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Effects of small delays on stability of singularly perturbed systems☆

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Abstract

A small delay in the feedback loop of a singularly perturbed system may destabilize it; however, without the delay, it is stable for all small enough values of a singular perturbation parameter ε. Sufficient and necessary conditions for preserving stability, for all small enough values of delay and ε, are obtained in two cases: in the case of delay proportional to ε and in the case of independent delay and ε. In the second case, the sufficient conditions are given in terms of an LMI. A delay-dependent LMI criterion for the stability of singularly perturbed differential–difference systems is derived.

Keywords: Time-delays; Singular perturbations; LMI; Stability

1. Introduction

It is well known that if an ordinary differential system of equations is asymptotically stable, then this property is robust with respect to small delays (see e.g. El'sgol'ts & Norkin, 1973; Hale & Lunel, 1993). Examples of the systems, where small delays change the stability of the system, are given in Hale and Lunel (1999) (see also references therein). All these examples are infinite-dimensional systems, e.g. difference systems, neutral-type systems with unstable difference operator or systems of partial differential equations. Another example of a system, sensitive to small delays, is a descriptor system (Logemann, 1998). Small delays in the descriptor system may lead to a system with advanced argument, whose solution is not defined for $t \to \infty$. Necessary and sufficient conditions for robust stability with respect to small delays are given in Logemann (1998) in terms of the spectral radius of a certain transfer matrix.

In the present note we give a new example of a finite-dimensional system that may be destabilized by the introduction of a small delay in the loop. This is a singularly perturbed system. Consider the following simple example:

$$\varepsilon\dot{x}(t) = u(t), \quad u(t) = -x(t-h), \tag{1}$$

where $x(t) \in R$ and $\varepsilon > 0$ is a small parameter. Eq. (1) is stable for $h = 0$; however, for small delays $h = \varepsilon g$ with $g > \pi/2$ this system becomes unstable (see e.g. El'sgol'ts & Norkin, 1973).

Two main approaches have been developed for the treatment of the effects of small delays: frequency domain techniques and direct analysis of characteristic equation. We suggest here a new approach of the second Lyapunov method, that leads to effective sufficient conditions for stability via LMIs. Note that the stability of singularly perturbed systems with delays in the frequency domain has been studied by Luse (1987), Pan, Hsiao, and Teng (1996) (see also references therein). However, the method of LMIs is more suitable for robust stability of systems with uncertainties and for other control problems (see e.g. Li & de Souza, 1997; de Souza & Li, 1999). Moreover, LMI conditions may be easily verified by using LMI toolbox of Matlab.

2. Problem formulation

Let R^m be an Euclidean space and $C^m[a,b]$ be the space of continuous functions $\phi : [a,b] \to R^m$ with the supremum norm $|\cdot|$. Denote $x_t(\theta) = x(t+\theta)$ $(\theta \in [-h,0])$.

Consider the following singularly perturbed system:

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t), \\ \varepsilon\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t), \end{aligned} \tag{2}$$

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where $x_1(t) \in R^{n_1}$, $x_2(t) \in R^{n_2}$ are the state vectors, $u(t) \in R^q$ is the control input, A_{ij}, B_i $(i = 1,2,\ j = 1,2)$ are the matrices of the appropriate dimensions, and ε is a small positive parameter. An ε-independent state-feedback

$$u(t) = K_1 x_1(t) + K_2 x_2(t), \tag{3}$$

robustly stabilizes (2) for all small enough ε, i.e. for the closed-loop system (2) and (3)

$$E_\varepsilon \dot{x}(t) = (A + H)x(t), \tag{4}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad E_\varepsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad H_{ij} = B_i K_j, \quad i = 1,2, j = 1,2, \ldots$$

is robustly asymptotically stable under the following assumptions (see e.g. Kokotovic, Khalil, & O'Reilly, 1986):

(A1) The "fast" matrix $A_{22} + H_{22}$ is Hurwitz.
(A2) The "slow" matrix

$$A_0 = A_{11} + H_{11} - (A_{12} + H_{12})(A_{22} + H_{22})^{-1}(A_{21} + H_{21})$$

is Hurwitz.

