Robust $H_\infty$ minimum entropy static output-feedback control of singularly perturbed systems

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Abstract

The problem of designing static output-feedback $\varepsilon$-independent controllers for linear time-invariant singularly perturbed systems is considered. The controller is required to satisfy a prescribed $H_\infty$-norm bound and to minimize the closed-loop entropy at $s = \infty$ for all small enough $\varepsilon$. The optimal controller gain is designed on the basis of generalized Riccati and Lyapunov equations with symmetric block (2,2), that are coupled via a projection. This gain is either purely fast, purely slow or a composite one, depending on the structure of the output coupling matrix. A well-posed problem with a finite value of entropy for $\varepsilon \to 0$ is obtained by assuming that the entropy of the fast subproblem is zero or by scaling the matrices of the system. In the first case the optimal controller is the one that minimizes the entropy of the corresponding descriptor system.

1. Introduction

Robust state-feedback and dynamic output-feedback $H_\infty$ control for singularly perturbed systems have been considered by Khalil and Chen (1992), Pan and Basar (1993, 1994), Dragan (1993), Tuan and Hosoe (1997) in the standard case, and by Xu and Mizukami (1996), Tan, Leung and Tu (1998) in the non-standard case (information on non-standard singularly perturbed systems is found in Khalil, 1989). In the present paper we investigate the problem of achieving minimum entropy by static output-feedback $\varepsilon$-independent $H_\infty$ controller for non-standard singularly perturbed systems. We denote this controller as the 'robust optimal controller'. This controller should minimize, for all small enough values of $\varepsilon$, the closed-loop entropy while ensuring a prescribed $H_\infty$-norm bound. For each $\varepsilon > 0$, the minimizing controller gain can be designed by solving a pair of $\varepsilon$-dependent Riccati and Lyapunov equations. We shall show that the robust optimal controller is the formal first-order approximation to the above minimizing controller. Unlike the conventional approaches we shall prove the optimality of the obtained robust controller directly without considering its closeness to the optimal $\varepsilon$-dependent one (the proof that relies on the closeness arguments requires additional restrictive assumptions).

2. Problem formulation

Consider the following linear time-invariant system

$$ E_\varepsilon \dot{x} = Ax + B_1 w + B_2 u, \quad (1a) $$

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\[ y = C_2 x, \quad (1b) \]
\[ z = C_1 x + D_{12} u, \quad (1c) \]

where
\[
x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad E_x = \begin{bmatrix} I_{n_1} & 0 \\ 0 & e I_{n_2} \end{bmatrix},
\]
\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix},
\]

\[ C_i = [C_{1i}, C_{2i}], \quad i = 1, 2, \quad x_1 \in \mathbb{R}^{n_1}, \quad x_2 \in \mathbb{R}^{n_2}, \]

and where \( e \) is a small positive scalar parameter. We assume that \( C_i \) is of full row rank, and \( D_{12}[D_{12} \ C_i] = [R \ 0], \ R > 0 \). We note that \( A_{22} \) may be singular.

Denoting by \( T_{zw} \) the transfer function from the exogenous input \( w \) to the objective vector \( z \), for a given scalar \( \gamma > 0 \), the problem is to find, of all \( \varepsilon \)-independent static output-feedback controllers \( u = Ky \) that satisfy, for all small enough \( \varepsilon \),

\[ \|T_{zw}\|_\infty < \gamma, \quad (2) \]

the one that minimizes, for small enough \( \varepsilon \), the entropy of the closed-loop transfer-function matrix

\[ T_{zw}(s) = T_{zw}^*(s) = (C_1 + D_{12} KC_2) \]
\[ [sE_x - A - B_2 KC_2]^{-1} B_1, \quad (3) \]

where the entropy is given by
\[ \delta(T_{zw}, \gamma) = \frac{\gamma^2}{2n} \int_{-\infty}^{\infty} \ln \det[I - (I - 2\gamma^2 T_{zw}(jw) T_{zw}(jw))] \, dw, \quad (4) \]

and where \( T_{zw}^*(s) = T^*(s) \). By minimizing the entropy, we push \( T_{zw} \) away from the upper bound \( \gamma \) in the magnitude Bode plot. We are thus looking for robust optimal controller gain \( K^* \) for which there is \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) (2) is satisfied and

\[ K^* = \text{argmin}_K \delta(T_{zw}, \gamma). \quad (5) \]

We begin by denoting the following:
\[ A_k = E_k^{-1} A, \quad B_{1k} = E_k^{-1} B_1, \quad \bar{A}_k = A + B_2 KC_2, \]
\[ \bar{K} = C_1 + D_{12} KC_2. \quad (6) \]

