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Robust H_2 filtering of linear systems with time delays

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SUMMARY

The problem of robust H_2 estimation of a combination of states of a stationary linear system with time delays is considered. Since the problem is infinite dimensional in nature, an attempt is being made to develop finite dimensional methods that will guarantee a preassigned estimation accuracy. The approach of minimizing the trace of a matrix that overbounds the exact covariance of the estimation error is considered. Sufficient conditions are given in the form of linear matrix inequalities (LMIs). The results are illustrated by a numerical example. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: time-delay systems; robust H_2 filtering; linear matrix inequalities

1. INTRODUCTION

Estimation of linear processes with time-delays has attracted much attention in the past (see, e.g. References [1–7]). Solutions that are infinite dimensional in nature have been obtained there which were based on perfectly known models of the estimated processes. With the development of the H_∞ control theory it has been recognized that it provides efficient tools to deal with system uncertainties. H_∞ filtering problems have then been solved that guarantee a prescribed estimation accuracy in spite of large parameter uncertainty (see, e.g. References [8–11]).

The above mentioned works on H_∞ filtering consider systems without time delays. In Reference [7] a H_∞ filter design for precisely known systems with a single time delayed measurement was introduced. The H_∞ observer design for precisely known systems with state delays was considered in Reference [12], where a sufficient condition based on an algebraic Riccati equation was derived. Robust H_∞ filtering of uncertain systems with state delays has been considered in References [13–15] where both delay-independent and delay-dependent sufficient conditions have been derived.

For retarded type linear systems robust H_2-filtering problem was considered in Reference [17], where delay-independent conditions were derived in the form of LMIs. In the present work we consider more general neutral type systems and we derive less conservative delay-dependent results, which become delay-independent for a particular choice of the parameters. It is

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well-known (see e.g. References [18–20]) that delay-dependent methods for stability and control are based on different model transformations. In our approach we combine neutral with parameterizing model transformations (see Reference [20]). Results are demonstrated in a simple example.

2. PROBLEM FORMULATION

Consider the following asymptotically stable system:

$$\dot{x}(t) - \sum_{i=1}^{2} D_i \dot{x}(t-h_i) = \sum_{j=0}^{2} A_j x(t-h_j) + Bw(t) \tag{1a}$$

$$z(t) = Lx(t) \tag{1b}$$

where $x(t) \in \mathcal{R}^n$ is the system state vector, $0 < h_i,\ i = 1, 2$ are known time delays and $h_0 = 0$, $w(t) \in \mathcal{R}^q$ is the exogenous disturbance signal which is assumed to be Gaussian standard, zero-mean, unit covariance, white noise, independent of $x(s),\ s \in [-h,\ 0],\ h = \max\{h_1,\ h_2\}$, and $z(t) \in \mathcal{R}^p$ is the combination of states to be estimated. We assume, for simplicity, that the initial value $x(s),\ s \in [-h,\ 0]$ is zero. We define

$$P(t) \triangleq \mathcal{E}\{x(t)x^{\mathrm{T}}(t)\} \tag{2}$$

where $\mathcal{E}$ denotes expectation.

To guarantee robustness of the results with respect to small changes of delay we assume:

(A1) The difference equation $x(t) - \sum_{i=1}^{2} D_i x(t-h_i) = 0$ is asymptotically stable for all values of h_i.

A sufficient LMI condition for A1 may be found in Reference [21]. The measurement is given by

$$y(t) = \mathrm{col}\{C_0 x(t), C_1 x(t-h_1), C_2 x(t-h_2)\} + D_{21} w(t) \tag{3}$$

where $y(t) \in \mathcal{R}^r$. The matrices $A_i,\ C_i,\ i = 0, 1, 2\ D_i,\ i = 1, 2$ and D_{21} are constant matrices of appropriate dimensions, $C_i \in \mathcal{R}^{r_i \times n},\ i = 0, 1, 2$.

We first treat the case where these matrices are all known. We then consider the case where the parameters of these matrices are uncertain.

In the first case, where the matrices and the time delays are all known, we consider a Luenberger type observer:

$$\dot{\hat{x}}(t) - \sum_{i=1}^{2} D_i \dot{\hat{x}}(t-h_i) = \sum_{j=0}^{2} A_j \hat{x}(t-h_j) + \sum_{j=0}^{2} K_j(\Pi_j y(t) - C_j \hat{x}(t-h_j)),$$

$$\hat{x}(t) \equiv 0,\ t \in [-h\ 0] \tag{4}$$

where $\Pi_0 = [I_{r_0}\ 0\ 0],\ \Pi_1 = [0\ I_{r_1}\ 0],\ \Pi_2 = [0\ 0\ I_{r_2}]$, and define

$$e(t) = x(t) - \hat{x}(t) \tag{5}$$

We seek the filter gains K_j that minimize an upper-bound on the variance of Le defined by

$$J = \mathrm{trace}\{LP_e L^{\mathrm{T}}\} \tag{6a}$$

where

$$P_e \triangleq \mathscr{E}\{e(t)e(t)^{\mathrm{T}}\} \tag{6b}$$

We begin our analysis by considering the propagation of w through the system of (1).

3. THE PROPAGATION OF w

Denoting the fundamental matrix of (1) ($B = 0$) by $\Phi(t)$ we have that

$$\dot{\Phi}(t) - \sum_{i=1}^{2} D_i \dot{\Phi}(t - h_i) = \sum_{j=0}^{2} A_j \Phi(t - h_j), \;\; \Phi(s) \equiv 0, \;\; \forall\ s < 0, \;\; \Phi(0) = I_n \tag{7}$$

The signal $x(t)$ that corresponds to a specific $w(t)$ is

$$x(t) = \int_0^t \Phi(t - s) B w(s)\,\mathrm{d}s \tag{8}$$

Thus

$$\begin{aligned} P(t) &= \mathscr{E}\left\{\int_0^t \int_0^t \Phi(t - s) B w(s) w^{\mathrm{T}}(\bar{s}) B^{\mathrm{T}} \Phi^{\mathrm{T}}(t - \bar{s})\,\mathrm{d}s\,\mathrm{d}\bar{s}\right\} \\ &= \int_0^t \int_0^t \Phi(t - s) B \delta(s,\ \bar{s}) B^{\mathrm{T}} \Phi^{\mathrm{T}}(t - \bar{s})\,\mathrm{d}s\,\mathrm{d}\bar{s} \\ &= \int_0^t \Phi(t - s) B B^{\mathrm{T}} \Phi^{\mathrm{T}}(t - s)\,\mathrm{d}s \end{aligned} \tag{9}$$

Note also that due to the properties of $\Phi(s)$ we have

$$P(t - h) = \int_0^t \Phi(t - s - h) B B^{\mathrm{T}} \Phi^{\mathrm{T}}(t - s - h)\,\mathrm{d}s \tag{10}$$

For $t \to \infty$ we have

$$P = \int_0^\infty \Phi(\tau) B B^{\mathrm{T}} \Phi(\tau)^{\mathrm{T}}\,\mathrm{d}\tau \tag{11}$$

It follows from (7) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[\Phi(t) + \sum_{i=1}^{2} F_i \int_{t-h_i}^{t} \Phi(s)\,\mathrm{d}s - \sum_{i=1}^{2} D_i \Phi(t - h_i)\right] \\ &= \left(A_0 + \sum_{i=1}^{2} F_i\right)\Phi(t) + \sum_{i=1}^{2} (A_i - F_i)\Phi(t - h_i) \end{aligned} \tag{12}$$

The latter combines the neutral type and the parameterized model transformations of Reference [20]. For $F_i = 0,\ i = 1, 2$ we obtain below delay-independent results and for $F_i = A_i$ we have the standard neutral type representation.

The result of (12) holds for all $F_i \in \mathscr{R}^{n \times n}$. In the sequel we restrict F_i as follows:

(A2) The eigenvalues of the matrix $A_0 + \sum_{i=1}^{2} F_i$ all reside in the left half of the complex plane.

Applying the model transformation of (12) we consider the following:

$$Q = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\tau}\Bigg[\left(\Phi(\tau) + \sum_{i=1}^{2} F_i \int_{\tau-h_i}^{\tau} \Phi(s)\,\mathrm{d}s - \sum_{i=1}^{2} D_i\Phi(\tau-h_i)\right)BB^{\mathrm{T}} \times \left(\Phi^{\mathrm{T}}(\tau) + \sum_{i=1}^{2}\int_{\tau-h_i}^{\tau}\Phi^{\mathrm{T}}(s)\,\mathrm{d}sF_i^{\mathrm{T}} - \sum_{i=1}^{2}\Phi^{\mathrm{T}}(\tau-h_i)D_i^{\mathrm{T}}\right)\Bigg]\mathrm{d}\tau \tag{13}$$

Due to the properties of $\Phi(t)$, bearing in mind that system (1) is asymptotically stable, it is readily found that $Q = -BB^{\mathrm{T}}$ and we thus obtain the following:

$$\begin{aligned} -BB^{\mathrm{T}} = & \int_0^\infty \left[\left(A_0 + \sum_{i=1}^{2} F_i\right)\Phi(\tau) + \sum_{i=1}^{2}(A_i - F_i)\Phi(\tau-h_i)\right] \\ & \times BB^{\mathrm{T}}\left[\Phi^{\mathrm{T}}(\tau) + \sum_{i=1}^{2}\int_{\tau-h_i}^{\tau}\Phi^{\mathrm{T}}(s)\,\mathrm{d}sF_i^{\mathrm{T}} - \sum_{i=1}^{2}\Phi^{\mathrm{T}}(\tau-h_i)D_i^{\mathrm{T}}\right]\mathrm{d}\tau \\ & + \int_0^\infty \left[\Phi(\tau) + \sum_{i=1}^{2} F_i\int_{\tau-h_i}^{\tau}\Phi(s)\,\mathrm{d}s - \sum_{i=1}^{2} D_i\Phi(\tau-h_i)\right] \\ & \times BB^{\mathrm{T}}\left[\Phi^{\mathrm{T}}(\tau)\left(A_0^{\mathrm{T}} + \sum_{i=1}^{2} F_i^{\mathrm{T}}\right) + \sum_{i=1}^{2}\Phi^{\mathrm{T}}(\tau-h_i)(A_i^{\mathrm{T}} - F_i^{\mathrm{T}})\right]\mathrm{d}\tau \end{aligned} \tag{14}$$

