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Brief Paper

A descriptor system approach to nonlinear singularly perturbed optimal control problem☆

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Abstract

This paper considers the infinite horizon nonlinear quadratic optimal control problem for a singularly perturbed system which is nonlinear in both, the slow and the fast variables. The relationship between this problem and the analogous one for a descriptor system is investigated. Parameter-independent controllers are constructed that solve the problem for the descriptor system and lead the full-order system to the near-optimal performance. Estimates on the closeness of the cost under near-optimal controllers to the optimal one are obtained. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

By *standard* singularly perturbed problem, we mean such a problem that the algebraic equation of the limit problem (i.e. the problem, where $\varepsilon = 0$) is solvable with respect to the fast variable. Optimal control of a class of standard nonlinear singularly perturbed system, being nonlinear only on the slow variable, has been studied by Chow and Kokotovic (1978, 1981), where a two-stage procedure for design of ε-independent composite controller has been suggested. In the case of general standard problem, nonlinear in both the fast and the slow variables, a composite controller has been designed by Saberi and Khalil (1985) and the limit of the optimal cost as $\varepsilon \to 0$ has been found by Bensoussan (1987).

A descriptor system approach has been introduced by Wang, Shi, and Zhang (1988) for the case of *non-standard* LQ problem. It has been shown that optimal (ε-independent) regulators for the descriptor system are near-optimal regulators for the corresponding singularly perturbed system. For the full-order system the values of the cost under these regulators are $O(\varepsilon)$-close to the optimal one. Xu, Mukaidani, and Mizukami (1997) have shown that only the composite controller, which is $O(\varepsilon)$-close to the optimal regulator, achieves $O(\varepsilon^2)$ near-optimal cost. In the present paper we extend the results of Wang et al. (1988) and Xu et al. (1997) to the non-standard problem, which is nonlinear in both, the slow and the fast variables.

Our results are based on the geometric approach of Van der Schaft (1991), Isidori and Astolfi (1992) and Byrnes (1998), which relates Hamilton–Jacobi equations with special invariant manifolds of Hamiltonian systems. We apply results of Fridman (2000) on the existence of the solution to Hamilton–Jacobi equation and its asymptotic approximation. Proofs of the theorems are given in appendix.

2. Problem formulation

Consider the optimal control problem for the system

$$E_\varepsilon \dot{x} = F(x) + B(x)u, \quad E_\varepsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}, \tag{1}$$

with respect to the functional

$$J = \int_0^\infty [k'(x)k(x) + u'R(x)u]\,\mathrm{d}t, \tag{2}$$

where $x = \mathrm{col}\{x_1, x_2\}$, $x_1(t) \in \mathbf{R}^{n_1}$ and $x_2(t) \in \mathbf{R}^{n_2}$ are the state vectors, $u(t) \in \mathbf{R}^m$ is the control input, and $\varepsilon > 0$ is

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a small parameter. Prime denotes the transposition of a matrix. The functions

$$F(x)=\begin{bmatrix}F_1(x_1,x_2)\\F_2(x_1,x_2)\end{bmatrix},\quad B(x)=\begin{bmatrix}B_1(x_1,x_2)\\B_2(x_1,x_2)\end{bmatrix},$$

$R(x)$ and $k(x)$ are differentiable with respect to x a sufficient number of times. We assume also that $F(0)=0, k(0)=0$ and $R(x)=R'(x)>0$.

System (1)–(2) has a *non-standard* singularly perturbed form in the sense that we do not require that the algebraic equation

$$F_2(x_1,x_2)+B_2(x_1,x_2)u=0 \tag{3}$$

has a solution of the form $x_2=h(x_1,u)$. In the *standard* case, such an assumption is a crucial one (see e.g. Chow et al. 1978, 1981; Saberi et al., 1985; Kokotovic, Khalil, & O'Reilly, 1986; Bensoussan, 1987; Pan & Basar, 1996).

