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Systems & Control Letters 40 (2000) 121–131

SYSTEMS & CONTROL LETTERS

www.elsevier.com/locate/sysconle

Exact slow–fast decomposition of the nonlinear singularly perturbed optimal control problem

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Received 6 June 1999; received in revised form 15 November 1999; accepted 20 January 2000

Abstract

We study the infinite horizon nonlinear quadratic optimal control problem for a singularly perturbed system, which is nonlinear in both, the slow and the fast variables. It is known that the optimal controller for such problem can be designed by finding a special invariant manifold of the corresponding Hamiltonian system. We obtain exact slow–fast decomposition of the Hamiltonian system and of the special invariant manifold into the slow and the fast ones. On the basis of this decomposition we construct high-order asymptotic approximations of the optimal state-feedback and optimal trajectory. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Singular perturbations; Nonlinear optimal control; Order reduction; Hamiltonian systems; Invariant manifolds

1. Introduction

Exact slow–fast decomposition of the infinite horizon LQ singularly perturbed optimal control problem has been obtained in [11]. Such decomposition leads to new effective algorithms for numerical and asymptotic approximations of optimal solutions [11,5,3]. For a class of singularly perturbed systems being nonlinear only on the slow variable a formal expansion of the optimal controller in the powers of ε has been found in [2,7] (see also references therein). For this class of systems the exact slow–fast decomposition has been obtained in [4] by applying invariant manifolds approach [8]. On the basis of this decomposition a new algorithm for expansion of the optimal controller has been introduced and near-optimality of the high-order approximation to the optimal controller (in the sense of its closeness to the optimal one) has been proved.

In the present paper we extend results of [4] to the general singularly perturbed system, being affine in the control and nonlinear in both, the slow and the fast variables. Under suitable assumptions on the linearized system, we obtain the exact decomposition of the problem and construct a higher-order approximation to the optimal controller and optimal trajectory.

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PII: S0167-6911(00)00008-6

2. Problem formulation

Consider the nonlinear singularly perturbed system

$$\dot{x}_1 = F_1(x_1,x_2) + B_1(x_1,x_2)u, \quad \varepsilon\dot{x}_2 = F_2(x_1,x_2) + B_2(x_1,x_2)u \tag{1}$$

with the functional

$$J = \int_0^\infty [k'(x_1,x_2)k(x_1,x_2) + u'R(x_1,x_2)u]\,\mathrm{d}t, \tag{2}$$

where $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$ are the state vectors, $x = \mathrm{col}\{x_1,x_2\}$, $u(t) \in \mathbb{R}^m$ is the control input, and $\varepsilon > 0$ is a small parameter. Prime denotes the transposition of a matrix. The functions F_i, B_i $(i = 1,2)$, R and k are differentiable with respect to x a sufficient number of times. We assume also that $F_i(0,0)=0$, $k(0,0)=0$ and $R = R' > 0$. We consider a nonstandard singularly perturbed problem in the sense that we do not require the solvability with respect to x_2 of the algebraic equation $F_2(x_1,x_2) + B_2(x_1,x_2)u = 0$. In the standard case the assumption that the latter equation is solvable with respect to x_2 is a crucial one (see e.g. [2,7]).

We are looking for a nonlinear state feedback

$$u = \beta(x), \quad \beta(0) = 0, \tag{3}$$

that minimizes the cost (2), where $x(t)$ satisfies (1) with the initial condition $x(0) = x_0$. For each $\varepsilon > 0$ the control law (3) is locally optimal on $\Omega \subset R^{n_1} \times R^{n_2}$ if there exists Ω_1, $0 \in \Omega \subset \Omega_1$, such that the closed-loop trajectories for initial data in Ω remain in Ω_1 and if for any initial value $x_0 \in \Omega$ and for any control $u(t)$ such that

$$\text{(i)} \quad x(t) \in \Omega_1,\ t \geqslant 0, \quad \text{(ii)}\ J(x_0,u) < \infty, \quad \text{(iii)}\ \lim_{t\to\infty} x(t) = 0$$

we have $J(x_0,\beta) \leqslant J(x_0,u)$ [1].

Consider the Hamiltonian function

$$\mathscr{H}(x_1,x_2,p_1,p_2) = p_1'F_1(x_1,x_2) + p_2'F_2(x_1,x_2) - \frac{1}{2}(p_1'\,p_2')\begin{pmatrix} S_{11}(x) & S_{12}(x) \\ S_{21}(x) & S_{22}(x)\end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \tfrac{1}{2}k'(x_1,x_2)k(x_1,x_2), \tag{4}$$

where p_1 and εp_2 play the role of the costate variables and $S_{ij} = B_iR^{-1}B_j'$, $i=1,2$, $j=1,2$. The corresponding Hamiltonian system has the form

$$\dot{x}_1 = f_1(x_1,p_1,x_2,p_2), \tag{5a}$$

$$\dot{p}_1 = f_2(x_1,p_1,x_2,p_2), \tag{5b}$$

$$\varepsilon\dot{x}_2 = f_3(x_1,p_1,x_2,p_2), \tag{5c}$$

$$\varepsilon\dot{p}_2 = f_4(x_1,p_1,x_2,p_2), \tag{5d}$$

where $f_1 = (\partial \boldsymbol{H}/\partial p_1)'$, $f_2 = -(\partial \boldsymbol{H}/\partial x_1)'$, $f_3 = (\partial \boldsymbol{H}/\partial p_2)'$, $f_4 = -(\partial \boldsymbol{H}/\partial x_2)'$.

