) \in \Omega$ for all $t > t_0$. Consequently, (49) holds for all $t > t_0$. A sliding motion will thus be attained in finite time.

Example 1: The following model of a liquid monopropellant rocket motor has been considered in [35]. It is assumed that the variable $\kappa = 0.8$ in this case, where $A_d(1,1) = -\kappa$ and $A(1,1) = \kappa - 1$. The outputs have been chosen to be the second and fourth states so that in (1)

$$A = \begin{bmatrix} -0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad A_d = \begin{bmatrix} -0.8 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{50}$$

Clearly, the Kimura–Davison conditions are not met. Here, the rate at which the delay varies with time has been examined with $d = 0$, a slow varying delay. The gain from the LMI tool solver with $\delta = 50$, $\epsilon = 0.5$, $h = 0.45s$, $\bar{P}_2 = 1$, $\gamma = 2.9$, and $M = [5 \;\; 0.2]$, yields $F = [-1 \;\; -2.0754]$. The poles of the sliding-mode dynamics are $\{-2.67, \; -0.4, \; -0.2\}$. A simulation was performed with the initial state values $[1 \;\; 1 \;\; 1 \;\; 1]$. As shown, the LMI solver gave a feasible result for stability for $h \leq 0.45s$, but in simulation, the closed-loop system only became unstable for $h \geq 1s$ (Fig. 1); this is due to the conservativeness of the method. The LMI solver gave feasible closed-loop stability results for controller gain $\gamma \geq 2.9$ while a choice of $\gamma \geq 1$ in simulation was able to stabilize the system with the compensation of longer settling time. The switching function for $h = 0.45s$, $\gamma = 2.9$ is shown in Fig. 2.

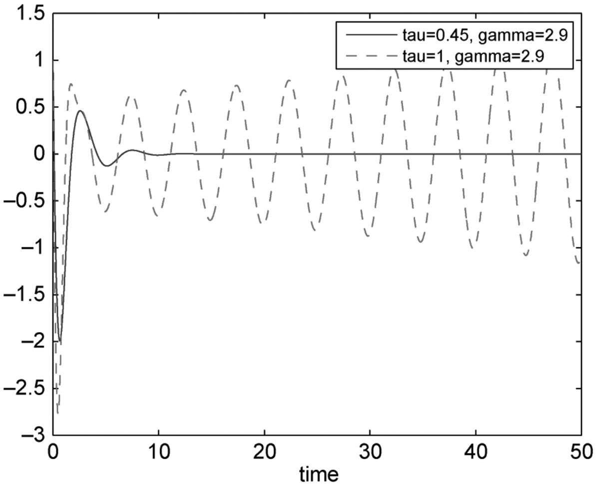

Fig. 1. Output against time.

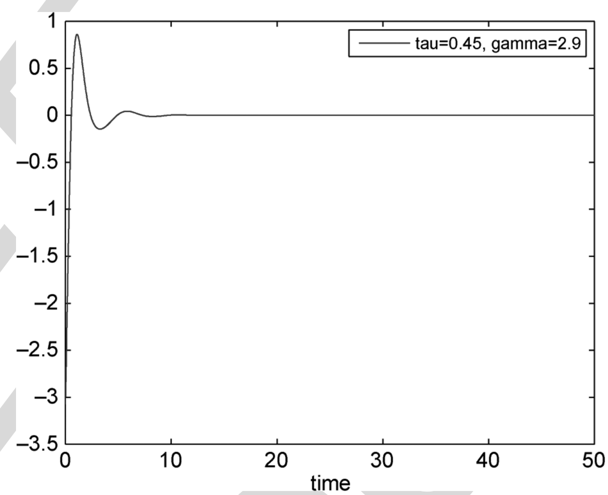

Fig. 2. Switching function.

VI. Compensator-Based Existence Problem

For certain system triples (A_{11}, A_{12}, C_1), LMI (24) is known to be infeasible. In this case, consider a dynamic compensator similar to that of El-Khazali and DeCarlo [36]

$$\dot{z}_c(t) = H z_c(t) + D y(t) \tag{51}$$

where the matrices $H \in \mathcal{R}^{q \times q}$ and $D \in \mathcal{R}^{q \times p}$ are to be determined. Define a new hyperplane in the augmented state space, formed from the plant and compensator state spaces, as

$$S_c = \left\{(z(t), z_c(t)) \in \mathcal{R}^{n+q} \; : \; F_c z_c(t) + F C z(t) = 0\right\} \tag{52}$$

where $F_c \in \mathcal{R}^{m \times q}$ and $F \in \mathcal{R}^{m \times p}$. Define $D_1 \in \mathcal{R}^{q \times (p-m)}$ and $D_2 \in \mathcal{R}^{q \times m}$ as

