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New Lyapunov–Krasovskii functionals for stability of linear retarded and neutral type systems ☆

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Abstract

A new (descriptor) model transformation and a corresponding Lyapunov–Krasovskii functional are introduced for stability analysis of systems with delays. Delay-dependent/delay-independent stability criteria are derived for linear retarded and neutral type systems with discrete and distributed delays. Conditions are given in terms of linear matrix inequalities and for the first time refer to neutral systems with discrete and distributed delays. The proposed criteria are less conservative than other existing criteria (for retarded type systems and neutral systems with discrete delays) since they are based on an equivalent model transformation and since they require bounds for fewer terms. Examples are given that illustrate advantages of our approach. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Time-delay systems; Stability; Linear matrix inequalities; Delay-dependent/delay-independent criteria

1. Introduction

The choice of an appropriate Lyapunov–Krasovskii functional is the key-point for deriving of stability criteria. It is known that the general form of this functional leads to a complicated system of partial differential equations (see e.g. [12]). That is why many authors considered special forms of Lyapunov–Krasovskii functional and thus derived simpler (but more conservative) sufficient conditions. Among the latter there are delay-independent and delay-dependent conditions.

Three main transformations of the original system have been used for delay-dependent stability analysis of retarded type systems [7]. One of these transformations (the second one in [7]—the "neutral type representation" of system) has been used also for neutral type systems [9,13]. The conservatism of approaches based on these transformations is two-fold: the transformed system is not equivalent to the original one (see [1] concerning the first transformation, while the second transformation requires additional assumptions as mentioned in [13]) and bounds should be obtained (completion to the squares) for certain terms.

In the present paper, we introduce a new type of Lyapunov–Krasovskii functional inspired by [3] which is based on equivalent augmented model—a "descriptor form" representation of the system. Our approach essentially reduces the conservatism of the existing methods.

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2. Delay-dependent stability of linear systems: discrete delays

Let R^n be Euclidean space and $C[a,b]$ be the space of continuous functions $\phi: [a,b] \to R^n$ with the supremum norm $|\cdot|$. Denote by $x_t(\theta) = x(t+\theta)$ $(\theta \in [-h, 0])$.

2.1. Neutral systems

Given the following system:

$$\dot{x}(t) - \sum_{i=0}^{m} D_i \dot{x}(t-h_i) = \sum_{i=0}^{m} A_i x(t-h_i), \quad x(t) = \phi(t), \quad t \in [-h, 0], \tag{1}$$

where $x(t) \in \mathscr{R}^n$, $h_0 = 0$, $0 < h_i \leqslant h$, $i = 1, \ldots, m$, A_i and D_i are constant $n \times n$-matrices, ϕ is a continuously differentiable initial function.

To guarantee that the difference operator $\mathscr{D}: C[-h, 0] \to R^n$ given by $\mathscr{D}(x_t) = x(t) - \sum_{i=1}^{m} D_i x(t-h_i)$ is stable (i.e. difference equation $\mathscr{D}x_t = 0$ is asymptotically stable) we assume [4,5]:

A1. Let

$$\sum_{i=1}^{m} |D_i| < 1,$$

where $|\cdot|$ is any matrix norm.

Note that for neutral type systems there exist two types of stability results: those corresponding to continuous initial functions and continuously differentiable initial functions. Under A1 both types of stability are equivalent [2].

We represent (1) in the equivalent descriptor form:

$$\dot{x}(t) = y(t), \quad y(t) = \sum_{i=1}^{m} D_i y(t-h_i) + \sum_{i=0}^{m} A_i x(t-h_i). \tag{2}$$

The latter can be represented in the form of descriptor system with discrete and distributed delay in the "fast variable" y:

$$\dot{x}(t) = y(t), \quad 0 = -y(t) + \sum_{i=1}^{m} D_i y(t-h_i) + \left(\sum_{i=0}^{m} A_i\right) x(t) - \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} y(s)\,\mathrm{d}s. \tag{3}$$