The solution of the optimal control problem is related to the *invariant manifold*

$$p_1 = Z_1(x_1,x_2), \quad p_2 = Z_2(x_1,x_2), \tag{6}$$

of (5) [8]. For each $\varepsilon > 0$ Eqs. (6) define the invariant on Ω_1 manifold of (5) if for any $x_1^0, x_2^0 \in \Omega_1$, there exists $t_1 < 0 < t_2$ such that a solution of (5) with the initial conditions

$$x_1(0) = x_1^0, \quad x_2(0) = x_2^0, \quad p_1(0) = Z_1(x_1^0,x_2^0), \quad p_2(0) = Z_2(x_1^0,x_2^0)$$

satisfies (6) for $t \in (t_1, t_2)$. The restriction of $(2n_1 + 2n_2)$-dimensional system (5) to (6) (i.e. the *flow* on the invariant manifold (6)) is governed by the $(n_1 + n_2)$-dimensional system

$$\dot{x}_1 = f_1(x_1, Z_1, x_2, Z_2), \quad \varepsilon\dot{x}_2 = f_3(x_1, Z_1, x_2, Z_2). \tag{7}$$

Eqs. (7) result after substitution of (6) into (5a) and (5c).

Denote by (V_{x_1}, V_{x_2}) the Jacobian matrix of V. For each $\varepsilon > 0$ the problem is locally solvable on $\Omega \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ if there exists Ω_1, $0 \in \Omega \subset \Omega_1$, and a C^2 nonnegative solution $V : \Omega_1 \to \mathbb{R}$ to the Hamilton–Jacobi (HJ) partial differential equation

$$V_{x_1} F_1(x_1, x_2) + \frac{1}{\varepsilon} V_{x_2} F_2(x_1, x_2)$$

$$- \frac{1}{2} \begin{pmatrix} V_{x_1} & \frac{1}{\varepsilon} V_{x_2} \end{pmatrix} \begin{pmatrix} S_{11}(x) & S_{12}(x) \\ S_{21}(x) & S_{22}(x) \end{pmatrix} \begin{pmatrix} V'_{x_1} \\ \frac{1}{\varepsilon} V'_{x_2} \end{pmatrix} + \tfrac{1}{2} k'(x_1, x_2) k(x_1, x_2), \quad V(0) = 0 \tag{8}$$

with the property that system (7), where

$$V_{x_1} = Z'_1, \quad V_{x_2} = \varepsilon Z'_2, \tag{9}$$

is asymptotically stable [1]. The latter is equivalent to the existence of the stable invariant manifold (6) of (5) with asymptotically stable flow (7), such that $V \geqslant 0$, $V(0) = 0$ (that implies $V_x(0) = 0$). Then Ω is the set of all initial conditions that give rise to asymptotically stable trajectories of (7) that are restricted to Ω_1. The optimal controller is given by

$$u = -R^{-1}[B'_1, \varepsilon^{-1} B'_2]\; V'_x = -R^{-1} B'_1 Z_1 - R^{-1} B'_2 Z_2. \tag{10}$$

Remark 2.1. In [4] we used the other form of Hamiltonian (4), where coefficients $-\frac{1}{2}$ and $\frac{1}{2}$ were replaced by $-\frac{1}{4}$ and 1 correspondingly. Therefore, the optimal controller was given by the right-hand sides of (10) multiplied by $\frac{1}{2}$.

Note that the flow on the special manifold (6) is governed by system of coupled slow and fast equations (7) and thus this manifold is a slow–fast one. We obtain the exact decomposition of the special slow–fast manifold into the reduced-order slow submanifold of the Hamiltonian system and the fast manifold of an auxiliary system.

3. Exact decomposition of the special invariant manifold

3.1. Assumptions

Consider the linearization of (1) at $x = 0$:

$$E_\varepsilon \dot{x} = Ax + B_0 u \tag{11}$$

with the quadratic functional

$$J = \int_0^\infty [x' C' C x + u' R(0) u] \, \mathrm{d}t, \tag{12}$$

where

$$E_\varepsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{10} \\ B_{20} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

$$A_{ij} = \frac{\partial F_i}{\partial x_j}(0,0), \quad B_{i0} = B_i(0,0), \quad C_i = \frac{\partial k}{\partial x_i}(0,0), \quad i = 1,2; \; j = 1,2.$$

Denote $E_0 = E_{\varepsilon|_{\varepsilon=0}}$. The Hamiltonian system that corresponds to (11), (12) can be written in the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{bmatrix} = \mathrm{Ham} \begin{bmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix}, \tag{13a}$$

$$\mathrm{Ham} = \begin{bmatrix} T_{11} & T_{12} \\ \varepsilon^{-1} T_{21} & \varepsilon^{-1} T_{22} \end{bmatrix}, \tag{13b}$$

$$T_{ij} = \begin{bmatrix} A_{ij} & -S_{ij}(0) \\ -C_i' C_j & -A_{ji}' \end{bmatrix}. \tag{13c}$$

To guarantee that for all small ε this LQ problem is solvable we assume [12]:

A1. The exponential modes of descriptor system (11), where $\varepsilon = 0$, are controllable-observable, i.e. both pencils $[sE_0 - A; B]$ and $[sE_0' - A'; C]$ are of full row rank for any finite s.