It is known (Doyle, Glover, Khargonekar & Francis, 1989) that for each \( \varepsilon > 0 \) the matrix \( \bar{A}_k \) is stable and the transfer-function matrix \( T_{zw} = \bar{C}(sI - \bar{A}_k)^{-1} B_{1k} \) satisfies (2) iff there exists a matrix \( X \geq 0 \) that satisfies the following Riccati equation:
\[ \bar{A}_k^T X_k + X_k \bar{A}_k + \gamma^{-2} X_k B_{1k} B_{1k}^T X_k + \bar{C}^T \bar{C} = 0, \quad (7) \]

so that \( \bar{A}_k + \gamma^{-2} B_{1k} B_{1k}^T X_k \) is asymptotically stable. If such \( X_k \) exists, then \( \delta(T_{zw}, \gamma) \) is given by Stoorvogel

\[ \delta(T_{zw}, \gamma) = \text{Tr}[B_{1k} X_k B_{1k}^T]. \quad (8) \]

Considering next the following Lyapunov (with respect to \( Y_e \) equation
\[ (A_e + B_2 K C_2 + \gamma^{-2} B_{1e} B_{1e}^T X_e) Y_e + Y_e (A_e + C_2 K B_2^T + \gamma^{-2} X_e B_{1e} B_{1e}^T) + B_{1e} B_{1e}^T = 0. \quad (9) \]

It has been shown (Yaesh & Shaked, 1997) that for each \( \varepsilon > 0 \), the following gain matrix:
\[ K = K_e = - R^{-1} B_{2e}^T Y_e, \quad (10) \]
solves the \( H_\infty \) minimum entropy static output-feedback control problem. Denote
\[ v_e = Y_e C_2 (C_2 Y_e C_2)^{-1} C_1, \quad v_{e\perp} = I - v_e, \quad (11) \]

where \( C_1 \) is the right inverse of \( C_2 \) (i.e. \( C_1 C_2 = I \)). It has been also found that \( v_e^2 = v_e \), \( K_{C_2} = - R^{-1} B_{2e}^T X_e v_e \), and that (7) can be written in the form
\[ A_k^T X_k + X_k A_k + \gamma^{-2} X_k B_{1k} B_{1k}^T X_k + C_1 C_1^T X_k B_{1e} B_{1e}^T X_e v_{e\perp} = 0. \quad (12) \]

The above are summarized in the following lemma (Yaesh & Shaked, 1997):

**Lemma 2.1.** For each \( \varepsilon > 0 \), if there exist \( X_e, Y_e \) and \( K_e \) that satisfy (9)–(12) with the following properties:

(a) \( X_e \geq 0, \quad C_2 Y_e C_2 > 0 \) and \( A_e + \gamma^{-2} B_{1e} B_{1e}^T X_e - B_{2e} R^{-1} B_{2e}^T X_e v_e \) is stable, then (2) holds and the gain \( K_e \) achieves (5).

Note that for \( v_{e\perp} = 0 \) (this corresponds to the state-feedback case) (12) and (9) are decoupled Riccati and Lyapunov equations coupled via projection that are highly nonlinear in \( X_e, Y_e \) and \( K_e \). For each \( \varepsilon \), this system has been successfully solved in (Yaesh & Shaked, 1997) by applying the homotopy method (Richter, Hodel & Pruet, 1993).

3. Main results

3.1. System transformation (diagonalization of \( C_2 \))

Since \( C_2 \) is of full row rank, we assume, without loss of generality, that \( C_2 \) possesses one of the following three forms:

(i) \( C_2 = [C_{1i} \ 0] \), where \( C_{2i}, i = 1, 2 \) are of full row rank,
(ii) \( C_2 = [C_3 \ C_{22}] \), where \( C_{22} \) is of full row rank,
(iii) \( C_2 = [C_{21} \ 0] \), where \( C_{21} \) is of full row rank.
Cases (ii) and (iii) are degenerate cases of (i), where (ii) corresponds to \( \mathcal{C}_{21} \) with zero number of rows, and (iii) corresponds to \( \mathcal{C}_{22} \) with zero number of rows. Cases (ii) and (iii) physically mean that linear independent combinations of the fast or the slow variables are, respectively, observed.

Let \( \mathcal{C}_2 \) be in the form of (i) and let \( L \) be a matrix that transforms \( \mathcal{C}_2 \) to block-diagonal form \( \hat{\mathcal{C}}_2 \) as follows:

\[
\hat{\mathcal{C}}_2 = \begin{bmatrix} \mathcal{C}_{21} & 0 \\ 0 & \mathcal{C}_{22} \end{bmatrix} = \begin{bmatrix} \mathcal{C}_{21} & 0 \\ \mathcal{C}_3 & \mathcal{C}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ L & 1 \end{bmatrix}
\]  

(13)

We introduce the following nonsingular transformation of the state variables:

\[
\hat{x}_1 = x_1, \quad \hat{x}_2 = x_2 - Lx_1, \quad \hat{x} = \text{col}(\hat{x}_1, \hat{x}_2).
\]  

(14)

Then, from (1a)-(1c), (14) and (13), we obtain the following system for \( \hat{x} \):

\[
\dot{\hat{x}} = \hat{A}_i\hat{x} + \hat{B}_{1i}w + \hat{B}_{2i}u,
\]  

(15a)

\[
y = \hat{C}_2\hat{x},
\]  

(15b)

\[
z = \hat{C}_1\hat{x} + D_{1i}u,
\]  

