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On regional nonlinear H^∞-filtering[1]

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Abstract

The structure of the nonlinear H^∞-filter in the neighborhood of the estimated trajectory is investigated and a bound on the size of the neighborhood that allows this structure is determined, both for finite and infinite horizons. Riccati inequalities that depend on the estimated trajectory are derived for finding the filter gain matrix and an algorithm for calculating the bound on the size of the above neighborhood is presented. Explicit formulas are obtained in the infinite horizon case for the minimum achievable disturbance attenuation level, the size of the neighborhood, and the corresponding filter gain.

Keywords: H_∞-filtering; Nonlinear systems; Hamilton–Jacobi inequalities; Riccati inequalities

1. Introduction

In the H^∞-filtering problem an estimator is looked for that achieves a given bound on the ratio between the energy of the estimation error and the energy of the exogenous inputs to the estimated process. Conditions for the existence of such an estimator and formulas for its derivation have recently been obtained for nonlinear processes using game theory and the theory of dissipative systems [1–6]. Sufficient conditions for solving the regional H^∞-filtering problem in the neighborhood of the estimated trajectory have been derived in [1, 5]. These conditions incorporate a solution of the Hamilton–Jacobi inequality.

The objectives of the present note are:

(i) Derivation of simplified sufficient conditions for the existence of a solution to the regional H^∞ problem that are equivalent to those of [1, 5]. Our conditions will be based on solutions of Riccati inequalities that depend on the estimated trajectory.

(ii) Determining a bound on the size of the maximum possible domain that allows a solution to the regional filtering problem by the derived filter.

(iii) Deriving explicit formulas for the minimum achievable disturbance attenuation level, for the size of the domain and the corresponding filter gain.

Throughout this note we denote by $|\cdot|$, the Euclidean norm of a vector or the appropriate norm of a matrix. Let $L_2[0, T]$ be the space of the square integrable functions with the norm $\|\cdot\|_{L_2}$.

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2. Problem formulation

Consider the system

$$\dot{x} = f(x, t) + g_1(x, t)w, \tag{1a}$$

$$y = h_2(x, t) + k_{21}(x, t)w, \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^q$ is the disturbance, $y \in \mathbb{R}^p$ is the observation, and f, g_1, h_2 and k_{21} are smooth functions. We define the following penalty function:

$$z = h_1(x, t) - h_1(\bar{x}, t), \quad z \in \mathbb{R}^s \tag{2}$$

and a positive-definite function $N(x)$, such that $N_x(0) = 0$. The objective is to synthesize a filter for estimating $h_1(x(t))$ from the available observation at time $\tau \leqslant t$ so that the filter state $\bar{x}$ satisfies an "H^∞-type requirement" for a preassigned value of γ:

$$\|z\|^2_{L_2} \leqslant \gamma^2[N(x(0) - \bar{x}(0)) + \|w\|^2_{L_2}] \tag{3}$$

for all $w \in L_2[0, T]$, $x(0) \in \mathbb{R}^n$, $\bar{x}(0) \in \mathbb{R}^n$.

Assume that

A.1. $k_{21}g_1^{\mathrm{T}} = 0$ and $k_{21}k_{21}^{\mathrm{T}} > 0$ for all $x \in \mathbb{R}^n$.

Our attention is focused on the following family of estimators:

$$\dot{\bar{x}} = f(\bar{x}, t) + G(\bar{x}, t)[y - h_2(\bar{x}(t), t)], \tag{4}$$

where G is a matrix of appropriate dimensions and desired smoothness.

By regional nonlinear H^∞-filtering problem we understand the following: Given a set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ find an estimator (4) such that (3) holds for all $(x(0), \bar{x}(0)) \in \Omega$, $t \in [0, T]$ and all $w \in L_2[0, T]$ that leave the trajectory $(x(t), \bar{x}(t))$ within Ω.