Using (11) we thus have

$$\Gamma(P) \triangleq BB^{\mathrm{T}} + \left(A_0 + \sum_{i=1}^{2} F_i\right)P + P\left(A_0^{\mathrm{T}} + \sum_{i=1}^{2} F_i^{\mathrm{T}}\right) + \sum_{i=1}^{2}\sum_{j=0}^{2}\mu_{i,j} + \sum_{i=0}^{2}\sum_{j=1}^{2}\eta_{i,j} = 0 \tag{15a}$$

where

$$\begin{aligned} \mu_{i,0} = & -\left(A_0 + \sum_{j=1}^{2} F_j\right)\int_0^\infty \Phi(\tau)BB^{\mathrm{T}}\Phi^{\mathrm{T}}(\tau-h_i)D_i^{\mathrm{T}}\mathrm{d}\tau \\ & - D_i\int_0^\infty \Phi(\tau-h_i)BB^{\mathrm{T}}\Phi^{\mathrm{T}}(\tau)\mathrm{d}\tau\left(A_0^{\mathrm{T}} + \sum_{j=1}^{2} F_j^{\mathrm{T}}\right) \\ & + (A_i - F_i)\int_0^\infty \Phi(\tau-h_i)BB^{\mathrm{T}}\Phi^{\mathrm{T}}(\tau)\mathrm{d}\tau + \int_0^\infty \Phi(\tau)BB^{\mathrm{T}}\Phi^{\mathrm{T}}(\tau-h_i)\mathrm{d}\tau(A_i^{\mathrm{T}} - F_i^{\mathrm{T}}) \end{aligned} \tag{15b}$$

$$\begin{aligned} \mu_{i,j} = & -(A_i - F_i)\int_0^\infty \Phi(\tau-h_i)BB^{\mathrm{T}}\Phi^{\mathrm{T}}(\tau-h_j)D_j^{\mathrm{T}}\mathrm{d}\tau \\ & - D_j\int_0^\infty \Phi(\tau-h_j)BB^{\mathrm{T}}\Phi^{\mathrm{T}}(\tau-h_i)\mathrm{d}\tau(A_i^{\mathrm{T}} - F_i^{\mathrm{T}}), \end{aligned} \tag{15c}$$

$$i = 1, 2, \quad j = 1, 2$$

$$\eta_{0,j} = \left(A_0 + \sum_{i=1}^{2} F_i\right) \int_0^\infty \Phi(\tau)BB^{\mathrm{T}} \int_{\tau-h_j}^{\tau} \Phi^{\mathrm{T}}(s)\mathrm{d}s\, \mathrm{d}\tau F_j^{\mathrm{T}} + F_j \int_0^\infty \times \int_{\tau-h_j}^{\tau} \Phi(s)\mathrm{d}sBB^{\mathrm{T}}\Phi^{\mathrm{T}}(\tau)\mathrm{d}\tau \left(A_0^{\mathrm{T}} + \sum_{i=1}^{2} F_i^{\mathrm{T}}\right) \quad (15d)$$

$$\eta_{i,j} = (A_i - F_i) \int_0^\infty \Phi(\tau - h_i)BB^{\mathrm{T}} \int_{\tau-h_j}^{\tau} \Phi^{\mathrm{T}}(s)\mathrm{d}s\, \mathrm{d}\tau F_j^{\mathrm{T}} + F_j \int_0^\infty \times \int_{\tau-h_j}^{\tau} \Phi(s)\mathrm{d}sBB^{\mathrm{T}}\Phi^{\mathrm{T}}(\tau - h_i)\mathrm{d}\tau(A_i^{\mathrm{T}} - F_i^{\mathrm{T}}) \quad (15e)$$

Using the fact that for any two matrices X and Y that are of the appropriate dimensions the following holds for any positive scalar ε:

$$XY^{\mathrm{T}} + YX^{\mathrm{T}} \leqslant \varepsilon XX^{\mathrm{T}} + \varepsilon^{-1}YY^{\mathrm{T}} \quad (16)$$

Applying the latter to (15a) the following inequality is obtained for any positive scalars $q_{i,j}$, $\bar{r}_i$ and $r_{j,i}$, $i = 1, 2,\ j = 0, 1, 2$:

$$\begin{aligned} 0 = \Gamma(P) \leqslant\ & BB^{\mathrm{T}} + \left(A_0 + \sum_{i=1}^{2} F_i\right)P + P\left(A_0^{\mathrm{T}} + \sum_{i=1}^{2} F_i^{\mathrm{T}}\right) \\ & + \sum_{j=1}^{2}\sum_{i=1}^{2} \left[q_{i,j}(A_i - F_i)P(A_i^{\mathrm{T}} - F_i^{\mathrm{T}}) + q_{i,j}^{-1}D_jPD_j^{\mathrm{T}}\right] \\ & + \sum_{i=1}^{2} [q_{i,0}(A_i - F_i)P(A_i^{\mathrm{T}} - F_i^{\mathrm{T}})] + q_{i,0}^{-1}P] \\ & + \sum_{j=1}^{2} h_j\left[r_{0,j}(A_0 + \sum_{i=1}^{2} F_i)P\left(A_0^{\mathrm{T}} + \sum_{i=1}^{2} F_i^{\mathrm{T}}\right) + r_{0,j}^{-1}F_jPF_j^{\mathrm{T}}\right] \\ & + \sum_{i=1}^{2}\sum_{j=1}^{2} h_j\left[r_{i,j}(A_i - F_i)P(A_i^{\mathrm{T}} - F_i^{\mathrm{T}}) + r_{i,j}^{-1}F_jPF_j^{\mathrm{T}}\right] \\ & + \sum_{j=1}^{2}\left[\bar{r}_j\left(A_0 + \sum_{i=1}^{2} F_i\right)P\left(A_0^{\mathrm{T}} + \sum_{i=1}^{2} F_i^{\mathrm{T}}\right) + \bar{r}_j^{-1}D_jPD_j^{\mathrm{T}}\right] \\ = \bar{\Psi} & + \sum_{i=1}^{2} \sigma_i^2(A_i - F_i)P(A_i^{\mathrm{T}} - F_i^{\mathrm{T}}) + \beta^2\left(A_0 + \sum_{i=1}^{2} F_i\right)P\left(A_0^{\mathrm{T}} + \sum_{i=1}^{2} F_i^{\mathrm{T}}\right) \\ & + \sum_{j=1}^{2} \bar{\sigma}_j^2 F_jPF_j^{\mathrm{T}} \triangleq \bar{\Gamma}(P) \end{aligned} \quad (17)$$

where

$$\bar{\Psi} = BB^{\mathrm{T}} + A_0 P + PA_0^{\mathrm{T}} + \sum_{i=1}^{2} \left[F_i P + PF_i^{\mathrm{T}} \right]$$

$$+ \sum_{j=1}^{2} \left(\bar{r}_j^{-1} + \sum_{i=1}^{2} q_{i,j}^{-1} \right) D_j P D_j^{\mathrm{T}} + P \sum_{i=1}^{2} q_{i,0}^{-1} \tag{18a}$$

$$\sigma_i^2 = \sum_{j=0}^{2} q_{i,j} + \sum_{j=1}^{2} h_j r_{i,j}, \quad i = 1, 2 \tag{18b}$$

$$\beta^2 = \sum_{j=1}^{2} (h_j r_{0,j} + \bar{r}_j), \tag{18c}$$

$$\bar{\sigma}_j^2 = \sum_{i=0}^{2} r_{i,j}^{-1} h_j, \quad j = 1, 2. \tag{18d}$$

We represent $\bar{\Gamma}(P)$ in the form

$$\bar{\Gamma}(P) = \left(A_0 + \sum_{i=1}^{2} F_i \right) P + P \left(A_0^{\mathrm{T}} + \sum_{i=1}^{2} F_i^{\mathrm{T}} \right) + \sum_{i=1}^{8} M_i P M_i^{\mathrm{T}} + BB^{\mathrm{T}}$$

Considering next the inequality

$$\bar{\Gamma}(\hat{P}) < 0 \tag{19}$$

we show that for any solution $\hat{P}$ of (19) $\Delta = \hat{P} - P > 0$, i.e. $\hat{P}$ constitutes an upper-bound for the covariance matrix P. Since $\bar{\Gamma}(P) \geqslant 0$, we have

$$\bar{\Gamma}(\hat{P}) - \bar{\Gamma}(P) = \left(A_0 + \sum_{i=1}^{2} F_i \right) \Delta + \Delta \left(A_0^{\mathrm{T}} + \sum_{i=1}^{2} F_i^{\mathrm{T}} \right) + \sum_{i=1}^{8} M_i \Delta M_i^{\mathrm{T}} < 0$$

Assume to the contrary that Δ has a negative eigenvalue, and consider the family $S_t = (1-t)\hat{P} + t\Delta, \; 0 \leqslant t \leqslant 1$, which, clearly satisfies

$$\left(A_0 + \sum_{i=1}^{2} F_i \right) S_t + S_t \left(A_0^{\mathrm{T}} + \sum_{i=1}^{2} F_i^{\mathrm{T}} \right) + \sum_{i=1}^{8} M_i S_t M_i^{\mathrm{T}} < 0, \;\; \forall t \in [0, 1]$$

Then there exists $t = a \in (0, 1)$ such that $S_a \geqslant 0$ and has a zero eigenvalue. Let ξ be the eigenvector that corresponds to the zero eigenvalue. Then

$$\xi^{\mathrm{T}} \left[\left(A_0 + \sum_{i=1}^{2} F_i \right) S_a + S_a \left(A_0^{\mathrm{T}} + \sum_{i=1}^{2} F_i^{\mathrm{T}} \right) + \sum_{i=1}^{8} M_i S_a M_i^{\mathrm{T}} \right] \xi = \xi^{\mathrm{T}} \left(\sum_{i=1}^{8} M_i S_a M_i^{\mathrm{T}} \right) \xi \geqslant 0$$

which contradicts to the previous inequality. Hence $\hat{P} > P$.

