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H_∞-state-feedback control of linear systems with small state delay

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Abstract

In this note, a H_∞ controller for systems with state delay is presented. For a prechosen γ the controller is obtained by solving Riccati partial differential inequalities (RPDIs). For small time delays, an asymptotic approximation of the controller is achieved by expanding the solution in powers of the delay. It is shown that a higher-order accuracy controller improves the performance. An example is brought where the results of the new method provide a satisfactory solution for a delay length comparable with the system 'bandwidth'. The performance of the system under the zero-order accuracy controller, which corresponds to systems without delay, is studied. Explicit formula for the guaranteed performance level is obtained for the delay lengths that preserve the internal stability of the system.

Keywords: Time-delay systems; H_∞-control; Riccati partial differential inequalities; Asymptotic controller

1. Problem formulation

Throughout this note we denote by $|\cdot|$ the Euclidean norm of a vector, or the appropriate norm of a matrix. The space of the square integrable functions on $[0,\infty)$ with the norm $\|\cdot\|_{L_2}$ is denoted by $L_2[0,\infty)$. By $C[a,b]$ we denote the space of the continuous functions on $[a,b]$ with the norm $|\cdot|$. We also denote $x_t = x(t+\theta),\ \theta\in[-h,0]$.

We consider the system

$$\dot{x}(t) = A_1x(t) + A_2x(t-h) + Bu(t) + Dw(t), \tag{1.1a}$$

$$z(t) = \text{column}\{Cx(t), u(t)\}, \tag{1.1b}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^k$ is the control, $w(t)\in\mathbb{R}^q$ is the disturbance, $z(t)\in\mathbb{R}^p$ is the observation, and A_1, A_2, B, D and C are matrices of the appropriate dimensions. Given $\gamma>0$, and assuming that $w\in L_2[0,\infty)$, we consider the performance index

$$J = \|z\|^2_{L_2} - \gamma^2\|w\|^2_{L_2}. \tag{1.2}$$

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The H_∞-control problem for a performance level γ is to find a controller that internally stabilizes the system (1.1a) and ensures that $J \leqslant 0$ for all $w \in L_2[0,\infty)$ and for the zero initial conditions $x(\tau)=0,\ \tau \leqslant 0$. Such a controller has been obtained in [1, 6] by solving Riccati operator equations. Numerical solutions to such equations have been obtained in [5]. In [7] the delay-independent controller has been derived. The LQ optimal-control problem for a system with small time delay has been studied in [9] where asymptotic series solution to the Hamiltonian system has been constructed.

In the present note we obtain the controller by solving RPDIs that are similar to those obtained in [8] where the LQ optimal-control problem has been studied. We derive an asymptotic approximation to the solution of the RPDIs by expanding in powers of the delay. We show that the higher-order terms in the approximation lead to an improved performance. We also study the system performance under the suboptimal controller that corresponds to the system without delay. For small delays it satisfies the γ performance level. For all the delays that preserve the internal stability of the system we obtain an upper bound to the resulting performance level in terms of the coefficients of the system. We bring a numerical example that demonstrates the effectiveness of the results. The proofs of the lemmas and the theorems are given in the appendix.

2. Main results

Let $S=\gamma^{-2}DD'-BB'$, where prime denotes the transpose of a matrix. We consider the following RPDIs:

$$A_1'P+PA_1+PSP+Q'(0)+Q(0)+C'C \leqslant 0, \tag{2.1a}$$

$$\dot{Q}(\xi)=(A_1'+PS)Q(\xi)+R(0,\xi), \tag{2.1b}$$

$$\frac{\partial}{\partial\xi}R(\xi,s)+\frac{\partial}{\partial s}R(\xi,s)-Q'(\xi)SQ(s)=0, \tag{2.1c}$$

$$A_2'P=Q'(-h), \tag{2.1d}$$

$$A_2'Q(\xi)=R(-h,\xi)=R'(\xi,-h). \tag{2.1e}$$

A solution of Eqs. (2.1a)–(2.1e) is a triple of $n\times n$-matrices $\{P,Q(\xi),R(\xi,s)\}$, $\xi\in[-h,0]$, $s\in[-h,0]$, where Q and R are continuous and piecewise continuously differentiable functions of their arguments, satisfying Eqs. (2.1a)–(2.1e).

We also consider the equation

$$\dot{x}_1(t)=[A_1+SP]x_1(t)+A_2x_1(t-h)+S\int_{-h}^{0}Q(\xi)x_1(t+\xi)\,\mathrm{d}\xi. \tag{2.2}$$

We show that the controller

$$u^*(t)=-B'\left[Px(t)+\int_{-h}^{0}Q(\xi)x(t+\xi)\,\mathrm{d}\xi\right], \tag{2.3}$$

that internally stabilizes the system of (1.1a), i.e.

$$\dot{x}_2(t)=[A_1-BB'P]x_2(t)+A_2x_2(t-h)-BB'\int_{-h}^{0}Q(\xi)x_2(t+\xi)\,\mathrm{d}\xi \tag{2.4}$$

is asymptotically stable, also solves the H_∞-control problem.

