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A new Lyapunov–Krasovskii methodology for coupled delay differential and difference equations

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In this paper a new Lyapunov–Krasovskii methodology for both the (local) asymptotic stability and the global asymptotic stability of non-linear coupled delay differential and difference equations is proposed. This methodology is based on the concept of input-to-state stability applied to the difference equation, for which a sufficient Lyapunov criterion is given, and on previous methodologies developed in the literature for linear delay descriptor systems.

1. Introduction

Coupled delay differential and difference equations describe, for instance, lossless propagation phenomena (see Niculescu (2001) and Rasvan and Niculescu (2002)) and internal dynamics of recently studied non-linear delay control systems; see Germani *et al*. (2003) and Pepe (2004) and references therein. Neutral equations in the Hale's form, which describe many engineering systems (consider for instance the model of partial element equivalent circuits in Bellen *et al*. (1999)), can be rewritten as coupled delay differential and difference equations; see Niculescu (2001), Fridman (2002) and Pepe (2005) and references therein). Therefore stability criteria for coupled delay differential and difference equations can also be successfully used for neutral equations in Hale's form. Recently, a Lyapunov–Krasovskii methodology for the (local) asymptotic stability of general coupled delay differential and difference equations has been proposed in Pepe and Verriest (2003) and Pepe (2005). The methodology presented there consists of two steps: the first step leads to the L_2 asymptotic stability; the second one leads to the Lyapunov asymptotic stability. In Fridman (2002) a Lyapunov–Krasovskii methodology for linear delay descriptor systems is proposed. Since coupled delay differential and difference equations can be written as descriptor systems, the methodology proposed there can be applied for studying the stability of the class of systems considered in this paper, at least in the linear case. In the context of linear systems, different conditions, in terms of linear matrix inequalities, for the delay-independent asymptotic stability have been obtained by the Lyapunov–Krasovskii methodologies proposed in Fridman (2002) and in Pepe and Verriest (2003) and Pepe (2005) respectively; see Fridman (2002 Theorem 1), Pepe (2005, Corollary 5) Pepe and Verriest (2003, Corollary 3.4).

The purpose of this paper is to extend the methodology proposed in Fridman (2002) from the linear case to the general non-linear case. In order to carry out the extension, Sontag's concept of input-to-state stability (ISS) for finite-dimensional continuous-time systems (Sontag 1989) is borrowed and our main results are based on the discrete-time version of ISS and its Lyapunov characterization in Jiang and Wang (2001). In particular, a Lyapunov criterion for the input-to-state stability of continuous-time difference equations, based on the one for non-linear finite-dimensional discrete-time systems given in Jiang and Wang (2001), is first proposed. Then, using this criterion, it can be stated that the variable of the continuous-time difference part of the equations can be guaranteed arbitrarily small if the

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variable of the differential part of the equations is sufficiently small. After this result is established, the descriptor methodology in Fridman (2002) can be adapted to the current case of general non-linear coupled delay differential and difference equations, leading to a new Lyapunov–Krasovskii methodology, for both the (local) asymptotic stability and the global asymptotic stability. It is worth noting that the ISS of a continuous-time difference equation is automatically guaranteed by the asymptotic stability (input equal to zero), as long as the equation in question is linear; see Hale and Verduyn Lunel (1993, Theorem 3.5, pp. 275).

We believe that the presented stability results together with Pepe and Verriest (2003) and Pepe (2005) will provide a solid foundation for analysis and synthesis of non-linear coupled delay differential and difference equations. In particular, they should prove useful for delay-dependent stability analysis.

The delay-independent global asymptotic stability of an electrical circuit containing a LC transmission line is here studied, showing the effectiveness of the proposed methodology.

The paper is organized as follows: in § 2 continuous-time difference equations are briefly described; in § 3 the input-to-state stability for continuous-time difference equations is studied; in § 4 coupled delay differential and difference equations are briefly introduced; in § 5 the asymptotic stability of coupled delay differential and difference equations is addressed, and two Lyapunov–Krasovskii theorems are proved for the global and local case respectively; in § 6 two examples are studied; § 7 contains the conclusions.

Notation: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of non-negative real numbers, $\mathbb{R}^*$ denotes the extended real line $[-\infty, +\infty]$. For any given positive integer l, $\mathbb{R}^l$ denotes the set of real vectors of length l. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. For any $a, b \in \mathbb{R}$, $a < b$, $C([a, b]; \mathbb{R}^l)$ denotes the set of continuous functions defined on $[a, b]$ and taking values in $\mathbb{R}^l$, endowed with the supremum norm. For any set $V \subset \mathbb{R}^l$, $B((a, b]; V)$ denotes the set of the essentially bounded functions defined on $(a, b]$ and taking values in V. The essential supremum norm of an essentially bounded function is denoted with the symbol $\|\cdot\|$. For $\phi \in B((a,b]; V)$, $\|\phi\| = \operatorname{ess\,sup}_{\tau \in (a,b]} |\phi(\tau)|$. A function u: $[0, +\infty) \to R^l$ is said to be locally essentially bounded if, for any positive real T the function u_T: $[0, +\infty) \to R$, given by $u_T(t) = u(t)$ for all $t \in [0, T)$ and $= 0$ elsewhere, is essentially bounded on $[0, +\infty)$. A function w: $[0, c) \to R$, $0 < c \le +\infty$, is said to be locally absolutely continuous if it is absolutely continuous in any interval $[0, d]$, $0 < d < c$. A function γ: $\mathbb{R}^+ \to \mathbb{R}^+$ is said to be of class K if it is continuous, strictly increasing and satisfies $\gamma(0) = 0$. It is of class K_∞ if, additionally, it is unbounded. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class KL if for each fixed t, the function $\beta(\cdot, t)$ is of class K and for each fixed s, the function $\beta(s, t)$ decreases to 0 as $t \to +\infty$. For any positive integer j, the symbol I_j denotes the identity matrix of dimension j. For any positive integers i, j, the symbol $0_{i,j}$ denotes a matrix of zeros in $\mathbb{R}^{i\times j}$.

