An Improved Stabilization Method for Linear Time-Delay Systems

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***Abstract*—In this note, we combine a new approach for linear time-delay systems based on a descriptor representation with a recent result on bounding of cross products of vectors. A delay-dependent criterion for determining the stability of systems with time-varying delays is obtained. This criterion is used to derive an efficient stabilizing state-feedback design method for systems with parameter uncertainty, of either the polytopic or the norm-bounded types.**

***Index Terms*—Delay-dependent stability, linear matrix inequality (LMI), stabilization, time-delay systems, time-varying delay.**

I. Introduction

The problem of reducing the conservatism entailed in applying finite-dimensional techniques to asses the stability of linear systems with time delay has attracted much attention in the past few years [1]–[6]. All these techniques provide sufficient conditions only for the asymptotic stability of these systems and they entail a considerable conservatism which stems from two main sources. The first cause for conservatism is the model transformation used to describe the system which makes it more amenable for analysis [7], [8] and the second reason for conservatism is the bounding method used to derive the bounds on weighted cross products of the state and its delayed version while trying to secure a negative value to the derivative of the corresponding Lyapunov–Krasovskii functional. The search for the most appropriate model transformation has led to four main approaches [9]–[11]. The most recent one [9], the one that is based on a descriptor representation of the system, which is equivalent to the original system, minimizes the overdesign that stems from the model transformation source of conservatism [11].

The conservatism that stems from the bounding of the cross terms has also been significantly reduced in the past few years. An important result for improving the standard bounding technique of, e.g., [2], has been proposed in [12]. Indeed, combining the later with the descriptor model transformation lead in [10] and [11] to an efficient delay-dependent stability criterion that was also used in synthesis for stabilization and optimal performance. Only recently, an improvement of the bounding technique has been proposed [13]. The latter generalizes the one in [12] and the resulting criteria that are obtained in [13] are, therefore, more efficient than those found in [12].

It is the purpose of this note to combine the bounding method of [13] with the descriptor model transformation of [9] and [11] in order to derive a most efficient stability criterion for systems with time-varying delays. This criterion is then applied to solve the problem of robust stabilizing the system in presence of either norm-bounded or polytopic uncertainties by means of state-feedback control. The resulting criterion is applied to an example taken from [13], and its superiority to the results of the latter is demonstrated.

Notation: Throughout this note, the superscript T stands for matrix transposition, $\mathcal{R}^n$ denotes the n-dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathcal{R}^{n\times n}$, means that P is symmetric and positive definite. The space of vector functions that are square integrable over $[0\quad\infty)$ is denoted by $\mathcal{L}_2$.

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II. A New Stabilization Method

We consider the following linear system with time-varying delays:

$$\dot{x}(t)=\sum_{i=0}^{2}A_i x(t-\tau_i(t))+Bu(t), \qquad x(t)=\phi(t),\ t\in[-h,0] \tag{1}$$

where $x(t)\in\mathcal{R}^n$ is the system state, $u(t)\in\mathcal{R}^q$ is the control input, $\tau_0\equiv 0$, A_i and B are constant $n\times n$ matrices, ϕ is a continuously differentiable initial function, and h is an upper-bound on the time-delays τ_i, $i=1,2$. For simplicity only, we took two delays τ_1 and τ_2. The results of this section can be easily applied to the case of multiple delays $\tau_1,\ldots,\tau_m$.

The matrices of the system are not exactly known. Denoting

$$\Omega=[\,A_0\quad A_1\quad A_2\quad B\,]$$

we assume that

$$\Omega=\sum_{j=1}^{N}f_j\Omega_j, \quad \text{for some} \quad 0\le f_j\le 1,\ \sum_{j=1}^{N}f_j=1 \tag{2}$$

where the N vertices of the polytope are described by

$$\Omega_j=[\,A_0^{(j)}\quad A_1^{(j)}\quad A_2^{(j)}\quad B^{(j)}\,].$$

In Section III, we extend our results to the case where the uncertainty in the system parameters obeys the norm-bounded model [17].

As in [11], we consider two different cases for time-varying delays

$\tau_i(t)$ are differentiable functions, satisfying for all $t\ge 0$:

$$0\le\tau_i(t)\le h_i, \qquad \dot{\tau}_i(t)\le d_i<1, \qquad i=1,2. \tag{3}$$

$\tau_i(t)$ are continuous functions, satisfying for all $t\ge 0$, $0\le\tau_i(t)\le h_i$, $i=1,2$.

Note that in the past, the Razumikhin's approach was the only one that was to cope with Case I) of fastly varying delays. The Krasovskii approach for this case was introduced recently in [11].

We seek a control law

$$u(t)=Kx(t) \tag{4}$$

that will asymptotically stabilize the system.

