Sampled-data $H_\infty$ state-feedback control of systems with state delays

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The finite horizon piecewise-constant state-feedback $H_\infty$ control of time-invariant linear systems with a finite number of point and distributed time-delays is considered. For the controller, coupled Riccati type partial differential equations between sampling and updating the terminal conditions are derived. For small time delays the solutions and the resulting controllers are approximated by series expansions in powers of the largest delay. Unlike the infinite horizon case, these approximations possess both regular and boundary layer terms. It is shown that the controller obtained by high-order approximations improves the performance of the system. The performance of the closed-loop system under the memory-less zero-approximation controller is analysed. Bounded Real Lemmas for state-delay systems with jumps are obtained.

1. Introduction

Continuous-time $H_\infty$ control problem of state-delay systems has been studied by Bensoussan et al. (1992), van Keulen et al. (1993), van Keulen (1993), McMillan and Triggiani (1993), Lee et al. (1994), Ge et al. (1996), Fridman and Shaked (1998, 2000), Bensoussan et al. (1992), van Keulen et al. (1993), van Keulen (1993) and McMillan and Triggiani (1993) have obtained the controller by solving Riccati operator equations. Lee et al. (1994) and Ge et al. (1996) have designed a delay-independent controller. Fridman and Shaked (1998, 2000) have derived the controller from Riccati type partial differential equations (RPDEs) or inequalities, and the solution of the RPDEs has been approximated by expansions in the powers of the delay. In Shaked et al. (1998) a bounded real lemma has been obtained in terms of differential linear matrix inequalities.

Sampled-data $H_\infty$ control of systems without delay has been studied by Bamieh and Pearson (1992), Toivonen (1992), Khargonekar and Khargonekar (1994), Basar and Bernaì (1995), Sagfors and Toivonen (1997). Two main approaches have been used. The first one is based on the lifting technique in which the problem is transformed into equivalent finite-dimensional discrete $H_\infty$ problem (Bamieh and Pearson 1992, Toivonen 1992). The second, more direct, approach is based on the representation of the system in the form of hybrid discrete/continuous one and the solution is obtained in terms of Riccati equations with jumps (Khargonekar et al. 1993, Sivasahand and Khargonekar 1994, Basar and Bernard 1995, Sagfors and Toivonen 1997).

In the present paper, we adopt the second approach to the finite horizon sampled-data state-delay $H_\infty$-control problem. Note that in the case of state-delays the lifting technique leads to complicated discrete system with infinite-dimensional state variable. We obtain the piecewise-constant state-feedback controllers by solving coupled RPDEs between sampling and updating the terminal conditions at the sampling instants. We derive an asymptotic approximation to the solution of these RPDEs by expanding it in the powers of the largest delay. The resulting approximation is obtained by solving uncoupled low-order partial differential equations. The performance of the system with the controller that has been obtained using the zero approximation (the one that corresponds to zero delay) is analysed when the open-loop system possesses a non-zero delay. Bounded real lemmas for systems with state delays and jumps are obtained.

2. Problem formulation

Throughout this paper we denote by $\|\cdot\|$ the Euclidean norm of a vector or the appropriate norm of a matrix. Given $t_f > 0$, let $L_2[0,t_f]$ be the space of the square integrable functions with the norm $\|\cdot\|_{L_2}$ and let $C[a,b]$ be the space of the continuous functions on $[a,b]$ with the norm $\|\cdot\|_C$. We denote $x_i = x(t+\theta)$, $y_i = y(t-\theta)$, $\theta \in [-h,0]$ and $x(t) = \lim_{\theta \to -0} x(t+s)$, and $x_{\infty} = x((t+\theta)\cdots)$. Prime denotes the transpose of a matrix and col $[x,y]$ denotes a column vector with components $x$ and $y$.

Consider the system

$$\begin{align*}
\dot{x}(t) &= L(x(t)) + Bu(t) + Dw(t) \\
z(t) &= \text{col} \{Cx(t),u(t)\}
\end{align*}$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^l$ is the control signal, $w(t) \in \mathbb{R}^q$ is the exogenous disturbance, and $z(t) \in \mathbb{R}^p$ is the observation vector, $B$, $C$ and $D$ are constant matrices of appropriate dimensions. The $\mathbb{R}^q$-valued
function \( L(\cdot) \) which carries \( \mathbb{R}^n \)-valued functions on \([-h,0)\) into \( \mathbb{R}^n \) is defined as
\[
L(x_t(\cdot)) = \sum_{i=0}^{r} A_i x_t(-h_i) + \int_{-h}^{0} A_0(s)x_t(s) \, ds
\]
where \(-h = -h_r < -h_{r-1} < \cdots < -h_1 < -h_0 = 0\), \( A_0, A_1, \ldots, A_r \) are constant matrices and \( A_0(\cdot) \) is a smooth enough matrix function.

Consider the 'structural operator' \( F : L_2[-h,0) \rightarrow L_2[-h,0) \) given by
\[
F(x_t)(\xi) = \sum_{i=1}^{r} A_i x_t(-h_i - \xi) \chi_i(\xi) + \int_{-h}^{\xi} A_0(p)x_t(p-\xi) \, dp
\]
where \( \chi_i \) is the indicator function for the set \([-h_i,0)\), i.e. \( \chi_i(\xi) = 1 \) if \( \xi \in [-h_i,0] \) and \( \chi_i(\xi) = 0 \) otherwise. This operator was introduced by Delfour (1986) and the use of this operator in the Lyapunov–Krasovskii functional leads to simplified PDEs for \( H_2 \) and \( H_\infty \) design (Delfour 1986; Fridman and Shaked 2000).

Given \( \gamma > 0 \), and assuming that \( w \in L_2[0,t_f] \), we consider the following performance index
\[
J = \|z\|^2_{L_2} - \gamma^2 \|w\|^2_{L_2} + E(x_t)
\]
where
\[
E(x_t) = x'(t)P_f x(t) + 2x'(t) \int_{-h}^{0} Q_f(\xi)F(x_t)(\xi) \, d\xi + \int_{-h}^{\xi} F'(x_t)(s)R_f(s,\xi)F(x_t)(\xi) \, ds \, d\xi
\]
and \( P_f = P_f^t \geq 0 \). The form of \( E(x_t) \) stems from the form of Lyapunov–Krasovskii functional. We assume that the matrix-functions \( Q_f \) and \( R_f \) are continuous and piecewise continuously differentiable functions of their arguments, that satisfy the relations:
\[
P_f = Q_f(0), \quad Q_f(\xi) = R_f(0,\xi)
\]
We are looking for the piecewise-constant state-feedback controller of the form
\[
u(t) = \mu(x_{t_k}, \cdots, x_{t_l}, x_0), \quad t_k \leq t < t_{k+1},
\]
where \( 0 < t_1 < \cdots < t_K = t_f \) are the sampling instants. This type of control is encountered in many practical problems where a zero-order-hold is used to provide an analog input signal at the output of a digital controller.

The problem is to find a state-feedback controller of (7) which ensures that \( J \leq 0 \) for all \( w \in L_2[0,t_f] \) and for the zero initial conditions \( x(\tau) = 0 \), \( \tau \leq 0 \). This means that the \( H_\infty \)-norm of (1), which is defined by the supremum over \( w \in L_2[0,t_f] \) of the ratio between \( \|z\|^2_{L_2} + E(x_t) \) and \( \|w\|^2_{L_2} \), is not greater than \( \gamma \).