We are looking for a nonlinear state feedback

$$u=\beta(x),\quad \beta(0)=0, \tag{4}$$

that locally minimizes the cost (2). Consider the Hamiltonian function

$$\mathscr{H}(x,p)=p'F(x)-\tfrac{1}{2}p'S(x)p+\tfrac{1}{2}k'(x)k(x),\quad S=BR^{-1}B', \tag{5}$$

where $p=\mathrm{col}\{p_1,p_2\}$, p_1 and εp_2 play the role of the costate variables. The corresponding Hamiltonian system has the form

$$\dot{x}_1=f_1(x_1,p_1,x_2,p_2), \tag{6a}$$

$$\dot{p}_1=f_2(x_1,p_1,x_2,p_2), \tag{6b}$$

$$\varepsilon\dot{x}_2=f_3(x_1,p_1,x_2,p_2), \tag{6c}$$

$$\varepsilon\dot{p}_2=f_4(x_1,p_1,x_2,p_2), \tag{6d}$$

where

$$f_1=\left(\frac{\partial \mathbf{H}}{\partial p_1}\right)',\quad f_2=-\left(\frac{\partial \mathbf{H}}{\partial x_1}\right)',\quad f_3=\left(\frac{\partial \mathbf{H}}{\partial p_2}\right)',$$

$$f_4=-\left(\frac{\partial \mathbf{H}}{\partial x_2}\right)'.$$

Denote by $V_x=[V_{x_1}\ \ V_{x_2}]$ the Jacobian matrix of V. For each $\varepsilon>0$, the problem is locally solvable on $\Omega\subset\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}$, $0\in\mathbf{\Omega}$ if there exists a C^2 non-negative solution $V:\Omega\to\mathbf{R}$ to the Hamilton–Jacobi (HJ) partial differential equation

$$V_xE_\varepsilon^{-1}F(x)-\tfrac{1}{2}V_xE_\varepsilon^{-1}S(x)E_\varepsilon^{-1}V_x'+\tfrac{1}{2}k'(x)k(x)=0,\quad V(0)=0, \tag{7}$$

with the property that the system

$$E_\varepsilon\dot{x}=F(x)-S(x)E_\varepsilon^{-1}V_x' \tag{8}$$

is asymptotically stable (Byrnes, 1998). The latter is equivalent to the existence of the invariant manifold of (6)

$$p_1=Z_1(x_1,x_2)=V'_{x_1},\quad p_2=Z_2(x_1,x_2)=\varepsilon^{-1}V'_{x_2} \tag{9}$$

with asymptotically stable flow (8), such that $V\geq 0$, $V(0)=0$ (that implies $V_x(0)=0$). The optimal controller that solves the problem is given by

$$u=-R^{-1}B'E_\varepsilon^{-1}V'_x=-R^{-1}B'_1Z_1-R^{-1}B'_2Z_2. \tag{10}$$

We shall find ε-independent controllers that near-optimally solve the local problem on some ε-independent neighborhood Ω_0 for all small enough ε.

3. Main results

3.1. Composite controller design

Consider the linearization of (1) at $x=0$:

$$E_\varepsilon\dot{x}=Ax+B_0u \tag{11}$$

with the quadratic functional

$$J=\int_0^\infty[x'C'Cx+u'R(0)u]\,\mathrm{d}t, \tag{12}$$

where

$$A=\begin{bmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{bmatrix},\quad B=\begin{bmatrix}B_{10}\\B_{20}\end{bmatrix},\quad C=[C_1\quad C_2],$$

$$A_{ij}=\frac{\partial F_i}{\partial x_j}(0{,}0),\quad B_{i0}=B_i(0{,}0),\quad C_i=\frac{\partial k}{\partial x_i}(0),$$

$$i=1{,}2,\ j=1{,}2.$$

Denote

$$E_0=E_{\varepsilon|\varepsilon=0},\quad S_{ij}=B_iR^{-1}B'_j,$$

$$T_{ij}=\begin{bmatrix}A_{ij}&-S_{ij}(0)\\-C'_iC_j&-A'_{ji}\end{bmatrix},\qquad i=1{,}2,\ j=1{,}2.$$

To guarantee that for all small ε this LQ problem is solvable we assume (Xu et al., 1997):

A1. The descriptor system (11), where $\varepsilon=0$, is stabilizable-detectable, i.e. both pencils $[sE_0-A;\ B]$ and $[sE_0'-A';\ C]$ are of full row rank for all s with non-negative real parts.

A2. The triple $\{A_{22},B_{20},C_2\}$ is stabilizable-detectable.