By applying Schur formula to $\bar{\Gamma}(\hat{P})<0$ and denoting $F_i\hat{P}\triangleq W_i,\ i=1,2$ we thus obtain the following.

Lemma 1

Consider the asymptotically stable system (1). If, for some positive scalars $q_{i,j},\ \bar{r}_i$ and $r_{j,i}$, $i=1,2,\ j=0,1,2$, there exists a solution $(\hat{P},\ W_1,\ W_2)$ to the following LMI

$$\begin{bmatrix} \Psi & \sigma_1(A_1\hat{P}-W_1) & \sigma_2(A_2\hat{P}-W_2) & \bar{\sigma}_1W_1 & \bar{\sigma}_2W_2 & \beta(A_0\hat{P}+W_1+W_2) \\ * & -\hat{P} & 0 & 0 & 0 & 0 \\ * & * & -\hat{P} & 0 & 0 & 0 \\ * & * & * & -\hat{P} & 0 & 0 \\ * & * & * & * & -\hat{P} & 0 \\ * & * & * & * & * & -\hat{P} \end{bmatrix}<0 \tag{20a}$$

where

$$\Psi = BB^{\mathrm{T}} + A_0\hat{P} + \hat{P}A_0^{\mathrm{T}} + \sum_{i=1}^{2}[W_i+W_i^{\mathrm{T}}] + \sum_{j=1}^{2}\left(\bar{r}_j^{-1}+\sum_{i=1}^{2} q_{i,j}^{-1}\right)D_j\hat{P}D_j^{\mathrm{T}} + \hat{P}\sum_{i=1}^{2} q_{i,0}^{-1} \tag{20b}$$

then

$$P<\hat{P} \tag{21}$$

where P is defined in (11).

Remark 1

In the case where $D_i=0,\ i=1,2$ the above inequality still holds, omitting the term that contains D_i in Ψ and deleting $\bar{r}_i$ and $q_{i,j},\ i=1,2,\ j=1,2$ in the definitions of (18b,c).

A simpler, although to some extent more conservative, result can be obtained by substituting in Lemma 1 $F_i=A_i,\ i=0,1,2$. The result, which corresponds to the neutral model transformation [18, 20] is the following.

Corollary 1

Consider the asymptotically stable system (1). If, for some positive scalars $\bar{r}_j$ and $r_{0,j},\ j=1,2$, there exists a solution $\hat{P}$ to the following LMI:

$$\begin{bmatrix} \Psi_S & \bar{\sigma}_1A_1\hat{P} & \bar{\sigma}_2A_2\hat{P} & \beta(\sum_{i=0}^{2} A_i)\hat{P} \\ * & -\hat{P} & 0 & 0 \\ * & * & -\hat{P} & 0 \\ * & * & * & -\hat{P} \end{bmatrix}<0 \tag{22a}$$

where

$$\Psi_S = BB^{\mathrm{T}} + \sum_{i=0}^{2} A_i\hat{P} + \hat{P}\sum_{i=0}^{2} A_i^{\mathrm{T}} + \sum_{j=1}^{2} \bar{r}_j^{-1} D_j\hat{P}D_j^{\mathrm{T}} \tag{22b}$$

$$\bar{\sigma}_j^2 = r_{0,j}^{-1} h_j$$

and

$$\beta^2 = \sum_{j=1}^{2} (h_j r_{0,j} + \bar{r}_j) \tag{22c}$$

then (22) is satisfied where P is defined in (11).

The delay-independent inequality: The delay-independent result is obtained taking $F_i = 0,\ i = 1,2$ and $r_{i,j} = 0,\ i = 0,1,2,\ j = 1,2$ in the above. The resulting delay-independent upper bound to the covariance matrix of the stationary state-vector x of (1) is given by the following.

Corollary 2
If, for some positive scalars $q_{i,j},\ \bar{r}_i,\ j = 0,1,2,\ i = 1,2$, there exists a solution $\hat{P}$ to the following LMI:

$$\begin{bmatrix} \Psi & \hat{\sigma}_1 A_1\hat{P} & \hat{\sigma}_2 A_2\hat{P} & \hat{\beta}A_0\hat{P} \\ * & -\hat{P} & 0 & 0 \\ * & * & -\hat{P} & 0 \\ * & * & * & -\hat{P} \end{bmatrix} < 0 \tag{23}$$

where Ψ is defined in (20b), and where $W_i = 0,\ \hat{\sigma}_i^2 = \sum_{j=0}^{2} q_{i,j},\ i = 1,2$ and $\hat{\beta}^2 = \bar{r}_1 + \bar{r}_2$, then (21) holds independently of the delay lengths h_1 and h_2.

The affinity of (20a) in the system matrices allows bounding the covariance of $x(t)$ also in the case where these matrices and the time delays h_1 and h_2 are uncertain.

Denoting

$$\Omega = [A_0 \quad A_1 \quad A_2 \quad D_1 \quad D_2 \quad B \quad L]$$

we assume that $\Omega \in \mathcal{C}o\{\Omega_j,\ j = 1,\ldots,N_\Omega\}$, namely,

$$\Omega = \sum_{j=1}^{N_\Omega} f_j\Omega_j \quad \text{for some} \quad 0 \leqslant f_j \leqslant 1, \ \sum_{j=1}^{N_\Omega} f_j = 1 \tag{24}$$

where the N_Ω vertices of the polytope are described by

$$\Omega_j = [A_0^{(j)} \quad A_1^{(j)} \quad A_2^{(j)} \quad D_1^{(j)} \quad D_2^{(j)} \quad B^{(j)} \quad L^{(j)}]$$

Assuming also that

$$h_i \in [0\ \bar{h}_i], \quad i = 1, 2 \tag{25}$$

we obtain the following:

Theorem 1

Consider the asymptotically stable system (1) where the time delays h_1 and h_2 satisfy (25) and the system matrices reside anywhere in the uncertainty polytope Ω of (24). The matrix $\bar{H}$ is an upper-bound to the covariance of $z(t)$ if for some positive scalars $q_{i,j},\ \bar{r}_i$ and $r_{j,i},\ i = 1, 2,\ j = 0, 1, 2$, there exists a solution $(\hat{P},\ W_1^{(j)},\ W_2^{(j)},\ j = 1, 2, \ldots, N_\Omega, \bar{H})$ to the following set of $2N_\Omega$ LMIs:

$$\begin{bmatrix} \Psi^{(j)} & \sigma_1(A_1^{(j)}\hat{P} - W_1) & \sigma_2(A_2^{(j)}\hat{P} - W_2) & \bar{\sigma}_1 W_1 & \bar{\sigma}_2 W_2 & \beta(A_0^{(j)}\hat{P} + W_1 + W_2) \\ * & -\hat{P} & 0 & 0 & 0 & 0 \\ * & * & -\hat{P} & 0 & 0 & 0 \\ * & * & * & -\hat{P} & 0 & 0 \\ * & * & * & * & -\hat{P} & 0 \\ * & * & * & * & * & -\hat{P} \end{bmatrix} < 0 \tag{26a}$$

and

$$\begin{bmatrix} H & L^{(j)}\hat{P} \\ \hat{P}L^{(j)T} & \hat{P} \end{bmatrix} \geqslant 0 \ \ j = 1, 2, \ldots, N_\Omega \tag{26b}$$

where

$$\begin{aligned} \Psi^{(j)} = {} & B^{(j)}B^{(j)T} + A_0^{(j)}\hat{P} + \hat{P}A_0^{(j)T} + \sum_{i=1}^{2}[W_i^{(j)} + W_i^{(j)T}] \\ & + \sum_{j=1}^{2}\left(\bar{r}_j^{-1} + \sum_{i=1}^{2} q_{i,j}^{-1}\right) D_j^{(j)}\hat{P}D_j^{(j)T} \\ & + P\sum_{i=1}^{2} q_{i,0}^{-1} \end{aligned} \tag{27}$$

and where σ_i, β, and $\bar{\sigma}_i$, $i = 1, 2$, are defined in (18b)–(18d).

Corollaries 1 and 2 can be similarly generalized to the case of polytopic uncertainty.

4. A SIMPLE OBSERVER

The results of Section 3 correspond, in a way to the expressions that have to be solved in the mixed H_∞/H_2 BRL when the bound on the disturbance attenuation level tends to infinity [23]. Similarly to the BRL, they can be used, after some modifications, to minimize a bound on the covariance of the error that is encountered while estimating z of (1b).

The above was based on the assumption that the system considered is asymptotically stable. In the sequel we shall apply the results of Lemma 2 to the system that describes the evolution of the estimation error. Verification of the stability of the latter system can be made by applying the known conditions for stability (see e.g. References [18, 16, 20, 21]). In the remark below we give one of such sufficient conditions for stability.

Remark 2
The following result provides a sufficient condition for stability.

Consider system (1). Assume that for some positive scalars $q_{i,j}$, $\bar{r}_i$ and $r_{j,i}$, $i = 1, 2$, $j = 0, 1, 2$, there exists a solution $(\hat{P}, W_1, W_2)$ to (20a) where β, $\bar{\sigma}_i$, $i = 1, 2, \sigma_i$ and Ψ are defined in (18b–d) and (20b). System (1) is asymptotically stable (and thus inequality (22) is satisfied) if the equation

$$\mathscr{D}(x_t) \triangleq x(t) - \sum_{i=1}^{2} D_i^{\mathrm{T}} x(t-h_i) + \sum_{i=1}^{2} \hat{P}^{-1} W_i^{\mathrm{T}} \int_{t-h_i}^{t} x(s)\,\mathrm{d}s = 0 \tag{28}$$

is asymptotically stable.