Lyapunov–Krasovskii functional for the latter system has the form introduced in [3]:

$$V(t) = [x^{\mathrm{T}}(t)\; y^{\mathrm{T}}(t)] EP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + V_1 + V_2, \tag{4}$$

where

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad P_1 = P_1^{\mathrm{T}} > 0, \tag{5}$$

$$V_1 = \sum_{i=1}^{m} \int_{t-h_i}^{t} y^{\mathrm{T}}(s) Q_i y(s)\,\mathrm{d}s, \quad Q_i > 0 \tag{6}$$

and

$$V_2 = \sum_{i=1}^{m} \int_{-h_i}^{0} \int_{t+\theta}^{t} y^{\mathrm{T}}(s) R_i y(s)\,\mathrm{d}s\,\mathrm{d}\theta, \quad R_i > 0. \tag{7}$$

The first term of (4) corresponds to the descriptor system, V_1 corresponds to the delay-independent stability with respect to the discrete delays and V_2—to delay-dependent stability with respect to the distributed delays. The functional (4) is *degenerated* (i.e. nonpositive-definite) as it is usual for descriptor systems (see e.g. [14]).

We obtain the following:

Theorem 1. *Under* A1 (1) *is stable if there exist* $0<P_1=P_1^{\mathrm{T}}, P_2, P_3,$ *and* $Q_i=Q_i^{\mathrm{T}},\ R_i=R_i^{\mathrm{T}},\ i=1,\dots,m$ *that satisfy the following linear matrix inequality* (*LMI*):

$$\begin{bmatrix} \left(\sum_{i=0}^m A_i^{\mathrm{T}}\right)P_2+P_2^{\mathrm{T}}\left(\sum_{i=0}^m A_i\right) & P_1-P_2^{\mathrm{T}}+\left(\sum_{i=0}^m A_i^{\mathrm{T}}\right)P_3 & h_1P_2^{\mathrm{T}}A_1 & \cdots & h_mP_2^{\mathrm{T}}A_m & P_2^{\mathrm{T}}D_1 & \cdots & P_2^{\mathrm{T}}D_m \\ P_1-P_2+P_3^{\mathrm{T}}\left(\sum_{i=0}^m A_i\right) & -P_3-P_3^{\mathrm{T}}+\sum_{i=1}^m(Q_i+h_iR_i) & h_1P_3^{\mathrm{T}}A_1 & \cdots & h_mP_3^{\mathrm{T}}A_m & P_3^{\mathrm{T}}D_1 & \cdots & P_3^{\mathrm{T}}D_m \\ h_1A_1^{\mathrm{T}}P_2 & h_1A_1^{\mathrm{T}}P_3 & -h_1R_1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 & 0 & \cdots & 0 \\ h_mA_m^{\mathrm{T}}P_2 & h_mA_m^{\mathrm{T}}P_3 & \cdot & \cdots & -h_mR_m & 0 & \cdots & 0 \\ D_1^{\mathrm{T}}P_2 & D_1^{\mathrm{T}}P_3 & \cdot & \cdots & 0 & -Q_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ D_m^{\mathrm{T}}P_2 & D_m^{\mathrm{T}}P_3 & \cdot & \cdots & 0 & 0 & \cdots & -Q_m \end{bmatrix}<0. \tag{8}$$

Proof. We represent (1) in the equivalent form (3). Note that

$$[x^{\mathrm{T}}\ y^{\mathrm{T}}]EP\begin{bmatrix} x \\ y \end{bmatrix}=x^{\mathrm{T}}P_1x$$

and, hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}[x^{\mathrm{T}}(t)\ y^{\mathrm{T}}(t)]EP\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}=2x^{\mathrm{T}}(t)P_1\dot{x}(t)=2[x^{\mathrm{T}}(t)\ y^{\mathrm{T}}(t)]P^{\mathrm{T}}\begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix}.$$

Due to (3) the latter relations imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}[x^{\mathrm{T}}(t)\ y^{\mathrm{T}}(t)]EP\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$
$$=2[x^{\mathrm{T}}(t)\ y^{\mathrm{T}}(t)]P^{\mathrm{T}}\begin{bmatrix} y(t) \\ -y(t)+\sum_{i=1}^m D_iy(t-h_i)+\left(\sum_{i=0}^m A_i\right)x(t)-\sum_{i=1}^m A_i\int_{t-h_i}^t y(s)\,\mathrm{d}s \end{bmatrix}. \tag{9}$$