### 3. Piecewise-constant controller design

Following Basar and Bernard (1995) and Toivonen and Sagfors (1997) we denote
\[
P = \begin{bmatrix} A_0 & B \\ 0 & 0 \end{bmatrix}, \quad \tilde{P} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}
\]
\[
\bar{D} = \begin{bmatrix} D & 0 \\ 0 & I_f \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0 \\ 0 & I_f \end{bmatrix}
\]
\[
A_d = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 \\ I_f \end{bmatrix}
\]
The system of (1) attains then the form
\[
\ddot{x}(t) = \sum_{i=0}^{r} \tilde{A}_i \dot{x}(t-h_i) + \int_{-h}^{0} \tilde{A}_0(s)\dot{x}(t+s) \, ds + \bar{D}w(t),
\]
\[
z(t) = \bar{C}\dot{x}(t), \quad t_k \leq t < t_{k+1}
\]
\[
\ddot{x}(t_k) = A_d \dot{x}(t_k) + B_d u_k
\]
where \( \ddot{x}(t) = \{x(t), \dot{u}(t)\} \). Similarly to (3) we denote
\[
\bar{F}(\ddot{x}_t)(\xi) = \sum_{i=1}^{r} \tilde{A}_i \dot{x}_t(-h_i - \xi) \chi_i(\xi)
\]
\[
+ \int_{-h}^{\xi} \tilde{A}_0(p)\dot{x}_t(p-\xi) \, dp
\]
clearly \( \bar{F}(\ddot{x}_t)(\xi) = \{\{F(x_t)(\xi), 0\} \). The Lyapunov–Krasovskii functional for the system of (9) is given by Delfour (1986)
\[
\bar{V}(t,\ddot{x}_t) = \dot{x}(t)\tilde{P}(t)\ddot{x}(t) + 2\ddot{x}'(t) \int_{-h}^{0} \tilde{Q}(t,\xi)\bar{F}(\ddot{x}_t)(\xi) \, d\xi + \int_{-h}^{\xi} \tilde{F}'(\ddot{x}_t)(s)\tilde{R}(t,s,\xi)\bar{F}(\ddot{x}_t)(\xi) \, ds \, d\xi
\]
where
\[
\tilde{P} = \begin{bmatrix} P & M \\ M' & U \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} Q & 0 \\ N' & 0 \end{bmatrix}, \quad \tilde{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}
\]
In the latter formulas the zero-blocks stem from the zero delay in \( u \).

Then \( E(x_{t_f}) = \tilde{E}(\ddot{x}_{t_f}) \), where
\[
\tilde{E}(\ddot{x}_{t_f}) = \ddot{x}'(t_f)\tilde{P}(t_f)\ddot{x}(t_f)
\]
\[
+ 2\ddot{x}'(t_f) \int_{-h}^{0} \tilde{Q}(\xi)\tilde{F}(\ddot{x}_t)(\xi) \, d\xi + \int_{-h}^{\xi} \tilde{F}'(\ddot{x}_t)(s)\tilde{R}(t,s,\xi)\tilde{F}(\ddot{x}_t)(\xi) \, ds \, d\xi
\]
The supremum value with respect to \( w \) of \( f \) is given by
\[
J_K = E(x_f) + \int_{t_{k-1}}^{t_k} [z(t)^2 - \gamma_w^2 w(t)^2] dt
\]
for the system of (9) is given by
\[
\sup_w J_K = J_K = V(t_{k-1}, x_{t_{k-1}})
\]
where the matrix-functions \( \tilde{P}, \tilde{Q} \) and \( \tilde{R} \) of the form (13) satisfy (16)–(20).

The proof of Lemma 1 is given in the Appendix. Writing (16)–(18) for the various blocks of (13), we obtain
\[
\begin{align*}
\frac{\partial}{\partial t} \tilde{Q}(t, \xi) + \frac{\partial}{\partial \xi} \tilde{Q}(t, \xi) &= - [\tilde{A}_0 + \tilde{P} \tilde{S}] \tilde{Q}(t, \xi) \\
&\quad - \sum_{i=1}^{r} \tilde{A}_i \tilde{R}(t, -h_i, \xi) \\
&\quad - \int_{-h}^{0} \tilde{A}_0(s) \tilde{R}(t, s, \xi) ds
\end{align*}
\]
(17)

\[
\begin{align*}
\frac{\partial}{\partial t} \tilde{R}(t, \xi, s) + \frac{\partial}{\partial \xi} \tilde{R}(t, \xi, s) + \frac{\partial}{\partial s} \tilde{R}(t, \xi, s)
&= - \tilde{Q}(t, \xi) \tilde{S} \tilde{Q}(t, \xi)
\end{align*}
\]
(18)

A solution of (16)–(20) on \( \{t_{k-1}, t_f\} \) is a triple of \((n + l) \times (n + l)\)-matrices \( \{P(t), Q(t, \xi), R(t, \xi, s)\} \) \( t \in [t_{k-1}, t_f], \xi \in [-h, 0], s \in [-h, 0] \), where \( \tilde{P}(t), \tilde{Q}(t, \xi) \) and \( \tilde{R}(t, \xi, s) \) are continuous and piecewise continuously differentiable functions of their arguments that have a form of (13), satisfy initial and terminal conditions (19), (20) for every \( t, \xi \) and \( s \) and satisfy partial differential equations (16)–(18) for almost every \( t, \xi \) and \( s \).

**Lemma 1:** The supremum value with respect to \( w \) of

\[
J_K = E(x_f) + \int_{t_{k-1}}^{t_k} [z(t)^2 - \gamma^2 w(t)^2] dt
\]

for the system of (9) is given by
\[
\sup_w J_K = J_K = V(t_{k-1}, x_{t_{k-1}})
\]
where the matrix-functions \( \tilde{P}, \tilde{Q} \) and \( \tilde{R} \) of the form (13) satisfy (16)–(20).

The proof of Lemma 1 is given in the Appendix. Writing (16)–(18) for the various blocks of (13), we obtain
\[
\begin{align*}
\frac{\partial}{\partial t} Q(t, \xi) + \frac{\partial}{\partial \xi} Q(t, \xi) &= - [A_0^* + P(t)] Q(t, \xi) \\
&\quad - \sum_{i=1}^{r} A_i^* R(t, -h_i, \xi) \\
&\quad - \int_{-h}^{0} A_0(s) R(t, s, \xi) ds
\end{align*}
\]
(17)

\[
\begin{align*}
\frac{\partial}{\partial t} R(t, \xi, s) + \frac{\partial}{\partial \xi} R(t, \xi, s) + \frac{\partial}{\partial s} R(t, \xi, s)
&= - Q(t, \xi) S Q(t, \xi)
\end{align*}
\]
(18)

A solution of (16)–(20) on \( \{t_{k-1}, t_f\} \) is a triple of \((n + l) \times (n + l)\)-matrices \( \{P(t), Q(t, \xi), R(t, \xi, s)\} \) \( t \in [t_{k-1}, t_f], \xi \in [-h, 0], s \in [-h, 0] \), where \( \tilde{P}(t), \tilde{Q}(t, \xi) \) and \( \tilde{R}(t, \xi, s) \) are continuous and piecewise continuously differentiable functions of their arguments that have a form of (13), satisfy initial and terminal conditions (19), (20) for every \( t, \xi \) and \( s \) and satisfy partial differential equations (16)–(18) for almost every \( t, \xi \) and \( s \).

**Lemma 1:** The supremum value with respect to \( w \) of
\[ P(t_k^-) = P(t_k) - M(t_k)U(t_k)M'(t_k) \]
\[ U(t_k^-) = 0, \quad N(t_k^-, \xi) = 0, \quad M(t_k^-) = 0 \]
\[ Q(t_k^-, \xi) = Q(t_k, \xi) - M(t_k)U(t_k)N'(t_k, \xi) \]
\[ R(t_k^-, s, \xi) = R(t_k, s, \xi) - N(t_k,s)U(t_k)N'(t_k, \xi) \]
\[
\left\{ \begin{array}{l}
\inf_{u} J_{K-1} = V(t_{k-1}, x_{k-1}) - x'(t_{k-1})M(t_{k-1}) \\
\times U^{-1}(t_{k-1})M'(t_{k-1})x(t_{k-1}) \\
\times \int_{-h}^{0} F'(x_{k-1})(\xi)N(t_{k-1}, \xi) \, d\xi U^{-1}(t_{k-1}) \\
\times \int_{-h}^{0} N'(t_{k-1}, \xi)F(x_{k-1})(\xi) \, d\xi \\
+ 2M'(t_{k-1})x(t_{k-1}) \end{array} \right. \]
\[
\text{where } k = K. \text{ Therefore at the stage } K - 1 \text{ we obtain the performance cost}
\]
\[ J_{K-1} = \inf_{u} J_{K} + \int_{t_{k-1}}^{t_{k}} [\gamma^2 - \gamma |w|^2] \, dt \]