Under A2 a fast Riccati equation

$$A'_{22}X_\mathrm{f}+X_\mathrm{f}A_{22}+C_2'C_2-X_\mathrm{f}S_{22}(0)X_\mathrm{f}=0 \tag{13}$$

has a solution $X_\mathrm{f}=X_\mathrm{f}'\geq 0$, such that the matrix $\Lambda_\mathrm{f}=A_{22}-S_{22}(0)X_\mathrm{f}$ is Hurwitz. Under A1 and A2 a slow algebraic Riccati equation

$$X_0A_0+A_0'X_0-X_0S_0X_0+Q_0=0, \tag{14}$$

where

$$\begin{bmatrix} A_0 & -S_0 \\ -Q_0 & -A'_0 \end{bmatrix} = T_{11} - T_{12}T_{22}^{-1}T_{21} = T_0, \tag{15}$$

has a solution $X_0 = X'_0 \geq 0$ such that the matrix $\Lambda_s = A_0 - S_0 X_0$ is Hurwitz. It is known (Wang et al., 1988; Xu et al., 1997) that for all small enough ε the linear controller

$$u_l = -R^{-1}(0)B'_{10}X_0 x_1 - R^{-1}(0)B'_{20}(X_c x_1 + X_f x_2),$$

$$X_c = [X_f, -I]T_{22}^{-1}T_{21}\begin{bmatrix} I \\ X_0 \end{bmatrix} \tag{16}$$

solves the LQ problem.

Note that under A1 and A2 the Hamiltonian matrices T_{22} and T_0 have no eigenvalues on the imaginary axis. Then by implicit function theorem the system of equations

$$f_3(x_1, p_1, x_2, p_2) = 0, \quad f_4(x_1, p_1, x_2, p_2) = 0$$

has an isolated solution

$$x_2 = \phi(x_1, p_1), \quad p_2 = \psi(x_1, p_1) \tag{17}$$

in a small enough neighborhood of $\mathbf{R}^{n_2} \times \mathbf{R}^{n_2}$ containing 0, and the matrix

$$\begin{pmatrix} \dfrac{\partial f_3}{\partial x_2} & \dfrac{\partial f_3}{\partial p_2} \\ \dfrac{\partial f_4}{\partial x_2} & \dfrac{\partial f_4}{\partial p_2} \end{pmatrix}\Bigg|_{(x_2,p_2)=(\phi(x_1,p_1),\psi(x_1,p_1))}$$

has n_2 stable eigenvalues λ, $\operatorname{Re}\lambda < -\alpha < 0$, and n_2 unstable ones λ, $\operatorname{Re}\lambda > \alpha$.

Consider the *reduced* Hamiltonian system

$$\dot{x}_1 = f_1(x_1, p_1, \phi(x_1, p_1), \psi(x_1, p_1)), \tag{18a}$$

$$\dot{p}_1 = f_2(x_1, p_1, \phi(x_1, p_1), \psi(x_1, p_1)). \tag{18b}$$

This system results after substitution of (17) into (6a) and (6b). From the theory of nonlinear differential equations (Kelley, 1967) it follows that this system has a stable manifold

$$p_1 = N_0(x_1) \tag{19}$$

with asymptotically stable flow

$$\dot{x}_1 = f_1(x_1, N_0(x_1), \phi(x_1, N_0(x_1)), \psi(x_1, N_0(x_1))) \tag{20}$$

for x_1 from small enough neighborhood of 0. Note that (20) results from substitution of (19) into (18a). Function $N_0 = N_0(x_1)$ satisfies the *slow* partial differential equation (PDE)

$$\frac{\partial N_0}{\partial x_1} f_1(x_1, N_0, \phi(x_1, N_0), \psi(x_1, N_0)) = f_2(x_1, N_0, \phi(x_1, N_0), \psi(x_1, N_0)). \tag{21}$$

This PDE can be derived by differentiating on t of (19), where $p_1 = p_1(t)$, $x_1 = x_1(t)$ and by substituting for $\dot{x}_1$ the right-hand side of (20). The function N_0 can be approximated by

$$N_0(x_1) = X_0 x_1 + O(|x_1|^2), \tag{22}$$

where $|\cdot|$ denotes the Euclidean norm of a vector.