A. Stability Issue

In this section, we consider $B=0$. Representing (1) in an equivalent descriptor form [9]

$$\dot{x}(t)=y(t) \tag{5a}$$

$$0=-y(t)+\left\{\sum_{i=0}^{2}A_i\right\}x(t)-\sum_{i=1}^{2}A_i\int_{t-\tau_i(t)}^{t}y(s)ds \tag{5b}$$

or

$$\begin{aligned}E\dot{\bar{x}}(t)&=\begin{bmatrix}\dot{x}(t)\\0\end{bmatrix}\\&=\begin{bmatrix}0&I\\\sum_{i=0}^{2}A_i&-I\end{bmatrix}\bar{x}(t)-\sum_{i=1}^{2}\begin{bmatrix}0\\A_i\end{bmatrix}\int_{t-\tau_i(t)}^{t}y(s)ds\end{aligned} \tag{5c}$$

with $\bar{x}(t) = \mathrm{col}\{x(t),\ y(t)\}$, $E = \mathrm{diag}\{I, 0\}$, the following Lyapunov–Krasovskii functional is applied:

$$V(t) = \bar{x}^T(t)EP\bar{x}(t) + V_2 + V_3 \tag{6}$$

where

$$\begin{aligned} P &= \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \qquad P_1 > 0 \quad EP = P^T E \geq 0 \\ V_2 &= \sum_{i=1}^{2} \int_{-h_i}^{0} \int_{t+\theta}^{t} y^T(s) R_i y(s) ds d\theta \\ V_3 &= \sum_{i=1}^{2} \int_{t-\tau_i(t)}^{t} x^T(\tau) S_i x(\tau) d\tau. \end{aligned} \tag{7a-e}$$

The following result is obtained for Case I).

Lemma 1: Under Case I), (1), with $B = 0$, is asymptotically stable if there exist $n \times n$ matrices $0 < P_1,\ P_2,\ P_3,\ S_i,\ Y_{i1},\ Y_{i2},\ Z_{i1},\ Z_{i2}, Z_{i3}$, and $R_i > 0$, $i = 1, 2$ that satisfy the following linear matrix inequalities (LMIs):

$$\Gamma = \begin{bmatrix} \Psi & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - Y_1^T & P^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} - Y_2^T \\ * & -S_1(1-d_1) & 0 \\ * & * & -S_2(1-d_2) \end{bmatrix} < 0$$

$$\text{and} \begin{bmatrix} R_i & Y_i \\ * & Z_i \end{bmatrix} \geq 0, \qquad i = 1, 2 \tag{8a,b}$$

where

$$\begin{aligned} Y_i &= [Y_{i1} \quad Y_{i2}] \quad Z_i = \begin{bmatrix} Z_{i1} & Z_{i2} \\ * & Z_{i3} \end{bmatrix}, \qquad i = 1, 2 \\ \Psi &= P^T \begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix}^T P \\ &\quad + \sum_{i=1}^{2} h_i Z_i + \begin{bmatrix} \sum_{i=1}^{2} S_i & 0 \\ 0 & \sum_{i=1}^{2} h_i R_i \end{bmatrix} \\ &\quad + \sum_{i=1}^{2} \begin{bmatrix} Y_i \\ 0 \end{bmatrix} + \sum_{i=1}^{2} \begin{bmatrix} Y_i \\ 0 \end{bmatrix}^T. \end{aligned} \tag{9a-c}$$

Proof: Note that

$$\bar{x}^T(t)EP\bar{x}(t) = x^T(t)P_1x(t)$$

and, hence, differentiating the first term of (6) with respect to t gives

$$\frac{d}{dt}\left\{\bar{x}^T(t)EP\bar{x}(t)\right\} = 2x^T(t)P_1\dot{x}(t) = 2\bar{x}^T(t)P^T \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix}. \tag{10}$$

Substituting (5) into (10), we obtain (11), as shown at the bottom of the page, where

$$\begin{aligned} \bar{\Gamma} &\triangleq P^T \begin{bmatrix} 0 & I \\ \sum_{i=0}^{2} A_i & -I \end{bmatrix} + \begin{bmatrix} 0 & \sum_{i=0}^{2} A_i^T \\ I & -I \end{bmatrix} P \\ &\quad + \begin{bmatrix} \sum_{i=1}^{2} S_i & 0 \\ 0 & \sum_{i=1}^{2} h_i R_i \end{bmatrix} \\ \eta_i(t) &\triangleq -2\int_{t-\tau_i}^{t} \bar{x}^T(t)P^T \begin{bmatrix} 0 \\ A_i \end{bmatrix} y(s) ds. \end{aligned} \tag{12}$$

Since, by [13], for any $a \in R^n, b \in \mathcal{R}^{2n}, \mathcal{N} \in \mathcal{R}^{2n\times n}, R \in \mathcal{R}^{n\times n}$, $Y \in \mathcal{R}^{n\times 2n}$, $Z \in \mathcal{R}^{2n\times 2n}$, the following holds:

$$-2b^T\mathcal{N}a \leq \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} R & Y - \mathcal{N}^T \\ Y^T - \mathcal{N} & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\text{where} \quad \begin{bmatrix} R & Y \\ Y^T & Z \end{bmatrix} \geq 0 \tag{13}$$

we apply the latter on the expression we have previously obtained for η_i. From (13), taking $\mathcal{N} = \mathcal{N}_i = P^T \begin{bmatrix} 0 \\ A_i \end{bmatrix}$, $R = R_i$, $Z = Z_i$, $Y = Y_i$, $a = y(s)$ and $b = \bar{x}(t)$, we obtain, for $i = 1, 2$, (14) found at the bottom of the page. Substituting the latter and (12) into (11), we obtain that

$$\frac{dV(t)}{dt} \leq \xi^T(t)\Gamma_1\xi(t)$$

where the first equation shown at the bottom of the next page holds, and where $\xi(t) = \mathrm{col}\{x(t),\ y(t),\ x(t-\tau_1),\ x(t-\tau_2)\}$. Since $\Gamma_1 = \Gamma$ the LMIs in (8) lead to $\dot{V} < 0$, while $V \geq 0$ and, thus, (1) with $B = 0$ is asymptotically stable [5], [15]. □

Choosing in Lemma 1 $S_i \to 0$ and $Y_i = [0 \quad A_i^T]P^T$ we obtain the following result for the case B.