By the same arguments and due to (28) for \( k = K - 1 \) we see that \( \inf_{u} J_{K-1} \) is given by (30), where \( k = K - 1 \). Similarly it can be shown that (30) holds for all \( k = K - 1, \ldots, 1 \). Then \( \inf_{u} J = \inf_{u} J_{1} = 0 \) since \( x_0 = 0 \).}

4. Asymptotic approximation of the \( H_{\infty} \) controller

4.1. Asymptotic solutions to the RPDEs

As we have seen the \( H_{\infty} \) controller has been found by solving a set of coupled RPDEs. Finding a solution to the latter is not an easy task and we are, therefore, looking for a solution to the RPDEs on \( [t_k, t_{k+1}], k = K - 1, \ldots, 0 \) in a form of asymptotic expansion in the powers of the delay \( h \)

\[
\begin{aligned}
\hat{P}(t) &= \hat{P}_0(t) + h[\hat{P}_1(t) + \Pi_1 \hat{P}(\tau)] \\
&\quad + h^2[\hat{P}_2(t) + \Pi_2 \hat{P}(\tau)] + \cdots \\
\hat{Q}(t, h \zeta) &= \hat{Q}_0(t, \zeta) + h[\hat{Q}_1(t, \zeta) + \Pi_1 \hat{Q}(\tau, \zeta)] \\
&\quad + h^2[\hat{Q}_2(t, \zeta) + \Pi_2 \hat{Q}(\tau, \zeta)] + \cdots \\
R(t, h \zeta, h \theta) &= R_0(t, \zeta, \theta) + h[R_1(t, \zeta, \theta) \\
&\quad + \Pi_1 R(\tau, \zeta, \theta)] + h^2[R_2(t, \zeta, \theta) \\
&\quad + \Pi_2 R(\tau, \zeta, \theta)] + \cdots \\
t &\in [t_k, t_{k+1}], \; k = K - 1, \ldots, 0 \\
\tau &= \frac{t_{k+1} - t}{h}, \; \zeta \in [-1, 0], \; \theta \in [-1, 0] \\
\end{aligned}
\]

Note that we consider the asymptotic expansion of \( \hat{R} \) instead of \( \hat{R} \) since \( \hat{R} \) is the only non-zero block of \( \hat{R} \). The ‘outer expansion’ terms \( \{\hat{P}_i, \hat{Q}_i, R_i\}, \; i = 0, 1, \ldots \) constitute the major part of the solution that satisfies (16)–(18) for \( t \in [0, t_f], \; t \neq t_{k+1}, \; \theta \in [-1, 0], \; \zeta \in [-1, 0] \). The boundary-layer correction terms
\[ \Pi_{fP}, \Pi_{fQ} \text{ and } \Pi_{fR} \text{ will be chosen such that (31) satisfies} \]
\[ \text{the terminal conditions of (20) and that} \]
\[ |\Pi_{fP}(\tau)| + \sup_{\zeta \in [-1,0]} |\Pi_{fQ}(\tau, \zeta)| + \sup_{\zeta \in [-1,0]} |\Pi_{fR}(\tau, \zeta, \theta)| \to 0 \]
\[ \text{as } \tau \to \infty \quad (32) \]
Since \( \tau \) is a stretched-time variable around \( t = t_{k+1} \), (32) asserts that \( \Pi_{fP}, \Pi_{fQ} \) and \( \Pi_{fR} \) are essentially only around \( t = t_{k+1} \) and they thus provide a correction to the outer expansion at the terminal point \( t = t_{k+1} \).

In the sampled-data case we approximate the solution to the RPDEs on each sampling segment: \([t_{K-1}, t_f] \cup \ldots \cup [0, t_1]\). We assume that the terminal values \( P_f, Q_f \) and \( R_f \) are approximated as:
\[ P_f = P_f + \sum_{i=1}^{m+1} h' P_{fi} \]
\[ Q_f(h\zeta) = Q_f + \sum_{i=1}^{m+1} h' Q_{fi}(\zeta) \]
\[ Q_f(0, \theta) = Q_f(\theta) \]
\[ R_f(h\zeta, h\theta) = R_f + \sum_{i=1}^{m+1} h' R_{fi}(\zeta, \theta) \]
\[ R_f(0, 0) = Q_f(\theta) \]
where \( Q_{fi} \) and \( R_{fi} \) are continuous and piecewise-continuously differentiable functions of \( \zeta \) and \( \theta \). The remainders \( P_{f,m+1}, Q_{f,m+1} \) and \( R_{f,m+1} \) may depend on \( h \) and are uniformly bounded functions. We realize that, in practical cases there is a little sense in taking delay-dependent \( P_f \). We nevertheless consider the general case of nonzero \( P_{fi} \) since this will be required in our analysis on the sampling segments \([t_{k+1} - 1, t_k] \), \( k < K \).

For simplicity we assume that \( A_{01} = 0 \). Note that \( A_{01} \) appears in the integral terms of RPDEs and the latter terms are of the order of \( O(h) \). That is why in the case of \( A_{01} \neq 0 \) the asymptotic approximation is similar. We substitute (31) in (16)–(19) and equate, separately, outer expansion and boundary-layer correction terms with the same powers of \( h \). We notice that for \( t = t_{k+1} - h \tau, \zeta = h\zeta \) and \( s = h\theta \) we have
\[ \frac{\partial}{\partial t} = -h^{-1} \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial \zeta} = h^{-1} \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial s} = h^{-1} \frac{\partial}{\partial \theta} \]
Thus, for the zero-order terms we obtain from (23), (24) and (19)
\[ \tilde{Q}_0(t, \zeta) = \begin{bmatrix} P_0 & 0 \\ M_0 & 0 \end{bmatrix}, \quad R_0(t, \zeta, \theta) = P_0(t) \quad (34) \]
\[ P_0 = \begin{bmatrix} P_0 & M_0 \\ M_0 & U_0 \end{bmatrix} \]
Then, from (22), (25) and (27), we obtain the differential equations
\[ \dot{P}_0(t) + \sum_{i=0}^r A_i P_0(t) + \sum_{i=0}^r P_0(t) A_i \]
\[ + P_0(t) SP_0(t) + C'C = 0 \quad (35) \]
\[ \dot{M}_0(t) + \sum_{i=0}^r A_i(t) P_0(t) S M_0(t) + P_0(t) B = 0 \]
\[ \dot{U}_0(t) + B'M_0(t) + M_0(t) B + M_0(t) S M_0(t) + I_t = 0 \quad (36) \]
with the terminal conditions
\[ P_0(t_k) = P_{f0} \]
\[ P_0(t_k) - M_0(t_k) U_0(t_k) M_0(t_k) = 0 \quad (37) \]
\[ M_0(t_k) = 0, \quad U_0(t_k) = 0, \quad k = K, \ldots, 1 \]
These equations correspond to the sampled-data \( H_\infty \) control of systems without delay (Basar and Bernard 1995). We make here the following assumption:

**Assumption 1:** For a specified value of \( \gamma \), the DREs (35), (36) and (37) possess a bounded solution on \([0, t_f]\).

Assumption 1 means that the \( H_\infty \) state-feedback sampled-data control problem for (1) without delay has a solution. If this were not the case, even \( P_0 \), the zero-order term in (31), would not exist.