Consider now the controller

$$u(t) = K_1 x_1(t-h) + K_2 x_2(t-h) \tag{5}$$

with a small delay h. The closed-loop system (2), (5) may become unstable for some ε. Thus in example (1) the closed-loop system becomes unstable for all h and e.g. for $\varepsilon < 2h/\pi$. In the present note we obtain sufficient and necessary conditions for the stability of (2) for all small enough ε and h. We consider two different cases: (1) h is proportional to ε and (2) ε and h are independent. The first case, being less general than the second one, is encountered in many publications (see e.g. Glizer & Fridman, 2000 and references therein). In practical systems (see e.g. a model of a two-core nuclear reactor in Reddy and Sannuti, 1975), the fast states usually appear with a delay proportional to ε, since otherwise (as follows from Corollary 1 below) the fast system should be delay-independently stable. The latter is too restrictive for real systems.

3. The case of delay proportional to ε: invariant manifolds approach

We start with the case of $h = \varepsilon g$, $g \in (0, g_0]$, i.e.

$$u(t) = K_1 x_1(t - \varepsilon g) + K_2 x_2(t - \varepsilon g). \tag{6}$$

Consider the closed-loop system (2), (6):

$$E_\varepsilon \dot{x}(t) = Ax(t) + Hx(t - \varepsilon g). \tag{7}$$

Consider also the fast system

$$\dot{x}_2(t) = A_{22} x_2(t) + H_{22} x_2(t - g). \tag{8}$$

To ensure the stability of (7) for all $g \in (0, g_0]$ and all small enough ε we assume additionally

(A3) There exist $K > 0$ and $\alpha > 0$ such that for all $g \in [0, g_0]$ and for all $x_{20} \in C^{n_2}[-h, 0]$

$$|x_2(t)| \leqslant K e^{-\alpha t} |x_{20}|, \tag{9}$$

where $x_2(t)$ is a solution of (8).

Sufficient conditions for (9) can be found e.g. in Niculescu, de Souza, Dugard, and Dion (1998), Fridman and Shaked (1998). We obtain the following lemma

Lemma 1. *Under* A1–A3 *there exists* ε_0 *such that for all* $\varepsilon \in (0, \varepsilon_0]$ *the state-feedback* (6) *exponentially stabilizes* (2) *for all* $g \in (0, g_0]$.

Proof. For each $h \in (0, \varepsilon g_0]$ the result follows from Fridman (1996), Glizer and Fridman (2000). Proof of the result for all $g \in (0, g_0]$ follows by the same invariant manifold argument since (9) holds uniformly in $g \in [0, g_0]$. □

We obtain now a simple necessary condition for the stability of (2), (6) under A1 and A2. Note that this condition is weaker than A3.

Lemma 2. *Let there exist* $g_1 > 0$ *such that the fast characteristic equation*

$$\Delta(\lambda) \triangleq \det(\lambda I - A_{22} - H_{22} e^{-\lambda g_1}) = 0, \tag{10}$$

that corresponds to the fast system (8) *has at least one root with positive real part. Then, for all small enough* ε *and* $g = g_1$, *the closed-loop system* (2), (6) *is unstable.*

Proof. Consider $g = g_1$. Writing (7) in the fast time $\tau = t/\varepsilon$ and denoting $v(\tau) = x_1(\varepsilon\tau)$, $w(\tau) = x_2(\varepsilon\tau)$ we obtain

$$\begin{aligned} \dot{v}(\tau) &= \varepsilon[F_1 v_\tau + F_2 w_\tau], \\ \dot{w}(\tau) &= F_3 v_\tau + A_{22} w(\tau) + H_{22} w(\tau - g_1), \end{aligned} \tag{11}$$

where F_i: $C^{n_i}[-h, 0] \to R^{n_1}$, $i = 1,2$, $F_3 : C^{n_1}[-h, 0] \to R^{n_2}$ are given by