Corollary 1: Under Case II), (1), with $B = 0$, is asymptotically stable if there exist $n \times n$ matrices $0 < P_1,\ P_2,\ P_3,\ Z_{i1},\ Z_{i2},\ Z_{i3}$ and $R_i > 0$, $i = 1, 2$ that satisfy the following LMIs:

$$\Psi_1 < 0 \quad \text{and} \quad \begin{bmatrix} R_i & [0 \quad A_i^T]P^T \\ * & Z_i \end{bmatrix} \geq 0, \qquad i = 1, 2$$

$$\frac{dV(t)}{dt} \leq \bar{x}^T(t)\bar{\Gamma}\bar{x}(t) - \sum_{i=1}^{2}\left[(1-d_i)x^T(t-\tau_i)S_ix(t-\tau_i) + \int_{t-h_i}^{t} y^T(\tau)R_iy(\tau)d\tau - \eta_i\right] \tag{11}$$

$$\begin{aligned} \eta_i(t) &\leq \int_{t-\tau_i}^{t} [y^T(s) \quad \bar{x}^T(t)] \begin{bmatrix} R_i & Y_i - [0 \quad A_i^T]P \\ Y_i^T - P^T \begin{bmatrix} 0 \\ A_i \end{bmatrix} & Z_i \end{bmatrix} \begin{bmatrix} y(s) \\ \bar{x}(t) \end{bmatrix} ds \\ &= \int_{t-\tau_i}^{t} y^T(s)R_iy(s)ds + 2\int_{t-\tau_i}^{t} y^T(s)(Y_i - [0 \quad A_i^T]P)\bar{x}(t)ds + \int_{t-\tau_i}^{t} \bar{x}(t)^TZ_i\bar{x}(t)ds \\ &= \int_{t-\tau_i}^{t} y^T(s)R_iy(s)ds + 2\int_{t-\tau_i}^{t} \dot{x}^T(s)(Y_i - [0 \quad A_i^T]P)\bar{x}(t)ds + \tau_i\bar{x}(t)^TZ_i\bar{x}(t) \\ &\leq \int_{t-h_i}^{t} y^T(s)R_iy(s)ds + 2x^T(t)(Y_i - [0 \quad A_i^T]P)\bar{x}(t) - 2x^T(t-\tau_i)(Y_i - [0 \quad A_i^T]P)\bar{x}(t) + h_i\bar{x}(t)^TZ_i\bar{x}(t). \end{aligned} \tag{14}$$

where

$$Z_i = \begin{bmatrix} Z_{i1} & Z_{i2} \\ * & Z_{i3} \end{bmatrix}, \qquad i = 1, 2$$

$$\Psi_1 = P^T \begin{bmatrix} 0 & I \\ \sum_{i=0}^{2} A_i & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ \sum_{i=0}^{2} A_i & -I \end{bmatrix}^T P + \sum_{i=1}^{2} h_i Z_i + \begin{bmatrix} 0 & 0 \\ 0 & \sum_{i=1}^{2} h_i R_i \end{bmatrix}.$$

Remark 1: It follows from (8a) that the diagonal elements $-S_i(1-d_i)$, $i = 1$,2 are negative and, thus, $S_i > 0$, since by assumption $d_i < 1$.

Remark 2: A question may arise as to whether the standard Lyapunov criterion can be restored when letting h go to 0. Taking $R_i = I$, $Z_i = \rho^{-1/2}$, $\rho \to \infty$, $Y_i = [0 \quad A_i^T\,]P$ and $0 < S_i \to 0$ we obtain

$$\begin{aligned} \Psi =& P^T \begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix}^T P + \sum_{i=1}^{2} \begin{bmatrix} Y_i \\ 0 \end{bmatrix} + \sum_{i=1}^{2} \begin{bmatrix} Y_i \\ 0 \end{bmatrix}^T \\ =& \begin{bmatrix} P_2^T(\sum_{i=0}^{2} A_i) + (\sum_{i=0}^{2} A_i^T)P_2 & P_1 - P_2^T + (\sum_{i=0}^{2} A_i^T)P_3 \\ * & -P_3 - P_3^T \end{bmatrix}. \end{aligned}$$

For $P_3 = \lambda I,\ \lambda \to 0$ and $P_2 = P_1 > 0$ the requirement that $\Psi < 0$ becomes

$$P_1(\sum_{i=0}^{2} A_i) + (\sum_{i=0}^{2} A_i^T)P_1 < 0, \qquad P_1 > 0. \tag{15}$$

It follows from (15) that if the system with $h = 0$ is asymptotically stable, then there exists $P_1 > 0$ that solves (15) and, thus, (8a),(b) possess a solution for small enough $h > 0$.