A2. The triple $\{A_{22}, B_{20}, C_2\}$ is controllable-observable.

It is known [12] that under A1 and A2 the LQ problem is solvable for all small ε.

Lemma 3.1. *Under* A1 *and* A2

(i) *A fast Riccati equation*

$$A_{22}' X_f + X_f A_{22} + C_2' C_2 - X_f S_{22}(0) X_f = 0 \tag{14}$$

has a solution $X_f = X_f' \geqslant 0$, *such that the matrix* $\Lambda_f = A_{22} - S_{22}(0) X_f$ *is Hurwitz.*

(ii) *A slow algebraic Riccati equation*

$$X_0 A_0 + A_0' X_0 - X_0 S_0 X_0 + Q_0 = 0, \tag{15}$$

where

$$\begin{bmatrix} A_0 & -S_0 \\ -Q_0 & -A_0' \end{bmatrix} = T_{11} - T_{12} T_{22}^{-1} T_{21} = T_0 \tag{16}$$

has a solution $X_0 = X_0' \geqslant 0$ *such that the matrix* $\Lambda_s = A_0 - S_0 X_0$ *is Hurwitz.*

(iii) *The matrix* T_{22} *has* n_2 *eigenvalues with negative real parts and* n_2 *with positive ones.*

(iv) *The matrix* T_0 *has* n_1 *eigenvalues with negative real parts and* n_1 *with positive ones.*

(v) *In a small enough neighborhood of* $\mathbb{R}^{n_2} \times \mathbb{R}^{n_2}$ *containing* 0 *the system of equations*

$$f_3(x_1, p_1, x_2, p_2) = 0, \quad f_4(x_1, p_1, x_2, p_2) = 0$$

has an isolated solution

$$x_2 = \phi(x_1, p_1), \quad p_2 = \psi(x_1, p_1) \tag{17}$$

and the matrix

$$\begin{pmatrix} \dfrac{\partial f_3}{\partial x_2} & \dfrac{\partial f_3}{\partial p_2} \\ \dfrac{\partial f_4}{\partial x_2} & \dfrac{\partial f_4}{\partial p_2} \end{pmatrix}_{|(x_2, p_2) = (\phi(x_1, p_{tiny1}), \psi(x_1, p_1))}$$

has n_2 *stable eigenvalues* λ, $\mathrm{Re}\,\lambda < -\alpha < 0$, *and* n_2 *unstable ones* λ, $\mathrm{Re}\,\lambda > \alpha$.

Proof. Items (i)–(iii) follow from A1 [12]. To prove (iv) consider the matrix Ham. It has one group of $2n_1$ small eigenvalues $O(\varepsilon)$ close to those of T_0 and another group of $2n_2$ large eigenvalues $O(1)$ close to those of $\varepsilon^{-1}T_{22}$ [2]. Then (iv) follows from the symmetry of the eigenvalues of Ham, of T_{22} and thus of T_0 and from the relation

$$T_0 = \begin{pmatrix} I & 0 \\ X_0 & I \end{pmatrix} \begin{pmatrix} \Lambda_s & -S_0 \\ 0 & -\Lambda_s' \end{pmatrix} \begin{pmatrix} I & 0 \\ -X_0 & I \end{pmatrix}. \tag{18}$$

Item (v) follows from (i) to (iii) by the implicit function theorem. □

Remark 3.1. It follows from the proof of Lemma 3.1 that under A1 and A2 for all small enough ε the matrix Ham has no purely imaginary eigenvalues, i.e. the equilibrium point of the Hamiltonian system (5) is *hyperbolic*.

3.2. Decomposition of the slow–fast manifold (6)

Denote by $\Omega_{m_i} = \{x_i \in \mathbb{R}^{n_i}\colon |x_i| < m_i\}$, $i = 1,2$, where $|\cdot|$ is a Euclidean norm of a vector. From (iii) and (v) of Lemma 3.1 it follows that there exists $m_1 > 0$ such that for all small enough ε system (5) has the slow manifold [10,7]

$$\begin{pmatrix} x_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} L_3^*(x_1, p_1, \varepsilon) \\ L_4^*(x_1, p_1, \varepsilon) \end{pmatrix} = \begin{pmatrix} \phi(x_1, p_1) \\ \psi(x_1, p_1) \end{pmatrix} + O(\varepsilon), \tag{19}$$

defined on $\Omega_{m_1} \times \Omega_{m_1}$. The subscripts of L_3^* and L_4^* correspond to the third and the fourth variables in the system of (5). To avoid cumbersome notation we shall omit the argument ε in the functions below.