3. Riccati inequalities for regional H^∞-filtering

We denote:

$$R_1(x,t) = g_1(x,t)g_1^{\mathrm{T}}(x,t), \qquad R_2(x,t) = k_{21}(x,t)k_{21}^{\mathrm{T}}(x,t)$$

and

$$\begin{aligned} H_G(V) = {} & V_t + V_x f(x,t) + V_{\bar{x}} f(\bar{x},t) + z^{\mathrm{T}}z + b(x,\bar{x},t) + \frac{1}{4\gamma^2} V_x R_1(x,t) V_x^{\mathrm{T}} \\ & - \gamma^2 [h_2(x,t) - h_2(\bar{x},t)]^{\mathrm{T}} R_2^{-1}(x,t)[h_2(x,t) - h_2(\bar{x},t)], \end{aligned} \tag{5a}$$

where

$$\begin{aligned} b(x,\bar{x},t) = {} & \frac{1}{4\gamma^2}\{V_{\bar{x}}G + 2\gamma^2[h_2(x,t) - h_2(\bar{x},t)]^{\mathrm{T}} R_2^{-1}(x,t)\} R_2(x,t) \\ & \times \{G^{\mathrm{T}} V_{\bar{x}}^{\mathrm{T}} + 2\gamma^2 R_2^{-1}(x,t)[h_2(x,t) - h_2(\bar{x},t)]\}. \end{aligned} \tag{5b}$$

A sufficient condition for solving the regional H^∞-filtering problem is as follows [1–3, 5]:

Lemma 1. *Given $\gamma > 0$ and a region $\Omega = \Omega_0 \times \Omega_1 \subset \mathbb{R}^n \times \mathbb{R}^n$. Suppose there exists a twice continuously differentiable nonnegative function $V: \Omega \times [0, T] \to R$ and a function $G: \Omega_1 \times [0, T] \to \mathbb{R}^p$, satisfying for all $\bar{x} \in \Omega_1$, $t \in [0, T]$, the condition*

$$V(0,0,t) = 0, \quad V_{\bar{x}}|_{x=\bar{x}} = 0, \quad V_x|_{x=\bar{x}} = 0, \quad \textit{while } V_{\bar{x}x}|_{x=\bar{x}} \textit{ has full rank.} \tag{6a–d}$$

Also assume that the following Hamilton–Jacobi inequality:

$$H_G(V) \leqslant 0, \tag{7a}$$

holds for $t \in [0, T]$ and $(x, \bar{x}) \in \Omega$, with the initial condition requirement of

$$V(x, \bar{x}, 0) \geqslant N(x - \bar{x}). \tag{7b}$$

Then, the estimator of (4) solves the regional H^∞-filtering problem on Ω.

Denote

$$\xi = x - \bar{x}.$$

A natural choice of G for x close to $\bar{x}$ follows from [1, 2, 5]:

Lemma 2. *Given $\gamma > 0$. Suppose there exists a twice continuously differentiable nonnegative function V satisfying conditions (6a–d) and the following Hamilton–Jacobi inequality:*

$$V_t + V_x f(x,t) + V_{\bar{x}} f(\bar{x},t) + z^{\mathrm{T}} z + \frac{1}{4\gamma^2} V_x R_1(x,t) V_x^{\mathrm{T}}$$
$$- \gamma^2 [h_2(x,t) - h_2(\bar{x},t)]^{\mathrm{T}} R_2^{-1}(x,t) [h_2(x,t) - h_2(\bar{x},t)] + \gamma^2 \delta' \xi^{\mathrm{T}} \xi \leqslant 0, \tag{8a}$$

with the initial condition

$$V(x, \bar{x}, 0) \geqslant N(\xi) + \delta'_1 \xi^{\mathrm{T}} \xi, \tag{8b}$$

for $t \in [0, T]$, $(x, \bar{x}) \in \Omega$ and for some positive scalars δ' and δ'_1. Suppose, also, that $V_{x\bar{x}}$ and h_x are uniformly continuous on $\Omega \times [0, T]$ and $\Omega_1 \times [0, T]$, respectively. Then, there exists $r > 0$ such that the regional H^∞-filtering problem is solved by

$$G(\bar{x}, t) = -2\gamma^2 (V_{x\bar{x}})^{-1}_{|x=\bar{x}} h_{2x}^{\mathrm{T}}(\bar{x}, t) R_2^{-1}(\bar{x}, t) \tag{9}$$

on $\bar{x} \in \Omega_1$, $|x - \bar{x}| < r$.