Differentiating (4) in t and applying (9) we obtain

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t}=\xi^{\mathrm{T}}\begin{bmatrix} \Psi & P^{\mathrm{T}}\begin{bmatrix} 0 \\ D_1 \end{bmatrix} & \cdots & P^{\mathrm{T}}\begin{bmatrix} 0 \\ D_m \end{bmatrix} \\ [0\ D_1^{\mathrm{T}}]P & -Q_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ [0\ D_m^{\mathrm{T}}]P & 0 & \cdots & -Q_m \end{bmatrix}\xi+\sum_{i=1}^m\eta_i-\sum_{i=1}^m\int_{t-h_i}^t y^{\mathrm{T}}(s)R_iy(s)\,\mathrm{d}s, \tag{10}$$

where $\xi \triangleq \mathrm{col}\{x(t),\ y(t),\ y(t-h_1),\dots,y(t-h_m)\}$ and

$$\Psi \triangleq P^{\mathrm{T}}\begin{bmatrix} 0 & I \\ \left(\sum_{i=0}^{m}A_i\right) & -I \end{bmatrix} + \begin{bmatrix} 0 & \left(\sum_{i=0}^{m}A_i^{\mathrm{T}}\right) \\ I & -I \end{bmatrix} P + \begin{bmatrix} 0 & 0 \\ 0 & \sum_{i=1}^{m}(Q_i + h_iR_i) \end{bmatrix}, \tag{11}$$

$$\eta_i(t) \triangleq -2\int_{t-h_i}^{t} [x^{\mathrm{T}}(t)\ y^{\mathrm{T}}(t)]P^{\mathrm{T}}\begin{bmatrix} 0 \\ A_i \end{bmatrix} y(s)\,\mathrm{d}s. \tag{12}$$

For any $n\times n$-matrices $R_i>0$

$$\eta_i \leqslant h_i[x^{\mathrm{T}}\ y^{\mathrm{T}}]P^{\mathrm{T}}\begin{bmatrix} 0 \\ A_i \end{bmatrix} R_i^{-1}[0\ A_i^{\mathrm{T}}]P\begin{bmatrix} x \\ y \end{bmatrix} + \int_{t-h_i}^{t} y^{\mathrm{T}}(s)R_iy(s)\,\mathrm{d}s. \tag{13}$$

Eqs. (10) and (13) yield (by Schur complements) that $\mathrm{d}V(x,t)/\mathrm{d}t<0$ if the following LMI holds

$$\begin{bmatrix} \Psi & h_1P^{\mathrm{T}}\begin{bmatrix} 0 \\ A_1 \end{bmatrix} & \cdots & h_mP^{\mathrm{T}}\begin{bmatrix} 0 \\ A_m \end{bmatrix} & P^{\mathrm{T}}\begin{bmatrix} 0 \\ D_1 \end{bmatrix} & \cdots & P^{\mathrm{T}}\begin{bmatrix} 0 \\ D_m \end{bmatrix} \\ h_1[0\ A_1^{\mathrm{T}}]P & -h_1R_1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ h_m[0\ A_m^{\mathrm{T}}]P & 0 & \cdots & -h_mR_m & 0 & \cdots & 0 \\ [0\ D_1^{\mathrm{T}}]P & 0 & \cdots & 0 & -Q_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ [0\ D_m^{\mathrm{T}}]P & 0 & \cdots & 0 & 0 & \cdots & -Q_m \end{bmatrix} < 0, \tag{14}$$

where Ψ is given by (11). LMI (8) results from the latter LMI by multiplying the block matrices.

Functional V of (4) is degenerated and it has a negative derivative. This implies asymptotic stability of (1) in the space of continuous functions and thus, under A1, in the space of continuously differentiable functions [2]. □

Remark 1. Conservatism of our method is caused by bounding (13). We are bounding however fewer terms than in the other existing criteria. Thus, there are m^2 more terms of this kind when the first transformation of [7] is used (in the case of retarded system with $D_i=0$).

Remark 2. Criterion (8) is delay-independent with respect to delays in the difference operator $\mathscr{D}$. As we mentioned above A1 guarantees delay-independent stability of $\mathscr{D}$. The latter is necessary for robustness of stability of (1) with respect to small delays [10,5].