2. Continuous-time difference equations

Consider the following system of non-linear continuous-time difference equations:

$$\left.\begin{aligned} &x(t) = f(x(t-\Delta_1), x(t-\Delta_2), \ldots, x(t-\Delta_m), u(t)), \quad t \ge 0, \\ &x(\tau) = x_0(\tau), \qquad \tau \in [-\Delta, 0], \end{aligned}\right\} \tag{1}$$

where the (continuous) time variable $t \in [0, +\infty)$, $x(t) \in \mathbb{R}^n$, $0 < \Delta_1 < \Delta_2 < \cdots < \Delta_m = \Delta$ are the arbitrary (non-commensurate) delays, f is a continuous function defined on $\mathbb{R}^{nm} \times \mathbb{R}^p$ and taking values in $\mathbb{R}^n$, the input u is a locally essentially bounded function defined on $[0, +\infty)$ and taking values in $\mathbb{R}^p$, the initial condition x_0 is a continuous function defined on $[-\Delta, 0]$ and taking values in $\mathbb{R}^n$, m, n, p are positive integers. Assume that $f(0, \ldots, 0, 0) = 0$, thus ensuring that $x(t) = 0$ is the solution of system (1) with zero input and zero initial conditions. Note that system (1) admits a unique solution in $[0, +\infty)$ for any input function u and any initial condition x_0. It was shown in Germani *et al.* (2003) that continuous-time difference equations can be rewritten as discrete-time systems on a suitable normed linear space. In Pepe (2003) such transformation has been given in the case of multiple and not commensurate delays. In this general case, the involved space is $B((0, \Delta_{\min}]; \mathbb{R}^{(k_m+1)n})$, endowed with the essential supremum norm, where

$$\left.\begin{aligned} \Delta_{\min} &= \min\{\Delta_1, \Delta_2 - \Delta_1, \ldots, \Delta_m - \Delta_{m-1}\}, \\ \Delta_i &= k_i \Delta_{\min} + \tau_i, \quad i = 1, 2, \ldots, m, \end{aligned}\right\} \tag{2}$$

k_i; $i = 1, 2, \ldots, m$, are suitable positive integers, τ_i, $i = 1, 2, \ldots, m$ are reals such that $0 \le \tau_i < \Delta_{\min}$. More specifically, system (1) can be rewritten as a discrete-time system in the space $B((0, \Delta_{\min}]; \mathbb{R}^{(k_m+1)n})$ as follows (see Germani *et al.* (2003) and Pepe (2003) for the details):

$$X(k+1) = G(X(k), U(k)), \quad k = 0, 1, \ldots \tag{3}$$

where $X(k) \in B(0, \Delta_{\min}]; \mathbb{R}^{(k_m+1)n})$, $U(k) \in B(0, \Delta_{\min}]; \mathbb{R}^p)$ is defined as

$$U(k)(\tau) = u(k\Delta_{\min} + \tau), \quad k = 0, 1 \ldots. \tag{4}$$

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G is a suitable function defined on $B((0, \Delta_{\min}]; \mathbb{R}^{(k_m+1)n}) \times B((0, \Delta_{\min}]; \mathbb{R}^p)$ and taking values in $B((0, \Delta_{\min}]; \mathbb{R}^{(k_m+1)n})$.

Due to the particular discrete-time dynamics of system (3), which involves the function f just for the last components of the state $X(k)$, it is useful, for the application of the Lyapunov's second method, to consider the following discrete-time system, obtained by (3),

$$\mathcal{X}(k+1) = F(\mathcal{X}(k), \mathcal{U}(k)), \quad k = 0, 1, \ldots, \tag{5}$$

where

$$\mathcal{X}(k) = X(k(k_m+1)) \in B(0, \Delta_{\min}]; \quad \mathbb{R}^{(k_m+1)n}),$$

$$\mathcal{U}(k) = \begin{bmatrix} U((k(k_m+1)) \\ U((k(k_m+1)+1) \\ \vdots \\ U(k(k_m+1)+k_m) \end{bmatrix} \in B((0, \Delta_{\min}]; \quad \mathbb{R}^{p(k_m+1)}),$$

and the function F is suitably obtained by the function G in (3).

Remark 1: In the case of one single delay the above difficult procedure leading to the discrete-time system (5) is not necessary. In the case of one single delay Δ, the discrete-time system (5) is obtained with $\mathcal{X}(k) \in B((0, \Delta]; \mathbb{R}^n)$, and $\mathcal{U}(k) \in B((0, \Delta]; \mathbb{R}^p)$ (Germani *et al.* 2003). When the delays are multiple but commensurate, the equation (1) can be transformed into an equation with one single delay, by a state extension.

3. Input-to-state stability of continuous-time difference equations

In the following, for a given positive integer $\bar{k}$, we will indicate with $\mathcal{U}_{[\bar{k}]}$ the truncation of $\mathcal{U}(k)$ at $\bar{k}$, that is the sequence which is equal to $\mathcal{U}(k)$, for $k = 0, 1, \ldots, \bar{k}$, and is equal to zero for $k > \bar{k}$. We will indicate with $\|\mathcal{U}_{[\bar{k}]}\|_\infty$ the quantity $\sup_{0 \le k \le \bar{k}} \|\mathcal{U}(k)\|$.

For a given $\rho_s \in \mathbb{R}^*$, $\rho_s \in \mathbb{R}^*$, let $\mathcal{S}(\rho_s) = \{v \in \mathbb{R}^p, |v| \le \rho_s\}$, and let $\mathcal{M}_\mathcal{S}(\rho_s)$ be the set of the locally essentially bounded functions defined on $[0, +\infty)$ and taking values in $\mathcal{S}(\rho_s)$. By these notations, it results that $\mathcal{S}(+\infty) = \mathbb{R}^p$, and that $\mathcal{M}_\mathcal{S}(+\infty)$ is the set of the locally essentially bounded functions defined on $[0, +\infty)$ and taking values in $\mathbb{R}^p$.