(15c)

where \( \hat{A}_{1i} = A_{1i} + A_{12}L, \) \( i = 1, 2 \) and

\[
\hat{A}_i = \begin{bmatrix} A_{11} & A_{12} \\ -A^{-1}A_{21} - LA_{11} & -A^{-1}A_{22} - LA_{12} \end{bmatrix}.
\]  

(15d)

\[
\hat{B}_{1i} = \begin{bmatrix} B_{1i} \\ -B_{12} - LB_{11} \end{bmatrix}.
\]  

(15e)

\[
\hat{C}_i = \begin{bmatrix} \hat{C}_{1i} & \hat{C}_{12i} \end{bmatrix} = C_i \begin{bmatrix} I & 0 \\ L & 1 \end{bmatrix}.
\]  

Since the closed-loop transfer-function matrix \( T_{zw} \) of the new system of (15a)-(15c) is identical to the one defined by (3), the robust optimal control law \( u = Ky \) for the \( H_2 \) minimum entropy control of (15a)-(15c) is also optimal for the original system. For the system of (15a)-(15c) the optimal controller is derived by solving the coupled equations of (12), (9) and (10), where

\[
\hat{A}_i = \hat{A}_i, \quad \hat{B}_{1i} = \hat{B}_{1i}, \quad \hat{C}_i = \hat{C}_i, \quad i = 1, 2.
\]  

(16)

3.2. Generalized \( \epsilon \)-Independent Riccati and Lyapunov Equations

Denote

\[
E = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix},
\]  

(17a)

\[
\hat{A} = \begin{bmatrix} I & 0 \\ L & 1 \end{bmatrix},
\]  

(17b)

\[
\hat{A} = [\hat{A} + B_{2i}K\hat{C}_i].
\]  

(17c)

\[
\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}
\]  

(17d)

\[
\hat{C} = [\hat{C}_i + D_{1i}KC_i] = [\hat{C}_1 \quad \hat{C}_2].
\]  

(17e)

With Riccati and Lyapunov equations (7) and (9) with (16) we associate (similarly to Tan et al., 1998) the following generalized Riccati and Lyapunov equations:

\[
\hat{A}'X + X'\hat{A} + \gamma^{-2}X'B_1B_1'X + \hat{C}'\hat{C} = 0,
\]  

(18)

\[
(\hat{A} + B_2KC_2 + \gamma^{-2}B_1B_1')Y
+ Y'(\hat{A} + \hat{C}_1K'\hat{C}_2 + \gamma^{-2}B_1B_1') + B_1B_1' = 0,
\]  

(19)

where

\[
X = \begin{bmatrix} X_1(0) & 0 \\ X_2(0) & X_3(0) \end{bmatrix},
\]  

(20a)

\[
Y = \begin{bmatrix} Y_1(0) & 0 \\ Y_2(0) & Y_3(0) \end{bmatrix},
\]  

(20b)

\[
K = [K_1(0) \quad K_2(0)]',
\]  

(20c)

\[
X_1(0) = X_1(0) \geq 0,
\]  

(20d)

\[
X_2(0) = X_3(0) \geq 0,
\]  

(20e)

\[
Y_1(0) = Y_1(0),
\]  

(20f)

\[
Y_3(0) = Y_3(0).
\]  

(20g)

Consider

\[
K = - R^{-1}B_2XY\hat{C}_2(\hat{C}_2'Y\hat{C}_2)^{-1}.
\]  

(21)

Denoting

\[
v = Y\hat{C}_2(\hat{C}_2'Y\hat{C}_2)^{-1}\hat{C}_2, \quad v_\perp = I - v.
\]  

(22)

we find that \( v_\perp = v \) and (18) can be written in the form

\[
\hat{A}'X + X'\hat{A} + \gamma^{-2}X'B_1B_1'X + \hat{C}'\hat{C}_1
- X'B_2R^{-1}B_2'X + v_\perp X'B_2R^{-1}B_2'Xv_\perp = 0.
\]  

(23)

3.3. Fast and slow subsystems

Expanding (21) and noting that

\[
(\hat{C}_2'Y\hat{C}_2)^{-1} = \begin{bmatrix} (\hat{C}_{21}'Y_1(0)\hat{C}_{21})^{-1} & 0 \\ M' & (\hat{C}_{22}'Y_3(0)\hat{C}_{22})^{-1} \end{bmatrix},
\]  

(24a)

\[
M = -(\hat{C}_{21}'Y_1(0)\hat{C}_{21})^{-1}\hat{C}_{21}Y_3(0)\hat{C}_{21}(\hat{C}_{22}'Y_3(0)\hat{C}_{22})^{-1},
\]  

(24b)

we find

\[
K_1(0) = -[R^{-1}[B_{12}'X_1(0)Y_1(0) + B_{22}'X_2(0)Y_2(0)]
+ X_3(0)Y_3(0)] + K_2(0)\hat{C}_{22}(\hat{C}_{22}'Y_3(0)\hat{C}_{22})^{-1},
\]  

(25)

\[
K_2(0) = -R^{-1}B_{22}'X_3(0)Y_3(0)\hat{C}_{22}(\hat{C}_{22}'Y_3(0)\hat{C}_{22})^{-1}.
\]  