$$F_1 v_\tau \triangleq A_{11} v(\tau) + H_{11} v(\tau - g_1),$$

$$F_2 w_\tau \triangleq A_{12} w(\tau) + H_{12} w(\tau - g_1),$$

$$F_3 v_\tau \triangleq A_{21} v(\tau) + H_{21} v(\tau - g_1).$$

Let $T(t) : C^{n_2}[-g_1, 0] \to C^{n_2}[-g_1, 0]$, $t \geqslant 0$ be the semigroup of operators that corresponds to the fast system (8) (see e.g. Hale, 1971, p. 61). Denote

$$\Lambda_0 = \{\lambda \in C\text{: } \Delta(\lambda) = 0 \text{ and } \operatorname{Re}\lambda = 0\},$$

$$\Lambda_+ = \{\lambda \in C\text{: } \Delta(\lambda) = 0 \text{ and } \operatorname{Re}\lambda > 0\}.$$

All the other roots of (10) have negative real parts. It is well known (see e.g. Hale, 1971, p. 62; Hale & Lunel, 1993, p. 200) that according to this splitting we have the following decomposition of $C^{n_2}[-g_1, 0]$:

$$C^{n_2}[-g_1,0] = P_1 \oplus P_2 \oplus Q,$$

where P_1, P_2 and Q are invariant spaces for solutions of (8) in the sense that for all initial conditions from P_i (Q) solutions to (8) satisfy $x_{2t} \in P_i,\ i = 1,2$ ($x_{2t} \in Q$) for all $t \geqslant 0$. Moreover, P_1 (P_2) is finite-dimensional and corresponds to solutions of (8) of the form $p(t)e^{\lambda t}$, where $p(t)$ is a polynomial in t and $\lambda \in \Lambda_0$ ($\lambda \in \Lambda_+$). Denote by Φ_1, Φ_2 the matrices, the columns of which are basis vectors for P_1 and P_2. Let Ψ_1 (Ψ_2) be the matrices, the rows of which are basis for the initial values of those solutions to the transpose of (8)

$$\dot{\bar{x}}_2(s) = -\bar{x}_2(s)A_{22} - \bar{x}_2(s+g)H_{22}, \quad \bar{x}_2'(s) \in R^{n_2}, \quad s \leqslant 0,$$

which have the form of $p(s)e^{-\lambda s}$, where $p(s)$ is a polynomial in s and $\lambda \in \Lambda_0$ ($\lambda \in \Lambda_+$) (see Hale, 1971, p. 63).

Denote by Y_i, $i = 1,2$ such matrices that $T(t)\Phi_i = \Phi_i \exp(Y_i t)$. The spectrum of Y_1 (Y_2) is Λ_0 (Λ_+). Let $X(t)$ be the fundamental matrix of (8) and thus $X_0(s) = 0$, $s \in [-g_1, 0)$; $X_0(0) = I$. Following Hale (1971, pp. 62–64), we denote

$$X_0^Q = X_0 - \Phi_1\Psi_1(0) - \Phi_2\Psi_2(0),$$

$$T(t)X_0^Q = X_t - T(t)\Phi_1\Psi_1(0) - T(t)\Phi_2\Psi_2(0).$$

It is shown in Hale (1971, p. 64) that $T(t)z_0$ for $z_0 \in Q$ and $T(t)X_0^Q$ are exponentially decaying. Moreover, the solution of (11) $v(\tau), w_\tau = \Phi_1 y_1(\tau) + \Phi_2 y_2(\tau) + z_\tau,\ z_\tau \in Q$, with the initial conditions $v(0), w_0$ is a solution of the system

$$\begin{aligned}
&\dot{v}(\tau) = \varepsilon[F_1 v_\tau + F_2(\Phi_1 y_1(\tau) + \Phi_2 y_2(\tau) + z_\tau)],\\
&\dot{y}_1(\tau) = Y_1 y_1 + \Psi_1(0)F_3 v_\tau,\\
&\dot{y}_2(\tau) = Y_2 y_2 + \Psi_2(0)F_3 v_\tau,\\
&z_\tau = T(\tau)z_0 + \int_0^\tau T(\tau - s)X_0^Q F_3 v_s\,ds
\end{aligned} \tag{12}$$

and vice versa (Hale, 1971, p. 66).