Definition 1: Let $\rho_s \in \mathbb{R}^*$, $\rho_s > 0$. System (1) is said to be input-to-state stable (ISS) with respect to inputs $u \in \mathcal{M}_\mathcal{S}(\rho_s)$, if there exist a function β of class KL and a function γ of class K such that, for any essentially bounded initial condition x_0 and for any input function $u \in \mathcal{M}_\mathcal{S}(\rho_s)$, the following inequality holds for the solution of the equivalent discrete-time system (5)

$$\|\mathcal{X}(k)\| \le \beta(\|\mathcal{X}(0)\|, k) + \gamma(\|\mathcal{U}_{[k-1]}\|_\infty). \tag{6}$$

Theorem 1: *Let $\rho_s \in \mathbb{R}^*$, $\rho_s > 0$. Let there exist a continuous functional $V : B((0, \Delta_{\min}]; \mathbb{R}^{(k_m+1)n}) \to \mathbb{R}^+$ such that*

(i) *there exist functions α_1 and α_2, of class K_∞, such that, for any $\mathcal{X} \in B((0, \Delta_{\min}]; \mathbb{R}^{(k_m+1)n})$, the following inequalities hold*

$$\alpha_1(\|\mathcal{X}\|) \le V(\mathcal{X}) \le \alpha_2(\|\mathcal{X}\|); \tag{7}$$

(ii) *there exis a function α_3 of class K_∞ and a function σ of class K, such that, for any $\mathcal{X} \in B((0, \Delta_{\min}]; \mathbb{R}^{(k_m+1)n})$, and any*

$$\mathcal{U} \in B((0, \Delta_{\min}]; \mathcal{S}(\rho_s)^{(k_m+1)}),$$

the following inequality holds

$$V(F(\mathcal{X}, \mathcal{U})) - V(\mathcal{X}) \le -\alpha_3(\|\mathcal{X}\|) + \sigma(\|\mathcal{U}\|). \tag{8}$$

Then, system (1) *is input-to-state stable with respect to inputs $u \in \mathcal{M}_\mathcal{S}(\rho_s)$.*

Proof: The same proof given in Jiang and Wang (2001, Lemma 3.5), concerning finite-dimensional discrete-time systems, is applicable to the present case (5) of infinite-dimensional discrete-time systems. □

Remark 2: Theorem 1 is useful for studying the internal dynamics of full relative degree delay systems, when the output is driven, by means of a suitable feedback control law, to follow a prescribed reference signal bounded away from zero. That internal dynamics is often described by continuous-time difference equations (see Germani *et al.* (2003) and Pepe (2003, 2004))

$$x(t) = f(x(t-\Delta_1), x(t-\Delta_2), \ldots, x(t-\Delta_m), z(t)), \tag{9}$$

where $z(t)$ is the vector of the controlled system output and its time derivatives up to the order $(n-1)$, with n the length of the state vector. In this case, the ISS property with respect to $z(t)$ assures the desirable behaviour of the overall time-delay control system, when $z(t)$ is guaranteed to belong to a suitable compact set.

4. Coupled delay differential and difference equations

The following system of time invariant non-linear coupled delay differential and difference equations is

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considered

$$\left.\begin{aligned} \dot{\xi}(t) &= \mathcal{A}(x(t-\Delta_1),\ldots,x(t-\Delta_m),\xi(t),\\ &\qquad \xi(t-\Delta_1),\ldots,\xi(t-\Delta_m)), \quad t\geq 0,\\ x(t) &= \mathcal{B}(x(t-\Delta_1),\ldots,x(t-\Delta_m),\xi(t),\xi(t-\Delta_1),\ldots,\\ &\qquad \xi(t-\Delta_m)), \end{aligned}\right\} \tag{10}$$

$$\xi(\tau)=\xi_0(\tau), \quad x(\tau)=x_0(\tau), \quad \tau\in[-\Delta,0], \tag{11}$$

where $0<\Delta_1<\Delta_2<\cdots<\Delta_m=\Delta$ are the arbitrary (non-commensurate) delays; $t\in[0,+\infty)$; $x(t)\in\mathbb{R}^n$; $\xi(t)\in\mathbb{R}^d$; x_0 and ξ_0 are functions in $C([-\Delta,0];\mathbb{R}^n)$ and $C([-\Delta,0];\mathbb{R}^d)$, respectively; $\mathcal{A}$ is a continuous function from $\mathbb{R}^{d(m+1)+nm}$ to $\mathbb{R}^d$; $\mathcal{B}$ is a continuous function from $\mathbb{R}^{d(m+1)+nm}$ to $\mathbb{R}^n$, m, n, d are positive integers. Assume that $\mathcal{A}(0,\ldots,0)=0$, and $B(0,\ldots,0)=0$, thus ensuring that $\xi(t)=0$, $x(t)=0$, for every $t\geq 0$, is the trivial solution of system (10)–(11) corresponding to zero initial conditions.

We also impose the following hypothesis; see Pepe and Verriest (2003, Remark 2.1)

H_1: *The functional* $\bar{\mathcal{A}}: C([-\Delta,0];R^n)\times C([-\Delta,0];R^d)\to R^d$, *given, for* $\phi\in C([-\Delta,0];R^n)$, $\psi\in C([-\Delta,0];R^d)$, *by*

$$\begin{aligned}\bar{\mathcal{A}}(\phi,\psi) = \mathcal{A}(\phi(-\Delta_1),\ldots,\phi(-\Delta_m),\psi(0),\\ \psi(-\Delta_1),\ldots,\psi(-\Delta_m))\end{aligned} \tag{12}$$

is such that, for any $(\bar{\phi},\bar{\psi})\in C([-\Delta,0];R^n)\times C([-\Delta,0];R^d)$, *there exist a neighbourhood of* $(\bar{\phi},\bar{\psi})$ *and a positive real* $L_{(\bar{\phi},\bar{\psi})}$ *such that, for all* (ϕ,ψ_1), (ϕ,ψ_2) *in that neighbourhood, the inequality holds*

$$|\bar{\mathcal{A}}(\phi,\psi_1)-\bar{\mathcal{A}}(\phi,\psi_2)|\leq L_{(\bar{\phi},\bar{\psi})}\|\psi_1-\psi_2\|. \tag{13}$$

From the hypothesis H_1 it follows that system (10)–(11) admits a unique solution

$$\begin{bmatrix}\xi(t)\\ x(t)\end{bmatrix}$$

on a maximal time interval $[0, b)$, $0<b\leq+\infty$, with $\xi(t)$ locally absolutely continuous and $x(t)$ continuous. Moreover, if $b<+\infty$, then $\xi(t)$ is unbounded in $[0, b)$.