Remark 3. Note that (8) yields the following inequality:

$$\begin{bmatrix} -P_3 - P_3^{\mathrm{T}} + \sum_{i=1}^{m}Q_i & P_3^{\mathrm{T}}D_1 & \cdots & P_3^{\mathrm{T}}D_m \\ D_1^{\mathrm{T}}P_3 & -Q_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ D_m^{\mathrm{T}}P_3 & 0 & \cdots & -Q_m \end{bmatrix} <0. \tag{15}$$

If there exists a solution to (8) then there exists a solution to (15). If moreover $P_3=P_3^{\mathrm{T}}$, then $P_3>0$ and the "fast system"

$$\dot{y}(t) = -y(t) + \sum_{i=1}^{m} D_iy(t-g_i) \tag{16}$$

is asymptotically stable for all $g_i>0$ (see e.g. [6]). Note also that (15) implies by Schur complements that

$$-P_3 - P_3^{\mathrm{T}} + \sum_{i=1}^{m}Q_i + \sum_{i=1}^{m}P_3^{\mathrm{T}}D_iQ_i^{-1}D_i^{\mathrm{T}}P_3 < 0. \tag{17}$$

Remark 4. In the scalar case (17) implies A1 and therefore Theorem 1 holds without assumption A1. Really, since

$$Q_i + P_3^2 D_i^2/Q_i \geqslant 2P_3|D_i|.$$

We have from (17)

$$-2P_3\left(1-\sum_{i=1}^{m}|D_i|\right) \leqslant -2P_3 + \sum_{i=1}^{m}(Q_i + P_3^2 D_i^2/Q_i) < 0.$$

The latter yields A1 since $P_3 > 0$.

Remark 5. In the case of a single delay in the difference operator $\mathscr{D}x_t = x(t) - D_1 x(t-h_1)$ assumption A1 should be changed to the following standard assumption for neutral systems ([4,11,13,15]):

A1′. Assume that all eigenvalues of D_1 are inside of unit circle.

Under A1′ similarly to [2] the stability in the space of continuous and in the space of continuously differentiable functions is equivalent. As in the scalar case (17) implies A1′ and therefore Theorem 1 holds without assumption A1. Really, multiplying (17) by $y \in \mathscr{R}^n$ from the right and by y^{T} from the left we have

$$-2|y^{\mathrm{T}}P_3 y| + |Q_1^{1/2}y|^2 + |Q_1^{-1/2}D_1^{\mathrm{T}}P_3 y|^2 < 0.$$

Since

$$|Q_1^{1/2}y|^2 + |Q_1^{-1/2}D_1^{\mathrm{T}}P_3 y|^2 \geqslant 2|y^{\mathrm{T}}D_1^{\mathrm{T}}P_3 y|.$$

We obtain from the previous inequality that

$$-2|y^{\mathrm{T}}P_3 y| + 2|y^{\mathrm{T}}D_1^{\mathrm{T}}P_3 y| < 0.$$

Choose y to be an eigenvector of D_1 that corresponds to the eigenvalue λ. From the latter inequality we conclude that

$$-|y^{\mathrm{T}}P_3 y| + |\lambda||y^{\mathrm{T}}P_3 y| < 0$$

and thus $|\lambda| < 1$.

Remark 6. Comparing our approach with the "neutral type representation" of [7,9,13] in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) - \sum_{i=1}^{m} D_i x(t-h_i) + \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} x(s)\,\mathrm{d}s\right] = \sum_{i=0}^{m} A_i x(t),$$

we see that there is the same number of terms in the bounding of the type (13), though these terms are different. Our advantage is that unlike [7,9,13] we have no additional assumption on stability of $\bar{\mathscr{D}} : C[-h,0] \to R^n$ given by

$$\bar{\mathscr{D}}(x_t) = x(t) - \sum_{i=1}^{m} D_i x(t-h_i) + \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} x(s)\,\mathrm{d}s,$$

which is difficult to verify. Sufficient condition for stability of $\bar{\mathscr{D}}$ is as follows:

$$\sum_{i=1}^{m}|D_i| + \sum_{i=1}^{m} h_i|A_i| < 1,$$

but the latter may lead to conservative results (see Example 2 below).