The latter can be readily used to verify the stability of (1) over the uncertainty polytope (2) [1]:

$$\bar{\Omega} = \sum_{j=1}^{N} f_j \bar{\Omega}_j, \quad \text{for some} \quad 0 \le f_j \le 1,\ \sum_{j=1}^{N} f_j = 1$$

where the N vertices of the polytope are described by

$$\bar{\Omega}_j = [\,A_0^{(j)} \quad A_1^{(j)} \quad A_2^{(j)}\,]$$

by solving the LMI simultaneously for all the N vertices, applying the same P_1, P_i, S_i, Y_{i1}, Y_{i2}, and R_i, $i = 1$,2.

In the sequel, it will be important to determine the conditions for achieving H_∞ norm of (1) less than 1, where u is the input vector and the controlled output is given by

$$z(t) = Lx(t) + L_1 x(t-\tau_1) + L_2 x(t-\tau_2). \tag{16}$$

Similarly to the derivation of the bounded real lemma (BRL) in [11], we obtain the following.

Lemma 2: Under Case I) the H_∞ norm of (1) and (16) is less than one if there exist $n \times n$ matrices $0 < P_1,\ P_2,\ P_3,\ S_i,\ Y_{i1},\ Y_{i2},\ Z_{i1},\ Z_{i2}, Z_{i3}$ *and* R_i, $i = 1$,2 that satisfy (17), as shown at the bottom of the page, and (8b), where Ψ is given by (9c).

Proof: Adding the term $z^T(t)z(t) - w(t)^T w(t)$ to $dV(t)/dt$ in (11) and substituting for $z(t)$ from (16), the result follows from the arguments used to derive Lemma 1 where the last column and row blocks in (17) are obtained by applying the standard Schur's formula [1].

B. State-Feedback Stabilization

The results of Lemma 1 can also be used to verify the stability of the closed loop obtained by applying (4) to (1) (with $B \neq 0$) if we replace A_0 in (8a) by $A_0 + BK$ and verify that the resulting inequality is feasible over the polytope defined in (2) by solving the LMI simultaneously for all the N vertices, applying the same P_1, P_i, S_i, Y_{i1}, Y_{i2}, and R_i, $i = 1$,2.

The problem with (8a) is that it is linear in its variables, only when the state-feedback gain K is given. In order to find K, consider the inverse of P. It is obvious from the requirement of $0 < P_1$, and the fact that in (8)$-(P_3 + P_3^T)$ must be negative definite, that P is nonsingular. Defining

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \tag{18a}$$

$$\Delta = \mathrm{diag}\{Q,\ I\} \tag{18b}$$

we multiply (8a) by Δ^T and Δ, on the left and on the right, respectively, and (8b), on the left and on the right, by $\mathrm{diag}\{R_i^{-1}, \quad Q^T\}$ and $\mathrm{diag}\{R_i^{-1}, \quad Q\}$, respectively . Applying Schur formula to the emerging quadratic term in Q, denoting $\bar{S}_i = S_i^{-1}$, $\bar{Z}_i = \begin{bmatrix} \bar{Z}_{i1} & \bar{Z}_{i2} \\ \bar{Z}_{i2}^T & \bar{Z}_{i3} \end{bmatrix} = Q^T Z_i Q$ and $\bar{R}_i = R_i^{-1}$, $i = 1$,2 and choosing $[Y_{i1} \quad Y_{i2}] = \varepsilon_i A_i^T [P_2 \quad P_3]$, where $\varepsilon_i \in \mathcal{R}^{n \times n}$ is a diagonal matrix, we obtain, similarly to [14], the following.

Theorem 1: The control law of (4) asymptotically stabilizes (1) for all the delays that belong to Case I) and for all the system parameters that reside in the uncertainty polytope, if for some diagonal matrices

$$\Gamma_1 = \begin{bmatrix} \bar{\Gamma} + \sum_{i=1}^{2} \left[h_i Z_i + \begin{bmatrix} I \\ 0 \end{bmatrix} (Y_i - [0 \quad A_i^T\,]P) + \left(Y_i^T - P^T \begin{bmatrix} 0 \\ A_i \end{bmatrix} \right) [I \quad 0] \right] & -Y_1^T + P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & -Y_2^T + P^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \\ * & -S_1(1-d_1) & 0 \\ * & * & -S_2(1-d_2) \end{bmatrix}$$

$$\begin{bmatrix} \Psi & P^T \begin{bmatrix} 0 \\ B \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - Y_1^T & P^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} - Y_2^T & [L^T\,] \\ * & -I_q & 0 & 0 & 0 \\ * & * & -S_1(1-d_1) & 0 & L_1^T \\ * & * & * & -S_2(1-d_2) & L_2^T \\ * & * & * & 0 & -I_r \end{bmatrix} < 0 \tag{17}$$

$\varepsilon_1,\ \varepsilon_2 \in \mathcal{R}^{n\times n}$, there exist: $0 < Q_1,\ Q_{i+1}^{(j)},\ \bar{S}_i,\ \bar{R}_i > 0,\ \bar{Z}_{ij}^{(j)} \in \mathcal{R}^{n\times n}$, $i = 1,2$, $j = 1,2,3$ and $\bar{Y} \in \mathcal{R}^{q\times n}$ that satisfy the LMIs shown in (19) at the bottom of the page, where

$$\Xi^{(j)} = Q_3^{(j)} - Q_2^{(j)T} + Q_1\Big(A_0^{(j)T} + \sum_{i=1}^{2}\varepsilon_i A_i^{(j)T}\Big) + \sum_{i=1}^{2} h_i \bar{Z}_{i2}^{(j)} + \bar{Y}^T B^{(j)T},\ j = 1,2,\ldots,N.$$