For each x_1 such that (17) and (19) exist consider the ‘fast’ system

$$\dot{\bar{x}}_2(\tau) = \bar{f}_3(x_1, \bar{x}_2(\tau), \bar{p}_2(\tau)), \quad \dot{\bar{p}}_2(\tau) = \bar{f}_4(x_1, \bar{x}_2(\tau), \bar{p}_2(\tau)),$$
$$\tau = \frac{t}{\varepsilon}, \tag{23}$$

where

$$\bar{f}_i = f_i(x_1, N_0, \bar{x}_2(\tau) + \phi(x_1, N_0), \bar{p}_2(\tau) + \psi(x_1, N_0)),$$
$$i = 3, 4.$$

From the theory of nonlinear differential equations (Kelley, 1967), it follows that this system has a stable manifold $\bar{p}_2 = M_0(x_1, \bar{x}_2)$ with asymptotically stable flow

$$\dot{\bar{x}}_2 = \bar{f}_3(x_1, \bar{x}_2, M_0(x_1, \bar{x}_2)) \tag{24}$$

for x_1 and $\bar{x}_2$ from small enough neighborhood Ω of $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ containing 0. Function $M_0 = M_0(x_1, \bar{x}_2)$ satisfies the *fast* PDE

$$\frac{\partial M_0}{\partial \bar{x}_2} \bar{f}_3(x_1, \bar{x}_2, M_0) = \bar{f}_4(x_1, \bar{x}_2, M_0) \tag{25}$$

and

$$M_0(x_1, \bar{x}_2) = X_f \bar{x}_2 + O((|x_1| + |\bar{x}_2|)|\bar{x}_2|). \tag{26}$$

Define the *composite* controller as follows:

$$u_0(x_1, x_2) = -R^{-1}B'_1 N_0(x_1) - R^{-1}B'_2[\psi(x_1, N_0(x_1)) + M_0(x_1, x_2 - \phi(x_1, N_0(x_1)))]. \tag{27}$$

From (16), (22) and (26) it follows that

$$u_0 = u_l + O(|x_1|^2 + |x_2|^2). \tag{28}$$

We shall show that u_0 near-optimally solves the problem on some ε-independent neighborhood for all small enough ε. Let $\Omega_{m_i} = \{x_i \in \mathbf{R}^{n_i}: |x_i| < m_i\}$, $i = 1,2$. From Fridman (2000) we obtain the following result:

Lemma 3.1. *Under* A1 *and* A2 *there exist* $m_1 > 0$, $m_2 > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$ *the following holds:*

(i) *There exists a* C^2 *function* $V: \Omega_{m_1} \times \Omega_{m_2} \to [0, \infty)$, *satisfying the HJ equation* (7) *with the property that* (8) *is asymptotically stable.*

(ii) *The invariant manifold, the solution to HJ equation and the optimal controller have the following* *approximations*:

$$Z_1(x_1,x_2) = V'_{x_1} = N_0(x_1) + O(\varepsilon), \tag{29a}$$

$$Z_2(x_1,x_2) = \varepsilon^{-1}V'_{x_2} = \psi(x_1,N_0(x_1)) + M_0(x_1,x_2-\phi(x_1,N_0(x_1))) + O(\varepsilon), \tag{29b}$$

$$V(x_1,x_2) = V_0(x_1) + O(\varepsilon), \tag{29c}$$

$$\frac{\partial V_0}{\partial x_1} = N'_0(x_1), \tag{29d}$$

$$\beta(x) = u_0(x_1,x_2) + O(\varepsilon), \tag{29e}$$

where u_0 *is given by* (27). *Approximations are uniform on* $x_1, x_2 \in \Omega_{m_1}\times\Omega_{m_2}$. *The composite controller* (27) *achieves the cost* O(ε)*-close to the optimal one* J_{opt} *for* $x(0)\in\Omega_0$, *where* Ω_0 *is the set of all initial conditions which give rise to asymptotically stable trajectories that are restricted to* $\Omega_{m_1}\times\Omega_{m_2}$.

(iii) *The optimal trajectory* $x^*(t)$ *with the initial data* $x(0)=\text{col}\{x_{10},x_{20}\}\in\Omega_0$ *and the corresponding optimal open-loop control* $u^*(t)$ *are approximated for* $t\in[0,\infty)$ *by*

$$x^*(t) = x^{(0)}(t,\tau) + \varepsilon r_1(t,\varepsilon),$$

$$x^{(0)}(t,\tau) = \text{col}\{x_1(t);\ \phi(x_1(t),N_0(x_1(t)) + \Pi(\tau))\},\quad \tau = \frac{t}{\varepsilon},$$