2.2. Retarded type systems

Given the system

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t - h_i), \tag{18}$$

we represent it in the equivalent descriptor form:

$$\dot{x}(t) = y(t), \quad y(t) = \left(\sum_{i=0}^{m} A_i\right) x(t) - \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} y(s)\,\mathrm{d}s. \tag{19}$$

Since the fast variable y has no discrete delays, the Lyapunov–Krasovskii functional has the form (4), where $V_1 = 0$, i.e. $Q_i = 0,\ i = 1, \dots, m$. Theorem 1 implies the following

Corollary 1. *Eq.* (18) *is asymptotically stable if there exist* $0 < P_1 = P_1^{\mathrm{T}}$, P_2, P_3, *and* $R_i = R_i^{\mathrm{T}}$, $i = 1, \dots, m$ *that satisfy the following LMI*:

$$\begin{bmatrix} \left(\sum_{i=0}^{m} A_i^{\mathrm{T}}\right) P_2 + P_2^{\mathrm{T}} \left(\sum_{i=0}^{m} A_i\right) & P_1 - P_2^{\mathrm{T}} + \left(\sum_{i=0}^{m} A_i^{\mathrm{T}}\right) P_3 & h_1 P_2^{\mathrm{T}} A_1 & \cdots & h_m P_2^{\mathrm{T}} A_m \\ P_1 - P_2 + P_3^{\mathrm{T}} \left(\sum_{i=0}^{m} A_i\right) & -P_3 - P_3^{\mathrm{T}} + \sum_{i=1}^{m} h_i R_i & h_1 P_3^{\mathrm{T}} A_1 & \cdots & h_m P_3^{\mathrm{T}} A_m \\ h_1 A_1^{\mathrm{T}} P_2 & h_1 A_1^{\mathrm{T}} P_3 & -h_1 R_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 \\ h_m A_m^{\mathrm{T}} P_2 & h_m A_m^{\mathrm{T}} P_3 & \cdot & \cdots & -h_m R_m \end{bmatrix} < 0. \tag{20}$$

2.3. Examples

We applied our criteria to examples considered in [7–9] and [13]. We solved LMIs by using LMI Toolbox of Matlab. None of the results we obtained for these examples were more conservative than the existing results and in some examples our results were less conservative. Two of these examples (one for retarded and one for neutral type cases) are given below.

Example 1 (*Kolmanovskii and Richard* [7] (*retarded type case*)). Consider the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h_1)$$

with

$$A_0 = \begin{bmatrix} -1 & 1/2 \\ -1/2 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}.$$

This is Example 2.2 of [7] with $\alpha = 1$, $\beta = \frac{1}{2}$, $\gamma = \delta = 2$. In [7] it was found that the second transformation leads to the less restrictive (than the other two transformations) condition: $h_1 < 0.0369$. Our Corollary 1 improves this result and our condition is $h_1 \leqslant 0.271$. Thus for $h_1 = 0.271$ we obtain the following solution to LMI (20):

$$P_1 = \begin{bmatrix} 94.1609 & 0.1653 \\ 0.1653 & 94.0469 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 93.5589 & 0.1872 \\ 0.1872 & 94.6599 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 18.5170 & -0.0930 \\ -0.0930 & 18.4880 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 68.2748 & 0.0349 \\ 0.0349 & 68.1810 \end{bmatrix}.$$

Note that applying LMI conditions of [8] and [13] we obtained $h_1 \leqslant 0.268$ and $h_1 \leqslant 0.271$, respectively.

Example 2 (*Lien et al.* [9] (*neutral type case*)). Consider the following system:

$$\dot{x}(t) - D_1\dot{x}(t-h_1) = A_0x_1(t) + A_1x(t-h_1)$$

with

$$A_0 = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix}. \tag{21}$$

The stability condition of [9] is $h_1 \leqslant 0.3$. We obtain $h_1 \leqslant 0.74$ and for $h_1 = 0.74$ we have the following solution to (8):

$$P_1 = \begin{bmatrix} 7.6203 & 2.6912 \\ 2.6912 & 7.4063 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.7783 & 0.0259 \\ 0.0259 & 0.7739 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 3.8011 & 1.3542 \\ 1.3542 & 3.6974 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 0.6499 & -0.1261 \\ -0.1261 & 0.4085 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 4.2419 & 2.0425 \\ 2.0425 & 4.4669 \end{bmatrix}.$$

By applying LMI of [13] we find that LMI has a solution for $h_1 \leqslant 0.71$ (which is still less than 0.74), but the stability of $\bar{\mathscr{D}}$ should be verified. Sufficient condition for the latter $|D_1| + h_1|A_1| < 1$ (where similarly to [9] spectral norm is taken) leads to more restrictive condition $h_1 \leqslant 0.61$.