(26)
From (18) and (19) we obtain the following fast equations (Tan et al., 1998):
\[ A_{22}^X X_3^X + X_3^X A_{22} + \bar{C}_2 \bar{C}_2 + \gamma^{-2} X_3^X B_{12} B_{12}^\prime X_3^X = 0, \]
\[ F_{22}^X Y_3^X + Y_3^X F_{22} + B_{12} B_{12}^\prime = 0. \]
where
\[ F_{22} = A_{22} + \gamma^{-2} B_{12} B_{12} X_3^X + B_{22} K_{12}^X \bar{C}_{22}. \]

We write (27a) in the form
\[ A_{22}^X X_3^X + X_3^X A_{22} + \gamma^{-2} X_3^X B_{12} B_{12}^\prime X_3^X \]
\[ - X_3^X B_{22} R^{-1} B_{22}^\prime X_3^X + v_f \gamma^{-1} X_3^X B_{22} R^{-1} B_{22}^\prime X_3^X v_f + \bar{C}_{12} \bar{C}_{12} = 0, \]
\[ v_f = Y_3^X \bar{C}_{22} (C_2 X_3^X \bar{C}_{22})^{-1} \bar{C}_{22}, \]
\[ v_f = I - v_f. \]

Assume that

A1. The system of coupled equations (19), (21)–(23) has a solution \( X, Y, K \) of (20) with the following properties:
\( b \) \( X_3^X \geq 0, \bar{C}_2 \bar{C}_2^\prime X_{22} > 0 \) and \( F_{22} \), where \( K_{12}^X \bar{C}_{22} = - R^{-1} B_{22} X_3^X v_f \), is stable,
\( c \) \( X_1^X \geq 0, \bar{C}_2 \bar{C}_2 \bar{C}_2^\prime \bar{C}_2 \geq 0 \) and \( E, \bar{A} + \gamma^{-2} B_{12} B_{12}^\prime X - B_{22} R^{-1} B_{22}^\prime X v_{f} \) is stable.

Remark 3.1. The second property of (b) implies that \( B_{12} \neq 0 \) (otherwise \( Y_3^X = 0 \)).

The system of coupled equations (26), (27b) and (28) provides a solution to the \( H_{\infty} \) minimum entropy static output-feedback control problem for the fast subsystem:
\[ \hat{x}_2 = A_{22} \hat{x}_2 + B_{12} w + B_{22} u, \quad y_2 = \bar{C}_{22} \hat{x}_2, \]
\[ z_2 = \bar{C}_{12} \hat{x}_2 + D_{12} u. \]

The optimal controller for (29) \( u_f = K_{12}^X \bar{C}_{22} \hat{x}_2 \) leads to the stable matrix \( \bar{A}_{22} \) and to the minimum value of entropy
\[ \delta_f = Tr \{ B_{12}^X X_3^X B_{12} \}. \]

The slow subsystem is the descriptor one
\[ E \hat{x} = \bar{A} \hat{x} + B_{12} \hat{w} + B_{22} u, \quad y = \bar{C}_1 \hat{x}, \]
\[ z = \bar{C}_2 \hat{x} + D_{12} u. \]

It will be shown in Section 3.5 that the system of generalized Riccati and Lyapunov equations coupled by projection (19), (21)–(23) provide the optimal solution to (31), if \( \delta_f = 0 \).

3.4. Robust optimal controller design

We now are in a position to state our main result — the design of \( \varepsilon \)-independent robust optimal controller gain \( K^* \):

**Theorem 3.1.** Given \( \gamma > 0 \), for \( C_2 \) of the forms (i)–(iii) we have correspondingly the following results:
(i) Under A1 \( K^* = K \) is the robust optimal gain. The robust optimal controller \( u^* = K^* C_2 x \) (\( u^* = K^* C_2 \hat{x} \)) leads (1) (15) for all small enough values of \( \varepsilon \), to the \( H_{\infty} \) norm bound of \( \gamma \) and to the following minimum value of entropy:
\[ \delta^* = \varepsilon^{-1} \delta_f + Tr \{ B_{12}^X X_3^X B_{12} + 2 B_{12}^X X_3^X B_{12} \}
+ B_{12}^X X_3^X B_{12} \} + O(\varepsilon), \]
where \( \delta_f \) is defined in (30), and \( X_3^{(1)} \) is a solution of the following Lyapunov equation:
\[ F_{22}^X Y_3^{(1)} + X_3^{(1)} F_{22} + \bar{A}_{12} Y_3^{(1)} + Y_3^{(1)} \bar{A}_{12}
+ \gamma^{-2} X_3^X B_{12} B_{12}^X X_3^{(1)} + \gamma^{-2} X_3^X B_{12} B_{12}^X X_3^{(1)} \]
\[ + B_{12}^X X_3^X B_{12} = 0. \]

(ii) Assume that there exists a solution to the fast equations (26), (27b) and (28), with the properties of (b), and there exists a solution to the generalized Riccati equation (18), where \( K = K_{12}^0 \) of (26), such that \( E, \bar{A} + \gamma^{-2} B_{12} B_{12}^X X \) is stable. Then, \( K^* = K_{12}^0 \) and the robust optimal controller \( u^* = K_{12}^0 \bar{C}_2 x + \bar{C}_2 x \) (\( u^* = K_{12}^0 \bar{C}_2 \hat{x} + \bar{C}_2 \hat{x} \)) leads (1) (15), for all small enough \( \varepsilon \), to the \( H_{\infty} \) norm bound of \( \gamma \) and to the minimum entropy of entropy given by (32).