Denote $f_i^*(x_1, p_1) = f_i[x_1, p_1, L_3^*(x_1, p_1), L_4^*(x_1, p_1)]$, $i = 1,2,3,4$. Setting (19) into (5) and substituting v_1 and w_1 for x_1 and p_1, respectively, we get the $2n_1$-dimensional system for the flow on the slow manifold

$$\dot{v}_1 = f_1^*(v_1, w_1), \tag{20a}$$

$$\dot{w}_1 = f_2^*(v_1, w_1). \tag{20b}$$

The function $L^* = \mathrm{col}\{L_3^*, L_4^*\}$ satisfies the following partial differential equation (PDE):

$$\varepsilon \frac{\partial L^*}{\partial v_1} f_1^*(v_1, w_1) + \varepsilon \frac{\partial L^*}{\partial w_1} f_2^*(v_1, w_1) = \begin{pmatrix} f_3^*(v_1, w_1) \\ f_4^*(v_1, w_1) \end{pmatrix}. \tag{21}$$

This PDE can be derived by differentiating (19), where $x_2 = x_2(t)$, $p_2 = p_2(t)$, $x_1 = v_1(t)$, $p_1 = w_1(t)$, with respect to t and by substituting $\dot{v}_1$ and $\dot{w}_1$ from (20).

Consider the slow system (20). Its linearization in $(0,0)$ for $\varepsilon = 0$ is given by

$$\begin{bmatrix} \dot{v}_1 \\ \dot{w}_1 \end{bmatrix} = T_0 \begin{bmatrix} v_1 \\ w_1 \end{bmatrix}.$$

From (ii) and (iv) of Lemma 3.1 it follows that the latter system possesses the stable manifold of the form $w_1 = X_0 v_1$. Then (see, e.g. [6]) for small enough v_1 and ε the nonlinear system (20) possesses the *slow* (*stable*) *submanifold*

$$w_1 = N(v_1), \tag{22}$$

where the function $N = N(v_1, \varepsilon)$ is continuous on both arguments and uniformly bounded together with its first derivative on v_1, and $N(0) = 0$. Substituting (22) into (20a) we obtain the equation for the restriction of (20)–(22):

$$\dot{v}_1 = f_1^*(v_1, N(v_1)). \tag{23}$$

Moreover, for all small enough ε (23) is exponentially stable. Substituting (22) into (20b) and applying (23) we obtain the *slow PDE*:

$$\frac{\partial N}{\partial v_1} f_1^*(v_1, N(v_1)) = f_2^*(v_1, N(v_1)). \tag{24}$$

We shall construct the invariant manifold (6), with the stable flow, by means of the slow submanifold (22) and a fast manifold of an auxiliary system. To obtain the latter system we introduce the following change of variables:

$$\begin{pmatrix} v_2 \\ \bar{p}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ p_2 \end{pmatrix} - L^*(x_1, p_1), \quad \begin{pmatrix} \bar{x}_1 \\ \bar{p}_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ p_1 \end{pmatrix} - \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, \tag{25}$$

where v_1 and w_1 satisfy (20). For the new variables we get the system

$$\dot{\bar{x}}_1 = g_1(v_1, w_1, \bar{x}_1, \bar{p}_1, v_2, \bar{p}_2), \tag{26a}$$

$$\dot{\bar{p}}_1 = g_2(v_1, w_1, \bar{x}_1, \bar{p}_1, v_2, \bar{p}_2), \tag{26b}$$

$$\varepsilon \dot{v}_2 = g_3(v_1, w_1, \bar{x}_1, \bar{p}_1, v_2, \bar{p}_2), \tag{26c}$$

$$\varepsilon \dot{\bar{p}}_2 = g_4(v_1, w_1, \bar{x}_1, \bar{p}_1, v_2, \bar{p}_2), \tag{26d}$$

where for $i = 1, 2$,

$$g_i = f_i[\bar{x}_1 + v_1, \bar{p}_1 + w_1, v_2 + L_3^*(\bar{x}_1 + v_1, \bar{p}_1 + w_1), \bar{p}_2 + L_4^*(\bar{x}_1 + v_1, \bar{p}_1 + w_1)]$$
$$- f_i[v_1, w_1, L_3^*(v_1, w_1), L_4^*(v_1, w_1)],$$

and for $i = 3, 4$,

$$g_i = -\varepsilon \frac{\partial L_i^*(\bar{x}_1 + v_1, \bar{p}_1 + w_1)}{\partial x_1} \Delta f_1 - \varepsilon \frac{\partial L_i^*(\bar{x}_1 + v_1, \bar{p}_1 + w_1)}{\partial p_1} \Delta f_2 + \Delta f_i,$$

$$\Delta f_j = f_j[\bar{x}_1 + v_1, \bar{p}_1 + w_1, v_2 + L_3^*(\bar{x}_1 + v_1, \bar{p}_1 + w_1), \bar{p}_2 + L_4^*(\bar{x}_1 + v_1, \bar{p}_1 + w_1)]$$
$$- f_j[\bar{x}_1 + v_1, \bar{p}_1 + w_1, L_3^*(v_1 + \bar{x}_1, w_1 + \bar{p}_1), L_4^*(v_1 + \bar{x}_1, w_1 + \bar{p}_1)], \quad j = 1, 2, 3, 4.$$

From (iii), (v) and [10] it follows that there exists ε' such that for all $\varepsilon \in (0, \varepsilon']$ the system of (20) and (26) has the following *fast* (*stable*) *manifold* for $|v_2| < m'$

$$\begin{pmatrix} \bar{x}_1 \\ \bar{p}_1 \end{pmatrix} = \begin{pmatrix} \varepsilon L_1^+(v_1, w_1, v_2) \\ \varepsilon L_2^+(v_1, w_1, v_2) \end{pmatrix}, \quad \bar{p}_2 = L_4^+(v_1, w_1, v_2). \tag{27}$$