In the following it will be useful to consider the second equation in (10) rewritten as

$$x(t)=\mathcal{B}(x(t-\Delta_1),\ldots,x(t-\Delta_m),u(t)), \tag{14}$$

where the input $u(t)\in\mathbb{R}^{(m+1)d}$ takes the place of the terms $\xi(t)$, $\xi(t-\Delta_1),\ldots,\xi(t-\Delta_m)$. System (10) is a descriptor system with delays (Fridman 2002). Actually it can be rewritten as

$$E\begin{bmatrix}\dot{\xi}(t)\\ \dot{x}(t)\end{bmatrix} = \begin{bmatrix}\mathcal{A}(x(t-\Delta_1),\ldots,x(t-\Delta_m),\\ \quad \xi(t),\xi(t-\Delta_1),\ldots,\xi(t-\Delta_m))\\ -x(t)+\mathcal{B}(x(t-\Delta_1),\ldots,\\ \quad x(t-\Delta_m),\xi(t),\xi(t-\Delta_1),\ldots,\xi(t-\Delta_m))\end{bmatrix}, \tag{15}$$

where

$$E=\begin{bmatrix} I_d & 0_{d\times n}\\ 0_{n\times d} & 0_{n\times n}\end{bmatrix}.$$

In the following, the functions $\xi_t\in C([-\Delta,0];R^d)$ and $x_t\in C([-\Delta,0];R^n)$, $t\geq 0$, are given, as usual (Hale and Verduyn Lunel 1993), by $\xi_t(\tau)=\xi(t+\tau)$, $x_t(\tau)=x(t+\tau)$, $\tau\in[-\Delta,0]$.

5. Stability of coupled delay differential and difference equations

For stability and asymptotic stability definitions of coupled delay differential and difference equations; see Hale and Verduyn Lunel (1993), Niculescu (2001), Rasvan and Niculescu (2002), Pepe and Verriest (2003) and Pepe (2005). For global asymptotic stability we mean, as usual, stability and global attractivity.

For a continuous functional $V: C([-\Delta,0];\mathbb{R}^{n+d})\to\mathbb{R}^+$, define (see Hale and Verduyn Lunel (1993) and Fridman (2002))

$$\dot{V}\left(\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\right)=\limsup_{h\to 0+}\frac{1}{h}\left(V\left(\begin{bmatrix}\xi_h\\ x_h\end{bmatrix}\right)-V\left(\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\right)\right), \tag{16}$$

where

$$\begin{bmatrix}\xi_t\\ x_t\end{bmatrix}, \quad t\geq 0,$$

is the solution of system (10) with initial conditions $\xi_0=\phi_1\in C([-\Delta,0];\mathbb{R}^d)$, $x_0=\phi_2\in C([-\Delta,0];\mathbb{R}^n)$. Since the equations of system (10) are satisfied also for $t=0$, ϕ_1 and ϕ_2 must satisfy the matching condition

$$\begin{aligned}\phi_2(0)=\mathcal{B}(\phi_2(-\Delta_1),\ldots,\phi_2(-\Delta_m),\\ \phi_1(0)\ \phi_1(-\Delta_1),\ldots,\phi_1(-\Delta_m)).\end{aligned} \tag{17}$$

Theorem 2: *Assume that the continuous-time difference equation* (14) *is input-to-state stable with respect to inputs* $u\in\mathcal{M}_{\mathcal{S}}(+\infty)$. *Further assume there exist a continuous functional* $V: C([-\Delta,0];\mathbb{R}^{n+d})\to\mathbb{R}^+$, *functions* α, β *and*

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γ *of class* K_∞, such that

(i) *for every* $\phi_1 \in C([-\Delta, 0]; \mathbb{R}^d)$, *every* $\phi_2 \in C([-\Delta, 0]; \mathbb{R}^n)$, *with* $\phi_2(0) = \mathcal{B}(\phi_2(-\Delta_1), \ldots, \phi_2(-\Delta_m), \phi_1(0)\ \phi_1(-\Delta_1), \ldots, \phi_1(-\Delta_m))$, *the following inequalities hold*

$$\left.\begin{aligned} &\beta(|\phi_1(0)|) \le V\left(\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\right) \le \gamma\left(\left\|\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\right\|\right),\\ &\dot V\left(\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\right) \le -\alpha(|\phi_1(0)|);\end{aligned}\right\} \tag{18}$$

(ii) *the function*

$$w(t) = V\left(\begin{bmatrix}\xi_t\\ x_t\end{bmatrix}\right)$$

is locally absolutely continuous in $[0, b)$, *for*

$$\begin{bmatrix}\xi_t\\ x_t\end{bmatrix}$$

satisfying (10) *in a maximal time interval* $[0, b)$, $0 < b \le +\infty$.

Then, the origin of system (10) *is globally asymptotically stable.*

Proof: By Lemma A1 in Jiang *et al*. (1996), we may assume without any loss of generality that the function α is continuously differentiable. From (18), taking into account (ii), it follows that, for $t \in [0, b)$,

$$\begin{aligned}\beta(|\xi(t)|) &\le V\left(\begin{bmatrix}\xi_t\\ x_t\end{bmatrix}\right)\\ &\le V\left(\begin{bmatrix}\xi_0\\ x_0\end{bmatrix}\right) - \int_0^t \alpha(|\xi(s)|) \le \gamma\left(\left\|\begin{bmatrix}\xi_0\\ x_0\end{bmatrix}\right\|\right).\end{aligned} \tag{19}$$