The state-feedback gain is then given by

$$K = \bar{Y} Q_1^{-1}. \tag{20}$$

The previous result represents a delay-dependent sufficient condition for the controller of (4) to guarantee, for Case I), stability over the entire uncertainty polytope. The corresponding delay-independent result is obtained, still for Case I), by substituting $\varepsilon_i = 0$, $\bar{R}_i = \rho I_n$ and $\bar{Z}_i = 0$ in (19) and taking the limit where ρ tends to infinity. The last two row and column blocks of (19a) will disappear due to $\bar{R}_i \rightarrow \infty$. Considering, still in the delay-independent case, the more general control law

$$u(t) = \sum_{i=0}^{2} K_i x(t - \tau_i) \tag{21}$$

we replace $A_i^{(j)}$ in (19) by $A_i^{(j)} + B^{(j)} K_i$ and obtain the following.

Corollary 2: In Case I), the control law of (21) asymptotically stabilizes (1) independently of the delay lengths, for all the system parameters that reside in the uncertainty polytope, if there exist: $Q_1 > 0$, $\bar{S}_1$, $\bar{S}_2$, $Q_2^{(j)}$, $Q_3^{(j)} \in \mathcal{R}^{n\times n}$ *and* $\bar{Y}_i \in \mathcal{R}^{q\times n}$, $i = 0, 1, 2$ that satisfy the equations shown at the bottom of the page where

$$\Xi_g^{(j)} = Q_3^{(j)} - Q_2^{(j)T} + Q_1 A_0^{(j)T} + \bar{Y}_0^T B^{(j)T}, \qquad j = 1,2,\ldots,N. \tag{22a,b}$$

The state-feedback gains are then given by

$$K_0 = \bar{Y}_0 Q_1^{-1} \quad K_i = \bar{Y}_i \bar{S}_i^{-1}, \qquad i = 1, 2. \tag{23}$$

Remark 3: In Case I), the control law of (21) cannot be readily incorporated in the result of Theorem 1 because of the quadratic term that will emerge in (19b) when $A_i^{(j)}$ is replaced by $A_i^{(j)} + B^{(j)} K_i$. One can, however, solve the design problem with the feedback law of (21) by applying the method of [14] which converts the problem of dealing with delayed components in the input to one with the control law of (4) by adding, in series to the input, simple linear components. The transference of these components is almost I and the augmented system that results can be readily solved using the LMIs of Theorem 1.

The delay-dependent result for Case II) is obtained by deleting V_3 in (6) and choosing $\varepsilon_i = I_n$, $i = 1,2$. The theory then develops along the lines that led to Theorem 1. Thus, the result for Case II) is the following.

Corollary 3: The control law of (4) asymptotically stabilizes (1) for all the delays that belong to Case II) and for all the system parameters that reside in the uncertainty polytope, if there exist: $Q_1 > 0$, $Q_2^{(j)}$,

$$\begin{bmatrix} Q_2^{(j)} + Q_2^{(j)T} + \sum_{i=1}^{2} h_i \bar{Z}_{i1}^{(j)} & \Xi^{(j)} & 0 & 0 & Q_1 & Q_1 & h_1 Q_2^{(j)T} & h_2 Q_2^{(j)T} \\ * & -Q_3^{(j)} - Q_3^{(j)T} + \sum_{i=1}^{2} h_i \bar{Z}_{i3}^{(j)} & A_1^{(j)}(I_n - \varepsilon_1)\bar{S}_1 & A_2^{(j)}(I_n - \varepsilon_2)\bar{S}_2 & 0 & 0 & h_1 Q_3^{(j)T} & h_2 Q_3^{(j)T} \\ * & * & -(1-d_1)\bar{S}_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -(1-d_2)\bar{S}_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\bar{S}_1 & 0 & 0 & 0 \\ * & * & * & * & * & -\bar{S}_2 & 0 & 0 \\ * & * & * & * & * & * & -h_1\bar{R}_1 & 0 \\ * & * & * & * & * & * & * & -h_2\bar{R}_2 \end{bmatrix} < 0 \tag{19a}$$

$$\begin{bmatrix} \bar{R}_i & 0 & \bar{R}_i \varepsilon_i A_i^{(j)T} \\ * & \bar{Z}_{i1}^{(j)} & \bar{Z}_{i2}^{(j)} \\ * & * & \bar{Z}_{i3}^{(j)} \end{bmatrix} \geq 0, \qquad i = 1, 2 \tag{19b}$$

$$\begin{bmatrix} Q_2 + Q_2^T & \Xi_g^{(j)} & 0 & 0 & Q_1 & Q_1 \\ * & -Q_3^{(j)} - Q_3^{(j)T} & -A_1^{(j)}\bar{S}_1 + B^{(j)}\bar{Y}_1 & -A_2^{(j)}\bar{S}_2 + B^{(j)}\bar{Y}_2 & 0 & 0 \\ * & * & -(1-d_1)\bar{S}_1 & 0 & 0 & 0 \\ * & * & * & -(1-d_2)\bar{S}_2 & 0 & 0 \\ * & * & * & * & -\bar{S}_1 & 0 \\ * & * & * & * & * & -\bar{S}_2 \end{bmatrix} < 0$$