$$u^*(t) = -R^{-1}B'_1(x^{(0)})N_0(x_1(t)) - R^{-1}B'_2(x^{(0)}) \times[\psi(x_1(t),N_0(x_1(t))) + M_0(x_1(t),\Pi(\tau))] + \varepsilon r_2(t,\varepsilon), \tag{30}$$

where $x_1(t)$ *is a solution to* (20) *with the initial data* x_{10}. *The boundary layer term* $\Pi(\tau)$ *is exponentially decaying for* $\tau\to\infty$ *and satisfies the following initial value problem*:

$$\frac{\partial\Pi(\tau)}{\partial\tau} = \bar f_3[x_{10},\Pi(\tau),M_0(x_{10},\Pi(\tau))],\qquad \Pi(0) = x_{02} - \phi(x_{10},N_0(x_{10})). \tag{31}$$

The remainders satisfy inequality $|r_i(t,\varepsilon)|\le c e^{-\alpha t}$, $i=1,2$.

Note that (29c) and (29d) follow from (29a) and (29b).

3.2. Optimal controller for descriptor system.

Consider the corresponding to (1) descriptor system

$$E_0\dot x = F(x) + B(x)u. \tag{32}$$

A controller $u=u(x)$ is called an admissible, if the closed-loop system (32) has a unique solution for any initial condition $E_0x(0)$ from small enough neighborhood in $R^{n_1}\times\{0\}$ containing 0 as an interior point. The problem is to find, of all admissible locally asymptotically stabilizing controllers, the one that minimizes (2).

Theorem 3.1. (i) *Assume that* A1 *holds and that the linear descriptor system* (11), *where* ε = 0, *is impulsively controllable and observable, i.e. matrices* $[A_{22}\ B_{20}]$ and $[A'_{22}\ C'_2]$ *are of full row rank. Let there exists a twice continuously differentiable function* $V_{\text{d}}:\Omega_{m_1}\times\{0\}\to R$ *such that* $V_{\text{d}}(E_0x)\ge 0$,

$$\frac{\partial V_{\text{d}}(E_0x)}{\partial x} = W(x)E_0, \tag{33}$$

$$2W(x)F(x) - W(x)S(x)W'(x) + k'(x)k(x) = 0,\qquad S = BR^{-1}B', \tag{34}$$

with the property that $\tilde f_{x_2}(0,0)$ *is non-singular, where* $\tilde f = F_2 - B_2R^{-1}B'_2W'$, *and such that the system*

$$E_0\dot x = F(x) - S(x)W'(x)$$

is asymptotically stable. Then the controller

$$u_d(x) = -R^{-1}B'(x)W'(x) \tag{35}$$

solves the local optimal control problem for the descriptor system (32) *with respect to the functional* (2) *and leads to the optimal cost*

$$J_{\text{d}} = 2V_{\text{d}}(E_0x(0)). \tag{36}$$

(ii) *Under* A1 *and* A2 *the solution to* (33) *and* (34) *is given by*

$$V_{\text{d}}(E_0x) = V_0(x_1), \tag{37a}$$

$$W(x) = [N'_0(x_1)\quad \psi'(x_1,N_0(x_1))] + M'_0[x_1,x_2-\phi(x_1,N_0(x_1))] \tag{37b}$$

and the composite controller (27) *is locally optimal one for* (32).

(iii) *Assume that* A1 *and* A2 *hold. Let* $\bar M:\Omega\to R^{n_2}$ *be any continuously differentiable function that vanishes at* $x_2=0$ *and such that* $\tilde f_{x_2}(0,0)$ *is non-singular, where* $\tilde f = F_2 - B_2R^{-1}B'_2\bar M$. *Then the controller*

$$\bar u(x) = -R^{-1}B'_1N_0(x_1) - R^{-1}B'_2[\psi(x_1,N_0(x_1)) + \bar M(x_1,x_2-\phi(x_1,N_0(x_1)))] \tag{38}$$

is locally optimal one for (32). *The resulting optimal cost* $J_d = 2V_0(x_1(0))$ *is* O(ε)*-close to the optimal cost* J_{opt} *of singularly perturbed system* (1) *for all initial conditions* $x(0)$ *from small enough neighborhood* $\Omega_0\subset R^{n_1}\times R^{n_2}$ *containing* 0.

Note that relation (33) is analogous to one in Xu and Mizukami (1994).