(iii) Assume that (27a) with \( \bar{A}_{22} = A_{22} \) has a solution \( X_3^{(0)} \geq 0 \) such that \( A_{22} + \gamma^{-2} B_{12} B_{12} X_3^{(0)} \) is stable and assume that there exists a solution to the Eqs. (19), (21)–(23) with the properties of (c). Then, \( K^* = K_{12}^{X(0)} \) and the slow controller \( u^* = K_{12}^{X(0)} \bar{C}_2 X_3 \) leads (1) to the \( H_{\infty} \) norm bound of \( \gamma \) and to the minimum entropy of entropy (32).

**Proof.** Similarly to Yaesh and Shaked (1997), minimizing (8) with respect to \( K \) can be performed by forming the following Lagrangian:
\[ \mathcal{L}(K, X, Y_e) = Tr \{ B_{12}^X X_3^X B_{12} + [\bar{A} \bar{x}_e + X_e \bar{A}], \]
\[ + X_e \bar{B}_{12} \bar{B}_{12} \bar{x}_e + \bar{C}_1 \bar{C}_1 + \bar{C}_2 \bar{K} \bar{K} \bar{C}_2 \bar{C}_2 \} Y_e. \]

The stationarity of (8) with respect to \( X_e \) requires that \( \partial \mathcal{L}/\partial X_e = 0 \) that implies (9). As in Yaesh and Shaked (1997) we obtain that the part of \( \mathcal{L} \) that depends on \( K \) is given by \( Tr \{ T_3 \} \), where
\[ T_3 = [ K(\bar{C}_2 Y_e \bar{C}_2)^{1/2} + R^{-1} B_{12} X_e \bar{C}_2 (\bar{C}_2 Y_e \bar{C}_2)^{-1/2} \]
\[ + \bar{C}_2 \bar{K} \bar{K} \bar{C}_2 \bar{C}_2 \} Y_e. \]

Note that \( T_3 \geq 0 \). Since the generalized Riccati equations (18) has a stabilizing solutions (in the sense of Tan
et al., 1998), then by implicit function theorem the full-order Riccati equation (7) with $K = [K_1^{(0)} K_2^{(0)}]$ has a stabilizing solution and by standard arguments

$$X_{\varepsilon} = \begin{bmatrix} X_1^{(0)}(\varepsilon) + O(\varepsilon) & \varepsilon X_2^{(0)}(\varepsilon) + O(\varepsilon^2) \\ \varepsilon X_2^{(0)}(\varepsilon) + O(\varepsilon^2) & X_3^{(0)}(\varepsilon) + \varepsilon^2 X_4^{(0)}(\varepsilon) + O(\varepsilon^3) \end{bmatrix}$$  \hfill (35a)

$$Y_{\varepsilon} = \begin{bmatrix} Y_1^{(0)}(\varepsilon) + O(\varepsilon) \\ Y_2^{(0)}(\varepsilon) + O(\varepsilon) \end{bmatrix} e^{-1} Y_3^{(0)}(\varepsilon) + O(1)$$  \hfill (35b)

Substituting (35b) in $C_2 Y_{\varepsilon} C_2^{-1}$ we obtain that for all small enough $\varepsilon$ the matrix $C_2 Y_{\varepsilon} C_2^{-1}$ is invertible and

$$(C_2 Y_{\varepsilon} C_2^{-1})^{-1} = (C_{21} Y_1^{(0)} C_{21})^{-1} + O(\varepsilon)$$

and

$$(C_2 Y_{\varepsilon} C_2^{-1})^{-1} = (C_{21} Y_1^{(0)} C_{21})^{-1} + O(\varepsilon),$$

correspondingly. We obtain that $K^* = K$ of (21) leads (15) (and thus (1)) to $H_{\infty}$-norm bound of $\gamma$ and minimizes to $O(\varepsilon)$ the value of $Tr[T_3^{(0)}]$, for all small enough $\varepsilon$. Expanding (8) in the powers of $\varepsilon$ and applying (35) we obtain (32).

**Remark 3.2.** For $\nu_{\perp} = 0$ (23) and (19) are the well-known generalized Riccati and Lyapunov equations (for their solution see, e.g., Tan et al., 1998). For $\nu_{\perp} \neq 0$ the system of (19), (21)–(23) constitutes a set of highly nonlinear coupled equations with respect to $X, Y$ and $K$. One way of solving this system is to use the homotopy method (Richter et al., 1993).