To determine the first-order terms we start with \([t_{K-1}, t_f] \) and with the equations for \( \tilde{Q}_1 \):
\[ \frac{\partial}{\partial t} \tilde{Q}_1(t, \zeta) = -M(t) \tilde{Q}_0(t) - \tilde{Q}_0(t) \]
\[ \tilde{Q}_1(t, 0) = \tilde{P}_1(t) \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \]
\[ M = \sum_{i=0}^r A_i + S \tilde{P}_0 \]
Then
\[ \tilde{Q}_1(t, \zeta) = \tilde{Q}_1(t, 0) + (M(t) \tilde{Q}_0(t) + \tilde{Q}_0(t)) \zeta \]
Substituting this expression into the equation for \( \tilde{P}_1 \), we obtain
\[ \tilde{P}_1 + M' \tilde{P}_1 + \tilde{P}_1 M + \sum_{i=1}^r g_i A_i (\tilde{Q}_0 M + \tilde{Q}_0) \]
\[ + \sum_{i=0}^r g_i (M' \tilde{Q}_0 + \tilde{Q}_0) A_i = 0 \quad (39) \]
\[ \tilde{P}_1(t_f) + \Pi_{fP}(t) = \tilde{P}_{f1}, \quad g_i = h_i / h \]
It follows from (22) that \( \dot{1}_{P}(\tau) = 0 \). Since \( \Pi_{1, P} \)
vanishes for \( \tau \to \infty \), we have \( \Pi_{1, P}(\tau) \equiv 0, \tau \geq 0 \).
Hence, \( \dot{P}(t_f) = \dot{P}\bar{1}, and \( \dot{P}_1 \) is a solution to the linear
differential equation (39) with the latter terminal condition.

For \( \Pi_{1, Q} \), \( R_1 \) and \( \Pi_{1, R} \) we obtain from (17), (18) and (34)
\[
\begin{aligned}
\frac{\partial}{\partial \tau} \Pi_{1, Q}(\tau, \zeta) - \frac{\partial}{\partial \zeta} \Pi_{1, Q}(\tau, \zeta) &= 0 \\
\tilde{Q}_1(t_f, \zeta) + \Pi_{1, Q}(0, \zeta) &= \tilde{Q}_1(\zeta) \\
\Pi_{1, Q}(\tau, 0) &= \Pi_{1, P}(\tau) = 0 \\
\frac{\partial}{\partial \zeta} R_1(t, \zeta, \theta) + \frac{\partial}{\partial \theta} R_1(t, \zeta, \theta) &= -P_0(t)SP_0(t) - \dot{P}_0(t) \\
R_1(\tau, 0, \theta) &= Q_1(\tau, \theta)
\end{aligned}
\]
\[
(40)
\]
and
\[
\begin{aligned}
\frac{\partial}{\partial \tau} \Pi_{1, R}(\tau, \zeta, \theta) - \frac{\partial}{\partial \zeta} \Pi_{1, R}(\tau, \zeta, \theta) - \frac{\partial}{\partial \theta} \Pi_{1, R}(\tau, \zeta, \theta) &= 0 \\
\Pi_{1, R}(\tau, 0, \theta) &= 0
\end{aligned}
\]
\[
(41)
\]
where
\[
\tilde{Q}_1 = \begin{bmatrix} Q_1 & 0 \\
N_1 & 0 \end{bmatrix}, \quad \Pi_{1, i} = \begin{bmatrix} \Pi_{i} & 0 \\
\dot{\Pi}_{i} & 0 \end{bmatrix}
\]

Then, for \( \tau \geq 0 \) and \( t \in [t_k - 1, t_f] \), we find successively
\[
\Pi_{1, Q}(\tau, \zeta) = \begin{cases}
\tilde{Q}_1(\zeta + \tau) - \tilde{Q}_1(t_f, \zeta + \tau), & \text{if } \tau \leq -\zeta \\
0, & \text{if } \tau > -\zeta
\end{cases}
\]
\[
R_1(t, \zeta, \theta) = R_1(t, \theta, \zeta)
\]
\[
= -\zeta(P_0(t)SP_0(t) + \dot{P}_0(t)) + Q_1(t, \theta - \zeta), \quad \zeta \geq \theta
\]
\[
\Pi_{1, R}(0, \zeta, \theta) = R_1(\zeta, \theta) + \zeta(P_0SP_0 + \dot{P}_0(t_f))
\]
\[
- Q_1(t_f, \theta - \zeta)
\]
\[
\Pi_{1, R}(\tau, \zeta, \theta) = \Pi_{1, R}(\tau, \theta, \zeta) = \begin{cases}
\Pi_{1, R}(0, \zeta + \tau, \theta + \tau), & \text{if } \tau \leq -\zeta, \theta \leq \zeta \\
\Pi_{1, Q}(\tau + \zeta, \theta - \zeta), & \text{if } \tau > -\zeta, \theta \leq \zeta
\end{cases}
\]

Therefore
\[
\Pi_{1, Q}(\tau, \zeta) = 0, \quad \tau + \zeta > 0
\]
\[
\Pi_{1, R}(\tau, \zeta, \theta) = \Pi_{1, Q}(\tau + \zeta, \theta - \zeta) = 0
\]
\[
\tau + \theta > 0, \theta \leq \zeta
\]

The first-order and the higher order terms of the expansions can be similarly found on all sampling segments.

4.2. Near-optimal piecewise-constant \( H_{\infty} \)-control

For small enough values of \( h \) the approximations of the controller contain the outer expansion terms only, i.e. all the boundary-layer terms \( \Pi_{1} \) vanish. This is due to the fact that we need an approximation of the solution to RPDEs only on the left end of the sampling segment. In the Appendix we prove the following.

**Theorem 2:** Under Assumption 1 the following holds for all small enough delays \( h \):

1. The system of (16)–(20), (22)–(28) has a solution. This solution is approximated for any integer \( m \) by
\[
\begin{aligned}
\dot{P}(t) &= \dot{P}_0(t) + \sum_{i=1}^{m} h^i[\dot{P}_1(t)] \\
+ \Pi_{1, P}(\tau) + O(h^{m+1}) \\
\dot{Q}(t, h\zeta) &= \dot{Q}_0(t) + \sum_{i=1}^{m} h^i[\dot{Q}_1(t, \zeta)] \\
+ \Pi_{1, Q}(\tau, \zeta) + O(h^{m+1}) \\
R(t, h\zeta, h\theta) &= P_0(t) + \sum_{i=1}^{m} h^i[R_1(t, \zeta, \theta)] \\
+ \Pi_{1, R}(\tau, \zeta, \theta) + O(h^{m+1}) \\
\tau &= \frac{t_f - t}{h}, \zeta \in [-1, 0], \theta \in [-1, 0]
\end{aligned}
\]
\[
(42)
\]
where the boundary-layer terms satisfy
\[
\Pi_{1, P}(\tau) = 0, \quad \tau > i - 1; \quad \Pi_{1, Q}(\tau, \zeta) = 0, \quad \tau + \zeta > i - 1
\]
\[
\Pi_{1, R}(\tau, \zeta, \theta) = 0, \quad \tau + \theta > i - 1, \quad \theta \leq \zeta
\]
\[\text{and } |O(h^{m+1})| \leq ch^{m+1}, \text{ where } c \text{ is a positive scalar which is independent of } h, t, \zeta \text{ and } \theta. \text{ The matrices } \dot{P}_1 \text{ and } \dot{Q}_1 \text{ are taken with bars and have the structure of (13), where all the components are taken with index } i.
\]

2. Denote by \( Y_i, i = 0, \ldots, m \) the terms of the expansions of \( U^{-1} \) in the powers of \( h \), i.e.
\[
\left( \sum_{i=0}^{m} h^iU_i(t) \right)^{-1} = \sum_{i=0}^{m} h^iY_i(t) + O(h^{m+1}),
\]
\[
Y_0(t) = U_0^{-1}(t)
\]
Then the controller of (29) is approximated by

\[
\begin{align*}
\mathbf{u}(t) = \mathbf{u}_n(t) + O(h^{m+1}) \\
u_m(t) = - \sum_{i=0}^{m} \sum_{j=0}^{m-i} h^{j+i} Y_{ij}(t_k) \\
\times \left[ \mathbf{M}'(t_k)x(t_k) + \int_{t}^{0} N_{ij}(t_k,s)F(x_{ik})(hs) ds \right]
\end{align*}
\]

\[
t_k \leq t < t_{k+1}, \quad k = K - 1, \ldots, 0
\]

(43)

where \(N_{-1} = 0\) and \(N_0 = M_0\). The approximate controller \(u_m\) guarantees an attenuation level of \(\gamma + O(h^{m+1})\).