3.3. Near-optimal controllers for the singularly perturbed system

Theorem 3.2. *Under* A1 *and* A2 *for all small enough* ε *and for all initial conditions from small enough neighborhood* $\Omega_0\subset R^{n_1}\times R^{n_2}$ *containing* 0 *the following holds:*

(i) *The composite controller* (27), *where* M_0 *satisfies the fast PDE* (25), *leads the full-order system* (1) *to the cost* $O(\varepsilon^2)$*-close to the optimal one* J_{opt}.

(ii) *Let* $\bar{M}:\Omega \to R^{n_2}$ *be any continuously differentiable function that vanishes at* $x_2 = 0$ *and such that* (24) *with* $M_0 = \bar{M}$ *is exponentially stable uniformly on* x_1. *Then the controller* (38) *leads the full-order system* (1) *to the cost* $O(\varepsilon)$*-closeto* J_{opt}.

Thus, as in the linear case (Wang et al., 1988) and in the standard nonlinear case (Saberi et al., 1985), there exist many near-optimal ε-independent solutions (27) to the non-standard problem (1), (2), where the fast gain is any function that exponentially stabilizes (24) with $M_0 = \bar{M}$ (e.g. one can choose $\bar{M}(x_1, x_2) = Kx_2$ such that $A_{22} + B_{20}K$ is Hurwitz). These solutions lead to the values of the cost $O(\varepsilon)$-close to the optimal one. However, as well as in the linear case (Kokotovic et al., 1986; Xu et al., 1997), only the composite controller, being an $O(\varepsilon)$-approximation to the optimal controller, achieves $O(\varepsilon^2)$ near-optimal cost.

Example. Consider the system

$$\dot{x}_1 = \tan x_2 - u, \quad \varepsilon\dot{x}_2 = x_1 + u,$$

$$J = \int_0^\infty [\tan^2 x_2 + u^2]\mathrm{d}t, \quad x(0) = \mathrm{col}\{0.4, 1\}. \tag{39}$$

This is a non-standard problem since the algebraic equation $x_1 + u = 0$ is not uniquely solvable with respect to x_2. We obtain the following Hamiltonian function:

$$\mathscr{H} = -p_1 \tan x_2 + p_2 x_1 - 1/2p_1^2 + p_1 p_2 - 1/2p_2^2 + 1/2\tan^2 x_2$$

and the Hamiltonian system

$$\dot{x}_1 = \tan x_2 - p_1 + p_2, \quad \dot{p}_1 = -p_2,$$

$$\varepsilon\dot{x}_2 = x_1 + p_1 - p_2, \quad \varepsilon\dot{p}_2 = -(p_1 + \tan x_2)/\cos^2 x_2.$$

We find

$$\psi = x_1 + p_1, \quad \phi = -\arctan p_1, \quad N_0(x_1) = Kx_1,$$

$$K = 1 + \sqrt{2},$$

$$M_0(x_1, \bar{x}_2) = [Kx_1 + \tan[\bar{x}_2 - \arctan(Kx_1)]]/\cos^2(\bar{x}_2 - \arctan(Kx_1)).$$

The composite controller of (27) has a form

$$u_0 = -x_1 - (Kx_1 + \tan x_2)/\cos^2 x_2. \tag{40}$$

By choosing $\bar{M} = 3\bar{x}_2$, that stabilizes (24), we obtain another near-optimal controller given by the right-hand side of (38)

$$\bar{u} = -x_1 - 3x_2 - 3\arctan(Kx_1).$$

Table 1

ε	0.1	0.01	0.001	0.0001
$J(u_0)$	1.5509	0.4740	0.3948	0.3871
$J(\bar{u})$	1.5965	0.4830	0.3957	0.3872

Applying now u_0 and $\bar{u}$ to (39) we find the corresponding values of costs $J(u_0)$ and $J(\bar{u})$ for different values of ε (see Table 1). We find from Table 1 that for all ε under consideration $J(u_0) < J(\bar{u})$. The values of $J(u_0)$ and $J(\bar{u})$ approach the same limit as $\varepsilon \to 0$.