$Q_3^{(j)}$, $\bar{R}_1$, $\bar{R}_2 \in \mathcal{R}^{n\times n}$ and $\bar{Y} \in \mathcal{R}^{q\times n}$ that satisfy the LMIs shown in the equation at the bottom of the page, and (19b), where $\varepsilon_i = I$, and where

$$\hat{\Xi}^{(j)} = Q_3^{(j)} - Q_2^{(j)T} + Q_1(\sum_{i=0}^{2} A_i^{(j)T}) + \sum_{i=1}^{2} h_i \bar{Z}_{i2}^{(j)} + \bar{Y}^T B^{(j)T}, \qquad j = 1, 2, \ldots, N.$$

The state-feedback gain is then given by (20).

III. Stabilization of Systems With Norm-Bounded Uncertainties

The results of Section II were derived for the case where the unknown parameters of (1) lie in a given polytope. An alternative way of dealing with uncertain systems is to assume that the deviation of the system parameters from their nominal values is norm bounded [17]. In our case, consider the system

$$\begin{aligned}\dot{x}(t) =& \sum_{i=0}^{2}(A_i + H\Delta(t)E_i)x(t-\tau_i(t)) \\ &+ (B + H\Delta(t)E_3)u(t) \\ x(s) =& \phi(s) s \le 0 \end{aligned} \tag{24}$$

where $x(t)$ and $u(t)$ are defined in Section II and the time delays are defined in (3). The matrices $A_i, i = 0,1,2$, B, H and $E_i, i = 0, \ldots, 3$ are constant matrices of appropriate dimensions. The matrix $\Delta(t)$ is a time-varying matrix of uncertain parameters satisfying

$$\Delta^T(t)\Delta(t) \le I \qquad \forall\, t. \tag{25}$$

We consider also, for a given positive scalar $\hat{\varepsilon}$, the following augmented system:

$$\begin{aligned}\dot{\xi}(t) =& \sum_{i=0}^{2} A_i \xi(t-\tau_i(t)) + Bu(t) + \hat{\varepsilon}^{-1}Hw(t) \\ \xi(s) =& \phi(s) \;\; s \le 0 \\ z(t) =& \hat{\varepsilon}E_0\xi(t) + \sum_{i=1}^{2}\hat{\varepsilon}E_i\xi(t-\tau_i) + \hat{\varepsilon}E_3u(t) \end{aligned} \tag{26a,b}$$

with the performance index

$$J(w) = \int_0^\infty (z^Tz - w^Tw)d\tau \tag{27}$$

where $w \in \mathcal{L}_2$ is an exogenous signal.

It has been explicitly proved in [17], in the case without delays, that the existence of a solution to the Riccati equations or LMIs that are obtained when solving the H_∞ state-feedback control problem for the augmented system (26) with the index (27), without delays, guarantees the stability of (24), under the same feedback law, for all $\Delta(t)$ that satisfy (25). The proof follows, in fact, from the small gain theorem [16] which can also be applied to our case of retarded systems. The system (24) can be written as

$$\begin{aligned}\dot{x}(t) =& \sum_{i=0}^{2} A_i x(t-\tau_i) + Bu(t) \\ &+ \hat{\varepsilon}^{-1}H\Delta\hat{\varepsilon}[\,E_0 \quad E_1 \quad E_2 \quad E_3\,] \\ &\times \mathrm{col}\{x(t)\; x(t-\tau_1)\; x(t-\tau_2)\; u(t)\}.\end{aligned}$$

This system can be looked at as (26), where Δ of (25) is the feedback gain from z of (26b) to w in (26a). Consider the closed-loop system $\mathcal{G}$ that is obtained from (26a),(b) by applying the state-feedback controller. It follows from the existence of a solution to the above H_∞ state-feedback control problem, that $\mathcal{G}$ is asymptotically stable and that the H_∞ norm of the transference of $\mathcal{G}$ from w to z is less than 1. Applying, therefore, the feedback gain $\Delta(t)$, which satisfies (25), around $\mathcal{G}$, it follows from the small gain theorem [16] that the resulting closed-loop system will remain asymptotically stable. Since the latter closed-loop system is identical to the closed-loop obtained from (24) by applying the same state-feedback controller (in the sense that $x(t) \equiv \xi(t)$), this controller also stabilizes (24).

In order to apply the aforementioned argument to (24), one should use a BRL criterion that will guarantee a H_∞-norm less than one to the closed-loop system obtained from (26). An efficient delay-dependent BRL has recently been derived in [11]. The latter is based however on the bounding technique of [12]. In order to benefit from the new method in [13], we derive the following result from Lemma 2, applying the same transformation that was used in deriving Theorem 1 and taking $\bar{\delta} = \hat{\varepsilon}^2$.