5. The zero-order controller performance

We study the performance of the system under the zero-order controller

\[
u_0(t) = - U_0^{-1}(t_k) \mathbf{M}_0(t_k)x(t_k)
\]

which solves the \(H_\infty\)-control problem for (1) without delay.

5.1. \(L_2\)-gain of state-delay systems with jumps

In order to study the performance under the zero-order controller in the sampled-data case we note that the closed-loop system of (1), where \(u = u_0(t)\), has the form of (9) with the jumps condition

\[
\mathbf{x}(t_k) = \mathbf{G}_k \mathbf{x}(t_k), \quad k = 1, \ldots, K - 1
\]

(44)

where

\[
\mathbf{G}_k = \begin{bmatrix} I & 0 \\ - \mathbf{U}_0^{-1}(t_k) \mathbf{M}_0(t_k) & 0 \end{bmatrix}
\]

(45)

We derive first the relevant bounded real lemma for (9) with (44), for any given \(\mathbf{A}_j, \mathbf{A}_0, \mathbf{D}, \mathbf{C}\) and \(\mathbf{G}_k\) of the appropriate dimensions. Consider

\[
\mathbf{J} = ||x||^2_{L_2} - \gamma^2 ||w||^2_{L_2} + \bar{E}(\mathbf{x})
\]

(46)

where \(\bar{E}\) is given by (14), \(\bar{F}\) is defined by (11) and \(\mathbf{P}_f, \mathbf{Q}_f\) and \(\mathbf{R}_f\) are any continuous and piecewise continuously differentiable functions of their arguments. We then apply the lemma to the special matrices of (8) and (45).

**Lemma 2:** Assume that for every \(k = 1, \ldots, K\) there exists a solution on \([t_{k-1}, t_k]\) to (16)–(19) such that

\[
\mathbf{P}(t) = \mathbf{Q}(t,0), \quad \mathbf{Q}(t,\xi) = \mathbf{R}(t,0,\xi)
\]

with the following terminal conditions

\[
\mathbf{P}(t_K) = \mathbf{P}_f, \quad \mathbf{Q}(t_K,\xi) = \mathbf{Q}_f(\xi), \quad \text{and}
\]

\[
\mathbf{R}(t,K,\xi) = \mathbf{R}_f(\xi), \quad \mathbf{P}(t_k) = G_k^\dagger \mathbf{P}(t_k) G_k
\]

(47)

\[
\mathbf{Q}(t_k,\xi) = G_k^\dagger \mathbf{Q}(t_k,\xi), \quad \text{and}
\]

\[
\mathbf{R}(t_k,\xi) = \mathbf{R}(t_k,\xi), \quad k < K
\]

(48)

Then the performance index \(J\) of (46) for the system of (9) with the jumps condition (44) is non-negative for all \(w \in L_2[0, t_f]\) and \(\mathbf{x}_0 = 0\).

**Proof:** Similarly to Theorem 1 we use the dynamic programming argument

\[
\sup_w J = \sup_w \left\{ \int_0^{t_f} \|z\|^2 - \gamma^2 \|w\|^2 dt + \sup_w \left\{ \cdots + \sup_w \left\{ \bar{E}(\mathbf{x}_f) \right\} \cdots \right\} \right\}
\]

We first consider stage \(K\). By Lemma 1 the supremum value of

\[
J_K = \bar{E}(\mathbf{x}_f) + \int_{t_K-1}^{t_f} \|z\|^2 - \gamma^2 \|w\|^2 dt
\]

with respect to \(w\) is given by

\[
\sup_w J_K = \bar{V}(t_K-1, \mathbf{x}_{t_K-1})
\]

\[
= \mathbf{x}(t_K-1)' \mathbf{P}(t_K-1) \mathbf{x}(t_K-1) + 2 \mathbf{x}'(t_K-1)
\]

\[
\times \int_{-h}^{0} \mathbf{Q}(t_k,\xi) \bar{F}(\mathbf{x}_{t_k-1})(\xi) d\xi
\]

\[
+ \int_{-h}^{0} \bar{F}'(\mathbf{x}_{t_k-1})(\xi) \mathbf{R}(t_k,\xi,\xi) \bar{F}(\mathbf{x}_{t_k-1})(\xi) d\xi d\xi
\]

(49)

The matrix-functions \(\mathbf{P}, \mathbf{Q}\) and \(\mathbf{R}\) satisfy (16)–(18) and (47). Substituting now (44) into (49) we obtain the following performance cost for the stage \(K - 1\)

\[
J_{K-1} = \mathbf{x}(t_K-1)' G_{K-1}' \mathbf{P}(t_K-1) G_{K-1} \mathbf{x}(t_K-1)
\]

\[
+ 2 \mathbf{x}'(t_K-1) G_{K-1}' \int_{-h}^{0} \mathbf{Q}(t_k,\xi) \bar{F}(\mathbf{x}_{t_k-1})(\xi) d\xi
\]

\[
+ \int_{-h}^{0} \bar{F}'(\mathbf{x}_{t_k-1})(\xi) \mathbf{R}(t_k,\xi,\xi) \bar{F}(\mathbf{x}_{t_k-1})(\xi) d\xi d\xi
\]

\[
+ \int_{-h}^{0} \int_{-h}^{0} \|z\|^2 - \gamma^2 \|w\|^2 dt
\]

(49)

By Lemma 1 this cost has a supremum value given by (49), where \(k = K - 1\), since the corresponding RPDEs with terminal conditions (48) have no escape point.
Similarly it can be shown that (49) holds for all $k = K - 1, \ldots , 1$. Then $\sup_n J = \sup_n J_1 \leq 0$ due to $x_0 = 0$. 

Note that for $h \to 0$, $\bar{P} \to \bar{P}_0$ and $\bar{Q} \to \bar{Q}_0$ (cf. (42)). Therefore the latter lemma coincides with the corresponding results in Sivashankar and Khargonekar (1994).

### 5.2. Robustness of the performance under $u_0$

**Theorem 3:** Under Assumption 1 the controller $u_0$ for all small enough $h$ leads to a performance level of $\gamma$.

**Proof:** By Lemma 3, this closed-loop system has an induced $L_2$-gain less or equal to $\gamma$ if the corresponding RPDEs of (16)–(19) with the terminal conditions of (48) have a bounded solution. The existence of a solution to these RPDEs, approximated by (42) with $m = 0$, can be proved similarly to (i) of Theorem 2.

Given $\gamma > 0$ and $h$, one should verify that the corresponding RPDEs have a solution in order to make certain that $u_0$ leads to a performance level of $\gamma$. This is not an easy task. That is why one may resort to more conservative, but computationally simpler, conditions in terms of differential linear matrix inequalities (DLMI) or Riccati differential inequalities (RDI) that were formulated for the case of one delay in Shaked et al. (1998). Generalization of these lemmas to the case of $r$ delays and jumps in the system is given below in Lemmas 3 and 4.

### 5.3. Bounded real lemmas for state-delay systems with jumps using DLMI

Consider (9) with (44), where $\bar{A}_i$, $\bar{B}$, $\bar{C}$ and $G_k$ are any given matrices of appropriate dimensions. For simplicity we assume that $A_{01} = 0$. Consider

$$J = ||\tilde{z}||^2_{L^2} - \gamma^2 ||w||^2_{L^2} + \tilde{x}'(t_f)\bar{P}_f \tilde{x}(t_f)$$

where $\bar{P}_f$ is any matrix.