From (19) it follows that $b = +\infty$ (otherwise $\xi(t)$ would be unbounded in $[0, b)$), and that $\xi(t)$ can be as small as desired provided the initial conditions are sufficiently small. From the hypothesis of input-to-state stability of (14), it follows that the origin of system (10) is stable. As far as the global attractivity is concerned, let us note first that, since the inequalities (19) hold globally and since the continuous-time difference equation (14) is input-to-state stable with respect to inputs $u \in \mathcal{M}_{\mathcal{S}}(+\infty)$, for any initial conditions in (10), the correspondent solution is bounded in $[0, +\infty)$. From (19) it follows that

$$\lim_{t\to+\infty} \int_0^t \alpha(|\xi(s)|) \le V\left(\begin{bmatrix}\xi_0\\ x_0\end{bmatrix}\right). \tag{20}$$

From (20) it follows that the function

$$t \to \int_0^t \alpha(|\xi(s)|)\mathrm{d}s \tag{21}$$

admits a finite limit as $t \to +\infty$. We claim that $\lim_{t\to+\infty} \alpha(|\xi(t)|) = 0$. For, let us consider the derivative with respect to time of the function $t \to \alpha(|\xi(t)|)$. The following equality/inequality hold

$$\left|\frac{\mathrm{d}\alpha(|\xi(t)|)}{\mathrm{d}t}\right| = \left|\frac{\mathrm{d}\alpha(|\xi(t)|)}{\mathrm{d}|\xi(t)|}\frac{\xi^T(t)\dot\xi(t)}{\sqrt{\xi^T(t)\xi(t)}}\right| \le \left|\frac{\mathrm{d}\alpha(|\xi(t)|)}{\mathrm{d}|\xi(t)|}\right||\dot\xi(t)|. \tag{22}$$

From the boundedness of the solution and the continuity of the functional $\mathcal{A}$, it follows that $\dot\xi(t)$ is bounded in $[0, +\infty)$. Since the function α is continuously differentiable, and the solution is bounded, it follows that the function $t \to (\mathrm{d}\alpha(|\xi(t)|)/\mathrm{d}|\xi(t)|)$ is bounded in $[0, +\infty)$. Therefore, since its derivative is bounded in $[0, +\infty)$, the function $t \to \alpha(|\xi(t)|)$ is uniformly continuous in $[0, +\infty)$. From this fact, taking into account that the function (21) admits a finite limit, by invoking the Barbalat's Lemma, it follows that $\lim_{t\to+\infty} \alpha(|\xi(t)|) = 0$ and therefore that $\lim_{t\to+\infty} |\xi(t)| = 0$.

As far as the proof that $\lim_{t\to+\infty} |x(t)| = 0$ is concerned, taking into account the time invariant character of the equation (14) and its ISS property, the following inequality holds with suitable function $\bar\beta$ of class KL and function $\bar\gamma$ of class K

$$\|\mathcal{X}(k)\| \le \bar\beta(\|\mathcal{X}(k_0)\|, k - k_0) + \bar\gamma(\|\mathcal{U}_{[k_0, k-1]}\|_\infty), \tag{23}$$

where: k_0, k are positive integers, $k > k_0$; $\mathcal{X}$, $\mathcal{U}$ are the variables of the discrete-time system (5) equivalent to the continuous-time difference equation (14); $\mathcal{U}_{[k_0, k-1]}(\theta) = \mathcal{U}(\theta)$ for $k_0 \le \theta \le k-1$, and is $=0$ elsewhere. Now, let ϵ be a positive real. Since in the continuous-time difference equation (14) the role of the input u is played by the solution variable ξ, it follows that there exist a $\bar k_0$ such that $\bar\gamma(\|\mathcal{U}_{[\bar k_0, k]}\|_\infty) < \epsilon/2$, for any $k > \bar k_0$. Moreover, since $\bar\beta$ is a KL function, it follows that there exist a $\bar k_1 \ge \bar k_0$ such that $\bar\beta(\|\mathcal{X}(\bar k_0)\|, k - \bar k_0) < \epsilon/2$, for any $k > \bar k_1$. Therefore, for $k > \bar k_1$ the inequality $\|\mathcal{X}(k)\| < \epsilon$ holds and the proof of the theorem is accomplished. □

Remark 3: Note that the argument of the function β in the inequalities (18) involves only ϕ_1.

Analogously, a version of Theorem 2 concerning the (local) asymptotic stability can be obtained.

Theorem 3: *Assume that there exists a positive real* ρ_s *such that the continuous-time difference equation* (14) *is input-to-state stable with respect to inputs* $u \in \mathcal{M}_{\mathcal{S}}(\rho_s)$. *Further assume there exist a positive real* ρ, *a continuous functional* $V\colon C([-\Delta, 0]; \mathbb{R}^{n+d}) \to \mathbb{R}^+$, *functions* α, β *and* γ *of class* K_∞, *such that:*

(i) *for every* $\phi_1 \in C([-\Delta, 0]; \mathbb{R}^d)$ *with* $\|\phi_1\| < \rho$, *and every* $\phi_2 \in C([-\Delta, 0]; \mathbb{R}^n)$, *with* $\phi_2(0) = \mathcal{B}(\phi_2(-\Delta_1), \ldots, \phi_2(-\Delta_m),\ \phi_1(0), \phi_1(-\Delta_1), \ldots,$

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$\phi_1(-\Delta_m))$, *the following inequalities hold*

$$\left.\begin{aligned}&\beta(|\phi_1(0)|) \le V\left(\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\right) \le \gamma\left(\left\|\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\right\|\right),\\ &\dot V\left(\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\right) \le -\alpha(|\phi_1(0)|);\end{aligned}\right\} \tag{24}$$

(ii) *the function*

$$w(t) = V\left(\begin{bmatrix}\xi_t\\ x_t\end{bmatrix}\right)$$

is locally absolutely continuous in $[0, b)$, *for*

$$\begin{bmatrix}\xi_t\\ x_t\end{bmatrix}$$

satisfying (10) *in a maximal time interval* $[0, b)$, $0 < b \le +\infty$.