Note that the spectrum of the "main linear parts" of (12) is decomposed as follows: in the equations with respect to $\dot{v}, \dot{y}_1$ and z_τ it is on the imaginary axis and in the left-hand side of the plane, while in the equation with respect to y_2 it is on the right-hand side of the plane. By standard arguments (see e.g. Kelley, 1967, Theorem 1; Hale, 1971, Theorem 3.1), this system for all small enough $\varepsilon > 0$ has an unstable manifold

$$v_\tau = \varepsilon L_1(\theta, \varepsilon)y_2(\tau), \qquad y_1 = L_2(\varepsilon)y_2(\tau),$$

$$z_\tau = L_3(\theta, \varepsilon)y_2(\tau), \quad \theta \in [-g_1, 0],$$

where L_i, $i = 1,2,3$ are continuous and uniformly bounded functions. The flow on this manifold is governed by the equation

$$\dot{y}_2(\tau) = Y_2 y_2(\tau) + \varepsilon\Psi_2(0)F_3 L_1 y_2(\tau). \tag{13}$$

The solutions on the unstable manifold are unbounded solutions of (11). Therefore, (7) has unbounded solutions and thus (7) is unstable for small enough $\varepsilon > 0$ and $g = g_1$. □

4. The case of independent h and ε: an LMI approach

4.1. *On robustness of regular time-delay system with respect to small delay*

The closed-loop system (2), (5) has the form

$$E_\varepsilon \dot{x}(t) = Ax(t) + Hx(t-h). \tag{14}$$

For $\varepsilon = 1$ (14) is a regular system. It is well known that if $A + H$ is Hurwitz, then (14) with $\varepsilon = 1$ is asymptotically stable for all small enough h. The proof of this fact is usually based on Rousche's theorem (see e.g. El'sgol'ts & Norkin, 1973; Hale & Lunel, 1993). We mention here that this fact immediately follows from a delay-dependent LMI stability criterion (see e.g. Li & de Souza, 1997; Kolmanovskii, Niculescu, & Richard, 1999). Namely, (14) is stable if there exist symmetric positive-definite matrices P, R_1 and R_2 satisfying the following LMIs:

$$\begin{bmatrix} \Phi(h) & hPH & hPH \\ hH'P & -hR_1 & 0 \\ hH'P & 0 & -hR_2 \end{bmatrix} < 0, \tag{15}$$

where $\Phi(h) = (A+H)'P + P(A+H) + h(A'R_1A + H'R_2H)$. The fact that $A+H$ is Hurwitz implies the existence of $P > 0$ such that $\Phi(0) < 0$ and thus (15) holds e.g. for $R_1 = R_2 = I$ and for all small enough h. We develop an LMI approach to (14) for small ε and h. Note that in the case of $h = \varepsilon g$ one can apply (15) with $A = E_\varepsilon^{-1}A,\ H = E_\varepsilon^{-1}H$.

4.2. *Stability conditions for singularly perturbed system with delay*

From Lemma 2 we obtain the following *necessary condition*:

Corollary 1. *Let* (14) *be stable for all small enough* ε *and* h. *Then, for all* $g_1 \geqslant 0$, *the characteristic equation* (10) *has no roots with positive real parts.*

According to this corollary we derive a criterion for asymptotic stability which is delay-independent in the fast variables and delay-dependent in the slow ones by considering the following Lyapunov–Krasovskii functional:

$$V(x_t) = x'(t)E_\varepsilon P_\varepsilon x(t) + \int_{t-h}^t x_2'(\theta)Qx_2(\theta)\,d\theta + W(x_t), \tag{16}$$

where

$$P_\varepsilon = \begin{bmatrix} P_1 & \varepsilon P_2' \\ P_2 & P_3 \end{bmatrix}, \quad P_1 = P_1' > 0, \ P_3 = P_3' > 0, \tag{17}$$

and

$$W(x_t) = \int_{-h}^{0}\int_{t+\theta}^{t} x'(s)[A_{11} \quad A_{12}]' R_1 [A_{11} \quad A_{12}] x(s)\,\mathrm{d}s\,\mathrm{d}\theta$$
$$+ \int_{-h}^{0}\int_{t+\theta-h}^{t} x'(s)[H_{11} \quad H_{12}]'$$
$$\times R_2 [H_{11} \quad H_{12}] x(s)\,\mathrm{d}s\,\mathrm{d}\theta.$$

Note that we choose P_ε in the form of (17) similar to Xu and Mizukami (1997), such that for $\varepsilon = 0$ the functional V, with $E_\varepsilon = E_0$ and $P_\varepsilon = P_0$, corresponds to the descriptor case (i.e. $\varepsilon = 0$ in (14)).