4. Conclusions

We have designed ε-independent controllers for non-standard singularly perturbed systems being nonlinear in both, the slow and the fast state variables. We have shown that these controllers are optimal for the corresponding descriptor system. The slow gain of these controllers N_0 is uniquely defined from the slow PDE. The fast gain can be found either as a solution to the fast PDE or as a stabilizing gain for the fast system. In the first case the controller is $O(\varepsilon)$-close to the optimal controller and leads the singularly perturbed system to $O(\varepsilon^2)$ near-optimal cost, i.e. $J(u_0) = J_{\text{opt}} + O(\varepsilon^2)$. In the second case the controllers lead to $O(\varepsilon)$ near-optimal cost. The optimal cost of the descriptor system is $O(\varepsilon)$-close to J_{opt}.

The results are local. More general results under less restrictive assumptions than those of A1 and A2 is another interesting problem that remains open.

Appendix

Proof of Theorem 3.1. (i) Note that under assumptions of the theorem $\{E_0, F, B, k\}$ is locally stabilizable-detectable and locally impulsively controllable and observable. Let $x(t)$ satisfy (32) and start from $E_0x(0)$. Applying (33), (32) and (34) we find

$$2\frac{\mathrm{d}V_\mathrm{d}(E_0x)}{\mathrm{d}t} + k'k + u'Ru = 2W(x)(F(x) + B(x)u) + k'k + u'Ru = (u' + W(x)B(x)R^{-1}(x))R(x) \times (u + R^{-1}(x)B'(x)W'(x)). \tag{A.1}$$

For asymptotically stabilizing controllers $V_\mathrm{d}(E_0(x_\infty)) = 0$. Then, integrating (A.1) on t from 0 to ∞, we find

$$J_\mathrm{d}(x_0, u) \geq 2V_\mathrm{d}(E_0x_0) = J_\mathrm{d}(x_0, u_d), \tag{A.2}$$

i.e. u_d is a minimizing controller.

Consider the closed-loop system (32), (35). By the non-singularity of $\tilde{f}_{x_2}(0,0)$ and the implicit function theorem, the last n_2 algebraic equations of the closed-loop system (32), (35) can be solved with respect to x_2 in a small neighborhood of $x = 0$. Substituting the resulting x_2 into the first n_1 differential equations of (32) and (35) we see that the initial condition for x_1 defines the unique solution. Hence, u_d is admissible.

(ii) From (37) and (29d) we have

$$\frac{\partial V_d(E_0 x)}{\partial x} = \left[\frac{\partial V_0(x_1)}{\partial x_1} \quad 0\right] = [N_0'(x_1) \quad 0] = W(x)E_0.$$

Lemma 3.1 implies $V_x E_\varepsilon^{-1} = W + O(\varepsilon)$. Substituting this relation into (7) and neglecting the $O(\varepsilon)$-terms in the resulting equation, we find that W satisfies (34). Under A1 and A2, conditions of (i) of Theorem 3.1 are satisfied and therefore $u_d(x) = u_0(x_1, x_2)$.

(iii) Note that $M_0(x_1, 0) = \bar{M}(x_1, 0) = 0$ and for the descriptor system $x_2 = \phi(x_1, N_0(x_1))$. Then $\bar{M}[x_1, x_2 - \phi(x_1, N_0(x_1))] = M_0[x_1, x_2 - \phi(x_1, N_0(x_1))] = 0$ and $\bar{u}(x) = u_0(x_1, x_2)$ is locally optimal controller for (32). Moreover, descriptor system (32) is impulse free since $\bar{f}_{x_2}(0,0)$ is non-singular. From (36), (37a) and (29c) it follows that $J_d = J_{\text{opt}} + O(\varepsilon)$. □.

Proof of Theorem 3.2. (i) The closed-loop system (1), (2) and (27) has the form:

$$E_\varepsilon \dot{x} = F(x) - S(x)W'(x), \tag{A.3a}$$

$$J(u_0) = \int_0^\infty [k'(x)k(x) + W(x)S(x)W'(x)]\,dt, \tag{A.3b}$$

where W is given by (37b). Note that u_0 is asymptotically stabilizing controller and thus (A.3a) is asymptotically stable for small ε. Similarly to (36) it can be proved that for each ε $J(u_0) = 2U(x(0))$, where $U: \Omega_{m_1} \times \Omega_{m_2} \to R$ is a twice continuously differentiable function, such that $U(x) \geq 0, U(0) = 0, U_x(0) = 0$ and

$$2U_x E_\varepsilon^{-1}(F(x) - SW'(x)) + k'(x)k(x) + W(x)S(x)W'(x) = 0. \tag{A.4}$$

The optimal cost $J_{\text{opt}} = 2V(x(0))$, where V is a solution to HJ equation (7).