3. Delay-dependent/delay-independent stability: distributed delays

3.1. Main result

We generalize our results to systems with distributed delays:

$$\dot{x}(t) - \sum_{i=1}^{k} D_i\dot{x}(t-g_i) = \sum_{i=0}^{m} A_i x(t-h_i) + \sum_{i=1}^{m} A_{i0}\int_{t-\tau_i}^{t} x(s)\,\mathrm{d}s + \sum_{i=1}^{k} F_i x(t-g_i), \tag{22}$$

where $g_i \geqslant 0$. Similarly to [6,7], we are looking for stability criterion which is delay-dependent with respect to one part of delays (h_i and the distributed delays over $[-\tau_i, 0]$), and delay-independent with respect to the other delays (g_i). The descriptor form representation for this system has the form:

$$\dot{x}(t) = y(t), \quad y(t) = \sum_{i=1}^{k} D_i y(t-g_i) + \left(\sum_{i=0}^{m} A_i\right) x(t) - \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} y(s)\,\mathrm{d}s$$
$$+ \sum_{i=1}^{m} A_{i0}\int_{t-\tau_i}^{t} x(s)\,\mathrm{d}s + \sum_{i=1}^{k} F_i x(t-g_i). \tag{23}$$

The corresponding (degenerate) Lyapunov–Krasovskii functional is given by

$$V(t) = [x^{\mathrm{T}}(t)\ y^{\mathrm{T}}(t)]EP\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + V_1 + V_2 + V_3 + V_4, \quad P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad P_1 = P_1^{\mathrm{T}} > 0, \tag{24}$$

where V_2 is defined by (7), and where

$$V_1 = \sum_{i=1}^{k}\int_{t-g_i}^{t} y^{\mathrm{T}}(s)Q_i y(s)\,\mathrm{d}s, \quad V_3 = \sum_{i=1}^{k}\int_{t-g_i}^{t} x^{\mathrm{T}}(s)U_i x(s)\,\mathrm{d}s, \quad Q_i > 0,\ U_i > 0,$$

$$V_4 = \sum_{i=1}^{m}\int_{-\tau_i}^{0}\int_{t+\theta}^{t} x^{\mathrm{T}}(s)R_{i0}x(s)\,\mathrm{d}s\,\mathrm{d}\theta, \quad R_{i0} > 0. \tag{25}$$

The form of (24) (similarly to (4)) corresponds to discrete-delay independent/distributed-delay dependent conditions for descriptor system (23). We obtain the following result:

Theorem 2. *Under* A1 (22) *is stable for all* $g_i \geqslant 0,\ i=1,\ldots,k$ *if there exist* $0<P_1=P_1^{\mathrm{T}},\ P_2,\ P_3,\ Q_i=Q_i^{\mathrm{T}},\ U_i=U_i^{\mathrm{T}},\ i=1,\ldots,k$ *and* $R_j=R_j^{\mathrm{T}},\ R_{j0}=R_{j0}^{\mathrm{T}},\ j=1,\ldots,m$ *that satisfy the following LMI*:

$$
\left[\begin{array}{ccccc}
\Phi & P_1-P_2^{\mathrm{T}}+\left(\sum_{i=0}^{m}A_i^{\mathrm{T}}\right)P_3 & h_1P_2^{\mathrm{T}}A_1 & \cdots & h_mP_2^{\mathrm{T}}A_m \\
P_1-P_2+P_3^{\mathrm{T}}\left(\sum_{i=0}^{m}A_i\right) & -P_3-P_3^{\mathrm{T}}+\sum_{i=1}^{k}Q_i+\sum_{i=1}^{m}h_iR_i & h_1P_3^{\mathrm{T}}A_1 & \cdots & h_mP_3^{\mathrm{T}}A_m \\
h_1A_1^{\mathrm{T}}P_2 & h_1A_1^{\mathrm{T}}P_3 & -h_1R_1 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
h_mA_m^{\mathrm{T}}P_2 & h_mA_m^{\mathrm{T}}P_3 & \cdot & \cdots & -h_mR_m \\
D_1^{\mathrm{T}}P_2 & D_1^{\mathrm{T}}P_3 & \cdot & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
D_k^{\mathrm{T}}P_2 & D_k^{\mathrm{T}}P_3 & \cdot & \cdots & 0 \\
\tau_1A_{10}^{\mathrm{T}}P_2 & \tau_1A_{10}^{\mathrm{T}}P_3 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\tau_mA_{m0}^{\mathrm{T}}P_2 & \tau_mA_{m0}^{\mathrm{T}}P_3 & 0 & \cdots & 0 \\
F_1^{\mathrm{T}}P_2 & F_1^{\mathrm{T}}P_3 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
F_k^{\mathrm{T}}P_2 & F_k^{\mathrm{T}}P_3 & 0 & \cdots & 0
\end{array}\right.
$$