The functions L_i^+ $(i = 1, 2, 4)$ satisfy the inequalities

$$|L_i^+(v_1, w_1, v_2)| \leqslant c|v_2|, \quad |L_i^+(v_1, w_1, v_2) - L_i^+(v_1, w_1, \tilde{v}_2)| \leqslant c|v_2 - \tilde{v}_2|,$$
$$|L_i^+(v_1, w_1, v_2) - L_i^+(\tilde{v}_1, \tilde{w}_1, v_2)| \leqslant c|v_2|(|v_1 - \tilde{v}_1| + |w_1 - \tilde{w}_1|). \tag{28}$$

The flow on the latter manifold is governed by the decoupled system of the slow equations (20) and the fast equation

$$\varepsilon \dot{v}_2 = g_3(v_1, w_1, \varepsilon L_1^+, \varepsilon L_2^+, v_2, L_4^+), \tag{29}$$

where $L_i^+ = L_i^+(v_1, w_1, v_2)$ $(i = 1, 2, 4)$.

The function $L^+ = \mathrm{col}\{L_1^+, L_2^+, L_4^+\}$ satisfies the following *fast PDE*:

$$\varepsilon \frac{\partial L^+}{\partial v_1} f_1^*(v_1, w_1) + \varepsilon \frac{\partial L^+}{\partial w_1} f_2^*(v_1, w_1) + \frac{\partial L^+}{\partial v_2} g_3 = \mathrm{col}\{g_1, g_2, g_4\}, \tag{30}$$

where $g_k = g_k(v_1, w_1, \varepsilon L_1^+, \varepsilon L_2^+, v_2, L_4^+)$, $k = 1, \dots, 4$, $L^+ = L^+(v_1, w_1, v_2)$. Eq. (30) follows from substitution of (27) into (26a), (26b), (26d) and application of (29). Note that $L_4^+(v_1, w_1, v_2) = X_f v_2 + \mathrm{O}(|v_2|^2 + \varepsilon)$ and thus the right-hand side of (29) is given by $\Lambda_f v_2 + \mathrm{O}([|v_1| + |w_1| + |v_2| + \varepsilon]|v_2|)$, where $\Lambda_f = A_{22} - S_{22}(0)X_f$ is Hurwitz by (i) of Lemma 3.1. Hence, for small enough ε and in the small enough neighborhood of the origin the with the solutions of (26) lying on the fast manifold of (27) are rapidly exponentially decaying as t increases.

Substituting (22) and (27) into (25) we get the algebraic system

$$x_1 = v_1 + \varepsilon L_1^+[v_1, N(v_1), v_2], \tag{31a}$$

$$x_2 = v_2 + L_3^*[x_1, N(v_1) + \varepsilon L_2^+(v_1, N(v_1), v_2)], \tag{31b}$$

$$p_1 = N(v_1) + \varepsilon L_2^+[v_1, N(v_1), v_2], \tag{31c}$$

$$p_2 = L_4^*[x_1, N(v_1) + \varepsilon L_2^+(v_1, N(v_1), v_2)] + L_4^+(v_1, N(v_1), v_2). \tag{31d}$$

Considering (31a) and (31b) as the system with respect to v_1 and v_2 and using the contraction principle argument, one can prove that for all small enough ε, this system has a unique solution on $\Omega_{m_1} \times \Omega_{m_2}$

$$v_1 = U_1(x_1, x_2), \tag{32a}$$

$$v_2 = U_2(x_1, x_2), \tag{32b}$$

where the functions U_1 and U_2 are Lipschitzian on x_1 and x_2 and they vanish at $(x_1, x_2) = 0$. Further, applying the implicit function theorem one can show that U_1 and U_2 are continuously differentiable on x_1 and x_2. Substituting (32) into (31c) and (31d) we get (6), where

$$Z_1 = N(U_1) + \varepsilon L_2^+[U_1, N(U_1), U_2], \quad Z_2 = L_4^*(x_1, Z_1) + L_4^+(U_1, N(U_1), U_2). \tag{33}$$

Similarly to Theorem 2 of [4] we obtain the following

Theorem 3.1. *Under* A1 *and* A2 *there exist* m_1, m_2 *and* ε_1 *such that for all* $\varepsilon \in (0, \varepsilon_1]$ *the* $(2n_1 + 2n_2)$*-dimensional Hamiltonian system* (5) *has the invariant on* $\Omega_{m_1} \times \Omega_{m_2}$ *manifold* (6) *with* (7) *asymptotically stable, where continuously differentiable on* x_1 *and* x_2 *functions* Z_1 *and* Z_2 *are defined by* (33) *from the algebraic system* (31). *There exists a* C^2 *function* $V : \Omega_{m_1} \times \Omega_{m_2} \to [0, \infty)$, *satisfying the HJ equation* (8) *and relations* (9) (*therefore the optimal control problem is solvable by the controller of* (10)).