**Remark 3.3.** It follows from the proof of the theorem that $K^*$ defined by (20c), (25) and (26) is the formal $O(\varepsilon)$-approximation to the optimal gain $K_\varepsilon$ of (10).

**Remark 3.4.** In the case (ii) (in the case (iii)), where the linear independent combinations of the fast (slow) variables are observed, the robust optimal gain is purely fast (slow). Note that unlike Kokotovic, Khalil & O’Reilly (1986, Chapter 3), our static output-feedback control in the case, which is based on the slow model, is robust, in the sense that it cannot destabilize the original system. This is due to the structure (iii) of $C_2$ with $C_2 x = \hat{C}_{21} x_1$.

**Remark 3.5.** Note that if $\varepsilon f \neq 0$ the value of $\varepsilon^* f$ approaches infinity for $\varepsilon \to 0$ (see (32)), but still for all small values of $\varepsilon$ the controller gain $K^*$ minimizes the value of entropy among all static output-feedback controllers satisfying (2). A well-posed problem is obtained by assuming $\varepsilon f = 0$ or by scaling. We treat these cases in the next subsections.

### 3.5. Descriptor system approach to a well-posed problem

In this subsection we assume that $\varepsilon f$ given by (30) equals zero. A zero $\varepsilon f$ is encountered, for example when $B_{12} = 0$ (no disturbances in the fast equation) or when $C_{12} = 0$ with stable matrix $A_{22}$ (no fast variables in the objective vector $z$).

**Lemma 3.1.** If $\varepsilon f = 0$, then the following relations hold:

$$B_{12} X_3^{(0)} = 0,$$  \hfill (37a)

$$\tilde{C}_2 \tilde{A}_{22}^{-1} B_{12} = 0,$$  \hfill (37b)

$$\tilde{C}_2 B_{12} = 0,$$  \hfill (37c)

$$X_1^{(0)} B_{12} = -X_1^{(0)} \tilde{A}_{12} \tilde{A}_{22}^{-1} B_{12},$$  \hfill (37d)

$$X_1^{(0)} \tilde{A}_{22}^{-1} B_{12} = 0,$$  \hfill (37e)

and $X_1^{(0)}$ satisfies the following Riccati equation:

$$A_s X_1^{(0)} + X_1^{(0)} A_s + \gamma^{-2} X_1^{(0)} B_{12} B_{12}^* X_1^{(0)} + C_s C_s = 0,$$  \hfill (38)

where

$$A_s = \tilde{A}_{11} - \tilde{A}_{12} \tilde{A}_{22}^{-1} \tilde{A}_{21}, \quad B_{1s} = B_{11} - \tilde{A}_{12} \tilde{A}_{22}^{-1} B_{12},$$

$$C_s = \tilde{C}_1 - \tilde{C}_2 \tilde{A}_{22}^{-1} \tilde{A}_{21}. \hfill (39)$$

**Proof.** A zero $\varepsilon f$ implies (37a). It also means that the transfer function matrix of the fast subsystem (29) with $u = K_3^{(0)} C_{22} x_2$, is equal to zero and, hence, (37b) holds. Matrix $X_3^{(0)}$ satisfies the following equation (see e.g. Tan et al., 1998):

$$\tilde{A}_{21} X_3^{(0)} + X_3^{(0)} \tilde{A}_{12} + X_1^{(0)} \tilde{A}_{22} + \tilde{C}_1 \tilde{C}_2$$

$$+ \gamma^{-2} X_1^{(0)} B_{12} B_{12}^* X_3^{(0)} + \gamma^{-2} X_3^{(0)} B_{12}^* B_{12} X_3^{(0)} = 0.$$  \hfill (40)

Eqs. (37d) and (37e) follow from (40) and (27) multiplying, from the right, by $\tilde{A}_{22}^{-1} B_{12}$, while (37c) follows from (27a) multiplying it, from the left, by $B_{12}$ and, from the right, by $B_{12}$. Eq. (38) follows from (2.5)–(2.8) of Dragan (1993) and (37b). \qed
We consider the descriptor system (31) that corresponds to (15). The transfer-function matrix $T_d$ of (31), with $u = KC_2\dot{x}$, is given by

$$T_d = (\bar{C}_1 + D_{12}KC_2)[sE - \bar{A} - B_2KC_2]^{-1}B_1$$

$$= \bar{C}(sE - \bar{A})^{-1}B_1.$$  \hspace{1cm} (41)

We want to choose of all the $K$ that satisfy both $\|T_d\|_\infty < \gamma$ and $\gamma_f = 0$, the one that minimizes the entropy of $T_d$ given by (4), where $T_{2w} = T_d$.

**Lemma 3.2.** For the descriptor system of (31), with $u = B_2KC_2\dot{x}$ and $\gamma_f = 0$, where $\gamma_f$ is given by (30), the following holds:

(i) The transfer-function matrix is given by

$$T_d = C_d(sI - A_d)^{-1}B_{11}.$$  \hspace{1cm} (42)

(ii) $A_d$ is stable and $\|T_d\|_\infty < \gamma$ iff there exists a solution $X_1^{(0)} \geq 0$ to the Riccati equation (38) such that $A_d + \gamma^-2B_{11}B_{11}X_1^{(0)}$ is stable.