Under A1 and A2 there exists a non-negative twice continuously differentiable solution to (7) and this solution can be approximated in the form (Fridman, 2000):

$$V_x E_\varepsilon^{-1} = [Z_1(x_1, x_2) \quad Z_2(x_1, x_2)] = W(x) + \varepsilon V_1(x) + O(\varepsilon^2), \tag{A.5}$$

where V_1 is continuously differentiable and approximation is uniform on x from small enough neighborhood of 0. Analogously, under A1 and A2 there exists a non-negative twice continuously differentiable solution to (A.4) and this solution can be uniformly approximated in the form

$$U_x E_\varepsilon^{-1} = W(x) + \varepsilon U_1(x) + O(\varepsilon^2), \tag{A.6}$$

where U_1 is continuously differentiable.

Denote $\Delta(x) = U(x) - V(x)$. From (A.4) and (7) we obtain

$$2\Delta_x E_\varepsilon^{-1}(F(x) - SW'(x)) + (V_x E_\varepsilon^{-1} - W(x))S(x)(E_\varepsilon^{-1}V'_x - W'(x)) = 0. \tag{A.7}$$

Continuously differentiable in x functions $F - SW'$, $V_x E_\varepsilon^{-1} - W$ and Δ_x vanish at $x = 0$ and, therefore

$$F - SW' = \bar{A}(x)x, \tag{A.8a}$$

$$V_x E_\varepsilon^{-1} - W = \varepsilon x' Y(x, \varepsilon), \tag{A.8b}$$

$$\Delta_x(x) = x' K(x, \varepsilon), \tag{A.8c}$$

where the right-hand side of (A.8b) is multiplied by ε due to (A.5). The functions on the right-hand side of (A.8a)–(A.8c) are continuous in x. Substituting (A.8a)–(A.8c) into (A.7), we obtain

$$2K(x, \varepsilon)E_\varepsilon^{-1}\bar{A}(x) + \varepsilon^2 Y(x, \varepsilon)S(x)Y'(x, \varepsilon) = 0. \tag{A.9}$$

From (A.5) and (A.6) it follows that for all small enough ε function Δ_x can be approximated by

$$\Delta_x E_\varepsilon^{-1} = x'K(x, \varepsilon)E_\varepsilon^{-1} = x'K_0(x) + \varepsilon x' K_1(x) + O(\varepsilon^2), \tag{A.10}$$

where K_0 and K_1 are continuous. Denote $\bar{A}_{ij}(x)$, $i = 1,2$, $j = 1,2$ the corresponding blocks of the continuous matrix-function $\bar{A}(x)$ and $\bar{A}_0(x) = \bar{A}_{11}(x) - \bar{A}_{12}(x)\bar{A}_{22}^{-1}(x)\bar{A}_{12}(x)$. Matrices $\bar{A}_{22}(x)$ and $\bar{A}_0(x)$ are Hurwitz for x from small enough neighborhood of 0 ince matrices $\bar{A}_{22}(0) = A_{22} - S_{22}(0)X_f$ and $\bar{A}_0(0) = A_0 - S_0 X_0$ are Hurwitz. Substituting (A.10) into (A.9) and equating terms with ε^0 and ε^1 we obtain, similarly to Kokotovic et al. (1986, p. 118), $K_0 = K_1 = 0$ and $\Delta_x E_\varepsilon^{-1} = O(\varepsilon^2)$. Hence, for all $x(0)$ from small enough neighborhood of 0 and for all small enough ε the following holds: $J(u_0) - J_{\text{opt}} = 2\Delta(x(0)) = O(\varepsilon^2)$.

(ii) Consider the closed-loop system (1), (38), where $\bar{M}$ is any stabilizing function for (24) with $M_0 = \bar{M}$. Compare it with the closed-loop system (1), (27), where M_0 satisfies the fast PDE (23). The reduced problems for these systems, resulting after substitution 0 for ε, have the same solution $x_2 = \phi(x_1, N_0(x_1))$, where x_1 satisfies (20). Hence, solutions to these closed-loop systems have the same regular parts in the zero-order approximations. Therefore, the resulting values of J are $O(\varepsilon)$-close (boundary layer terms after integrating become $O(\varepsilon)$-terms). By (ii) of Lemma 3.1 these values are $O(\varepsilon)$-close to the optimal cost. □

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