4. High-order asymptotic approximations

4.1. High-order approximate controller

We shall find an asymptotic approximation to the controller (10) by expanding Z_1 and Z_2, defined by (33), into the powers of ε. It is known [10] that the functions L^*, N and L^+ can be found in the form of asymptotic approximations

$$\begin{gathered} L^*(x_1, p_1, \varepsilon) = \sum_{j=0}^{q} \varepsilon^j l_j^*(x_1, p_1) + \mathrm{O}(\varepsilon^{q+1}), \quad N(v_1, \varepsilon) = \sum_{j=0}^{q} \varepsilon^j N_j(v_1) + \mathrm{O}(\varepsilon^{q+1}), \\ L^+(v_1, w_1, v_2) = \sum_{j=0}^{q} \varepsilon^j l_j^+(v_1, w_1, v_2) + \mathrm{O}(\varepsilon^{q+1}). \end{gathered} \tag{34}$$

The terms of these approximations can be determined by substitution of (34) into the corresponding PDEs and equating terms with the same powers of ε. For l_j^* we obtain the algebraic equations, while for N_j and l_j^+ we get partial differential equations. Thus, $l_0^*(x_1, p_1) = \mathrm{col}\{\phi(x_1, p_1), \psi(x_1, p_1)\}$.

The function N_0 satisfies the following PDE:

$$\frac{\partial N_0}{\partial x_1} f_1(x_1, N_0, \phi(x_1, N_0), \psi(x_1, N_0)) = f_2(x_1, N_0, \phi(x_1, N_0), \psi(x_1, N_0)) \tag{35}$$

and defines the stable manifold $p_1 = N_0(x_1)$ of the *reduced* Hamiltonian system

$$\dot{x}_1 = f_1(x_1, p_1, \phi(x_1, p_1), \psi(x_1, p_1)), \tag{36a}$$

$$\dot{p}_1 = f_2(x_1, p_1, \phi(x_1, p_1), \psi(x_1, p_1)). \tag{36b}$$

The function N_0 can be approximated by $N_0(x_1) = X_0 x_1 + \mathrm{O}(|x_1|^2)$.

For l_j^+ we have

$$\frac{\partial l_j^+}{\partial v_2}[g_3(v_1, w_1, v_2, l_{40}^+)]_{|\varepsilon=0} = G_j(v_1, w_1, v_2, l_{4j}^+), \tag{37}$$

where $l_j^+ = \mathrm{col}\{l_{1j}^+, l_{2j}^+, l_{4j}^+\}$ and G_j is a known function such that $G_j[v_1, w_1, 0, l_{4j}^+(v_1, w_1, 0)] = 0$. Eq. (37) depends on v_1 and w_1 as on the parameters, and its solution can be found as the stable manifold $\mathrm{col}\{\bar{x}_1, \bar{p}_1, \bar{p}_2\} = l_j^+(v_1, w_1, v_2)$ of ε-independent system:

$$\dot{v}_2 = g_3(v_1, w_1, v_2, l_{40}^+)_{|\varepsilon=0}, \quad \mathrm{col}\{\dot{\bar{x}}_1, \dot{\bar{p}}_1, \dot{\bar{p}}_2\} = G_j(v_1, w_1, v_2, \bar{p}_2),$$

e.g. in the form of the expansions in the powers of v_2 with coefficients depending on parameters v_1 and w_1 (similar to [8]).

Next, we obtain from (32)

$$v_i = U_i(x_1, x_2, \varepsilon) = \sum_{j=0}^{q} \varepsilon^j U_{ij}(x_1, x_2) + \mathrm{O}(\varepsilon^{q+1}), \quad i = 1, 2, \tag{38}$$

where U_{ij} are differentiable with respect to x a sufficient number of times since L^*, N and L^+ are differentiable with respect to v_1, w_1 and v_2 a sufficient number of times. We substitute expansions (38) and (34) into (33):

$$Z_1 = \sum_{i=0}^{q} \varepsilon^i N_i\left(\sum_{j=0}^{q} \varepsilon^j U_{1j}\right) + \sum_{i=0}^{q} \varepsilon^{i+1} l_{2i}^+ \left[\sum_{j=0}^{q} \varepsilon^j U_{1j}, \sum_{i=0}^{q} \varepsilon^i N_i\left(\sum_{j=0}^{q} \varepsilon^j U_{1j}\right), \varepsilon^j U_{2j}\right] + \mathrm{O}(\varepsilon^{q+1}),$$

$$Z_2 = \sum_{i=0}^{q} \varepsilon^i l_{4i}^*[x_1, Z_1] + \sum_{i=0}^{q} \varepsilon^i l_{4i}^+ \left[\sum_{j=0}^{q} \varepsilon^j U_{1j}, \sum_{i=0}^{q} \varepsilon^i N_i\left(\sum_{j=0}^{q} \varepsilon^j U_{1j}\right), \varepsilon^j U_{2j}\right] + \mathrm{O}(\varepsilon^{q+1}).$$

Expanding the right-hand sides of the latter equations in the powers of ε we get the asymptotic approximations to Z_1 and Z_2:

$$Z_i(x_1, x_2, \varepsilon) = \sum_{j=0}^{q} \varepsilon^j Z_{ij}(x_1, x_2) + \mathrm{O}(\varepsilon^{q+1}), \quad i = 1, 2, \tag{39}$$

where Z_{ij} are differentiable with respect to x_1 and x_2 a sufficient number of times. Note that

$$Z_{10} = N_0(x_1), \; Z_{20} = \psi[x_1, N_0(x_1)] + l_{40}(x_1, x_2 - \phi(x_1, N_0(x_1))].$$