$$[D_1 \quad D_2] = DT \tag{53}$$

then the compensator can be written as

$$\dot{z}_c(t) = H z_c(t) + D_1 C_1 z_1(t) + D_2 z_2(t) \tag{54}$$

where C_1 is defined in (8). The sliding motion, obtained by eliminating the coordinates $z_2(t)$, can be written as

$$\begin{aligned}\dot{z}_1(t) &= (A_{11} - A_{12}KC_1)z_1(t) - A_{12}K_c z_c(t)\\ &\quad + (A_{d11} - A_{d12}KC_1)z_1(t-\tau) - A_{d12}K_c z_c(t-\tau)\\ \dot{z}_c(t) &= (D_1 - D_2K)C_1 z_1(t) + (H - D_2K_c)z_c(t) \qquad (55)\end{aligned}$$

where $K = F_2^{-1}F_1$ and $K_c = F_2^{-1}F_c$, then similar to [37], the design problem becomes one of selecting a compensator, represented by the matrices D_1, D_2, and H, and a hyperplane, represented by the matrices K and K_c, so that the system

$$\begin{aligned}\begin{bmatrix}\dot{z}_1(t)\\ \dot{z}_c(t)\end{bmatrix} &= \underbrace{\begin{bmatrix}A_{11} - A_{12}KC_1 & -A_{12}K_c\\ (D_1 - D_2K)C_1 & H - D_2K_c\end{bmatrix}}_{A_c}\begin{bmatrix}z_1(t)\\ z_c(t)\end{bmatrix}\\ &\quad + \underbrace{\begin{bmatrix}A_{d11} - A_{d12}KC_1 & -A_{d12}K_c\\ 0 & 0\end{bmatrix}}_{A_{cd}}\begin{bmatrix}z_1(t-\tau)\\ z_c(t-\tau)\end{bmatrix} \qquad (56)\end{aligned}$$

is stable. To obtain the compensator gains this problem can be shown to be a new output-feedback problem with

$$\begin{aligned}A_c &= \underbrace{\begin{bmatrix}A_{11} & 0\\ 0 & 0\end{bmatrix}}_{A_q} - \underbrace{\begin{bmatrix}A_{12} & 0\\ D_2 & -I_q\end{bmatrix}}_{B_q}\underbrace{\begin{bmatrix}K & K_c\\ D_1 & H\end{bmatrix}}_{K_q}\underbrace{\begin{bmatrix}C_1 & 0\\ 0 & I_q\end{bmatrix}}_{C_q}\\ A_{cd} &= \underbrace{\begin{bmatrix}A_{d11} & 0\\ 0 & 0\end{bmatrix}}_{A_{qd}} - \underbrace{\begin{bmatrix}A_{d12} & 0\\ 0 & 0\end{bmatrix}}_{B_{qd}}\underbrace{\begin{bmatrix}K & K_c\\ D_1 & H\end{bmatrix}}_{K_q}\underbrace{\begin{bmatrix}C_1 & 0\\ 0 & I_q\end{bmatrix}}_{C_q}. \qquad (57)\end{aligned}$$

The existence problem represented by system (56), where A_c and A_{cd} are partitioned as in (57) and D_2 is a tuning parameter, can be solved as for the noncompensated case (13). Similarly to (27),

$$\begin{aligned}K_qC_qQ_2 &= K_q\left[0_{(p-m+q)\times(n-p)} \quad I_{p-m+q}\right]\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{22}\mathcal{M} & \delta Q_{22}\end{bmatrix}\\ &= [\,K_qQ_{22}\mathcal{M} \quad \delta K_qQ_{22}\,]\\ &= [Y\mathcal{M} \quad \delta Y] \qquad (58)\end{aligned}$$

where $Y = K_qQ_{22}$, $\mathcal{M} \in \mathcal{R}^{(p-m+q)\times(n-p)}$ is a tuning matrix.

Example 2: Consider the delay system

$$A = \begin{bmatrix}0 & 25 & -1\\ 1 & 0 & 0\\ -5 & 0 & 1\end{bmatrix} \quad A_d = \begin{bmatrix}0.1 & 0.1 & 0\\ 0 & 0.3 & -0.1\\ 0 & 0.2 & 0\end{bmatrix}$$
$$B = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \quad C = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad (59)$$

from which

$$A_{11} = \begin{bmatrix}0 & 25\\ 1 & 0\end{bmatrix} \quad A_{12} = \begin{bmatrix}-1\\ 0\end{bmatrix} \quad C_1 = [1 \quad 0].$$