This claim can be proved by considering the adjoint of (1a) with $B = 0$

$$\dot{x}(t) - \sum_{i=1}^{2} D_i^{\mathrm{T}} \dot{x}(t-h_i) = \sum_{j=0}^{2} A_j^{\mathrm{T}} x(t-h_j)$$

and applying to the latter system the neutral type and the parameterized model transformations [20] (similar to (12)):

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - \sum_{i=1}^{2} D_i^{\mathrm{T}} x(t-h_i) + \sum_{i=1}^{2} F_i^{\mathrm{T}} \int_{t-h_i}^{t} x(s)\,\mathrm{d}s\right] \\ &= \left(A_0^{\mathrm{T}} + \sum_{i=1}^{2} F_i^{\mathrm{T}}\right) x(t) + \sum_{i=1}^{2} (A_i^{\mathrm{T}} - F_i^{\mathrm{T}}) x(t-h_i) \end{aligned} \tag{29}$$

The Lyapunov–Krasovskii functional that corresponds to (33) may be chosen in the form

$$\begin{aligned} V(t) \triangleq{} & \mathscr{D}^{\mathrm{T}}(x_t) \hat{P} \mathscr{D}(x_t) + \sum_{i=1}^{2} \left(\bar{r}_i^{-1} + \sum_{j=1}^{2} q_{j,i}^{-1}\right) \int_{t-h_i}^{t} x^{\mathrm{T}}(s) D_i \hat{P} D_i^{\mathrm{T}} x(s)\,\mathrm{d}s \\ & + \sum_{i=1}^{2} \sum_{j=0}^{2} q_{i,j}^{-1} \int_{t-h_i}^{t} x^{\mathrm{T}}(s)(A_i - F_i)\hat{P}(A_i^{\mathrm{T}} - F_i^{\mathrm{T}}) x(s)\,\mathrm{d}s \\ & + \sum_{i=1}^{2} \left(\sum_{j=0}^{2} r_{ji}^{-1}\right) \int_{-h_i}^{0} \int_{t+\theta}^{t} x^{\mathrm{T}}(s) F_i \hat{P} F_i^{\mathrm{T}} x(s)\,\mathrm{d}s\,\mathrm{d}\theta \end{aligned} \tag{30}$$

Applying standard arguments it can be shown that $\dot{V}<0$ if (20a) with $W_i = F_i\hat{P}$ is feasible. Therefore, under the assumption of stability of (28), (1) is asymptotically stable [18, 20].

A sufficient condition for the stability of (32) is

$$\sum_{i=1}^{2}[|D_i| + h_i|W_i\hat{P}^{-1}|] < 1 \tag{31}$$

where $|\cdot|$ is any matrix norm.

Considering delay-independent conditions (i.e. $W_i = 0$), A1 is equivalent to the stability of (28) and therefore the LMI (20a) implies the asymptotic stability of (1) and the fact that A_0 is Hurwitz. In this case, therefore, there is no need to check the stability of the resulting system.

In the case of neutral model transformation (i.e. $A_i = F_i, W_i = \hat{P}A_i$), the LMI (20a) implies that $\sum_{i=0}^{2} A_i$ is Hurwitz (this follows from $\Psi<0$, where Ψ is given by (20b)). The latter fact will be used in Section 5 to guarantee the stability of the estimation error equation.

Multiplying (20a) by diag$\{Q, Q, Q, Q, Q, Q\}$, from both sides, where $Q \stackrel{\triangle}{=} \hat{P}^{-1}$ and denoting $\hat{W}_i = QF_i$, $i = 1, 2$, we obtain

$$\begin{bmatrix} \hat{\Psi} & \sigma_1(QA_1-\hat{W}_1) & \sigma_2(QA_2-\hat{W}_2) & \bar{\sigma}_1\hat{W}_1 & \bar{\sigma}_2\hat{W}_2 & \beta(QA_0+\hat{W}_1+\hat{W}_2) & QB & \hat{\sigma}_1QD_1 & \hat{\sigma}_2QD_2 \\ * & -Q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -Q & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -Q & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -Q & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & * & -Q & 0 \\ * & * & * & * & * & * & * & * & -Q \end{bmatrix} < 0 \tag{32a}$$

where

$$\hat{\Psi} = QA_0 + A_0^{\mathrm{T}}Q + \sum_{i=1}^{2}[\hat{W}_i + \hat{W}_i^{\mathrm{T}}] + Q\sum_{i=1}^{2} q_{i,0}^{-1} \tag{32b}$$

and

$$\hat{\sigma}_i^2 = \bar{r}_i^{-1} + \sum_{j=1}^{2} q_{j,i}^{-1}, \quad i = 1, 2 \tag{32c}$$

Similarly to the derivation of the delay-independent result of Corollary 2 the delay-independent equivalent of (32) is obtained by substituting in the latter $W_i = 0$ and changing σ_i^2 and β^2 to $\sum_{j=0}^{2} q_{i,j}$ and $\sum_{j=1}^{2} \bar{r}_j$, respectively.

Assuming that the parameters of the system in (1) are all known we seek a Luenberger type observer in the form of (4). The estimation error signal $e(t)$ satisfies then

$$\dot{e}(t) - \sum_{i=1}^{2} D_i \dot{e}(t-h_i) = \sum_{j=0}^{2} (A_j - K_j C_j) e(t-h_j) + (B - [K_0 \quad K_1 \quad K_2] D_{21}) w(t) \tag{33}$$

and we seek $K_j,\ j = 0,1,2$ that minimize an upper-bound on the covariance of $Le(t)$.

The result of Lemma 1 and (32) can be readily applied to the system of (33). We thus obtain the following two filters.

Delay-independent filter:

Theorem 2

Assume A1. Consider the stationary system of (1) and (3) and the filter of (4). For a prescribed positive scalar δ the filter achieves $variance\{e(t)\} < \delta$, independently of the delays in (1), if for some positive scalars $q_{i,j}$, $\bar{r}_i$ and $r_{j,i}$, $i = 1,2$, $j = 0,1,2$, there exists a solution (Q, Y_i, $i = 0,1,2$, $\bar{H}$) to the following 3 LMIs:

$$\begin{bmatrix} \Psi_{\mathrm{ind}} & \sigma_1(QA_1 - Y_1C_1) & \sigma_2(QA_2 - Y_2C_2) & \beta(QA_0 - Y_0C_0) & QB - [Y_0 \ Y_1 \ Y_2]D_{21} & \hat{\sigma}_1 QD_1 & \hat{\sigma}_2 QD_2 \\ * & -Q & 0 & 0 & 0 & 0 & 0 \\ * & * & -Q & 0 & 0 & 0 & 0 \\ * & * & * & -Q & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -Q & 0 \\ * & * & * & * & * & * & -Q \end{bmatrix} < 0 \tag{34a}$$

$$\begin{bmatrix} \bar{H} & L \\ L^{\mathrm{T}} & Q \end{bmatrix} > 0 \tag{34b}$$

$$\mathrm{trace}\{\bar{H}\} < \delta \tag{34c}$$

where

$$\Psi_{\mathrm{ind}} = QA_0 - Y_0C_0 + A_0^{\mathrm{T}}Q - C_0^{\mathrm{T}}Y_0^{\mathrm{T}} + Q\sum_{i=1}^{2} q_{i,0}^{-1} \tag{35a}$$

$$\sigma_i^2 = \sum_{j=0}^{2} q_{i,j}, \quad i = 1,2 \tag{35b}$$

$$\beta^2 = \sum_{j=1}^{2} \bar{r}_j \tag{35c}$$

and where $\hat{\sigma}_i,\ i = 1,2$ are defined in (32c).

If a solution to (34a–c) exists, then the filter gains that achieve the prescribed bound on the variance of the estimation error are given by

$$K_j = Q^{-1} Y_j, \;\; j = 0, 1, 2 \tag{36}$$

and the estimation error equation (33) is asymptotically stable.

The above solution, if it exists, will guarantee the required error variance bound independently of h_1 and h_2. This independency on the delay length may lead to a considerable overdesign. To reduce the possible conservatism of the filter of Theorem 2 we address the general result of Lemma 1 and (32), once again for the case where the parameters of the system are all known.

Delay-dependent filter:

Theorem 3

Assume A1. Consider the stationary system of (1) and (3) and the filter of (4). For a prescribed positive scalar δ assume that for some positive scalars $q_{i,j}$, $\bar{r}_i$ and $r_{j,i}$, $i = 1, 2$, $j = 0, 1, 2$, there exists a solution (Q, W_1, W_2 Y_i, $i = 0, 1, 2$, $\bar{H}$) where Q, W_1, $W_2 \in \mathcal{R}^{n\times n}$, $Y_j \in \mathcal{R}^{n\times r_j}$, $j = 0, 1, 2$, and $\bar{H} \in \mathcal{R}^{p\times p}$ to the following three LMIs:

$$\begin{bmatrix} \Psi_{dep} & \sigma_1(QA_1 - \hat{W}_1 - Y_1C_1) & \sigma_2(QA_2 - \hat{W}_2 - Y_2C_2) & \bar{\sigma}_1\hat{W}_1 & \bar{\sigma}_2\hat{W}_2 & \beta(QA_0 + \hat{W}_1 - Y_0C_0 + \hat{W}_2) & QB - [Y_0\;\; Y_1\;\; Y_2]D_{21} & \hat{\sigma}_1 QD_1 & \hat{\sigma}_2 QD_2 \\ * & -Q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -Q & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -Q & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -Q & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q & 0 & 0 & 0 \\ * & * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & * & -Q & 0 \\ * & * & * & * & * & * & * & * & -Q \end{bmatrix} < 0 \tag{37a}$$

$$\begin{bmatrix} \bar{H} & L \\ L^{\mathrm{T}} & Q \end{bmatrix} \geqslant 0 \tag{37b}$$

$$\mathrm{trace}\{\bar{H}\} < \delta \tag{37c}$$

where

$$\Psi_{\mathrm{dep}} = QA_0 + A_0^{\mathrm{T}} Q - Y_0 C_0 - C_0^{\mathrm{T}} Y_0^{\mathrm{T}}$$

$$+ \sum_{i=1}^{2} [\hat{W}_i + \hat{W}_i^{\mathrm{T}}] + Q \sum_{i=1}^{2} q_{i,0}^{-1} \tag{38a}$$

$$\sigma_i^2 = \sum_{j=0}^{2} q_{i,j} + \sum_{j=0}^{2} h_j r_{i,j}, \quad i = 1, 2 \tag{38b}$$

$$\beta^2 = \sum_{j=1}^{2} (h_j r_{0,j} + \bar{r}_j) \tag{38c}$$

$\bar{\sigma}_i,\ i = 1, 2$ are defined in (18d) and $\hat{\sigma}_i$ is given in (32c).