Substituting (39) into (10) we get the following $\mathrm{O}(\varepsilon^{q+1})$-approximation to the optimal controller:

$$u = u_q + \mathrm{O}(\varepsilon^{q+1}), \quad u_q = -\sum_{k=1}^{2} \sum_{j=0}^{q} \varepsilon^j R^{-1} B_k' Z_{kj}(x_1, x_2). \tag{40}$$

4.2. Asymptotic expansion of optimal trajectory and open-loop control

It can be found from (31), (23), (29) and the relation

$$u(t) = -\tfrac{1}{2}R^{-1}B_1' p_1(t) - \tfrac{1}{2}R^{-1}B_2' p_2(t). \tag{41}$$

Applying standard asymptotic methods (see, e.g., [9]) to the decoupled exponentially stable equations (23) and (29), where $w_1 = N(v_1)$, we obtain correspondingly

$$v_1(t) = \sum_{i=0}^{q} \varepsilon^i v_1^{(i)}(t) + \varepsilon^{q+1} r_{1q}(t,\varepsilon), \quad v_2(t) = \sum_{i=0}^{q} \varepsilon^i v_2^{(i)}(\tau) + \varepsilon^{q+1} r_{2q}(\tau,\varepsilon), \tag{42}$$

where $\tau = t/\varepsilon$ and r_{1q} and r_{2q} satisfy the following inequalities for $t \geqslant 0$, $\tau \geqslant 0$:

$$|r_{1q}(t,\varepsilon)| < c\mathrm{e}^{-\alpha t}, \quad |r_{2q}(\tau,\varepsilon)| < c\mathrm{e}^{-\alpha\tau}, \quad \alpha > 0,\ c > 0.$$

Substituting (42a) into (23) and equating coefficients of equal powers of ε we find differential equations for $v_1^{(i)}$ with initial values defined by (32a), where $t = 0$. Similarly from (29), (22), (42b) and (32b) we obtain initial value problems for $v_2^{(i)}$.

Finally substituting expansions of v_1, v_2, L^+ and L^* into (31) and (41) and expanding right-hand sides of the resulting equations in the powers of ε we find the following approximations:

$$\begin{aligned} x(t) &= \sum_{i=0}^{q} \varepsilon^i x^{(i)}(t) + \sum_{i=0}^{q} \varepsilon^i \Pi_1^{(i)}(\tau) + \varepsilon^{q+1} R_{1q}(t,\varepsilon), \\ u(t) &= \sum_{i=0}^{q} \varepsilon^i u^{(i)}(t) + \sum_{i=0}^{q} \varepsilon^i \Pi_2^{(i)}(\tau) + \varepsilon^{q+1} R_{2q}(t,\varepsilon), \end{aligned} \tag{43a,b}$$

where Π_1 and Π_2 are boundary layer terms exponentially decaying when $\tau \to \infty$, $x^{(i)}, u^{(i)}$ and the remainders R_{1q} and R_{2q} are exponentially decaying when $t \to \infty$:

$$|x^{(i)}(t)| + |u^{(i)}(t)| \leqslant c\mathrm{e}^{-\alpha t}, \quad |\Pi_1^{(i)}(\tau)| + |\Pi_2(\tau)| \leqslant c\mathrm{e}^{-\alpha\tau}, \quad |R_{1q}(t,\varepsilon)| + |R_{2q}(t,\varepsilon)| \leqslant c\mathrm{e}^{-\alpha t}. \tag{44}$$

Note that (43b) can be found also by substitution of (43a) into the expansion of the optimal feedback (40). Similar to Theorem 3 of [4], it can be proved that $\mathrm{O}(\varepsilon^{q+1})$-approximate controller u_q leads to the value of the cost $\mathrm{O}(\varepsilon^{q+1})$-close to the optimal one. We summarize our results in the following

Theorem 4.1. *Under* A1 *and* A2 *there exist* m_1, m_2, ε_1 *such that for all* $\varepsilon \in (0, \varepsilon_1]$ *the following holds*:

(i) *The invariant manifold* (6) *and the optimal controller* (10) *can be approximated by* (39) *and* (40), *where approximation is uniform on* $x_1, x_2 \in \Omega_{m_1} \times \Omega_{m_2}$. *The optimal trajectory with the initial conditions from* $\Omega_{m_1} \times \Omega_{m_2}$ *and the corresponding optimal open-loop control can be approximated by* (43) *such that inequalities* (44) *are valid.*

(ii) *The controller* u_q *leads to the value of the cost* $\mathrm{O}(\varepsilon^{q+1})$-*close to the optimal one for all initial conditions from* $\Omega_{m_1} \times \Omega_{m_2}$.