Theorem 2: In case A, (24) is stabilized via the control law of (4) for all $\Delta(t)$ that satisfy (25), if for some diagonal $\bar{\varepsilon}_1, \bar{\varepsilon}_2 \in \mathcal{R}^{n\times n}$ and a scalar $0 < \hat{\delta}$, there exist $0 < Q_1, Q_{i+1}, \bar{S}_i, \bar{R}_i > 0, \bar{Z}_{ij}, \in \mathcal{R}^{n\times n}$, $i = 1,2$, $j = 1,2,3$ and $\bar{Y} \in \mathcal{R}^{q\times n}$ that satisfy the LMIs, shown in the equation at the bottom of the next page. where

$$\Xi_g = Q_3 - Q_2^T + Q_1(A_0^T + \sum_{i=1}^{2}\bar{\varepsilon}_i A_i^T) + \sum_{i=1}^{2} h_i \bar{Z}_{i2} + \bar{Y}^T B^T. \tag{28a-c}$$

The state-feedback gain is then given by

$$K = \bar{Y}Q_1^{-1}. \tag{29}$$

Remark 4: The delay-independent version in Case I) is obtained by solving (28a) and (28c), where $\bar{\varepsilon}_i = 0$, $\bar{Z}_i = 0$ and where the eighth and the ninth row and column blocks are omitted.

In Case II), the corresponding delay-dependent result is obtained by solving the LMIs of Theorem 2 for $\bar{\varepsilon}_i = I$, $i = 1, 2$, where in (28a) the fourth, fifth, sixth, and seventh row and column blocks are deleted.

Remark 5: The results of Theorems 1 and 2 apply the tuning parameters ε_1 and ε_2. The question arises how to find the optimal combination of these parameters. One way to address the tuning issue is to choose for a cost function the parameter $t_{\min}$ that is obtained while solving the feasibility problem using Matlab's LMI toolbox [18]. This scalar parameter is positive in cases where the combination of the tuning parameters is one that does not allow a feasible solution to the set of LMIs considered. Applying a numerical optimization algorithm, such as the program **fminsearch** in the optimization toolbox of Matlab

$$\begin{bmatrix} Q_2^{(j)} + Q_2^{(j)T} + \sum_{i=1}^{2} h_i \bar{Z}_{i1}^{(j)} & \hat{\Xi}^{(j)} & h_1 Q_2^{(j)T} & h_2 Q_2^{(j)T} \\ * & -Q_3^{(j)} - Q_3^{(j)T} + \sum_{i=1}^{2} h_i \bar{Z}_{i3}^{(j)} & h_1 Q_3^{(j)T} & h_2 Q_3^{(j)T} \\ * & * & -h_1\bar{R}_1 & 0 \\ * & * & * & -h_2\bar{R}_2 \end{bmatrix} < 0$$

[19], to the above cost function, a locally convergent solution to the problem is obtained. If the resulting minimum value of the cost function is negative, the tuning parameters that solve the problem are found.

The latter optimization procedure is time consuming. Our experience shows that taking $\varepsilon_1 = \varepsilon_2 = \bar{\varepsilon} I$, where $\bar{\varepsilon}$ is a scalar, a one-dimensional search for $\bar{\varepsilon}$ is easily performed. The cost function, or the bound on delay that still maintains stability, exhibit then a convex behavior with respect to $\bar{\varepsilon}$ and a clear optimum value of the latter is obtained. In the examples we solved, the single tuning parameter $\bar{\varepsilon}$ achieved results that are quite close to those obtained by the fminsearch program.

IV. Examples

We demonstrate the applicability of the above theory by solving the second example in [13] for a system with norm-bounded uncertainty and the third example from [2], where we neglect uncertainties.

Example 1: The problem in [13] is one where a state-feedback control is sought that stabilized (24) for one delay with

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ A_1 &= \begin{bmatrix} -2 & -.5 \\ 0 & -1 \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ H &= 0.2 I_2 \\ E_0 &= I_2. \end{aligned}$$

For the case of $d = 0$, the maximum bound h for which the system is stabilized by a state-feedback was found in [13], after 99 iterations, to be 0.45. Applying the result of Theorem 2, a maximum bound of $h = 0.5865$ is obtained using $\varepsilon = 0.7507 I_2$ and $\delta = 0.8$. The corresponding feedback-gain matrix is $K = -[0.3155 \quad 4.4417]$. Using the method of [11] (an improved version of [10]), which is based on the bounding method of [12], a maximum value of $h = 0.55$ was achieved with $\delta = 0.3$, $\varepsilon = -0.2$ and $K = -[0.0229 \quad 52.8656]$.

It is noted that the computational complexity of the solutions in [10] and [11] and the present method is the same. Comparing to [13], the iterations required there almost compensate the increase in the dimension of the LMIs that is caused by using the descriptor approach.

In Case II) (fastly varying delays), the corresponding results are: $h = 0.496$, $K = -[0.34 \quad 5.168]$ and $\delta = 0.8$ by Remark 2 and $h = 0.489$, $K = -[0.2884 \quad 13.8558]$ and $\delta = 0.1$ by [11].

It follows from this that the theory of [11] provides stabilization results that are superior to those obtained in [13]. This is true in spite of the fact that the former applies the old bounding method of [12] and that it handles time-delays that can vary very fast. The results that are obtained using the theory of the present note surpass those found by the methods of [11]. The combination of the descriptor approach and the new bounding method of [13] is shown to be superior to all other solutions that were proposed in the literature.