To guarantee that nondelay system (14), where $h = 0$, is asymptotically stable for all small enough ε we assume:

(A4) There exists P_0 of (17) such that $P_0'(A + H) + (A' + H')P_0 < 0$.

A4 implies the robust asymptotic stability of (14) with $h = 0$ since, choosing $V_0(x) = x'P_\varepsilon x$, we have, for all small enough ε and $x(t)$ satisfying (14) with $h = 0$, that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_0(x(t)) = x'(t)[P_\varepsilon'(A + H)(A' + H')P_\varepsilon]x(t)$$
$$= x'(t)[P_0'(A + H) + (A' + H')P_0$$
$$+ O(\varepsilon)]x(t) < 0.$$

Denote

$$\Psi(\varepsilon, h)$$
$$= \left[A' + \begin{pmatrix} H_{11}' & H_{21}' \\ 0 & 0 \end{pmatrix}\right] P_\varepsilon + P_\varepsilon' \left[A + \begin{pmatrix} H_{11} & 0 \\ H_{21} & 0 \end{pmatrix}\right]$$
$$+ \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} + h \begin{bmatrix} A_{11}' \\ A_{12}' \end{bmatrix} R_1 [A_{11} \quad A_{12}] + h \begin{bmatrix} H_{11}' \\ H_{12}' \end{bmatrix}$$
$$\times R_2 [H_{11} \quad H_{12}]. \tag{18}$$

We obtain the following *sufficient conditions*.

Theorem 1. *Under* A4 *the following holds*:

(i) *Given* $\varepsilon > 0$, $h > 0$ (14) *is asymptotically stable if there exist* P_ε *of* (17) *such that* $E_\varepsilon P_\varepsilon > 0$ *and* $n_2 \times n_2$*-matrix* Q *and* $n_1 \times n_1$*-matrices* $R_1 > 0$ *and* $R_2 > 0$ *that satisfy the "full-order" LMI*

$$\begin{bmatrix} \Psi(\varepsilon,h) & P_\varepsilon'\begin{pmatrix} H_{12} \\ H_{22} \end{pmatrix} & hP_\varepsilon'\begin{pmatrix} H_{11} \\ H_{21} \end{pmatrix} & hP_\varepsilon'\begin{pmatrix} H_{11} \\ H_{21} \end{pmatrix} \\ (H_{12}' \ H_{22}')P_\varepsilon & -Q & 0 & 0 \\ h(H_{11}' \ H_{21}')P_\varepsilon & 0 & -hR_1 & 0 \\ h(H_{11}' \ H_{21}')P_\varepsilon & 0 & 0 & -hR_2 \end{bmatrix} < 0. \tag{19}$$

(ii) *If there exist* P_0 *of* (17) *and* $n_2 \times n_2$*-matrix* Q *such that the LMI*

$$\Gamma = \begin{bmatrix} \Psi(0,0) & P_0'\begin{pmatrix} H_{12} \\ H_{22} \end{pmatrix} \\ (H_{12}' \ H_{22}')P_0 & -Q \end{bmatrix} < 0 \tag{20}$$

holds, then (14) *is asymptotically stable for all small enough* $\varepsilon > 0$ *and* $h \geqslant 0$.