**Lemma 3:** The performance index $J$ of (50) for the system of (9) with the jumps condition of (44) is non-positive for all $w \in L_2[0, t_f]$, and $\tilde{x}_0 = 0$, if there exist square integrable matrices $Q_i(t) = Q_i'(t)$, $i = 1, \ldots , r$ that are positive definite on $[t_f - h_i, t_f]$ and allow an absolutely continuous solution $P(t) = P(t)' \geq 0$ on $t \in [t_{k-1}, t_k]$, $k = 1, \ldots , K$ to the following DLMI

$$N(t) = \begin{bmatrix}
\psi(t) & P(t)A_1 & P(t)A_2 & \cdots & P(t)A_r & P(t)D
A_1'P(t) & -Q_1(t-h_1) & 0 & \cdots & 0 & 0
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots
A_r'P(t) & 0 & 0 & \cdots & -Q_r(t-h_r) & 0
\bar{B}'P(t) & 0 & 0 & \cdots & 0 & -\gamma^2 I
\end{bmatrix} \leq 0$$

with the terminal conditions

$$P(t_{K}) = \bar{P}_f$$

and

$$P(t_k) = G_k'P(t_k)G_k, \ k = K - 1, \ldots , 1$$

**Proof:** We consider the function

$$V(t, \tilde{x}) = \tilde{x}'(t)P(t)\tilde{x}(t) + \sum_{i=1}^{r} \int_{t-h_i}^{t} \tilde{x}'(\tau)Q_i(\tau)\tilde{x}(\tau) d\tau$$

Using (9), it can be easily established that for $t \neq t_k$

$$\frac{d}{dt} V(t, \tilde{x}_t) = \tilde{x}'(t)[\bar{P}_f + \bar{A}_0 P(t)+ P(t)\bar{A}_0 + \bar{C}'\bar{C} + \sum_{i=1}^{r} Q_i(t)] + \sum_{i=1}^{r} Q_i(t)[\bar{x}'(t)P(t) + P(t)\bar{x}(t)] + 2\bar{x}'(t)P(t)\bar{D}w(t)$$

where $y_i(t) \equiv x(t - h_i)$. Denoting $\xi \equiv [x', y_1', \ldots , y_r', w]'$ it follows from (54) that

$$\frac{dV}{dt} = \xi'N\xi + \gamma^2 |w|^2 - |x|^2, \quad t \in [t_k, t_{k+1})$$

From (52), (53) and the fact that $\tilde{x}(t) = 0, \forall t \leq 0$, it is readily obtained that

$$\int_{0}^{t_f} \frac{dV}{d\tau} d\tau = V(t_f, \tilde{x}_f) - V(t_{K-1}, \tilde{x}_{k-1}) + V(t_{K-1}, \tilde{x}_{k-1}) - \cdots - V(t_1, \tilde{x}_{1}) = \tilde{E}(\tilde{x}_f) + \sum_{i=1}^{r} \int_{t_i-h_i}^{t_i} \tilde{x}'(\tau)Q_i(\tau)\tilde{x}(\tau) d\tau$$

By considering (55), this implies that

$$J = \int_{0}^{t_f} \xi'N\xi d\tau - \sum_{i=1}^{r} \int_{t_i-h_i}^{t_i} \tilde{x}'Q_i\tilde{x}d\tau \leq 0 \quad \forall w \in L_2[0, t_f]$$

Lemma 3 provides a bounded real criterion in terms of the DLMI of (51). Note that if $Q(i) > 0$, $t \in [0, t_f]$, we obtain using Schur complements, that (51) is equivalent to the RDI
\( \dot{P}(t) + \sum_{i=0}^{r} \dot{A}_i P(t) + P(t) \sum_{i=0}^{r} \dot{A}_i \)

\[ + \sum_{i=1}^{r} h_i \left[ \dot{A}_{i1} P_{11} A_0 + \left( \sum_{j=1}^{r} \dot{A}_j \right) P_{2i} \left( \sum_{j=1}^{r} \dot{A}_j \right) \right] + \dot{C}' \dot{C} \]

\[ + P(t) \left[ \gamma^{-2} \bar{D} \left( I - \sum_{i=1}^{r} h_i P_{3i} \right)^{-1} \bar{D}' \right] \]

\[ + \sum_{i=1}^{r} \left[ T_i + \gamma^{-2} \bar{A}_i H_i \bar{A}_i' + P_{4i} \chi_{ik}(t) \right] P(t) \leq 0 \]

\( t \in [t_k, t_{k+1}) \) (57)

and to the following RDI for \( k = 0 \)

\[ \dot{P}(t) + \sum_{i=0}^{r} \dot{A}_i P(t) + P(t) \sum_{i=0}^{r} \dot{A}_i \]

\[ + \sum_{i=1}^{r} h_i \left[ \dot{A}_{i1} P_{11} A_0 + \left( \sum_{j=1}^{r} \dot{A}_j \right) P_{2i} \left( \sum_{j=1}^{r} \dot{A}_j \right) \right] + \dot{C}' \dot{C} \]

\[ + P(t) \left[ \gamma^{-2} \bar{D} \left( I - \sum_{i=1}^{r} h_i P_{3i} \right)^{-1} \bar{D}' \right] \]

\[ + \sum_{i=1}^{r} \left[ T_i + \gamma^{-2} \bar{A}_i H_i \bar{A}_i' + P_{4i} \chi_{ik}(t) \right] P(t) \leq 0 \]

\( t \in [t_k, t_{k+1}) \) (58)

The result of Lemma 3 is most powerful, in the sense that it establishes a condition for the performance index \( J \) to be non-positive almost independently of the delay length. It is thus too conservative. A less conservative condition which explicitly depends on the length of the delay is derived in the next lemma.

**Lemma 4:** Assume that \( h \leq \min_{k=1,\ldots,K} (t_k - t_{k-1}) \). The performance index \( J \) of (50) for the system of (9) with the jumps condition of (44) is non-positive for all \( w \in L_2[0,t_f] \) if for every \( k = K - 1,\ldots,0 \) there exist constant symmetric positive definite matrices \( P_{11}, P_{21}, P_{3i} \) and \( P_{4i}, i = 1,\ldots,r \) with \( I - \sum_{i=1}^{r} h_i P_{3i} > 0 \), that allow a solution \( P(t) = P(t) \geq 0 \), that is absolutely continuous for \( t \neq t_k \), to the following RDI for \( k = K - 1,\ldots,1 \)

\( \dot{P}(t) + \sum_{i=0}^{r} \dot{A}_i P(t) + P(t) \sum_{i=0}^{r} \dot{A}_i \)

\[ + \sum_{i=1}^{r} h_i \left[ \dot{A}_{i1} P_{11} A_0 + \left( \sum_{j=1}^{r} \dot{A}_j \right) P_{2i} \left( \sum_{j=1}^{r} \dot{A}_j \right) \right] + \dot{C}' \dot{C} \]

\[ + P(t) \left[ \gamma^{-2} \bar{D} \left( I - \sum_{i=1}^{r} h_i P_{3i} \right)^{-1} \bar{D}' \right] \]

\[ + \sum_{i=1}^{r} \left[ T_i + \gamma^{-2} \bar{A}_i H_i \bar{A}_i' + P_{4i} \chi_{ik}(t) \right] P(t) \leq 0 \]

\( t \in [t_k, t_{k+1}) \) (57)
Remark 1: In the case when $A_i(I - G_k) = 0$ for some $i > 0$ we set $P_{4i} = P_{4i}^T = 0$ in (57) and (59) and we delete the row and the column containing $P_{4i}, i = 1, \ldots, r$ in (61).

The above result is obtained for any matrices in (9), (44). In the special case of matrices of the form (8), (45) we obtain that $A_i(I - G_k) = 0$ for all $i = 1, \ldots, r$ and hence for all $k = K - 1, \ldots, 0$ we solve the RDI of (58) with the terminal conditions given by (52). We obtain the following corollary.

Corollary 1: Assume that for every $k = K - 1, \ldots, 0$ there exist constant symmetric positive definite matrices $P_{1i}, P_{2i}$ and $P_{3i}, i = 1, \ldots, r$ with $I - \sum_{i=1}^r h_i P_{3i} > 0$, that allow a solution $P(t) = P^T(t) \geq 0$, that is absolutely continuous for $t \in [t_k, t_{k+1}]$, to the RDI of (58) (to the DLMi of (61), where the row and the column containing $P_{4i}, i = 1, \ldots, r$ is deleted), with the terminal conditions of (52), where $A_i, D, C, G_k$ are given by (8), (45) and $P_f = 0$. Then the performance index $J$ of (4) with $E = 0$ for the closed-loop system of (1), where $u = u_0(x)$, is nonpositive for all $w \in L_2[0, t_f]$ and $x_0 = 0$.