Example 2 [2]: We address the problem of finding a state-feedback stabilizing controller for (24) with one delay and without uncertainties ($H = E_i = 0$), where

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ A_1 &= \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Applying the method of [2, Cor. 3.2], it was found that, for $\dot{\tau} \equiv 0$, the system is stabilizable for all $\tau < 1$. For, say, $\tau = .999$ a minimum

$$\begin{bmatrix} Q_2 + Q_2^T + \sum_{i=1}^{2} h_i \bar{Z}_{i1} & \Xi_g & 0 & 0 & 0 & Q_1 & Q_1 & h_1 Q_2^T & h_2 Q_2^T & Q_1 E_0^T + \bar{Y}^T E_3^T \\ * & -Q_3 - Q_3^T + \sum_{i=1}^{2} h_i \bar{Z}_{i3} & \hat{\delta} H & A_1(I_n - \bar{\varepsilon}_1)\bar{S}_1 & A_2(I_n - \bar{\varepsilon}_2)\bar{S}_2 & 0 & 0 & h_1 Q_3^T & h_2 Q_3^T & 0 \\ * & * & -\hat{\delta} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -(1-d_1)\bar{S}_1 & 0 & 0 & 0 & 0 & 0 & \bar{S}_1 E_2^T \\ * & * & * & * & -(1-d_2)\bar{S}_2 & 0 & 0 & 0 & 0 & \bar{S}_2 E_3^T \\ * & * & * & * & * & -\bar{S}_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\bar{S}_2 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -h_1 \bar{R}_1 & 0 & 0 \\ * & * & * & * & * & * & * & * & -h_2 \bar{R}_2 & 0 \\ * & * & * & * & * & * & * & * & * & -\hat{\delta} I \end{bmatrix} < 0$$

$$\begin{bmatrix} \bar{R}_i & 0 & \bar{R}_i \bar{\varepsilon}_i A_i^T \\ * & \bar{Z}_{i1} & \bar{Z}_{i2} \\ * & * & \bar{Z}_{i3} \end{bmatrix} \geq 0, \qquad i = 1, 2$$

value of $\gamma = 1.8822$ results for $K = -[.104\,52 \quad 749\,058]$. By [11], the system is stabilizable for $h \leq 1.408.$. By Theorem 1, the corresponding value is $h = 1.51$ for $\varepsilon = 0.59$ and $K = -[58.31 \quad 294.935]$.

V. Conclusion

The problem of finding a state-feedback controller that asymptotically stabilizes a linear time-delay system with either polytopic or norm-bounded uncertainty has been solved. A delay-dependent solution has been derived using a special Lyapunov–Krasovskii functional. The result is based on a sufficient condition and it thus entails an overdesign. This overdesign is considerably reduced due to the fact that is based on the descriptor representation and since it applies a new bounding method.

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Ultimate Periodicity of Orbits for Min–Max Systems

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***Abstract*—The ultimate periodicity theorem is an important result in min–max systems theory. It was first proved by Olsder and Perennes in their unpublished work. In this note, we present a new proof. This proof is also based on two important theorems: the existence of cycle time for any min–max function and the Nussbaum–Sine theorem. However, two different techniques, pure min–max function and conditional redundancy, are used to obtain two important intermediate results. The purpose of this note is to provide a simple alternate proof to the ultimate periodicity theorem.**

***Index Terms*—Discrete-event systems, min–max functions, ultimate periodicity.**

I. Introduction

Min–max functions (e.g., [1] and [2]) arise in modeling the dynamic behavior of discrete-event systems with maximum and minimum constraints, such as digital circuits, computer networks, manufacturing plants, etc. Mathematically, a min–max function $F : \mathbb{R}^n \to \mathbb{R}^n$ is built from terms of the form $x_i + a$, where $1 \leq i \leq n$ and $a \in \mathbb{R}$, by application of finitely many max and min operations in each component. In such a model, if we denote the time of the kth occurrence of event i by $x_i(k)$, then $x(k+1) = F(x(k))$.

Min–max functions are homogeneous

$$\forall x \in \mathbb{R}^n, \forall h \in \mathbb{R}, F(x+h) = F(x) + h$$

monotonic with respect to the usual product ordering on $\mathbb{R}^n$

$$\forall x, y \in \mathbb{R}^n, x \leq y \Rightarrow F(x) \leq F(y)$$

and nonexpansive in the sup norm

$$\forall x, y \in \mathbb{R}^n, \|F(x) - F(y)\| \leq \|x - y\|.$$

In this note, we study a min–max function as a dynamical system, and we are concerned with the behavior of the orbits of a min–max system F, namely the sequences $x(0)$, $x(1)$, and $x(2)$, $\ldots$, where $x(0) \in \mathbb{R}^n$ and $x(k+1) = F(x(k))$. Therefore, we shall be using "function" and "system" interchangeably in the sequel, depending on the context.

In studying the behavior of min–max functions, one is tempted to find out whether all or some min–max functions exhibit the following properties.

Property $\mathbf{C}(F)$: The limit $\chi(F, \xi) = \lim_{k \to \infty} F^k(\xi)/k$ exists.

It is to be noted that if for some ξ the limit $\chi(F, \xi)$ exists, then for all ξ this limit exists and is independent of ξ, because F is nonexpansive in the sup norm. This limit is called cycle time of F, and we will denote it by $\chi(F)$ in the sequel.

Property $\mathbf{I}(F)$: F has a cycle time with identical coordinates, i.e., there is a $\lambda \in \mathbb{R}$ such that $\chi(F) = (\lambda, \ldots, \lambda)$.

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