Then the filter gains that achieve the variance $(e(t)) < \delta$ are given by:

$$K_j = Q^{-1} Y_j, \quad j = 0, 1, 2 \tag{39}$$

if the resulting estimation error equation (33) is asymptotically stable.

A simpler result that corresponds to neutral model transformation ($F_i = 0$) is derived from the latter theorem by choosing $\hat{W}_i = QA_i - Y_i C_i,\ i = 0, 1, 2$.

Corollary 1

[Assume A1] Consider the stationary system of (1) and (3) and the filter of (4). For a prescribed positive scalar δ the filter achieves *variance*$(e(t)) < \delta$ if for some positive scalars $\bar{r}_j$ and $r_{0,j},\ j = 1, 2$, there exists a solution $(Q,\ Y_i,\ i = 0, 1, 2,\ \bar{H})$ where $Q \in \mathscr{R}^{n \times n}$, $Y_j \in \mathscr{R}^{n \times r_j}$,

$j=0,1,2$, and $\bar{H} \in \mathcal{R}^{p\times p}$ to the following three LMIs:

$$\begin{bmatrix} \Psi_{\text{dep},S} & \bar{\sigma}_1(QA_1 - Y_1C_1) & \bar{\sigma}_2(QA_2 - Y_2C_2) & \beta Q \sum_{i=0}^{2} (A_i - Y_iC_i) & QB - [Y_0\ \ Y_1\ \ Y_2]D_{21} & \hat{\sigma}_1 QD_1 & \hat{\sigma}_2 QD_2 \\ * & -Q & 0 & 0 & 0 & 0 & 0 \\ * & * & -Q & 0 & 0 & 0 & 0 \\ * & * & * & -Q & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -Q & 0 \\ * & * & * & * & * & * & -Q \end{bmatrix} < 0 \tag{40a}$$

$$\begin{bmatrix} \bar{H} & L \\ L^{\mathrm{T}} & Q \end{bmatrix} \geqslant 0 \tag{40b}$$

$$\operatorname{trace}\{\bar{H}\} < \delta \tag{40c}$$

where

$$\Psi_{\text{dep},S} = \sum_{i=0}^{2} (QA_i + A_i^{\mathrm{T}}Q - Y_iC_i - C_i^{\mathrm{T}}Y_i^{\mathrm{T}}) \tag{41a}$$

$$\bar{\sigma}_j^2 = r_{0,j}^{-1}h_j, \quad j=1,2 \tag{41b}$$

$$\beta^2 = \sum_{j=1}^{2} (h_j r_{0,j} + \bar{r}_j) \tag{41c}$$

$$\hat{\sigma}_i^2 = \bar{r}_i^{-1}, \quad i=1,2 \tag{41d}$$

If a solution to (40) exists, then the filter gains that achieve the prescribed bound on the variance of the estimation error are given by (39) under the condition that (33) is asymptotically stable.

The problem with the delay-dependent and the delay-independent results of Theorems 3 and 2 is that they are aimed at systems with perfectly known parameters and for given delay lengths. The affinity of the results of Section 3 in the system matrices and the delay length cannot be used to obtain robust filtering designs due to the special filter structure we used in (4) that enables an exact cancellation of the system states in the equation that describes the error dynamics (see (33)). When uncertainty is encountered a different filter is used.

5. ROBUST H_2 FILTERING

We consider the stationary system of (1) and (3) where the time delays h_1 and h_2 satisfy (25). We assume that the system matrices reside anywhere in the following uncertainty polytope $\bar{\Omega}$ of

$$\bar{\Omega} = [A_0 \quad A_1 \quad A_2 \quad D_1 \quad D_2 \quad C_0 \quad C_1 \quad C_2 \quad B \quad D_{21} \quad L]$$

$\bar{\Omega} \in \mathscr{C}o\{\bar{\Omega}_j,\ j = 1, \ldots, N\}$, namely,

$$\bar{\Omega} = \sum_{j=1}^{N} f_j \bar{\Omega}_j \quad \text{for some} \quad 0 \leqslant f_j \leqslant 1, \ \sum_{j=1}^{N} f_j = 1 \tag{42}$$

where the N vertices of the polytope are described by

$$\bar{\Omega}_j = [A_0^{(j)} \quad A_1^{(j)} \quad A_2^{(j)} \quad D_1^{(j)} \quad D_2^{(j)} \quad C_0^{(j)} \quad C_1^{(j)} \quad C_2^{(j)} \quad B^{(j)} \quad D_{21}^{(j)} \quad L^{(j)}]$$

We seek a filter of the form

$$\dot{\hat{x}}(t) = A_f \hat{x}(t) + B_f y(t), \quad \hat{x}(0) = 0; \quad \hat{z}(t) = C_f \hat{x}(t) \tag{43}$$

which ensures, for prescribed $\bar{\delta}$, $\bar{h}_1$ and $\bar{h}_2$ that

$$\text{variance}\{z(t) - \hat{z}(t)\} < \bar{\delta} \tag{44}$$

over the entire uncertainty polytope and for any h_1 and h_2 that satisfy (25).

Defining $\xi = \text{col}\{x,\ \hat{x}\}$ and

$$\tilde{z}(t) = z(t) - \hat{z}(t) \tag{45}$$

we derive from (1), (45) and (3) the following augmented model:

$$\dot{\xi}(t) - \sum_{i=1}^{2} \bar{D}_i \dot{\xi}(t - h_i) = \bar{A}_0 \xi(t) + \sum_{1}^{2} \bar{A}_i \xi(t - h_i) + \bar{B} w(t) \tag{46a}$$

$$\tilde{z}(t) = \bar{L} \xi(t) \tag{46b}$$

where

$$\bar{A}_0 = \begin{bmatrix} A_0 & 0 \\ B_f \begin{bmatrix} C_0 \\ 0 \end{bmatrix} & A_f \end{bmatrix} \tag{47a}$$

$$\bar{A}_1 = \begin{bmatrix} A_1 & 0 \\ B_f \begin{bmatrix} 0 \\ C_1 \\ 0 \end{bmatrix} & 0 \end{bmatrix} \tag{47b}$$

$$\bar{A}_2 = \begin{bmatrix} A_2 & 0 \\ B_f \begin{bmatrix} 0 \\ 0 \\ C_2 \end{bmatrix} & 0 \end{bmatrix} \tag{47c}$$

$$\bar{B} = \begin{bmatrix} B \\ B_f D_{21} \end{bmatrix} \tag{47d}$$

$$\bar{L} = [L \quad -C_f] \tag{47e}$$

and

$$\bar{D}_i = \mathrm{diag}\{D_i,\ 0\} \tag{47f}$$

Note that (46) is asymptotically stable if (1) is asymptotically stable and A_f is Hurwitz. The filtering problem thus becomes one of finding the filter parameters such that variance of $\tilde{z}(t)$ of (46b) will be less than $\bar{\delta}$ for all the points in the uncertainty polytope and for all the delays that satisfy (25).

To the system of (43) and (47) we can apply the results of Theorems 2 and 3. We begin with the delay-dependent case.

The delay-dependent robust filter: Applying the result of Theorem 3 (for $Y_i = 0,\ i = 1, 2$) to the latter system, changing the dimension of the matrices to comply with those of (47), a bilinear matrix inequality is obtained in (37a) due to the product of Q with $\bar{A}_j,\ j = 0, 1, 2$ and $\bar{B}$. To make the problem convex we apply the following transformation.

Denoting

$$Q = \begin{bmatrix} X & M \\ M^{\mathrm{T}} & \bar{U} \end{bmatrix} \tag{48a}$$

$$Q^{-1} = \begin{bmatrix} \bar{Y} & \bar{N} \\ \bar{N}^{\mathrm{T}} & V \end{bmatrix} \tag{48b}$$

and

$$R = \bar{Y}^{-1} \tag{48c}$$

we consider the following matrix:

$$T = \begin{bmatrix} I_n & R\bar{N} \\ I_n & 0 \end{bmatrix} \tag{49}$$

Multiplying (37a), from the left by $\Gamma_1 = \text{diag}\{T, T, T, T, T, T, I_q, T, T\}$ and from the right by Γ_1^{T} substituting for $\bar{A}_i$ and $\bar{B}$ from (47), taking $Y_i = 0,\ i = 1, 2$ and denoting

$$U_i^{(j)} = \begin{bmatrix} U_{i,1}^{(j)} & U_{i,2}^{(j)} \\ U_{i,3}^{(j)} & U_{i,4}^{(j)} \end{bmatrix} = T\hat{W}_i^{(j)}T^{\mathrm{T}} \tag{50}$$

the condition of (37a) is equivalent to the following:

$$\begin{bmatrix} \bar{\Psi}_{\text{dep}} & \begin{bmatrix} RA_1^{(j)} - U_{1,1}^{(j)} & RA_1^{(j)} - U_{1,2}^{(j)} \\ XA_1^{(j)} + MB_f\begin{bmatrix} 0 \\ C_1^{(j)} \\ 0 \end{bmatrix} - U_{1,3}^{(j)} & XA_1^{(j)} + MB_f\begin{bmatrix} 0 \\ C_1^{(j)} \\ 0 \end{bmatrix} - U_{1,4}^{(j)} \end{bmatrix}\sigma_1 & \begin{bmatrix} RA_2^{(j)} - U_{2,1}^{(j)} & RA_2^{(j)} - U_{2,2}^{(j)} \\ XA_2^{(j)} + MB_f\begin{bmatrix} 0 \\ 0 \\ C_2^{(j)} \end{bmatrix} - U_{2,3}^{(j)} & XA_2^{(j)} + MB_f\begin{bmatrix} 0 \\ 0 \\ C_2^{(j)} \end{bmatrix} - U_{2,2}^{(j)} \end{bmatrix}\sigma_2 \\ * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 \\ * & * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