Proof. Under the assumptions of Lemma 2, for x close to $\bar{x}$,

$$V_{\bar{x}}(x, \bar{x}, t) = (x - \bar{x})^{\mathrm{T}} V_{x\bar{x}}(x, \bar{x}, t) + \mathrm{o}(|\xi|),$$

where by $\mathrm{o}(|\xi|^k)$, $k \geqslant 1$ we denote a continuous function of x, $\bar{x}$, and t that satisfies

$$\lim_{\xi \to 0} \frac{\mathrm{o}(|\xi|^k)}{|\xi|^k} = 0, \quad \text{uniformly on } x, t.$$

In other words, for a given δ_0 there exists r_0 such that $|\mathrm{o}(|\xi|)| < \delta_0 |\xi|$ for all $\bar{x} \in \Omega_1$, $|x - \bar{x}| < r_0$ and $t \in [0, T]$. Since

$$h_2(x,t) - h_2(\bar{x},t) = h_{2x}(\bar{x},t)\xi + \mathrm{o}(|\xi|),$$

we obtain due to (5a) and (9) that

$$b(x, \bar{x}, t) = \mathrm{o}(|\xi|^2).$$

Hence, by (7a), for x close to $\bar{x}$ we get

$$H_G(V) = -\gamma^2 \delta' |\xi|^2 + \mathrm{o}(|\xi|^2) \leqslant 0. \quad \square$$

In the present note, we first derive sufficient conditions for the solvability of the regional H^∞-filtering problem with simple expressions for $G(\bar{x}, t)$ that are equivalent to the conditions of Lemma 2. These results are valid for x close enough to $\bar{x}$. We shall then investigate how big can Ω be to allow a solution of the regional H^∞-filtering problem with the derived filter. Under (6), for x close to $\bar{x}$, the function V can be represented in the form

$$V = V_0 + \gamma^2 (x-\bar{x})^{\mathrm{T}} U^{-1}(\bar{x},t)(x-\bar{x}) + \mathrm{o}(|x-\bar{x}|^2), \tag{10}$$

where $U = U^{\mathrm{T}} > 0$. Clearly,

$$V_{x\bar{x}}|_{x=\bar{x}} = -2\gamma^2 U^{-1}(\bar{x},t)$$

and $V_0 = V(\bar{x},\bar{x})$ does not depend on $\bar{x}$ due to the conditions (6b) and (6c).

We also obtain that

$$V_x = 2\gamma^2 \xi^{\mathrm{T}} U^{-1}(\bar{x},t) + \mathrm{o}(|\xi|), \quad \text{and} \quad V_{\bar{x}} = -2\gamma^2 \xi^{\mathrm{T}} U^{-1}(\bar{x},t) + \mathrm{o}(|\xi|).$$

Substitution of (10) into (9) and (8) leads to the following expression for G:

$$G(\bar{x},t) = U(\bar{x},t) h_{2x}^{\mathrm{T}}(\bar{x},t) R_2^{-1}(\bar{x},t), \tag{11}$$

and, since $(U^{-1})_t = -U^{-1}U_tU^{-1}$, it also leads to the following Hamilton–Jacobi inequality:

$$H_G(V) = \gamma^2 \xi^{\mathrm{T}} U^{-1}(\bar{x},t) F[U(\bar{x},t),\bar{x},t] U^{-1}(\bar{x},t)\xi + \gamma^2\delta'\xi^{\mathrm{T}}\xi + \mathrm{o}(|\xi|^2) \leqslant 0, \tag{12a}$$

with the initial condition

$$\gamma^2 \xi^{\mathrm{T}} U^{-1}(\bar{x},0)\xi \geqslant \xi^{\mathrm{T}}(N_{xx}(0) + \delta_1')\xi + \mathrm{o}(|\xi|^2), \tag{12b}$$

where

$$F(U,x,t) = -U_t + Uf_x^{\mathrm{T}} + f_xU + U(\gamma^{-2}h_{1x}^{\mathrm{T}}h_{1x} - h_{2x}^{\mathrm{T}}R_2^{-1}h_{2x})U + R_1. \tag{13}$$