$$
\left.\begin{array}{ccccccccc}
P_2^{\mathrm{T}}D_1 & \cdots & P_2^{\mathrm{T}}D_k & \tau_1P_2^{\mathrm{T}}A_{10} & \cdots & \tau_mP_2^{\mathrm{T}}A_{m0} & P_2^{\mathrm{T}}F_1 & \cdots & P_2^{\mathrm{T}}F_m \\
P_3^{\mathrm{T}}D_1 & \cdots & P_3^{\mathrm{T}}D_k & \tau_1P_3^{\mathrm{T}}A_{10} & \cdots & \tau_mP_3^{\mathrm{T}}A_{m0} & P_3^{\mathrm{T}}F_1 & \cdots & P_3^{\mathrm{T}}F_m \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
-Q_1 & \cdots & 0 & 0 & .. & 0 & 0 & \cdots & 0 \\
\cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & \cdots & -Q_k & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & -\tau_1R_{10} & \cdots & 0 & 0 & \cdots & 0 \\
\cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & \cdots & 0 & 0 & \cdots & -\tau_mR_{m0} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & -U_1 & \cdots & 0 \\
\cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & -U_k
\end{array}\right] < 0. \qquad (26)
$$

where

$$
\Phi = \left(\sum_{i=0}^{m}A_i^{\mathrm{T}}\right)P_2 + P_2^{\mathrm{T}}\left(\sum_{i=0}^{m}A_i\right) + \sum_{i=1}^{k}U_i + \sum_{i=1}^{m}\tau_iR_{i0}.
$$

Proof. Similar to the proof of Theorem 1. Note that in this case we have

$$\frac{dV(t)}{dt} = \xi^T \begin{bmatrix} \Psi_1 & P^T\begin{bmatrix}0\\ D_1\end{bmatrix} & \cdots & P^T\begin{bmatrix}0\\ D_k\end{bmatrix} & P^T\begin{bmatrix}0\\ F_1\end{bmatrix} & \cdots & P^T\begin{bmatrix}0\\ F_k\end{bmatrix} \\ [0\ D_1^T]P & -Q_1 & \cdots & 0 & \cdots & 0 & \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ [0\ D_k^T]P & 0 & \cdots & -Q_k & 0 & \cdots & 0 \\ [0\ F_1^T]P & 0 & \cdots & 0 & -U_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ [0\ F_k^T]P & 0 & \cdots & 0 & 0 & \cdots & -U_k \end{bmatrix} \xi$$

$$+\sum_{i=1}^m \eta_i + \sum_{i=1}^m \eta_{i0} - \sum_{i=1}^m \int_{t-h_i}^t y^T(s)R_i y(s)\,ds - \sum_{i=1}^m \int_{t-\tau_i}^t x^T(s)R_{i0}x(s)\,ds, \tag{27}$$

where $\xi \triangleq \text{col}\{x(t),\ y(t),\ y(t-g_1),\dots,y(t-g_k), x(t-g_1),\dots,x(t-g_k)\}, \eta_i$ is given by (12) and

$$\Psi_1 \triangleq P^T \begin{bmatrix} 0 & I \\ \left(\sum_{i=0}^m A_i\right) & -I \end{bmatrix} + \begin{bmatrix} 0 & \left(\sum_{i=0}^m A_i^T\right) \\ I & -I \end{bmatrix} P + \begin{bmatrix} \sum_{i=1}^k U_i + \sum_{i=1}^m \tau_i R_{i0} & 0 \\ 0 & \sum_{i=1}^k Q_i + \sum_{i=1}^m h_i R_i \end{bmatrix},$$

$$\eta_{i0}(t) \triangleq -2\int_{t-h_i}^t [x^T(t)\ y^T(t)]P^T \begin{bmatrix} 0 \\ A_{i0} \end{bmatrix} x(s)\,ds. \qquad \square$$

3.2. Delay-independent stability

Consider the system

$$\dot{x}(t) - \sum_{i=1}^k D_i \dot{x}(t-g_i) = A_0 x(t) + \sum_{i=1}^k F_i x(t-g_i). \tag{28}$$