$$\begin{matrix} \beta\begin{bmatrix} RA_0^{(j)} + \sum_{i=1}^{2} U_{i,1}^{(j)} & RA_0^{(j)} + \sum_{i=1}^{2} U_{i,2}^{(j)} \\ XA_0^{(j)} + MB_F\begin{bmatrix} C_0^{(j)} \\ 0 \\ 0 \end{bmatrix} + MA_F\bar{N}^{\mathrm{T}}R + \sum_{i=1}^{2} U_{i,3}^{(j)} & XA_0^{(j)} + MB_F\begin{bmatrix} C_0^{(j)} \\ 0 \\ 0 \end{bmatrix} + \sum_{i=1}^{2} U_{i,4}^{(j)} \end{bmatrix} & \begin{bmatrix} RB^{(j)} \\ XB^{(j)} + MB_f D_{21}^{(j)} \end{bmatrix} & \hat{\sigma}_1\begin{bmatrix} RD_1^{(j)} & 0 \\ XD_1^{(j)} & 0 \end{bmatrix} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 & 0 \\ * & -I & 0 \\ * & * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} \\ * & * & * \\ * & * & * \\ * & * & * \end{matrix}$$

$$\begin{array}{ccc} \hat{\sigma}_2\begin{bmatrix} RD_2^{(j)} & 0 \\ XD_2^{(j)} & 0 \end{bmatrix} & \bar{\sigma}_1\begin{bmatrix} U_{1,1}^{(j)} & U_{1,2}^{(j)} \\ U_{1,3}^{(j)} & U_{1,4}^{(j)} \end{bmatrix} & \bar{\sigma}_2\begin{bmatrix} U_{2,1}^{(j)} & U_{2,2}^{(j)} \\ U_{2,3}^{(j)} & U_{2,4}^{(j)} \end{bmatrix} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 & 0 \\ * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 \\ * & * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} \end{array}\Bigg] < 0 \tag{51}$$

where

$$\bar{\Psi}_{\text{dep}} = \begin{bmatrix} RA_0^{(j)} + A_0^{(j)\text{T}}R + \sigma^* R + \sum_{i=1}^{2} (U_{i,1}^{(j)} + U_{i,1}^{(j)\text{T}}) & RA_0^{(j)} + A_0^{(j)\text{T}}X + \sigma^* R + [C_0^{(j)\text{T}}\ 0\ 0]B_f^{\text{T}}M^{\text{T}} + R\bar{N}A_f^{\text{T}}M^{\text{T}} + \sum_{i=1}^{2} (U_{i,2}^{(j)} + U_{i,3}^{(j)\text{T}}) \\ * & XA_0^{(j)} + A_0^{(jT)}X + \sigma^* X + MB_F\begin{bmatrix} C_0^{(j)} \\ 0 \\ 0 \end{bmatrix} + [C_0^{(j)\text{T}}\ 0\ 0]B_f^{\text{T}}M^{\text{T}} + \sum_{i=1}^{2} (U_{i,4}^{(j)} + U_{i,4}^{(j)\text{T}}) \end{bmatrix}$$

$\hat{\sigma}_i^2,\ i = 1, 2$ are defined in (32c) and

$$\sigma_i^2 = \sum_{j=0}^{2} q_{i,j} + \sum_{j=0}^{2} \bar{h}_j r_{i,j}, \quad i = 1, 2 \tag{52a}$$

$$\beta^2 = \sum_{j=1}^{2} (\bar{h}_j r_{0,j} + \bar{r}_j) \tag{52b}$$

$$\sigma^* = \sum_{i=1}^{2} q_{i,0}^{-1} \tag{52c}$$

Multiplying (37b), from the left by $\Gamma_2 = \text{diag}\{I_p, T\}$ and from the right by Γ_2^{T} and substituting for $\bar{L}$, we obtain the following inequality:

$$\begin{bmatrix} H & [L^{(j)} - C_f \bar{N}^{\text{T}} R \quad L^{(j)}] \\ * & \begin{bmatrix} R & R \\ R & X \end{bmatrix} \end{bmatrix} > 0 \tag{53}$$

The latter two inequalities lead to the following result.

Theorem 4 Assume A1
Consider the stationary system of (1) and (3) where the time delays h_1 and h_2 satisfy (25) and the system matrices reside anywhere in the uncertainty polytope $\bar{\Omega}$ of (42). For a prescribed positive scalar δ and for some positive scalars $q_{i,j}$, $\bar{r}_i$ and $r_{j,i}$, $i = 1, 2$, $j = 0, 1, 2$, assume that there exists a solution X, R, Z_a, $U_{i,k}^{(j)}, = 1, 2$, $k = 1, \ldots, 4 \in \mathcal{R}^{n \times n}$, $Z_b \in \mathcal{R}^{n \times r}$, $Z_c \in \mathcal{R}^{p \times n}$ and $\bar{H} \in \mathcal{R}^{p \times p}$ that satisfy the following set of $2N + 1$ LMIs:

$$
\left[\begin{array}{ccccccccc}
\bar{\Psi}_{\rm dep} & \begin{bmatrix} RA_1^{(j)} - U_{1,1}^{(j)} & RA_1^{(j)} - U_{1,2}^{(j)} \\ XA_1^{(j)} + Z_b \begin{bmatrix} 0 \\ C_1^{(j)} \\ 0 \end{bmatrix} - U_{1,3}^{(j)} & XA_1^{(j)} + Z_b \begin{bmatrix} 0 \\ C_1^{(j)} \\ 0 \end{bmatrix} - U_{1,4}^{(j)} \end{bmatrix}\sigma_1 & \begin{bmatrix} RA_2^{(j)} - U_{2,1}^{(j)} & RA_2^{(j)} - U_{2,1}^{(j)} \\ XA_2^{(j)} + Z_b \begin{bmatrix} 0 \\ 0 \\ C_2^{(j)} \end{bmatrix} - U_{2,3}^{(j)} & XA_2^{(j)} + Z_b \begin{bmatrix} 0 \\ 0 \\ C_2^{(j)} \end{bmatrix} - U_{2,4}^{(j)} \end{bmatrix}\sigma_2 & \beta\begin{bmatrix} RA_0^{(j)} + \sum_{i=1}^2 U_{i,1}^{(j)} & RA_0^{(j)} + \sum_{i=1}^2 U_{i,2}^{(j)} \\ XA_0^{(j)} + Z_b \begin{bmatrix} C_0^{(j)} \\ 0 \\ 0 \end{bmatrix} + Z_a + \sum_{i=1}^2 U_{i,3}^{(j)} & XA_0^{(j)} + Z_b \begin{bmatrix} C_0^{(j)} \\ 0 \\ 0 \end{bmatrix} + \sum_{i=1}^2 U_{i,4}^{(j)} \end{bmatrix} & \begin{bmatrix} RB^{(j)} \\ XB^{(j)} + Z_b D_{21}^{(j)} \end{bmatrix} & \hat{\sigma}_1 \begin{bmatrix} RD_1^{(j)} & 0 \\ XD_1^{(j)} & 0 \end{bmatrix} & \hat{\sigma}_2 \begin{bmatrix} RD_2^{(j)} & 0 \\ XD_2^{(j)} & 0 \end{bmatrix} & \bar{\sigma}_1 \begin{bmatrix} U_{1,1}^{(j)} & U_{1,2}^{(j)} \\ U_{1,3}^{(j)} & U_{1,4}^{(j)} \end{bmatrix} & \bar{\sigma}_2 \begin{bmatrix} U_{2,1}^{(j)} & U_{2,2}^{(j)} \\ U_{2,3}^{(j)} & U_{2,4}^{(j)} \end{bmatrix} \\
* & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -I_q & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 & 0 & 0 \\
* & * & * & * & * & * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 & 0 \\
* & * & * & * & * & * & * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 \\
* & * & * & * & * & * & * & * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix}
\end{array}\right] < 0 \qquad (54a)
$$

where

$$\bar{\Psi}_{\mathrm{dep}} = \begin{bmatrix} RA_0^{(j)} + A_0^{(j)\mathrm{T}}R + \sigma^* R + \sum_{i=1}^{2} (U_{i,1}^{(j)} + U_{i,1}^{(j)\mathrm{T}}) & RA_0^{(j)} + A_0^{(j)\mathrm{T}}X + [C_0^{(j)\mathrm{T}}\ \ 0\ \ 0]Z_b^{\mathrm{T}} + Z_a^{\mathrm{T}} + \sigma^* R + \sum_{i=1}^{2} (U_{i,2}^{(j)} + U_{i,3}^{(j)}) \\ * & XA_0^{(j)} + A_0^{(j)\mathrm{T}}X + Z_b \begin{bmatrix} C_0^{(j)} \\ 0 \\ 0 \end{bmatrix} + [C_0^{(j)\mathrm{T}}\ \ 0\ \ 0]Z_b^{\mathrm{T}} + \sigma^* X + \sum_{i=1}^{2} (U_{i,4}^{(j)} + U_{i,4}^{(j)\mathrm{T}}) \end{bmatrix} \tag{54b}$$

$$\begin{bmatrix} \bar{H} & \begin{bmatrix} L^{(j)} - Z_c & L^{(j)} \end{bmatrix} \\ * & \begin{bmatrix} R & R \\ R & X \end{bmatrix} \end{bmatrix} \geqslant 0 \tag{54c}$$

$$j = 1, 2, \ldots, N \quad \text{and} \quad \operatorname{trace}\{\bar{H}\} < \delta \tag{54d}$$

and where σ_i^2 and $\hat{\sigma}_i^2$, $i = 1, 2$ are defined in (52a) and (32c), respectively, and β^2 and σ^* are defined in (52b,c).