(iii) The entropy of the system is given by

$$\delta_d = \operatorname{Tr}(B_{11}X_1^{(0)}B_{11}).$$  \hspace{1cm} (43)

This entropy $\delta_d$ is $O(\epsilon)$-close to the entropy of (15), where $u = KC_2\dot{x}$.

**Proof.** Denote

$$N_1 = \begin{bmatrix} I & -\bar{A}_{12}\bar{A}_{22}^{-1} \\ 0 & \bar{A}_{22}^{-1} \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} I & 0 \\ -\bar{A}_{22}^{-1}\bar{A}_{21} & I \end{bmatrix}$$

Then,

$$N_1\bar{A}N_2 = \begin{bmatrix} A_d & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad N_1EN_2 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

and thus

$$T_d = C_d(sI - A_d)^{-1}B_{11} - \bar{C}_2\bar{A}_{22}^{-1}B_{12}. \hspace{1cm} (44)$$

Relation (42) follows from (44) and (37b).

Item (i) implies (ii) and (43). It follows from (33) using $F_{22} = \bar{A}_{22}$, that $X_1^{(0)} = -X_2^{(0)}\bar{A}_{12}\bar{A}_{22}^{-2}$. The $O(\epsilon)$-closedness of $\delta_d$ to the entropy of (15) follows from (32), (43) and (37d). \hfill $\Box$

**Remark 3.6.** Note that if $\bar{C}_2\bar{A}_{22}^{-1}B_{12} \neq 0$ (and thus $\delta_f \neq 0$), then (44) implies $\delta_d = \infty$ since $T_d(\infty) \neq 0$ (Mustafa and Glover, 1990).

We obtain the following from the last lemma and Tan et al. (1998):

**Theorem 3.2.** The controller $u = KC_2\dot{x}$ that leads to $\delta_f = 0$ achieves a $H_\infty$-norm bound of $\gamma$ and minimizes the entropy of (31) iff it is the robust optimal controller for the singularly perturbed system (15). This controller is given by Theorem 3.1.

**Remark 3.7.** If $B_{12} = 0$, then only (iii) of Theorem 3.1 may hold (here $Y_i^{(0)} = 0$) and there is a purely-slow gain. If $C_{12} = 0$ and $A_{22}$ is stable, then from (i) we obtain $K_i^{(0)} = 0$ and thus also in this case the gain is purely slow.

### 3.6. Scaled well-posed problems

A well-posed problem with a finite entropy for $\epsilon \to 0$ can be also obtained by scaling the matrices of the system:

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + \epsilon^3B_{11}w + B_{21}u,$$  \hspace{1cm} (45a)

$$\epsilon\dot{x}_2 = \epsilon^\delta A_{21}x_1 + A_{22}x_2 + \epsilon^{1/2}B_{12}w + \epsilon^\delta B_{22}u.$$  \hspace{1cm} (45b)

The parameters $\alpha, \beta, \delta$ represent the relative size of the small parameters within the system, with respect to the small time constants of the fast subsystem. We multiply $B_{12}$ by $\epsilon^{1/2}$ to obtain a finite and positive value of entropy for $\epsilon \to 0$. For information on scaled LQG problem refer to Kokotovic, Khalil and Reilly (1986) and Saska and Basar (1986). The results below are obtained by using arguments similar to Theorem 3.1.

In the case of ‘uniform scaling’, when $\alpha = \frac{1}{2}, \beta = \delta = 0$ and $\gamma$ are non-scaled, we choose the $H_\infty$-norm bound of $\epsilon^{1/2}\gamma$. For each $\epsilon$ we obtain the same equations (7), (9) and (10) as in the case without scaling. Then the optimal robust controller of Theorem 3.1 achieves the $H_\infty$-norm bound of $\epsilon^{1/2}\gamma$ and the minimum value of entropy $\delta_f + O(\epsilon)$ for all small enough $\epsilon$.

Given $\gamma > 0$, consider next the case of $\alpha = 0$. The entropy of (45) with $u = KC_2\dot{x}$ satisfies relation

$$\lim_{\epsilon \to 0} \delta = \operatorname{Tr}(B_{11}X_1^{(0)}B_{11} + B_{12}X_2^{(0)}B_{12}).$$

A solution to Lyapunov equation (9) has the following form:

$$Y_\epsilon = \begin{bmatrix} Y_1 & Y_2 \\ Y_2' & Y_3 \end{bmatrix}$$

where since $B_{11}B_{12} = 0$ and $B_{12}B_{12} = 0$

$$\tilde{A}_{11}Y_1 + \tilde{A}_{12}Y_2 + Y_1\tilde{A}_{11} + Y_2\tilde{A}_{12} + B_{11}B_{11} = 0,$$

$$Y_2 = -Y_1(\tilde{A}_{21}\tilde{A}_{22}^{-1}) + O(\epsilon^{1/2}),$$

$$\tilde{A}_{21}Y_2 + \tilde{A}_{22}Y_3 + Y_2\tilde{A}_{22} + Y_3\tilde{A}_{22} = 0$$