Proof. (i) Differentiating (16) with respect to t we obtain

$$\frac{\mathrm{d}V(x_t)}{\mathrm{d}t} = 2x'P_\varepsilon'[Ax(t) + Hx(t-h)] + x_2'(t)Qx_2(t)$$
$$- x_2'(t-h)Qx_2(t-h) + \frac{\mathrm{d}W}{\mathrm{d}t}. \tag{21}$$

Considering the system of (14) we find that for $t \geqslant 0$

$$x_1(t-h) = x_1(t) - [A_{11} \quad A_{12}] \int_{t-h}^{t} x(\tau)\,\mathrm{d}\tau$$
$$- [H_{11} \quad H_{12}] \int_{t-h}^{t} x(\tau - h)\,\mathrm{d}\tau. \tag{22}$$

Then,

$$\frac{\mathrm{d}V(x_t)}{\mathrm{d}t} = 2x'(t)P_\varepsilon'\left[A + \begin{pmatrix} H_{11} & 0 \\ H_{21} & 0 \end{pmatrix}\right]x(t)$$
$$+ 2x'(t)P_\varepsilon'\begin{bmatrix} H_{12} \\ H_{22} \end{bmatrix} x_2(t-h)$$
$$+ x_2'(t)Qx_2(t) - x_2'(t-h)Qx_2(t-h)$$
$$+ \frac{\mathrm{d}W}{\mathrm{d}t} + \eta_1(t) + \eta_2(t), \tag{23}$$

where

$$\eta_1(t) \triangleq -2\int_{t-h}^{t} x'(t)P_\varepsilon'\begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix}[A_{11} \quad A_{12}]x(\tau)\,\mathrm{d}\tau,$$

$$\eta_2(t) \triangleq -2\int_{t-h}^{t} x'(t)P_\varepsilon'\begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix}[H_{11} \quad H_{12}]x(\tau - h)\,\mathrm{d}\tau.$$

Since for any $z, \ y \in R^n$ and for any symmetric positive-definite $n \times n$-matrix X,

$$-2z'y \leqslant z'X^{-1}z + y'Xy,$$

we find that for any $n_1 \times n_1$-matrices $R_1 > 0$ and $R_2 > 0$,

$$\eta_1 \leqslant hx'(t)P_\varepsilon'\begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix} R_1^{-1}[H_{11}' \quad H_{21}']P_\varepsilon x(t)$$
$$+ \int_{t-h}^{t} x'(\tau)\begin{bmatrix} A_{11}' \\ A_{12}' \end{bmatrix} R_1 [A_{11} \quad A_{12}]x(\tau)\,\mathrm{d}\tau,$$

$$\eta_2 \leqslant hx'(t)P'_\varepsilon \begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix} R_2^{-1}[H'_{11} \quad H'_{21}]P_\varepsilon x(t) + \int_{t-h}^{t} x'(\tau-h)\begin{bmatrix} H'_{11} \\ H'_{12} \end{bmatrix} R_2[H_{11} \quad H_{12}]x(\tau-h)\,\mathrm{d}\tau. \tag{24}$$

Then,

$$\frac{\mathrm{d}V(x_t)}{\mathrm{d}t} = x'(t)\Psi(\varepsilon,h)x(t) + 2x'(t)P'_\varepsilon \begin{bmatrix} H_{12} \\ H_{22} \end{bmatrix} x_2(t-h) - x'_2(t-h)Qx_2(t-h) + hx'(t)P'_\varepsilon \begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix} \times [R_1^{-1} + R_2^{-1}][H'_{11} \quad H'_{21}]P_\varepsilon x(t). \tag{25}$$

Eq. (25) and LMI (19) yield (by Schur complements) that $\mathrm{d}V/\mathrm{d}t < 0$ and therefore (14) is asymptotically stable.

(ii) If the "reduced-order" LMI (20) holds for some P_0 and Q, then e.g. for $R_1 = R_2 = I$ and for small enough ε and h the full-order LMI (19) holds for the same P_1, P_2, P_3 and Q and thus, due to (i) of Theorem 1, (14) is robustly asymptotically stable. □

Remark 1. LMI (20) implies A4, since from (20) it follows that for all $x = col\{x_1, x_2\} \in R^n$

$$[x' \quad x'_2]\Gamma col\{x\ x_2\} = x'[P'_0(A+H) + (A'+H')P_0]x < 0.$$

Remark 2. Item (ii) of Theorem 1 gives sufficient conditions for the robust stability of (14) with respect to small ε and h. Note that (20) yields the inequality

$$\begin{bmatrix} A'_{22}P_3 + P_3A_{22} + Q & P_3H_{22} \\ H'_{22}P_3 & -Q \end{bmatrix} < 0$$

that guarantees the stability of the fast system (8) for all delays $g \geqslant 0$.