If a solution to this set of LMIs exists then the matrices of the filter (43) that achieves *variance*$\{L\tilde{z}(t)\} < \delta$ are given by

$$A_f = -\bar{W}^{-1} Z_a \tag{55a}$$

$$B_f = -\bar{W}^{-1} Z_b \tag{55b}$$

$$C_f = Z_c \tag{55c}$$

where

$$\bar{W} = X - R \tag{55d}$$

under condition that A_f is Hurwitz.

Proof

The inequalities of (54a,c) follow from (51) and (53) by substituting

$$Z_a = MA_f \bar{N}^{\mathrm{T}} R, \quad Z_b = MB_f, \quad Z_c = C_f \bar{N}^{\mathrm{T}} R \tag{56}$$

The result of (55a–d) stem from the fact that the transfer function of the filter $S(-\bar{W}^{-1}Z_a,\ -\bar{W}^{-1}Z_b,\ Z_c)$ is identical, by (56), to the one of $S(A_f, B_f, C_f)$. The inverse of $\bar{W}$ exists since the (2,2) matrix block in (54a) must be positive definite.

Remark 3

The decision variable Z_c appears only in (54c). Using Schur formula, the latter LMI can be put in the form of

$$\begin{bmatrix} H - L^{(j)} X^{-1} L^{(j)T} & L^{(j)} - Z_c - L^{(j)} X^{-1} R \\ * & R - RX^{-1}R \end{bmatrix} > 0$$

In the case where there is no uncertainty in L, the latter inequality is satisfied iff

$$
\begin{bmatrix} H & L \\ L^{\mathrm{T}} & X \end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix} R & R \\ R & X \end{bmatrix} > 0 \tag{57}
$$

and the corresponding Z_c is then given by $Z_c = LX^{-1}\bar{W}$.

The delay-independent robust filter: By Remark 2, in the delay-independent case the feasibility of (54) implies the delay-independent stability of (46) and thus the fact that A_f is Hurwitz.

Substituting in Theorem 4 $U_{i,k}^{(j)} = 0,\ k = 1, \ldots, 4,\ r_{k,i} = 0,\ k = 0, 1, 2,\ i = 1, 2$ we obtain the following result for the delay-independent robust filter.

Corollary 4 Assume A1
Consider the stationary system of (1) and (3), where the system matrices reside anywhere in the uncertainty polytope $\bar{\Omega}$ of (42). For a prescribed positive scalar δ the filter (43) achieves variance $\{\tilde{z}(t)\} < \delta$, independently of the delays in (1), if for some positive scalars $q_{i,j}$, $\bar{r}_i$ and $r_{j,i}$, $i = 1, 2,\ j = 0, 1, 2$, there exists a solution X, R, $Z_a \in \mathcal{R}^{n\times n}$, $Z_b \in \mathcal{R}^{n\times r}$, $Z_c \in \mathcal{R}^{p\times n}$ and $\bar{H} \in \mathcal{R}^{p\times p}$ that satisfy the following set of $2N+1$ LMIs:

$$
\left[\begin{array}{cc}
\begin{bmatrix} RA_0^{(j)} + A_0^{(j)T}R + \sigma^* R & RA_0^{(j)} + A_0^{(j)T}X + [C_0^{(j)T}\ 0\ 0]Z_b^{\mathrm{T}} + Z_a^{\mathrm{T}} + \sigma^* R \\ * & XA_0^{(j)} + A_0^{(j)T}X + Z_b \begin{bmatrix} C_0^{(j)} \\ 0 \\ 0 \end{bmatrix} + [C_0^{(j)T}\ 0\ 0]Z_b^{\mathrm{T}} + \sigma^* X \end{bmatrix} & \begin{bmatrix} RA_1^{(j)} & RA_1^{(j)} \\ XA_1^{(j)} + Z_b \begin{bmatrix} 0 \\ C_1^{(j)} \\ 0 \end{bmatrix} & XA_1^{(j)} + Z_b \begin{bmatrix} 0 \\ C_1^{(j)} \\ 0 \end{bmatrix} \end{bmatrix} \sigma_1 \\
* & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} \\
* & * \\
* & * \\
* & * \\
* & * \\
* & *
\end{array}\right.
$$

$$
\left.\begin{array}{ccc}
\begin{bmatrix} RA_2^{(j)} & RA_2^{(j)} \\ XA_2^{(j)} + Z_b \begin{bmatrix} 0 \\ 0 \\ C_2^{(j)} \end{bmatrix} & XA_2^{(j)} + Z_b \begin{bmatrix} 0 \\ 0 \\ C_2^{(j)} \end{bmatrix} \end{bmatrix} \sigma_2 & \beta \begin{bmatrix} RA_0^{(j)} & RA_0^{(j)} \\ XA_0^{(j)} + Z_b \begin{bmatrix} C_0^{(j)} \\ 0 \\ 0 \end{bmatrix} + Z_a & XA_0^{(j)} + Z_b \begin{bmatrix} C_0^{(j)} \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} & \begin{bmatrix} RB^{(j)} \\ XB^{(j)} + Z_b D_{21}^{(j)} \end{bmatrix} \\
0 & 0 & 0 \\
-\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 & 0 \\
* & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 \\
* & * & -I_q \\
* & * & * \\
* & * & *
\end{array}\right.
$$

$$\left.\begin{matrix} \hat{\sigma}_1 \begin{bmatrix} RD_1^{(j)} & 0 \\ XD_1^{(j)} & 0 \end{bmatrix} & \hat{\sigma}_2 \begin{bmatrix} RD_2^{(j)} & 0 \\ XD_2^{(j)} & 0 \end{bmatrix} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 \\ * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} \end{matrix}\right] < 0 \tag{58a}$$

$$\begin{bmatrix} \bar{H} & L^{(j)} - Z_c & L^{(j)} \\ * & R & R \\ * & * & X \end{bmatrix} > 0, \quad j = 1, 2, \ldots, N \tag{58b}$$

$$\text{trace}\{\bar{H}\} < \delta \tag{58c}$$

where σ_i^2, $\hat{\sigma}_i^2$, $i = 1, 2$, β^2 and σ^* are defined in (35b), (32c), (35c) and (52c), respectively.

If a solution to this set of LMIs exists then the matrices of the filter (43) that achieves the required bound on the error variance are given by (55).

Remark 4
In the retarded case, where $D_1 = D_2 = 0$, only $q_{1,0}$ and $q_{2,0}$ remain as the free tuning parameters in (58a). Consequently, $\beta = \hat{\sigma}_i = 0$, $\sigma_i^2 = q_{i,0}$, $i = 1, 2$ and $\sigma^* = q_{1,0}^{-1} + q_{2,0}^{-1}$ should be substituted in (58a) and the resulting inequality becomes

$$\left[\begin{matrix} \begin{bmatrix} RA_0^{(j)} + A_0^{(j)T} R + \sigma^* R & RA_0^{(j)} + A_0^{(j)T} X + [C_0^{(j)T} \ \ 0 \ \ 0] Z_b^{\mathrm{T}} + Z_a^{\mathrm{T}} + \sigma^* R \\ * & XA_0^{(j)} + A_0^{(j)T} X + Z_b + \sigma^* X + Z_b \begin{bmatrix} C_0^{(j)} \\ 0 \\ 0 \end{bmatrix} + [C_0^{(j)T} \ \ 0 \ \ 0] Z_b^{\mathrm{T}} \end{bmatrix} \\ * \\ * \\ * \end{matrix}\right.$$

$$\begin{matrix} \begin{bmatrix} RA_1^{(j)} & RA_1^{(j)} \\ XA_1^{(j)} + Z_b \begin{bmatrix} 0 \\ C_1^{(j)} \\ 0 \end{bmatrix} & XA_1^{(j)} + Z_b \begin{bmatrix} 0 \\ C_1^{(j)} \\ 0 \end{bmatrix} \end{bmatrix} \sqrt{q}_{1,0} \\ -\begin{bmatrix} R & R \\ R & X \end{bmatrix} \\ * \\ * \end{matrix}$$

$$\begin{bmatrix} \begin{bmatrix} RA_2^{(j)} & RA_2^{(j)} \\ XA_2^{(j)} + Z_b \begin{bmatrix} 0 \\ 0 \\ C_2^{(j)} \end{bmatrix} & XA_2^{(j)} + Z_b \begin{bmatrix} 0 \\ 0 \\ C_2^{(j)} \end{bmatrix} \end{bmatrix} \sqrt{q}_{2,0} & \begin{bmatrix} RB^{(j)} \\ XB^{(j)} + Z_b D_{21}^{(j)} \end{bmatrix} \\ 0 & 0 \\ -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 \\ * & -I_q \end{bmatrix} < 0 \qquad (59)$$

The delay-dependent robust filter via neutral model transformation: The result of Theorem 4 requires a search for the positive scalars $q_{i,j}$, $\bar{r}_i$ and $r_{j,i}$, $i = 1, 2$, $j = 0, 1, 2$. A much simpler sufficient condition for the existence of a filter that achieves the required estimation H_∞ norm is obtained by applying the neutral model (with $F_i = A_i$). This is achieved by substituting $U_i^{(j)} = TQA_i^{(j)}T^{\mathrm{T}}$, $\sigma_i = 0$, $\bar{\sigma}_i^2 = \bar{h}_i r_{0,i}^{-1}$, $i = 1, 2$, $\beta^2 = \sum_{k=1}^{2} (\bar{h}_j r_{0,k} + \bar{r}_k)$ and $\sigma^* = 0$ in (54a,b). By Remark 2 in this case the feasibility of the LMI implies that $\sum_{i=0}^{2} \bar{A}_i$ is Hurwitz and thus A_f is Hurwitz too. We obtain the following.