Inequalities (12a, b) imply the following Riccati differential inequality for some positive δ and δ_1:

$$-U_t + Uf_{\bar{x}}^{\mathrm{T}} + f_{\bar{x}}U + U(\gamma^{-2}h_{1\bar{x}}^{\mathrm{T}}h_{1\bar{x}} - h_{2\bar{x}}^{\mathrm{T}}R_2^{-1}h_{2\bar{x}} + \delta I)U + R_1 \leqslant 0, \tag{14a}$$

and

$$U^{-1}(\bar{x},0) \geqslant \gamma^{-2}(N_{\bar{x}\bar{x}}(0) + \delta_1 I). \tag{14b}$$

Conversely, (8) and (9) follow from (14), (10) and (11).

We thus get the following sufficient conditions for the solvability of the regional H^∞-filtering problem which are equivalent to Lemma 2.

Theorem 1. *Given $\gamma > 0$. Let the Riccati differential inequality of* (14) *possess, for fixed $\delta > 0$ and $\delta_1 > 0$, a continuously differentiable solution $U > 0$ for all $\bar{x} = \mathrm{col}(\bar{x}_1, \ldots, \bar{x}_n) \in \Omega_1$, where the functions $U, U^{-1}, U_{\bar{x}_i}$ $(i = 1, \ldots, n)$, U_t are bounded, uniformly on $\Omega_1 \times [0,T]$. Then, there exists $r > 0$ such that the regional H^∞-filtering problem is solved by G of* (11) *on $\bar{x} \in \Omega_1$, $|x-\bar{x}| < r$.*

We next address the issue of how big can the domain Ω be that still allows a solution of the regional filtering problem by the filter of (4) with G given in (11), where U solves (14). To obtain a lower bound on the maximum possible r we take V in the form of

$$V = \gamma^2(x-\bar{x})^{\mathrm{T}} U^{-1}(\bar{x},t)(x-\bar{x}). \tag{15}$$

Then

$$\begin{aligned} &V_t = -\gamma^2\xi^{\mathrm{T}}U^{-1}(\bar{x},t)U_t(\bar{x},t)U^{-1}(\bar{x},t)\xi, \qquad V_x = 2\gamma^2\xi^{\mathrm{T}}U^{-1}(\bar{x},t), \\ &V_{\bar{x}} = -2\gamma^2\xi^{\mathrm{T}}U^{-1}(\bar{x},t) - \gamma^2\xi^{\mathrm{T}}U^{-1}[U_{\bar{x}_1}U^{-1}\xi, \ldots, U_{\bar{x}_n}U^{-1}\xi]. \end{aligned} \tag{16}$$

We therefore obtain the following algorithm for the regional H^∞-filtering with the preassigned value of γ:

(i) For prechosen δ and δ_1 find a solution $U > 0$ to (14) on Ω_1.

(ii) Calculate G by (11).

(iii) Compute U^{-1}, U_t and $U_{\bar{x}_i}$, $i = 1, \dots, n$.

(iv) Substitute (16) into (7) to find r such that for $\bar{x} \in \Omega_1$, $|x - \bar{x}| < r$ the Hamilton–Jacobi inequality (7) holds. Then the H^∞-filtering problem is solved by G for $\bar{x} \in \Omega_1$, $|x - \bar{x}| < r$.

(v) Change, if necessary, δ or δ_1 and return to step (i).

Remark 1. All the results of this section have natural counterparts in the time-invariant infinite horizon case. In this case (7) becomes a time-invariant Hamilton–Jacobi inequality and (14) becomes a Riccati algebraic inequality for any $\bar{x} \in \Omega_1$. We look for a continuously differentiable solution $U > 0$ of (14) with U, U^{-1}, $U_{\bar{x}_i}$ $(i = 1, \dots, n)$ bounded on Ω_1 and such that the matrix $f_{\bar{x}}^{\mathrm{T}} + (\gamma^{-2} h_{1\bar{x}}^{\mathrm{T}} h_{1\bar{x}} - h_{2\bar{x}}^{\mathrm{T}} R_2^{-1} h_{2\bar{x}} + \delta I)U$ is Hurwitz.