Remark 3. Given $h > 0$, consider the descriptor system (14) with $\varepsilon = 0$. If (19) holds for $\varepsilon = 0$ (and thus (20) is feasible), then the Lyapunov–Krasovskii functional of (16) with $\varepsilon = 0$ is nonnegative and has a negative-definite derivative. It has been shown recently (Fridman, 2001) that the latter guarantees the asymptotic stability of the descriptor system provided that all the eigenvalues of $A_{22}^{-1}H_{22}$ are inside a unit circle. Moreover, (19) for $\varepsilon = 0$ or (20) implies that the spectrum of $A_{22}^{-1}H_{22}$ is inside a unit circle. Thus, Theorem 1 holds in fact for $\varepsilon \geqslant 0$ (and not only for $\varepsilon > 0$ as it is stated).

LMI (20) guarantees for all $h \geqslant 0$ the asymptotic stability of the following slow (descriptor) system (Fridman, 2001):

$$E_0\dot{x}(t) = \left[A + \begin{pmatrix} H_{11} & 0 \\ H_{21} & 0 \end{pmatrix}\right] x(t) + \begin{bmatrix} H_{12} \\ H_{22} \end{bmatrix} x_2(t-h).$$

Table 1

ε	0.1	0.2	0.3	0.4
h_0	0.142	0.131	0.006	0.001

4.3. Example

Consider (14) of the form

$$\dot{x}_1 = x_2(t) + x_1(t-h), \qquad \varepsilon\dot{x}_2 = -x_2(t) + 0.5x_2(t-h) - 2x_1(t). \tag{26}$$

For $h = 0$ this system is asymptotically stable for all small enough ε since A1 and A2 hold. It is well known (see e.g. Hale & Lunel, 1993) that the fast system $\dot{x}_2(t) = -x_2(t) + 0.5x_2(t-g)$ is asymptotically stable for all g. Thus, the necessary condition for robust stability with respect to small ε and h is satisfied. LMI (20) for this system has a solution. Hence, the system is robustly asymptotically stable with respect to small ε and h. Applying LMI (20) we find that (26) is asymptotically stable e.g. for the values of ε and $h \leqslant h_0$, where ε and h_0 are given in Table 1. For $\varepsilon = 0.5$ and $h = 0$ the system is not stable and LMI (20) has no solution.

The condition of (19) is conservative. Thus for $\varepsilon = 0$ (26) is delay-independently stable (see Fridman, 2001), while (19) for $\varepsilon = 0$ is feasible only for $h \leqslant 0.144$. The conservatism of (19), as well as in the regular case (see e.g. de Souza & Li, 1999; Kolmanovskii et al., 1999), is twofold: the transformed equation (22) is not equivalent to the corresponding differential equation of (14) and the bounds (24) placed upon η_1 and η_2 are wasteful.

5. Conclusions

A new example of a system which can be destabilized by a small delay is given. This system is a singularly perturbed one. An LMI approach has been introduced for singularly perturbed systems and a new delay-dependent stability criterion has been derived for such systems with delay. The results can be easily generalized to the case of multiple delays. Since the LMI conditions are affine in the system matrices, the LMI approach also allows solutions for the uncertain case where the system parameters lie within an uncertainty polytope. The convexity of the LMI with respect to the delays implies that a solution, if it exists, will hold for all delays less than or equal to the one solved for. The method is most suitable for robust stabilization and for other control applications.

The LMI approach of this paper may provide a new impact to stability, stabilization and H_∞ control of time-delay systems, as well as singularly perturbed and descriptor systems with delay. As in the case of regular systems, the LMI stability criterion of this note is conservative. A work is currently being carried out to obtain conditions that are less conservative.

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