Then, the origin of system (10) *is asymptotically stable.*

Proof: Let $0 < \epsilon < \rho$. Let $0 < \delta < \epsilon$ such that $\gamma(\delta) < \beta(\epsilon)$. Let the initial conditions such that

$$\left\|\begin{bmatrix}\xi_0\\ x_0\end{bmatrix}\right\| < \delta.$$

Then $|\xi_0(0)| < \delta$. We show that $|\xi(t)| < \epsilon$ for all $t \in [0, b)$. For, by contradiction, let $t_1 \in [0, b)$ be the first time such that $|\xi(t_1)| = \epsilon$. From (24), taking into account (ii) it would follow that

$$\begin{aligned}\beta(\in) &\le V\left(\begin{bmatrix}\xi_{t_1}\\ x_{t_1}\end{bmatrix}\right) \le V\left(\begin{bmatrix}\xi_0\\ x_0\end{bmatrix}\right) - \int_0^t \alpha(|\xi(s)|)\,\mathrm{d}s\\ &\le \gamma\left(\left\|\begin{bmatrix}\xi_0\\ x_0\end{bmatrix}\right\|\right) \le \gamma(\delta).\end{aligned} \tag{25}$$

Thus it would be $\beta(\epsilon) \le \gamma(\delta)$, while it was supposed $\gamma(\delta) < \beta(\epsilon)$. It follows that $b = +\infty$ (otherwise the solution $\xi(t)$ would be unbounded in $[0, b)$) and that $\xi(t)$ can be as small as desired provided the initial conditions are sufficiently small. From the hypothesis of input-to-state stability of (14) when the input (that is $\xi(\cdot)$) is sufficiently small, it follows that the origin of system (10) is stable.

As far as the local attractivity is concerned, the same reasoning in the proof of Theorem 2 can be used here. For, just consider that, if the initial conditions are sufficiently small, then the solution is arbitrarily small in $(0, +\infty)$. Consequently, since the input u in the equation (14), which takes the place of ξ, can be supposed to be bounded by the positive real ρ_s, the ISS property of the equation (14) holds and the reasoning in the proof of Theorem 2, inequality (20) on, can be repeated equal. □

6. Illustrative examples

Example 1: Let us consider the following coupled delay differential and continuous-time difference equation

$$\left.\begin{aligned}\dot\xi(t) &= -\xi^3(t) + \xi(t-\Delta_1)x(t-\Delta_2),\\ x(t) &= 0.5x(t-\Delta_1) + \xi(t)x(t-\Delta_2),\end{aligned}\right\} \tag{26}$$

where $\xi(t), x(t) \in R$.

Let us prove first that the difference equation in (26) is ISS with respect to suitable small $\xi(\cdot)$. Let us consider, for instance, the case $\Delta_{\min} = \Delta_1$ and $k_2 = 2$. Other cases can be treated analogously. Since $n = 1$ and $k_2 = 2$, the state vector $\mathcal{X}$ of system (5) consists of three scalar functions, that is

$$\mathcal{X}(k) = \begin{bmatrix}\chi_1(k)\\ \chi_2(k)\\ \chi_3(k)\end{bmatrix} \in B((0, \Delta_{\min}]; \mathbb{R}^3), \tag{27}$$

and $\mathcal{U}(k) \in B((0, \Delta_{\min}]; R^3)$ (take into account that the role of the input is here taken by $\xi(t)$). Let us choose the Lyapunov functional $V: B((0, \Delta_{\min}]; \mathbb{R}^3) \to R^+$ given, for $\chi = [\chi_1\ \chi_2\ \chi_3]^T \in B((0, \Delta_{\min}]; \mathbb{R}^3)$, by

$$V(\mathcal{X}) = \sup_{i=1,2,3} \|\chi_i\|, \tag{28}$$

and let us apply Theorem 1. Let $\xi(\cdot) \in \mathcal{M}_{\mathcal{S}}(\rho_s)$, with $0 < \rho_s < 0.1$. The following inequality holds

$$\begin{aligned}V(F(\mathcal{X},\mathcal{U})) &\le \sup\{(0.5 + \|\mathcal{U}\|)V(\mathcal{X}),\\ &(0.25 + 1.5\|\mathcal{U}\|)V(\mathcal{X}), (0.125 + 1.25\|\mathcal{U}\| + \|\mathcal{U}\|^2)V(\mathcal{X})\}.\end{aligned} \tag{29}$$

Therefore,

$$V(F(\mathcal{X},\mathcal{U})) \le 0.5V(\mathcal{X}) + 1.5\|\mathcal{U}\|V(\mathcal{X}) + \|\mathcal{U}\|^2 V(\mathcal{X}), \tag{30}$$

from which the following inequality is obtained

$$V(F(\mathcal{X},\mathcal{U})) - V(\mathcal{X}) \le -0.025\|\mathcal{X}\| + \|\mathcal{U}\|. \tag{31}$$

Then it follows that the difference equation in (26) is ISS with respect to the variable ξ when this variable is such that $|\xi(t)| \le \rho_s$, $t \ge 0$.

Choose now the following Lyapunov–Krasovskii functional V: $C([-\Delta, 0]; R^2) \to \mathbb{R}^+$ given by

$$\begin{aligned}V\left(\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\right) &= [\phi_1(0)\quad \phi_2(0)]EP\begin{bmatrix}\phi_1(0)\\ \phi_2(0)\end{bmatrix}\\ &+ \int_{-\Delta_1}^0 [\phi_1^2(\tau)\quad \phi_2(\tau)]Q\begin{bmatrix}\phi_1^2(\tau)\\ \phi_2(\tau)\end{bmatrix}\mathrm{d}\tau\\ &+ \int_{-\Delta_2}^0 [\phi_1^2(\tau)\quad \phi_2(\tau)]S\begin{bmatrix}\phi_1^2(\tau)\\ \phi_2(\tau)\end{bmatrix}\mathrm{d}\tau,\end{aligned} \tag{32}$$