Remark 2: Note that the case of $A_0 \neq 0$ requires modification of technique used in lemmas 3 and 4 and leads to more complicated DLMIs and RDIs.

Example: Consider the system
\[ \dot{x}(t) = x(t) - x(t-h) + u + w, \quad z = \col{x, u} \quad (62) \]
and $P_f = Q_f = R_f = 0$. For $h = 0$ this example coincides with the one in Basar and Bernard (1995, p. 135). We chose $t_f = 1\text{s}$, $K = 2$, $t_1 = 0.5\text{s}$ and $\gamma = 1$. We verified that (35) (and, hence, linear equations (36)) with terminal conditions (37) had bounded solutions on $[0, 1]$:
\[
P_0(t) = U_0(t) = \tan(1-t), \quad M_0(t) = \cos^{-1}(1-t) - 1
\]
\[0.5 \leq t \leq 1 \]
\[P_0(t) = \tan(0.9721 - t), \quad 0 \leq t < 0.5\]

Therefore, Assumption 1 holds and
\[u_0(t) = 0, \quad 0 \leq t < 0.5; \quad u_0(t) = -0.2553 \times (0.5)
\]
\[0.5 \leq t < 1\]

From (39) and (43) we obtained
\[P_1(0.5) = -1.1955, \quad M_1(0.5) = -0.3113 \]
\[U_1(0.5) = -0.1029 \]
\[u_1(t) = 0, \quad 0 \leq t < 0.5 \]
\[u_1(t) = u_0(t) - h \left[ 3.0253 \times (0.5) - 0.2553 \int_0^h x(0.5 + hs) \, ds \right], \quad 0.5 \leq t \leq 1 \]

Considering the system of (62), where $u = u_0$, and applying the RDI of Lemma 3, we were not able to find a time-invariant $Q > 0$ for which the RDI of (56) holds. If there existed such a $Q$, then $u_0$ would lead to a closed-loop performance level of $\gamma = 1$ for all $h$. Applying therefore Lemma 4 and choosing $P_{11} = P_{21} = P_{31} = P_{41} = I$ we find that the RDI of (58) with terminal condition of (59) has a solution for all $0 \leq h \leq 0.17\text{s}$. For $h = 0.18\text{s}$ the solution of (58), with the equality sign and (59), has an escape point (see Figure 1). The control law of $u = u_0$ thus guarantees $\gamma = 1$ for all $0 \leq h \leq 0.17\text{s}$, whereas for $h = 0.18\text{s}$ our theory, which only provides a sufficient condition cannot guarantee the performance level of $\gamma = 1$.

![Figure 1](image)

Figure 1. The (1, 2) component of the solution $P$ to (57) for $h = 0.17\text{s}$ and $h = 0.18\text{s}$. 
Appendix

Proof of Lemma 1: Let \( \tilde{x}(t) \) be a solution of (9). Then, differentiating \( \tilde{V}(t, \tilde{x}) \) with respect to \( t \) we obtain

\[
\frac{d}{dt} \tilde{V}(t, \tilde{x}) = 2[\tilde{L}(\tilde{x}(\cdot)) + \tilde{D}w'] \times \left[ \tilde{P}(t)\tilde{x}(t) + \int_{-h}^{0} \frac{\partial}{\partial \xi} \tilde{Q}(t, \xi)\tilde{F}(\tilde{x}(\xi)) d\xi \right] \\
+ \tilde{x}'(t) \left[ \tilde{P}(t)\tilde{x}(t) + 2\int_{-h}^{0} \frac{\partial}{\partial \xi} \tilde{Q}(t, \xi)\tilde{F}(\tilde{x}(\xi)) d\xi \right] \\
+ 2\tilde{x}'(t) \int_{-h}^{0} \frac{\partial}{\partial \xi} \tilde{Q}(t, \xi) d\xi \tilde{F}(\tilde{x}(\xi)) d\xi \\
+ \int_{-h}^{0} \frac{d}{dt} \tilde{F}'(\tilde{x}(\xi))(s)\tilde{R}(t, s, \xi)\tilde{F}(\tilde{x}(\xi)) d\xi d\xi
\]

(63)

where \( \tilde{L}(\tilde{x}(\cdot)) \) is defined by (2) with all matrices taken with bars. Denote by

\[
w' = \gamma^{-2} \tilde{P}(t) + \int_{-h}^{0} \tilde{Q}(t, \xi)\tilde{F}(\tilde{x}(\xi)) d\xi
\]

Then, integrating by parts in (63) e.g.

\[
\int_{-h}^{0} \tilde{Q}(t, \xi) \frac{d}{dt} \tilde{F}(\tilde{x}(\xi)) d\xi = -\int_{-h}^{0} \tilde{Q}(t, \xi)\left\{ \frac{d}{d\xi} \left[ \tilde{F}(\tilde{x}(\xi)) \right] - \tilde{A}_{01}(\xi)\tilde{x}(t) \right\} d\xi
\]

\[
= -\tilde{Q}(t, 0)[\tilde{L}(\tilde{x}(\cdot)) - \tilde{A}_{0}\tilde{x}(t)] - \sum_{i=1}^{r} \tilde{Q}(t, -h_{i})\tilde{A}_{i}\tilde{x}(t)
\]

\[
+ \int_{-h}^{0} \frac{\partial}{\partial \xi} \tilde{Q}(t, \xi)\tilde{F}(\tilde{x}(\xi)) d\xi \\
+ \int_{-h}^{0} \tilde{Q}(t, \xi)\tilde{A}_{01}(\xi) d\xi \tilde{x}(t)
\]

where due to (19)

\[
\tilde{Q}(t, 0)[\tilde{L}(\tilde{x}(\cdot)) - \tilde{A}_{0}\tilde{x}(t)] = \tilde{P}(t)[\tilde{L}(\tilde{x}(\cdot)) - \tilde{A}_{0}\tilde{x}(t)]
\]

and applying (16)–(19) we get (64)

\[
\frac{d}{dt} \tilde{V}(t, \tilde{x}) = \tilde{x}'(t)[\tilde{P}(t) + \tilde{A}_{0}\tilde{P} + \tilde{P}\tilde{A}_{0}]
\]

\[
+ \sum_{i=1}^{r} \tilde{A}_{i}\tilde{Q}'(t, -h_{i}) + \sum_{i=1}^{k} \tilde{Q}(t, -h_{i})\tilde{A}_{i}
\]

\[
+ \int_{-h}^{0} \tilde{A}_{01}\tilde{Q}'(t, \theta) d\theta + \int_{-h}^{0} \tilde{Q}(t, \theta)\tilde{A}_{01}(\theta) d\theta]\tilde{x}(t)
\]

\[
+ 2\tilde{x}'(t) \int_{-h}^{0} \frac{\partial}{\partial \xi} \tilde{Q}(t, \xi) + \frac{\partial}{\partial \xi} \tilde{Q}(t, \xi)
\]

\[
+ \tilde{A}_{0}\tilde{Q}(t, \xi) + \sum_{i=1}^{r} \tilde{A}_{i}\tilde{R}(t, -h_{i}, \xi)
\]

\[
+ \int_{-h}^{0} \tilde{A}_{01}(\theta)\tilde{R}(t, \theta, \xi) d\theta]\tilde{F}(\tilde{x}(\xi)) d\xi
\]
Proof of Theorem 2: The proof of (i) is similar to Fridman and Shaked (1998). Equation (43) follows from (42). We prove only that (9) and the jump conditions

\[ x(t) = W_k x_0, \quad x_m(t) = W_k + O(h^{m+1}) x_m(t) \]

where

\[ y_i = x_i - x_{mi}, \quad y_i = \frac{\partial}{\partial s} x_i(s) ds \]

Thus, \( y_i = x_i - x_{mi} \) satisfies

\[ y_i(t) = \sum_{i=0}^{r} A_i y_i(t-h_i) + \int_{-h}^{0} A_0(s) y_i(t+s) ds \]

\[ z(t) = C x(t), \quad t_k \leq t < t_{k+1} \]

\[ y_i(t_k) = W_k y_i + O(h^{m+1}) x_{mi} \]

We find that

\[ ||z_{mi} - z||_{L_2}^2 \leq c_0(||y_i||_{L_2}^2 + h^{m+1} ||x_i||_{L_2}^2) \]

\[ + \int_{-h}^{0} c_1(||y_i||_{L_2}^2 + h^{m+1} ||x_i||_{L_2}^2) dt \]

\[ \leq c_0(||y_i||_{L_2}^2 + h^{m+1} ||x_i||_{L_2}^2) \]

\[ + c_1(||y_i||_{L_2}^2 + h^{m+1} ||x_i||_{L_2}^2), \quad c_1 > 0 \]

where \( ||y_i||_{L_2} = \max_{t \in [-h,0]} ||y_i(t)||_{L_2} \). Evidently, \( ||x(t)||_{L_2} \leq ||x_i||_{L_2} \).