Delay-independent stability conditions can be derived by applying Lyapunov–Krasovskii functional of (24), where $V_2 = V_4 = 0$. Theorem 2 implies the following delay-independent stability criterion:

Corollary 2. *Under* A1 (28) *is stable for all* $g_i \geqslant 0,\ i=1,\dots,k$ *if there exist* $0 < P_1 = P_1^T, P_2, P_3,$ *and* $Q_i = Q_i^T,\ U_i = U_i^T,\ i=1,\dots,k$ *that satisfy the following LMI*:

$$\begin{bmatrix} A_0^T P_2 + P_2^T A_0 + \sum_{i=1}^k U_i & P_1 - P_2^T + A_0^T P_3 & P_2^T F_1 & \cdots & P_2^T F_k & P_2^T D_1 & \cdots & P_2^T D_k \\ P_1 - P_2 + P_3^T A_0 & -P_3 - P_3^T + \sum_{i=1}^k Q_i & P_3^T F_1 & \cdots & P_3^T F_k & P_3^T D_1 & \cdots & P_3^T D_k \\ F_1^T P_2 & F_1^T P_3 & -U_1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 & 0 & \cdots & 0 \\ F_k^T P_2 & F_k^T P_3 & \cdot & \cdots & -U_k & 0 & \cdots & 0 \\ D_1^T P_2 & D_1^T P_3 & \cdot & \cdots & 0 & -Q_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ D_k^T P_2 & D_k^T P_3 & \cdot & \cdots & 0 & 0 & \cdots & -Q_k \end{bmatrix} < 0. \tag{29}$$

Example 3. Consider a two-dimensional system

$$\dot{x}(t) - D_1\dot{x}(t-g_1) = A_0 x(t) + F_1 x(t-g_1) + A_{10}\int_{t-\tau_1}^{t} x(s)\,\mathrm{d}s \tag{30}$$

with

$$A_0 = \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix}, \quad F_1 = \begin{bmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{bmatrix}, \quad A_{10} = \begin{bmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{bmatrix}.$$

For $D_1 = 0$ this is Example 4.1 from [7]. From results of [7] it follows that for $a_1 = a_2 = 1.5$, $b_1 = b_2 = 1$ and $c_1 = 1$, $c_2 = 0.5$ the system (30) is stable for all delays g_1 and for $\tau_1 \leqslant 0.03$. Our Theorem 2 leads to less restrictive condition: $\tau_1 \leqslant 0.07$. For $D_1 = 0$ and

$$a_1 = 2, \quad a_2 = 15, \quad b_1 = 1, \quad b_2 = 3, \quad c_1 = 1, \quad c_2 = 0.5 \tag{31}$$

stability conditions of [7] do not hold (even for $\tau_1 = 0$). For this case by Theorem 2 we find that (30) is stable for all g_1 and $\tau_1 \leqslant 1.1$.

Choosing D_1 given by (21) and the other parameters given by (31) we obtain that (30) is stable for all g_1 and $\tau_1 \leqslant 1$. Thus for $\tau = 1$ we obtain the following solution to (26):

$$P_1 = \begin{bmatrix} 272.1194 & 47.1695 \\ 47.1695 & 254.1802 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 270.1438 & 41.3581 \\ 41.3581 & 192.5597 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 67.5927 & -4.2914 \\ -4.2914 & 15.8292 \end{bmatrix},$$

$$U_1 = \begin{bmatrix} 295.8 & 498.8 \\ 498.8 & 3101.4 \end{bmatrix}, \quad R_{10} = \begin{bmatrix} 276.4176 & 171.1430 \\ 171.1430 & 464.0256 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 62.7484 & 3.8686 \\ 3.8686 & 9.0791 \end{bmatrix}.$$

4. Conclusions

New Lyapunov–Krasovskii functionals have been introduced for stability of linear retarded and neutral type systems with discrete and distributed delays. These degenerate functionals are based on equivalent descriptor form of the original system and lead to new results (for neutral systems with distributed delays) and to results that are less conservative than existing results for both, retarded and neutral type systems. Delay-dependent/delay-independent conditions have been obtained in terms of LMI. The new model transformation and functionals can be applied further to H_∞ control of linear systems with delay and to analysis and synthesis of some nonlinear time-delay systems. This work is currently in progress.

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