4. Bounds on γ and r in the infinite horizon case

Consider the time-invariant infinite horizon case. We shall use the method of [7] to obtain some a priori bounds for the maximum possible r and for the minimum possible γ. We make the following assumption.

A2. For all $x_i \in \mathbb{R}^n$ $(i = 1, \dots, 5)$ the functions of the system of (1) satisfy the following inequalities:

$$
\begin{aligned}
&x_1^{\mathrm{T}} f_x(x_2) x_1 \leqslant -\alpha |x_1|^2, \quad |h_{1x}(x_2)|^2 \leqslant H_1,\\
&0 \leqslant E \leqslant |h_{2x}^{\mathrm{T}}(x_1) R_2^{-1}(x_2) h_{2x}(x_1)|, \quad |R_1(x_2)| \leqslant R_1',\\
&\left[0.5\,|h_{2xx}^{\mathrm{T}}(x_1)|\,|R_2^{-1}(x_2)| + |h_x(x_3)| \left|\frac{\mathrm{d}}{\mathrm{d}x}(R_2^{-1})\right|\Bigg|_{x = x_4}\right]^2 |R_2(x_5)| \leqslant H_2,
\end{aligned}
\tag{17}
$$

where H_1 and R_1' are positive numbers, and $H_2 \geqslant 0$. We consider two cases for α: (i) $\alpha > 0$ and (ii) $\alpha \leqslant 0$, $E > 0$.

Note that if f_x is a symmetric matrix, we first find $\lambda(x)$ – the maximal eigenvalue of $f_x(x)$. Then $-\alpha = \max_{x \in \mathbb{R}^n} \lambda(x)$. Similarly, H_1, H_2 and R_1' can be determined. We also notice that E corresponds to the minimal eigenvalue of $h_{2x}^{\mathrm{T}} R_2^{-1} h_{2x}$.

Theorem 2. (i) *Under* A1 *and* A2 *assume that* $\alpha > 0$. *Let* r_1 *satisfy the following inequality*:

$$\alpha^2 + R_1'(E - H_2 r_1^2) > 0, \tag{18}$$

and γ_1 *be given by*

$$\gamma_1^2 = \frac{R_1' H_1}{\alpha^2 + R_1'(E - H_2 r_1^2)}. \tag{19}$$

Then, for $\bar{x} \in \mathbb{R}^n$, $|x - \bar{x}| < r_1$, *the level of* γ_1 *is guaranteed by*

$$G(\bar{x}) = \frac{R_1'}{\alpha} h_{2x}^{\mathrm{T}}(\bar{x}) R_2^{-1}(\bar{x}). \tag{20}$$

(ii) *Under* A1 *and* A2 *assume* $\alpha \leqslant 0$, $E > 0$. *Let* r_2 *satisfy the inequality*

$$E - H_2 r_2^2 > 0, \tag{21}$$

and γ_2 be given by

$$\gamma_2^2 = \frac{H_1}{E - H_2 r_2^2}. \tag{22}$$

Then, for $\gamma > \gamma_2$ and $\bar{x} \in \mathbb{R}^n$, $|x - \bar{x}| < r_2$, the level of γ is guaranteed by

$$G = \gamma^2 p^{-1} h_{2x}^{\mathrm{T}}(\bar{x}) R_2^{-1}(\bar{x}), \tag{23}$$

where

$$0 < p \leqslant \frac{\alpha + \sqrt{\alpha^2 - R_1' H_1 \gamma^{-2} + R_1'(E - H_2 r_2^2)}}{\gamma^{-2} R_1'}. \tag{24}$$