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where

$$E=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad P=\begin{bmatrix}p_1&0\\0&p_2\end{bmatrix},$$
$$Q=\begin{bmatrix}q_1&0\\0&q_2\end{bmatrix},\quad S=\begin{bmatrix}s_1&0\\0&s_2\end{bmatrix},\quad p_1,p_2,q_1,q_2,s_1,s_2$$

are positive reals, and apply Theorem 3 (note that this functional satisfies the hypothesis ii in Theorem 3, because ϕ_2 appears only inside an integral which transforms a continuous time function into an absolutely continuous one). The derivative $\dot{V}$ of the Lyapunov–Krasovskii functional V is given by

$$\begin{aligned}\dot{V}\left(\begin{bmatrix}\phi_1\\\phi_2\end{bmatrix}\right)&=2[\phi_1(0)\ \ \phi_2(0)]\\&\quad\times P\begin{bmatrix}-\phi_1^3(0)+\phi_1(-\Delta_1)\phi_2(-\Delta_2)\\0\end{bmatrix}+\phi_1^4(0)q_1\\&\quad-\phi_1^4(-\Delta_1)q_1+\phi_2^2(0)q_2-\phi_2^2(-\Delta_1)q_2+\phi_1^4(0)s_1\\&\quad-\phi_1^4(-\Delta_2)s_1+\phi_2^2(0)s_2-\phi_2^2(-\Delta_2)s_2.\end{aligned}\quad(33)$$

Taking into account that $0=-\phi_2(0)+0.5\phi_2(-\Delta_1)+\phi_1(0)\phi_2(-\Delta_2)$, the following inequalities hold

$$\begin{aligned}\dot{V}\left(\begin{bmatrix}\phi_1\\\phi_2\end{bmatrix}\right)&\le-2p_1\phi_1^4(0)+\tfrac{1}{2}p_1\phi_1^4(0)+\tfrac{1}{2}p_1\phi_1^4(-\Delta_1)\\&\quad+p_1\phi_2^2(-\Delta_2)-2p_2\phi_2^2(0)+\tfrac{1}{2}p_2\phi_2^2(0)\\&\quad+\tfrac{1}{2}p_2\phi_2^2(-\Delta_1)+p_2|\phi_1(0)|\phi_2^2(0)\\&\quad+p_2|\phi_1(0)|\phi_2^2(-\Delta_2)+\phi_1^4(0)q_1-\phi_1^4(-\Delta_1)q_1\\&\quad+\phi_2^2(0)q_2-\phi_2^2(-\Delta_1)q_2+\phi_1^4(0)s_1-\phi_1^4(-\Delta_2)s_1\\&\quad+\phi_2^2(0)s_2-\phi_2^2(-\Delta_2)s_2.\end{aligned}\quad(34)$$

Taking into account that $|\phi_1(0)|$ can be sufficiently small, it results that the second inequality in (24), Theorem 3, is satisfied by choosing $s_2>p_1$, $2q_1>p_1$, $3p_1>2q1+2s_1$, $\frac{3}{2}p_2>q_2+s_2$, $\frac{1}{2}p_2<q_2$. A solution is $p_1=1$, $p_2=3$, $q_1=1$, $q_2=2$, $s_1=1/3$, $s_2=4/3$. Therefore, system (26) is (locally) asymptotically stable.

Remark 4: The asymptotic stability of system (26) cannot be checked by means of methods based on first order approximations, since the linear approximation of system (26) is not asymptotically stable.

Example 2: Let us consider the following coupled delay differential and difference equation, describing an electrical circuit containing a LC transmission line (see Rasvan and Niculescu (2002, Example 5) or Niculescu (2001, example 5.55, pp. 213))

$$\left.\begin{aligned}&\dot{\xi}(t)=A\xi(t)+\begin{pmatrix}-\dfrac{1}{C_1}f_1(\xi_1(t))\\0\end{pmatrix}+Bx(t-\Delta)\\&x(t)=D\xi(t)+Fx(t-\Delta)\\&\xi(\tau)=\xi_0(\tau),\quad x(\tau)=x_0(\tau),\quad \tau\in[-\Delta,0],\end{aligned}\right\}\quad(35)$$

where

$$\xi(t)=\begin{bmatrix}\xi_1(t)\\\xi_2(t)\end{bmatrix}\in R^2;\quad x(t)\in R^2;\quad \Delta=\sqrt{LC};$$
$$\xi_0,x_0\in C([-\Delta,0];\,R^2);$$

$$\left.\begin{aligned}&A=\begin{bmatrix}-\dfrac{1+R_1\sqrt{C/L}}{R_1C_1}&0\\0&-\dfrac{\sqrt{C/L}}{(1+R_2\sqrt{C/L})C_2}\end{bmatrix};\\&B=\begin{bmatrix}0&2\dfrac{\sqrt{C/L}}{C_1}\\2\dfrac{\sqrt{C/L}}{(1+R_2\sqrt{C/L})C_2}&0\end{bmatrix};\\&D=\begin{bmatrix}1&0\\0&\dfrac{1}{1+R_2\sqrt{C/L}}\end{bmatrix};\quad F=\begin{bmatrix}0&-1\\-\dfrac{1-R_2\sqrt{C/L}}{1+R_2\sqrt{C/L}}&0\end{bmatrix};\end{aligned}\right\}\quad(36)$$

R_1, R_2, C, C_1, C_2, L are (positive real) electrical parameters (resistors, capacitors, inductors); f_1 is a scalar continuous function describing a non-linear resistor. Here we hypothesize that the matching condition is verified, that is the initial conditions are such that $x_0(0)=D\xi_0(0)+Fx_0(-\Delta)$. The matrix F has eigenvalues inside the open unit circle, therefore the linear continuous-time difference part of system (35), with no forcing input ($\xi(t)=0$), is asymptotically stable (see Hale and Verduyn Lunel (1993, §9.6). From the asymptotic stability, it follows that the linear continuous-time difference part of system is input-to-state stable with respect to inputs ξ in $\mathcal{M}_S(+\infty)$. As far as general linear continuous-time difference equations forced by continuous inputs (which is actually our case since ξ is continuous) are concerned (see Hale and Verduyn Lunel (1993, Theorem 3.5, pp. 275)). Nevertheless, for locally essentially bounded inputs, a direct computation of the solution of system (5), equivalent to the simple linear continuous-time difference part of system (35), yields the following ISS inequality

$$\|\chi(k)\|\le M\lambda^k\|\chi(0)\|+M\frac{\lambda}{1-\lambda}|D|\,\|\mathcal{U}_{[k-1]}\|_\infty,\quad(37)$$

where: $\chi(k)\in B((0,\Delta];\,R^2)$, $\mathcal{U}(k)\in B((0,\Delta];\,R^2)$; M and λ are positive reals, $\lambda<1$, such that, $\forall k\ge 0$, $|F^k|\le M\lambda^k$ holds. If one uses Theorem 1, the ISS is proved by means of the functional $V(\chi)=\sup_{\tau\in(0,\Delta]}\chi^T(\tau)Q\chi(\tau)$, with Q suitable symmetric positive definite matrix in $\mathbb{R}^{2\times 2}$; see (Germani *et al.* 2003, proof of Lemma A1).