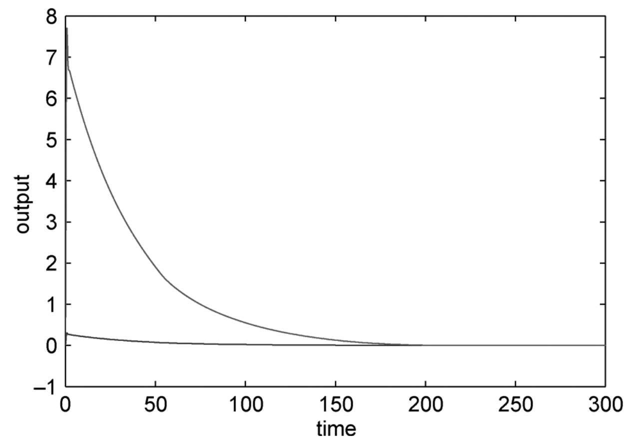

Fig. 3. Compensator-based controller design $h = 2.5s$.

It follows that

$$\lambda(A_{11} - A_{12}KC_1) = \pm\sqrt{(25 + K^2)}$$

and so (24) is infeasible. Now, consider designing a first-order compensator. Choosing $D_2 = 1$, it follows that

$$A_q = \begin{bmatrix}0 & 25 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad A_{qd} = \begin{bmatrix}0.1 & 0.1 & 0\\ 0 & 0.3 & 0\\ 0 & 0 & 0\end{bmatrix}$$
$$B_q = \begin{bmatrix}-1 & 0\\ 0 & 0\\ 1 & -1\end{bmatrix} \quad B_{qd} = \begin{bmatrix}-0.1 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$$
$$C_q = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Choosing $\delta = 50$, $\epsilon = 5$, $\mathcal{M} = [10 \quad 4]'$, and $d = 0$ (slowly varying delay) with the maximum allowable delay $h = 0.25s$, the LMITOOL solver returns

$$K_q = \begin{bmatrix}-25.38 & -0.37\\ -25.32 & -5.03\end{bmatrix}.$$

The augmented system with compensator given by

$$A_a = \begin{bmatrix}H & DC\\ 0 & A\end{bmatrix} \quad A_{da} = \begin{bmatrix}0 & 0\\ 0 & A_d\end{bmatrix}$$
$$B_a = \begin{bmatrix}0\\ B\end{bmatrix} \quad C_a = \begin{bmatrix}I_q & 0\\ 0 & C\end{bmatrix}$$

is asymptotically stablized by the controller

$$[F_c \quad F] = [-0.369 \quad -25.38 \quad 1].$$

Taking the controller in the form of (10) and (11) where $\gamma = 10$, $\rho = 10$. Simulation results with the switching gain and initial conditions $[0.5,\ 0,\ 0]$, as shown in Figs. 3 and 4.

VII. Conclusion

A descriptor Lyapunov–Krasovskii functional method has been introduced for SOF switching function design for systems

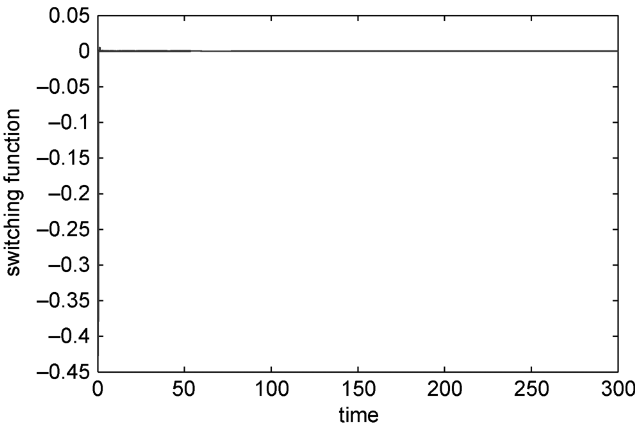

Fig. 4. Compensator-based controller design $h = 2.5s$.

with state time-varying delays. The delay is assumed bounded with a known upper bound, either slowly or fast varying. In addition, a novel stability analysis of the full-order closed-loop discontinuous time-delay system has been performed via the Krasovskii method, which is delay independent in $z_2(t)$ (and thus the delay is restricted to be slowly varying) and delay dependent in $z_1(t)$, i.e., in the state of the reduced-order system. The proposed SOF design approach also applies to compensator-based design. Examples show the effectiveness of the method. For future work, the results can be extended to the interval delay case, where the lower bound on the delay is taken into account. The Razumikin approach can be employed for the stability analysis of the full-order closed-loop system with fast varying delay.

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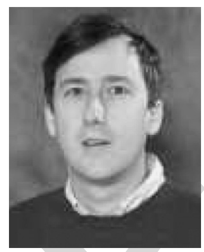

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