Let \( X(t) \) be a transition matrix for (9), \( X(0) = I \) and \( X(t) = 0 \) for \( t < 0 \). Denote \( T(t)x_0 = x_t \). By the variation of constants formula (Hale 1977)

\[ x_t = T(t-t_k) W_k x_{\hat{t}} + \int_{t_k}^{t} T(t-s)X_0 D w(s) ds \]

\[ t_k < t \leq t_{k+1} \]

From the latter equation and the condition \( x_{\hat{t}} = 0 \), \( k = 1, \ldots, K \), \( c' > 0 \)

Then from (68) and (69) we derive

\[ ||x_i||_{L_2}^2 = \sum_{k=0}^{K} \int_{t_k}^{t_{k+1}} ||D w(s)||_{L_2} \]

\[ \leq c ||w||_{L_2}^2 \]

Similarly to the latter inequality, one can derive \( ||x_{mi}||_{L_2}^2 \leq c ||w||_{L_2}^2 \), and \( ||y_{mi}||_{L_2}^2 \leq ch^{m+1} ||x_{mi}||_{L_2}^2 \leq c^2 h^{m+1} ||w||_{L_2}^2 \). The latter inequalities, together with (67), imply \( ||z_{mi}||_{L_2} \leq ||z_i||_{L_2} + O(h^{m+1}) ||w||_{L_2} \). Since \( ||z_i||_{L_2} \leq \gamma ||w||_{L_2} \), we derive \( ||z_{mi}||_{L_2} \leq \gamma^2 + O(h^{m+1}) ||w||_{L_2}^2 \).
\[
\dot{x}(t) = \sum_{i=0}^{r} \tilde{A}_i \tilde{x}(t) - \sum_{i=1}^{r} A_i \int_{t-h_i}^{t} \left[ \sum_{j=0}^{r} \tilde{A}_j \tilde{x}(\tau - h_j) \right] \, d\tau \\
- \sum_{i=1}^{r} \tilde{A}_i D \int_{t-h_i}^{t} w \, d\tau + \tilde{D}w(t) \\
+ \sum_{i=1}^{r} \tilde{A}_i (I - G_k) \tilde{x}(t_k) \chi_{\tilde{d}}(t)
\]

Defining the function \( V(t, \tilde{x}(t)) = \tilde{x}'(t)P(t)\tilde{x}(t) \), we obtain that for almost all \( t \)
\[
\frac{d}{dt} V(t, \tilde{x}(t)) = \tilde{x}'(t) \left[ \dot{P}(t) + \sum_{j=0}^{r} \tilde{A}_j P(t) + P(t) \sum_{j=0}^{r} \tilde{A}_j \right] \\
\times \tilde{x}(t) - 2(\eta_1 + \eta_2 + \eta_3 - \eta_4) 
\tag{70}
\]
where\[
\eta_1(t) = \sum_{i=1}^{r} \int_{t-h_i}^{t} \tilde{x}'(t)P(t) \dot{A}_i \tilde{A}_0 \tilde{x}(t) \, d\tau \\
\eta_2(t) = \sum_{i=1}^{r} \int_{t-h_i}^{t} \tilde{x}'(t)P(t) \tilde{A}_i \tilde{D}w(t) \, d\tau \\
\eta_3(t) = \sum_{i=1}^{r} \int_{t-h_i}^{t} \tilde{x}'(t)P(t) \tilde{A}_i D \tilde{x}(t) \, d\tau - \tilde{x}'(t)P(t) \dot{D}w(t) \\
\eta_4(t) = \tilde{x}'(t)P(t) \sum_{i=1}^{r} \left[ \tilde{A}_i (I - G_k) \tilde{x}(t_k) \chi_{\tilde{d}}(t) \right]
\]
Since for any \( z, y_i \in \mathbb{R}^n, i = 1, \ldots, r \) and for any symmetric positive definite matrices \( X_i \in \mathbb{R}^{n \times n} \)
\[
-2 \sum_{i=1}^{r} y_i' X_i \leq - \sum_{i=1}^{r} y_i' X_i y_i + \sum_{i=1}^{r} z' X_i z
\]
we find that for any \( (n+1) \times (n+1) \) constant symmetric matrices \( P_{1i} > 0, P_{2i} > 0, P_{3i}, P_{4i}, i = 0, \ldots, r \), and for any \( q \times q \) symmetric constant matrix \( P_5 > 0 \)
\[
-2\eta_1(t) \leq \sum_{i=1}^{r} h_i \tilde{x}' P_{1i}^{-1} \tilde{A}_i P \tilde{x} \\
+ \sum_{i=1}^{r} \int_{t-h_i}^{t} \tilde{x}' \tilde{A}_0 P_{1i} \tilde{A}_0 \tilde{x} \, d\tau \\
-2\eta_2 \leq \sum_{i=1}^{r} h_i \tilde{x}' P_{1i}^{-1} \tilde{A}_0 P \tilde{x} \\
+ \sum_{i=1}^{r} \int_{t-h_i}^{t} \tilde{x}' (\tau - h_i) \tilde{A}_j \tilde{x} \, d\tau \\
\tag{71}
\]
where \( S_1(t) = \dot{P} + \sum_{i=0}^{r} \tilde{A}_i P + P \sum_{i=0}^{r} \tilde{A}_i \)
\[
+ \sum_{i=1}^{r} \int_{t-h_i}^{t} \left[ \tilde{A}_j \tilde{x}(\tau - h_j) \right] \, d\tau \\
+ P \sum_{i=1}^{r} (I + \gamma^2 \tilde{A}_i H \tilde{A}_i) + \sum_{i=1}^{r} P_{4i} \chi_{\tilde{d}}(t) \\
+ \gamma^2 \tilde{D} P_5^{-1} \tilde{D}' P + \tilde{C}' \tilde{C} 
\tag{73}
\]
and where \( \Gamma_i \) is defined in (60). Due to (44) and (59) the term in the second line of (72) vanishes. Then from the inequality
\[
\int_0^\infty \int_{t-h}^t \xi'(\tau)\xi(\tau)d\tau dt \leq h\int_0^\infty \xi'(t)\xi(t)dt, \\
\forall \xi(t) : \xi(t) = 0, \forall t \leq 0
\]
and \( \dot{x}_0 = 0 \) and \( w_0 = 0 \), and from (72) and (59) it follows that
\[
J \leq \int_0^\infty \dot{x}'S(t)\dot{x}dt + \int_0^\infty \gamma^2w' \left( \sum_{i=1}^r h_iP_{3i} + P_5 - I \right)wdt
\]
(74)
where
\[
S(t) = S_1(t) + \sum_{i=1}^r h_i \left[ \tilde{A}_0 P_{1i} \tilde{A}_0 + \left( \sum_{j=1}^{r} \tilde{A}_j \right) P_{2i} \left( \sum_{j=1}^{r} \tilde{A}_j \right) \right]
\]
If \( S(t) \leq 0 \) and \( P_5 = I - \sum_{i=1}^r h_iP_{3i} \), due to (74) \( J \leq 0 \) for all \( w \in L_2[0, t_f] \). Finally we note that, in view of (74) the conditions of (74) are equivalent to the RDI (57).

References


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