Corollary 5 Assume A1

Consider the stationary system of (1) and (3) where the time delays h_1 and h_2 satisfy (25) and the system matrices reside anywhere in the uncertainty polytope $\bar{\Omega}$ of (42). For a prescribed positive scalar δ the filter (43) achieves *variance*$\{\tilde{z}(t)\} < \delta$, if for some positive scalars $\bar{r}_i$ and $r_{0,i}$, $i = 1, 2$ there exists a solution X, R, $Z_a \in \mathscr{R}^{n\times n}$, $Z_b \in \mathscr{R}^{n\times r}$, $Z_c \in \mathscr{R}^{p\times n}$ and $\bar{H} \in \mathscr{R}^{p\times p}$ that satisfy the following set of $2N+1$ LMIs:

$$
\left[\begin{matrix}
\bar{\Psi}_{\rm dep} & \begin{bmatrix} RA_1^{(j)} & RA_1^{(j)} \\ XA_1^{(j)} + Z_b \begin{bmatrix} 0 \\ C_1^{(j)} \\ 0 \end{bmatrix} & XA_1^{(j)} + Z_b \begin{bmatrix} 0 \\ C_1^{(j)} \\ 0 \end{bmatrix} \end{bmatrix} \bar{\sigma}_1 & \begin{bmatrix} RA_2^{(j)} & RA_2^{(j)} \\ XA_2^{(j)} + Z_b \begin{bmatrix} 0 \\ 0 \\ C_2^{(j)} \end{bmatrix} & XA_2^{(j)} + Z_b \begin{bmatrix} 0 \\ 0 \\ C_2^{(j)} \end{bmatrix} \end{bmatrix} \bar{\sigma}_2 \\
* & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 \\
* & * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} \\
* & * & * \\
* & * & * \\
* & * & * \\
* & * & * \\
* & * & * \\
* & * & *
\end{matrix}\right.
$$

$$
\begin{matrix}
\beta \begin{bmatrix} R \sum_{i=0}^{2} A_i^{(j)} & R \sum_{i=0}^{2} A_i^{(j)} \\ X \sum_{i=0}^{2} A_i^{(j)} + Z_b \begin{bmatrix} C_0^{(j)} \\ C_1^{(j)} \\ C_2^{(j)} \end{bmatrix} + Z_a & X \sum_{i=0}^{2} A_i^{(j)} + Z_b \begin{bmatrix} C_0^{(j)} \\ C_1^{(j)} \\ C_2^{(j)} \end{bmatrix} \end{bmatrix} \\
0 \\
0 \\
-\begin{bmatrix} R & R \\ R & X \end{bmatrix} \\
* \\
* \\
*
\end{matrix}
$$

$$
\left.\begin{matrix}
\begin{bmatrix} RB^{(j)} \\ XB^{(j)} + Z_b D_{21}^{(j)} \end{bmatrix} & \bar{r}_1^{-\frac{1}{2}} \begin{bmatrix} RD_1^{(j)} & 0 \\ XD_1^{(j)} & 0 \end{bmatrix} & \bar{r}_2^{-\frac{1}{2}} \begin{bmatrix} RD_2^{(j)} & 0 \\ XD_2^{(j)} & 0 \end{bmatrix} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-I_q & 0 & 0 \\
* & -\begin{bmatrix} R & R \\ R & X \end{bmatrix} & 0 \\
* & * & -\begin{bmatrix} R & R \\ R & X \end{bmatrix}
\end{matrix}\right] < 0 \qquad (60a)
$$

where

$$\bar{\Psi}_{\text{dep}} = \begin{bmatrix} \sum_{i=0}^{2} (RA_i^{(j)} + A_i^{(j)T} R) & \sum_{i=0}^{2} (RA_i^{(j)} + A_i^{(j)T} X) + [C_0^{(j)T} \; C_1^{(j)T} \; C_2^{(j)T}] Z_b^{\mathrm{T}} + Z_a^{\mathrm{T}} \\ * & \sum_{i=0}^{2} (XA_i^{(j)} + A_i^{(j)T} X) + Z_b \begin{bmatrix} C_0^{(j)} \\ C_1^{(j)} \\ C_2^{(j)} \end{bmatrix} + [C_0^{(j)T} \; C_1^{(j)T} \; C_2^{(j)T}] Z_b^{\mathrm{T}} \end{bmatrix} \tag{60b}$$

and (54c) for $j = 1, 2, \ldots, N$ and (54d) where

$$\bar{\sigma}_i^2 = r_{0,i}^{-1} \bar{h}_i, \quad i = 1, 2 \quad \text{and} \quad \beta^2 = \sum_{j=1}^{2} (\bar{h}_j r_{0,j} + \bar{r}_j)$$

If a solution to this set of LMIs exists then the matrices of filter (43) that achieves the required bound on the error variance are given by (55a–c).

Remark 5

The latter result is further simplified in the case where $D_i = 0, \; i = 1, 2$. In this case, only the scalars $r_{0,1}$ and $r_{0,2}$ should be searched for, in the LMI (60a) the last two column and row blocks should be deleted and in the definition of $\beta, \; \bar{r}_i, \; i = 1, 2$ are taken to be zeros.

It is of interest to examine in the latter case ($D_1 = D_2 = 0$) what form of LMIs the condition of Corollary 5 tends to when the bounds on the delay, $\bar{h}_1$ and $\bar{h}_2$ go to zero. In the latter case, we denote $A^{(j)} = \sum_{i=0}^{2} A_i^{(j)}$ and replace $[C_0^{(j)T} \; C_1^{(j)T} \; C_2^{(j)T}]^{\mathrm{T}}$ by $C^{(j)}$. Since a zero $\bar{h}_i$ leads to $\bar{\sigma}_i = 0, \; i = 1, 2$, (60a–b) become

$$\begin{bmatrix} RA^{(j)} + A^{(j)T} R & RA^{(j)} + A^{(j)T} X + C^{jT} Z_b^{\mathrm{T}} + Z_a^{\mathrm{T}} & RB^{(j)} \\ * & XA^{(j)} + A^{(j)T} X + Z_b C^{(j)} + C^{(j)T} Z_b^{\mathrm{T}} & XB^{(j)} + Z_b D_{21}^{(j)} \\ * & * & -I_q \end{bmatrix} < 0,$$

$$j = 1, 2, \ldots, N \tag{61}$$

Together with (54c,d), the LMIs of (61) are exactly those obtained in [24] for robust H_2 filter design for systems without delays.

6. EXAMPLE

Most recent examples that concerned robust H_∞ filtering design appear in Reference [15]. We apply our method to the first example that appears there.

Consider the system given in (1) where

$$A_0 = \begin{bmatrix} 0 & 2 \\ -3 & -4 + \rho \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.1 & 0 \\ 0.2 & -0.2 + \phi \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0.1 \\ -0.2 & -0.3+\phi \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_1 = 0, \quad C_2 = 0, \quad D_1 = D_2 = 0, \quad D_{21} = 1 \quad \text{and } L = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

The uncertain parameters satisfy $|\rho| \leqslant 2$ and $|\phi| \leqslant 0.1$ and we consider the case where $\bar{h}_1 = 0.4$ and $\bar{h}_2 = 0.5$.

Clearly, the uncertainty polytope Ω possesses $N = 4$ vertices. Applying Corollary 4 and using the LMI in Remark 4 we first derive the optimal robust delay-independent filter. Using $q_{0,1} = 3.1$ and $q_{0,2} = 2.8$ a near minimum value of $\delta = 1.066$ is obtained. The state-space matrices of this filter are given by:

$$A_f = \begin{bmatrix} -0.1900 & 1.5645 \\ -4.5541 & -2.5099 \end{bmatrix}, \quad B_f = \begin{bmatrix} 0.1841 \\ 1.6230 \end{bmatrix} \quad \text{and} \quad C_f = \begin{bmatrix} 0.6353 & 1.2132 \end{bmatrix}$$

The delay-dependent filter is sought next. Applying Corollary 5 and following Remark 5, a near minimum value of $\delta = 0.7319$ is obtained for $r_{0,1} = r_{0,2} = 0.12$. The corresponding filter is described by

$$A_f = \begin{bmatrix} 0.1189 & 1.8929 \\ -5.2583 & -4.0379 \end{bmatrix}, \quad B_f = \begin{bmatrix} 0.0156 \\ 1.7036 \end{bmatrix} \quad \text{and} \quad C_f = \begin{bmatrix} 0.7612 & 1.1481 \end{bmatrix}$$

We note that the resulting δ is 25% less than a corresponding result of $\delta = 0.9728$ that has been found in [22]. The latter has been obtained by applying a new H_∞ filtering result with a prescribed estimation error level that tends to infinity.

Assuming that the above system has a neutral part, say $D_1 = \text{diag}\{0.1, 0.1\}$, $D_2 = 0$, we apply Corollary 6 and find a near minimum value of $\delta = 9.263$. This value is obtained for $r_{0,1} = r_{0,2} = 0.11$ and $\bar{r}_1 = 0.064$. The corresponding filter is described by

$$A_f = \begin{bmatrix} -0.9608 & 1.6045 \\ -5.3586 & -2.9816 \end{bmatrix}, \quad B_f = \begin{bmatrix} 1.7051 \\ 0.8986 \end{bmatrix} \quad \text{and} \quad C_f = \begin{bmatrix} 0.3516 & 0.5061 \end{bmatrix}$$

7. CONCLUSIONS

A minimum variance filtering estimation of linear neutral type systems with multiple time delays and polytopic uncertainty has been considered. Our approach is based on the equations that the covariance matrix of the estimation error and its upper-bound satisfy. LMIs are obtained whose solution, if exists, provides a minimum upper-bound to the variance of the estimation error over the entire uncertainty polytope.

The general delay-dependent design requires an additional stability verification of the the resulting filter. In the delay-independent design and in the important delay-dependent synthesis that is based on the neutral model transformation, the stability of the resulting estimation process is guaranteed by the LMIs solved. These LMIs provide, in the case where the system is retarded and when the delays tend to zero, the LMIs that have been recently derived for general robust filtering design for systems without delays.

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