with

$$\tilde{A}_{11} = A_{11} + A_{12}L + B_{21}K_1^{(0)}C_{21},$$

$$\tilde{A}_{12} = A_{12} + \epsilon^3B_{21}K_2^{(0)}C_{22},$$

$$\tilde{A}_{21} = \epsilon^\delta A_{21} + A_{22}L + \epsilon^\delta B_{22}K_1^{(0)}C_{21},$$

$$\tilde{A}_{22} = A_{22} + \epsilon^\delta B_{22}K_2^{(0)}C_{22}.$$
decomposition of the problem. Thus, in case (ii)

\[ K^* = K_2^{(0)} = - R^{-1} \left[ B_2' X_1^{(0)} + B_2' X_2^{(0)} \right] Y_2^{(0)} + B_2' X_3^{(0)} Y_3^{(0)} C_22 (C_22 Y_2^{(0)} C_22)^{-1}, \]

where \( Y_3^{(0)} \) satisfy (46a)–(46c). In case (iii) the optimal controller is the purely slow one of Theorem 3.1, where \( Y_1^{(0)} \) and \( Y_2^{(0)} \) satisfy (46a) and (46b) with \( \varepsilon = 0 \).

If \( \beta = \frac{1}{2}, \delta = \frac{1}{2} \) and \( C_3 = 0 \), then both \( X_3^{(0)} \) and \( X_1^{(0)} \) do not depend on \( K_0^{(0)} \). Assuming that \( A_{22} \) is stable we find that the robust optimal controller gain is purely slow \( K^* = [K_1^{(0)}, 0] \), where \( K_1^{(0)} \) is the gain of the minimizing controller \( u = K_1^{(0)} y \), for the slow problem

\[
\dot{x} = A_{11} x_1 + B_{11} w + B_{21} u, \quad y = C_{21} x_1, \\
z = C_{11} x_1 + D_{12} u.
\]

3.7. Example

Consider (1a)–(1c) with the following matrices:

\[
A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
C_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

and \( D_{12} = \begin{bmatrix} 0 & 1 \end{bmatrix} \). This is the case of (i), where both \( C_{21} \) and \( C_{32} \) are full-rank, with singular \( A_{22} \). From (13) we find that \( L = \begin{bmatrix} 0 & -1 \end{bmatrix} \). The fast subproblem in this example is a state-feedback-type \( H_{\infty} \) control. Choosing \( \gamma = 8.5 \), which is close to the minimum possible value of \( \gamma \) for small values of \( \varepsilon \), we obtain from (19), (21)–(23)

\[ K^* = \begin{bmatrix} -1.0688 & -0.7153 \end{bmatrix}. \]

Applying robust optimal controller \( u = K^* y \), to (1a)–(1e) and choosing the values of \( \varepsilon \) that are given in Table 1, we find that (7) has a nonnegative stabilizing solution and thus (2) is satisfied. Using the solution of (7) we compute by (8) the resulting values of entropy \( \delta^* \) and bring them in Table 1.

For the same value of \( \gamma = 8.5 \), and for each value of \( \varepsilon \) under consideration, we obtain the values of the optimal \( \varepsilon \)-dependent gain \( K_\varepsilon \) by solving the full-order Eqs. (9)–(12). We see that for small \( \varepsilon \) the resulting \( K_\varepsilon = [K_1, K_2] \) is close to \( K^* \) (see Table 1). We also compute the corresponding values of \( \delta_\varepsilon \) using (8). It is seen from Table 1 that \( \delta_\varepsilon \) is close to \( \delta^* \).

In this example \( \delta_f = 9.619 \neq 0 \) and therefore \( \delta^* \) and \( \delta_\varepsilon \) are unbounded for small \( \varepsilon \). In order to obtain a finite entropy for \( \varepsilon \to 0 \) we apply a uniform scaling on the system, where instead of \( B_1 \) we take \( \sqrt{\varepsilon} B_1 \). Then choosing an \( H_{\infty} \)-norm bound of \( \sqrt{\varepsilon} 8.5 \), we find the same values of \( K^* \) and \( K_\varepsilon \). The resulting values of entropy are those given in the Table 1, multiplied now by \( \varepsilon \). We see that the resulting values of entropy are bounded for small values of \( \varepsilon \) and tend to \( \delta_f = 9.619 \).

### Table 1

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<th>( \delta_\varepsilon )</th>
<th>( K_1 )</th>
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4. Conclusions

In the present note we have designed \( \varepsilon \)-independent robust optimal static output-feedback controllers for non-standard singularly perturbed systems that satisfy given \( H_{\infty} \)-norm performance bounds and also minimize the entropy at \( s = \infty \). Our method is based on solving a generalized Riccati equation coupled via a projection with a generalized Lyapunov equation. Our solution yields a minimum value of the entropy which becomes unbounded when \( \varepsilon \) tends to zero. A well-posed problem with a finite value of entropy for \( \varepsilon \to 0 \) is obtained either when the entropy of the fast subproblem is zero or when the matrices of the system are scaled.

### References


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