Proof. We look for V in the form of $V = p|\xi|^2$, where $p > 0$ is a scalar. Then, $V_x = 2p\xi^{\mathrm{T}}$, and $V_{\bar{x}} = -2p\xi^{\mathrm{T}}$, and G, given by (9), is determined by (23). Therefore, (7a) has the form

$$\begin{aligned} H_G(V) = {} & 2p\xi^{\mathrm{T}}[f(x) - f(\bar{x})] + [h_1(x) - h_1(\bar{x})]^{\mathrm{T}}[h_1(x) - h_1(\bar{x})] + \gamma^2 c(x,\bar{x}) R_2(x) c^{\mathrm{T}}(x,\bar{x}) \\ & + \frac{1}{\gamma^2} p^2 \xi^{\mathrm{T}} R_1(x) \xi - \gamma^2 [h_2(x) - h_2(\bar{x})]^{\mathrm{T}} R_2^{-1}(x) [h_2(x) - h_2(\bar{x})] \leqslant 0, \end{aligned} \tag{25}$$

where

$$c(x,\bar{x}) = [h_2(x) - h_2(\bar{x}) - h_{2x}(\bar{x})\xi]^{\mathrm{T}} R_2^{-1}(\bar{x}) + [h_2(x) - h_2(\bar{x})]^{\mathrm{T}} [R_2^{-1}(x) - R_2^{-1}(\bar{x})].$$

Substituting the following formulas in (25):

$$f(x) - f(\bar{x}) = f_x(x_1)\xi, \qquad h_i(x) - h_i(\bar{x}) = h_{ix}(x_2)\xi, \quad i = 1,2,$$

$$R_2^{-1}(x) - R_2^{-1}(\bar{x}) = \frac{\mathrm{d}}{\mathrm{d}x}(R_2^{-1})\Big|_{x = x_3} \xi, \qquad h_2(x) - h_2(\bar{x}) - h_{2x}(\bar{x})\xi = \tfrac{1}{2}\xi^{\mathrm{T}} h_{2xx}(x_4)\xi,$$

where $x_j = \bar{x} + \theta_j \xi$, $\theta_j \in [0,1]$, $j = 1, \ldots, 4$, we estimate from the above the resulting left side by applying (17). Then since

$$|c(x,\bar{x}) R_2(x) c^{\mathrm{T}}(x,\bar{x})| \leqslant \left[0.5|\xi|^2 |h_{2xx}^{\mathrm{T}}(x_1)||R_2^{-1}(x_2)| + |h_x(x_3)| \left| \frac{\mathrm{d}}{\mathrm{d}x}(R_2^{-1}) \right|\Bigg|_{x = x_4} \right]^2 |R_2(x_5)| \leqslant H_2 |\xi|^4,$$

we get, that for $|x - \bar{x}| < r$,

$$|H_G(V)| \leqslant |\xi|^2 [-2\alpha p + H_1 + \gamma^2 H_2 r^2 + p^2 \gamma^{-2} R_1' - \gamma^2 E] \leqslant 0.$$

The latter inequality is equivalent to the following one:

$$-2\alpha p + H_1 + \gamma^2 H_2 r^2 + p^2 \gamma^{-2} R_1' - \gamma^2 E \leqslant 0. \tag{26}$$

In the case of (i), for $r = r_1$ and large enough γ, the scalar quadratic inequality of (26) possesses positive solutions p, and the discriminant Δ that corresponds to (26) is positive. If we reduce the value of γ we arrive at the case where $\Delta = 0$. We shall find this minimal value of $\gamma = \gamma_1$, equating the discriminant to 0:

$$\Delta = \alpha^2 - R_1' \gamma_1^{-2} [H_1 + \gamma_1^2 (H_2 r_1^2 - E)] = 0.$$

Evidently, we find (19). Then (26) has a solution $p = \gamma_1^2 \alpha / R_1'$ that implies, together with (23), the result of (20).

In the case of (ii) for $r = r_2$ (26) has positive solutions given by (24) if $E - H_2 r_2^2 - H_1 \gamma^{-2} > 0$, i.e. if $\gamma > \gamma_2$. □

Remark 2. (1) If h_2 is linear, then $H_2 = 0$ and $r = \infty$ in Theorem 2.
(2) In the case of (ii) for $\gamma \to \gamma_2$ the high-gain phenomenon can occur.