4.3. Example

Consider the system

$$\dot{x}_1 = -\tan x_2 + 2u, \quad \varepsilon\dot{x}_2 = \tan x_2 - u, \quad x(0) = [0.5, 1]', \quad J = \int_0^\infty [x_1^2(t) + u^2(t)]\,\mathrm{d}t. \tag{45}$$

Here A1 and A2 hold. We find the following Hamiltonian function (4):

$$\mathscr{H} = -p_1 \tan x_2 + p_2 \tan x_2 - 2p_1^2 + 2p_1 p_2 - 1/2 p_2^2 + 1/2 x_1^2$$

and the corresponding Hamiltonian system (5)

$$\dot{x}_1 = -\tan x_2 - 4p_1 + 2p_2, \quad \dot{p}_1 = -x_1, \quad \varepsilon\dot{x}_2 = \tan x_2 + 2p_1 - p_2, \quad \varepsilon\dot{p}_2 = p_1 - p_2 \cos^{-2} x_2. \tag{46}$$

Table 1

ε	0.01	0.05	0.1	0.15	0.2
$J(u_0)$	0.3134	0.6025	1.0563	1.6348	2.3678
$J(u_1)$	0.3133	0.5987	1.0346	1.5717	2.2267
K	[1; 2.030]	[1; 2.1536]	[1; 2.3136]	[1; 2479]	[1; 2.649]
$J(Kx)$	0.3171	0.6088	1.0505	1.5871	2.2266

Further we neglect terms of the order $O(\varepsilon^2)$. Then from (21), (24) and (30) we find

$$L^* = \left[-\arctan p_1 + \varepsilon \frac{2x_1}{(1+p_1^2)^2}; \, p_1 + \varepsilon \frac{x_1}{1+p_1^2}\right]', \quad N = x_1,$$

$$L^+ = [-3\varepsilon v_2; \; 0; \; 2\tan[v_2 - \arctan w_1] + 2w_1 + \varepsilon l_{41}^+]', \quad l_{41}^+ = O(|v_2|^2).$$

Eqs. (31) have the form

$$x_1 = v_1 - 3\varepsilon v_2, \tag{47a}$$

$$x_2 = v_2 - \arctan v_1 + \frac{2\varepsilon v_1}{(1+v_1^2)^2}, \tag{47b}$$

$$p_1 = v_1, \tag{47c}$$

$$p_2 = 3v_1 + 2\tan(v_2 - \arctan v_1) + \frac{\varepsilon v_1}{1+v_1^2} + \varepsilon l_{41}^+. \tag{47d}$$

From (47a) and (47b) we obtain

$$v_1 = x_1 + 3\varepsilon(x_2 + \arctan x_1), \quad v_2 = x_2 + \arctan(x_1 + 3\varepsilon(x_2 + \arctan x_1)) - \frac{2\varepsilon x_1}{(1+x_1^2)^2}. \tag{48}$$

Substituting (48) into (47c) and (47d) we find

$$p_1 = Z_1(x_1, x_2) = x_1 + 3\varepsilon(x_2 + \arctan x_1),$$

$$p_2 = Z_2(x_1, x_2) = 3Z_1(x_1, x_2) + 2\tan\left(x_2 - 2\varepsilon \frac{x_1}{(1+x_1^2)^2}\right) + \varepsilon l_{41}^+ + \frac{\varepsilon x_1}{1+x_1^2}.$$

By (10) $u = -2Z_1(x_1, x_2) + Z_2(x_1, x_2)$ and we obtain the following approximations to the optimal controller:

$$u_0 = x_1 + 2\tan x_2, \quad u_1 = u_0 + 3\varepsilon(x_2 + \arctan x_1) + \frac{\varepsilon x_1}{1+x_1^2} - 4\varepsilon \frac{x_1 \sec^2 x_2}{(1+x_1^2)^2} + \varepsilon l_{41}^+.$$

Applying now u_0 and u_1 (where we neglect l_{41}^+) to (45), we find the corresponding values of the cost $J(u_0)$ and $J(u_1)$ for $\varepsilon = 0.01, \ldots, 0.2$. The latter are given in the Table 1. Comparing the results we see that for all ε under consideration u_1 improves the performance incurred by u_0.

For each ε we find the gain K of the optimal controller $u = Kx$ for the full-order linearized problem (11) (which is $O(|x|^2)$-close to the optimal controller (10) [8]) and the corresponding values of the cost $J(Kx)$. For $\varepsilon < 0.1$ u_0 improves the performance incurred by the linear controller $u = Kx$. For the greater values of ε the higher order approximations u_q $(q \geqslant 1)$ should be taken in order to improve the performance incurred by the linear controller.

Eqs. (23) and (29) with neglected $O(\varepsilon^2)$ terms have the form

$$\dot v_1 = -v_1, \quad \varepsilon \dot v_2 = -\tan(v_2 - \arctan v_1) - v_1 + 2\varepsilon \frac{v_1}{1+v_1^2}\left[\frac{1}{\cos^2(v_2 - \arctan v_1)} - \frac{1}{\cos^2(\arctan v_1)}\right],$$
$$v_1(0) = 0.5 + 3\varepsilon(1 - \arctan 0.5), \quad v_2(0) = 1 - \arctan 0.5 - \varepsilon \tfrac{16}{25}. \tag{49}$$

Therefore, $O(\varepsilon^2)$ asymptotic approximations of v_1, v_2 and of the optimal trajectory x can be found from (49) and (47a), (47b).

5. Conclusions

We have developed a geometric approach to singularly perturbed optimal control problem, nonlinear in the state variables and affine in the control. We have obtained the exact decomposition of the slow–fast invariant manifold of the Hamiltonian system into the reduced-order slow manifold and a fast manifold. As a result, an asymptotic expansion of the optimal controller have been constructed by solving slow and fast partial differential equations. In the same time we have obtained decomposition of the Hamiltonian system to the slow and fast subsystems. This leads to asymptotic approximation to optimal trajectory and open-loop control. We have shown that a higher-order accuracy controller improves performance.

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