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Let us now apply Theorem 1 with the following Lyapunov–Krasovskii functional

$$V\left(\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\right) = \begin{bmatrix}\phi_1^T(0) & \phi_2^T(0)\end{bmatrix} EP \begin{bmatrix}\phi_1(0)\\ \phi_2(0)\end{bmatrix} + \int_{-\Delta}^{0} \phi^T(\tau) Q \phi(\tau)\,\mathrm{d}\tau, \qquad (38)$$

where

$$\phi = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}, \quad \phi_1 \in C([-\Delta, 0]; R^2),$$

$$\phi_2 \in C([-\Delta, 0]; R^2); \quad P = \begin{bmatrix}P_1 & 0\\ P_2 & P_3\end{bmatrix}, \quad P_1 \in R^{2\times 2}$$

is diagonal positive definite, $P_2, P_3 \in R^{2\times 2}$; $Q \in R^{4\times 4}$ is symmetric positive definite;

$$E = \begin{bmatrix}I_2 & 0_{2\times 2}\\ 0_{2\times 2} & 0_{2\times 2}\end{bmatrix} \in R^{4\times 4}.$$

The derivative of such functional is given by

$$\dot V(\phi) = \eta^T \begin{bmatrix} P^T G_0 + G_0^T P + Q & P^T G_1\\ G_1^T P & -Q\end{bmatrix}\eta - \frac{2}{C_1} P_1(1,1)\phi_1^1(0) f_1\left(\phi_1^1(0)\right), \qquad (39)$$

where $\eta^T = [\phi^T(0)\ \phi^T(-\Delta)]$; G_0 and G_1 are square matrices with dimension 4 given by

$$G_0 = \begin{bmatrix}A & 0_{2\times 2}\\ D & -I_2\end{bmatrix}, \quad G_1 = \begin{bmatrix}0_{2\times 2} & B\\ 0_{2\times 2} & F\end{bmatrix}; \qquad (40)$$

P_1 $(1,1)$ is the element first row first column of matrix ρ_1, $\phi_1^1(0)$ is the first component of $\phi_1(0)$. A global asymptotic stability condition is therefore given by the feasibility of the LMI

$$\begin{bmatrix} P^T G_0 + G_0^T P + Q & P^T G_1\\ G_1^T P & -Q\end{bmatrix} + \omega \begin{bmatrix} 1 & 0_{1,7}\\ 0_{7,1} & 0_{7,7}\end{bmatrix} < 0, \qquad (41)$$

and by the condition

$$\alpha f_1(\alpha) \geq -\frac{C_1 \omega}{2P_1(1,1)}\alpha^2, \quad \forall \alpha \in R, \qquad (42)$$

where ω is a suitable positive real.

An analysis of the electrical parameters, such that the LMI (41) is feasible, is beyond the aims of this paper. Nevertheless, it is worth pointing out that the condition (42) may be less restrictive than Rasvan and Niculescu (2002, Condition (14), pp. 163), which is sufficient for the global asymptotic (more, exponential) stability of system (6.10). For instance, the choice of the electrical parameters as $R_1 = 1$, $R_2 = 50$, $C = 10^{-7}$, $C_1 = 10^{-6}$, $C_2 = 10^{-6}$, $L = 10^{-3}$, yields the feasibility of the LMI (41) (checked by Matlab) with, in particular, $\omega = 9.107 \cdot 10^8$, $P_1(1,1) = 1.282 \cdot 10^3$. Therefore system (35), with the chosen values of the electrical parameters, results to be globally asymptotically stable provided that

$$\alpha f_1(\alpha) \geq -0.355\,\alpha^2, \quad \forall \alpha \in R. \qquad (43)$$

On the other hand, when the above values of the electrical parameters are chosen, Rasvan and Niculescu (2002, Condition (14)) yields the global asymptotic (exponential) stability of system (35) provided that

$$\alpha f_1(\alpha) \geq 0.9701\,\alpha^2, \quad \forall \alpha \in R \qquad (44)$$

The positive real 0.9701, which appears in the right-hand side of (44), is obtained by an analysis, validated by Matlab simulations, of the inequalities (17) and of the roots of equation (18) in Rasvan and Niculescu (2002), corresponding to the given values of the electrical parameters.

With the above values of the electrical parameters, the global asymptotic stability condition improvement, as far as the function f_1 describing the non-linear resistor is concerned, is considerable. The condition (44) does not allow to f_1 to describe negative resistors, while the condition (43) does. Moreover, the condition (43) proves that the circuit is globally asymptotically stable for any function f_1 describing positive resistors, while the condition (44) proves that the circuit is globally (exponentially) asymptotically stable only for a class of functions f_1 describing positive resistors.

Note that the choice of P_1 diagonal is instrumental to obtain the only condition (42) on the function f_1. Nevertheless, the matrix A is diagonal too, with negative elements on the diagonal, therefore the quantity $A^T P_1 + P_1 A$ inside the LMI (41) can be equal to any diagonal negative definite matrix for suitable diagonal positive definite P_1.

Remark 5: The result of (delay-independent) asymptotic stability given for example 2 is global, therefore it cannot be proved by means of methods based on first order approximations.

7. Conclusions

In this paper, we have employed the notion of input-to-state stability to establish a new Lyapunov–Krasovskii methodology for both the (local) asymptotic stability and the global asymptotic stability of a general class of coupled delay differential and difference equations. The obtained results are a significant extension to the non-linear case of earlier results on linear descriptor time-delay systems; see Fridman (2002).

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