Remark 3. In the present note we do not address the stability issue of the filter.

5. Example

Consider the following system:

$$\dot{x} = f(x) + [1,0]w, \quad y = x + \sin\tfrac{1}{2}x + [0, 0.5]w, \quad z = x - \bar{x},$$

where $x \in R$, and $w = [w_1, w_2]^{\mathrm{T}} \in \mathbb{R}^2$.

(i) Assume that $\dot{f} \leqslant -1$ for all $x \in R$. First, we apply Theorem 2. Here $\alpha = 1$, $H_1 = 1$, $E = 1$, $H_2 = \frac{1}{16}$ and $R_1' = 1$. Thus, choosing from (18) $r_1^2 = 8$, we obtain by (19) and (20) that

$$\gamma_1^2 = \tfrac{2}{3}, \qquad G = 4(1 + \tfrac{1}{2}\cos\tfrac{1}{2}\bar{x}). \tag{27}$$

Then for $\gamma_1^2 = \frac{2}{3}$ we find G and r by applying the algorithm of Section 3. We obtain

$$F(U,x) \leqslant -2U + (\gamma_1^{-2} - 1 + \delta)U^2 + 1 \leqslant 0. \tag{28}$$

As follows from the proof of Lemma 2, the greater values of r correspond to greater values of δ', and, hence, to the greater value of δ in (28). The maximal value of δ is $\delta = 0.5$ which leads to the positive solution $U = 1$ of (28). Then, by (11), we get the same expression (28) for G. Substitution of the above value of G and of $V_x = -V_{\bar{x}} = \frac{4}{3}\xi$ into (7) leads to the same value of $r^2 = 8$.

(ii) Assume that $\dot{f} \leqslant 0$ for all $x \in R$ (e.g. $f = -x^3$). First, we apply Theorem 2. Choosing from (21) $r_2^2 = 8$, we find from (22) $\gamma_2^2 = 2$. We take $\gamma = 1.5 > \gamma_2$. Then, by (24) and (23), we find that

$$G = 9(1 + \tfrac{1}{2}\cos\tfrac{1}{2}x)p^{-1}, \quad 0 < p \leqslant 0.5303.$$

For $p = 0.5$ we have $G = 18(1 + \frac{1}{2}\cos\frac{1}{2}x)$. Substitution of the latter value of G and of $V = 0.5\xi^2$ into (7) leads to $r^2 = 8.4928$. We thus get a value of r that slightly exceeds the a priori value of r_2.

Then for $\gamma = 1.5$ we find G and r by algorithm of Section 3. We obtain from (17) that

$$F(U,x) \leqslant U^2(2.25^{-1} - 1 + \delta) + 1 \leqslant 0.$$

The latter inequality has a positive solution for $\delta < 0.5556$. Thus, for $\delta = 0.2284$ we obtain $U = 4.5$ and G and r are as in the above. By increasing δ we achieve high gain, e.g. $G = 100(1 + \cos(0.5x))$ and $U = 100$, which leads to $r^2 = 8.8$ and slightly improves the value of r_2.

6. Conclusions

A new simplified method is presented for nonlinear H^∞-filtering in the domain $\bar{x} \in \Omega_1$, $|x - \bar{x}| \leqslant r$. Unlike the known results [1–3, 5], which are based on the Hamilton–Jacobi inequalities, our formula incorporates Riccati inequalities, depending on the estimated trajectory. We suggest an algorithm for obtaining a bound for maximal r. Moreover, in the infinite horizon case, we suggest some a priori simple evaluation of the bounds on the maximum value of r and on the minimum value of γ, together with a formula for the gain-matrix of the filter.

The results are simple to apply, however, the suggested algorithms lead to the bounds on the size of the domain Ω that are not maximal. To achieve a larger bound one may consider applying the method of [4] to the Hamilton–Jacobi inequality of (7), with G given by (11) and (14), and looking for a viscosity solution of (7) on Ω.

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