Lemma 1. *Let Eqs.* (2.1a)–(2.1e) *possess a solution such that Eqs.* (2.2) *and* (2.4) *are asymptotically stable. Then* $P\geqslant 0$ *and the controller of Eq.* (2.3) *solves the* H_∞*-control problem.*

In the present note we obtain an asymptotic expansion of the solution to Eq. (2.1) in powers of h. We look for the asymptotic solution in the form:

$$P = P_0 + hP_1 + \cdots + h^m P_m + O(h^{m+1}), \tag{2.5a}$$

$$Q(h\zeta) = Q_0 + hQ_1(\zeta) + \cdots + h^m Q_m(\zeta) + O(h^{m+1}), \tag{2.5b}$$

$$R(h\zeta, h\theta) = R_0 + hR_1(\zeta,\theta) + \cdots + h^m R_m(\zeta,\theta) + O(h^{m+1}), \quad \zeta \in [-1,0],\ \theta \in [-1,0], \tag{2.5c}$$

where $|O(h^{m+1})| \leqslant ch^{m+1}$ and where the constant $c > 0$ does not depend on h, ζ and θ. We consider the following equation:

$$A_1'P + PA_1 + PSP + Q'(0) + Q(0) + C'C + \Delta(h) = 0, \tag{2.6}$$

instead of (2.1a), where the matrix-function

$$\Delta(h) = \Delta'(h) = \Delta_0 + h\Delta_1 + \cdots + h^m \Delta_m \geqslant 0$$

will be chosen later.

We substitute Eqs. (2.5a)–(2.5c) in Eq. (2.6) and Eqs. (2.1b)–(2.1e) and equate terms with the same powers of h. We notice that for $\xi = h\zeta$ and $s = h\theta$ we have $\partial/\partial\xi = h^{-1}\partial/\partial\zeta$ and $\partial/\partial s = h^{-1}\partial/\partial\theta$. Thus, for the zero-order terms we obtain from Eqs. (2.1d) and (2.1e):

$$Q_0 = P_0 A_2, \qquad R_0 = A_2' P_0 A_2.$$

Then, from Eq. (2.6), we have

$$(A_1' + A_2')P_0 + P_0(A_1 + A_2) + P_0 S P_0 + C'C + \Delta_0 = 0. \tag{2.7}$$

The latter is the well-known algebraic Riccati equation [2], that corresponds to Eqs. (1.1a) and (1.1b) for $h = 0$. We assume the following:

(A1) Given $\gamma > 0$, Eq. (2.7) has a solution $P_0 > 0$ such that the matrices $A_1 + A_2 - BB'P_0$ and $M = A_1 + A_2 + SP_0$ are Hurwitz.

Assumption A1 means that the H_∞-state-feedback control problem for Eqs. (1.1a) and (1.1b) without delay has a solution [2]. If this is not the case, even P_0 the zero-order term in Eq. (2.5a) does not exist.

For the first-order terms we find:

$$(A_1 + SP_0)'P_1 + P_1(A_1 + SP_0) + Q_1'(0) + Q_1(0) + \Delta_1 = 0, \quad \dot{Q}_1(\zeta) = M'P_0A_2, \quad A_2'P_1 = Q_1'(-1)$$

and

$$\frac{\partial}{\partial\zeta}R_1(\zeta,\theta) + \frac{\partial}{\partial\theta}R_1(\zeta,\theta) - Q_0'SQ_0 = 0, \quad Q_1'(\zeta)A_2 = R_1(\zeta,-1).$$

Thus, $Q_1 = P_1A_2 + (\zeta + 1)M'P_0A_2$, P_1 is a unique solution of the following Lyapunov equation:

$$M'P_1 + P_1M + M'P_0A_2 + A_2'P_0M + \Delta_1 = 0, \tag{2.8}$$

and R_1 is defined by formula

$$R_1(\zeta,\theta) = \begin{cases} A_2'Q_1(\theta) + (\zeta + 1)Q_0'SQ_0 & \text{if } \zeta \leqslant \theta, \\ A_2'Q_1(\zeta) + (\theta + 1)Q_0'SQ_0 & \text{if } \zeta > \theta. \end{cases}$$

The higher-order terms of the expansions can be similarly found. The matrices $\Delta_0, \Delta_1, \ldots$ can be chosen such that the resulting approximation $P_0 + hP_1 + \cdots + h^m P_m$ to P remains nonnegative for all h under consideration.

Theorem 1. *For a prechosen γ, and for all small enough h, the following holds under* A1:
(i) *The system of Eqs.* (2.1a)–(2.1e) *has a solution such that Eqs.* (2.2) *and* (2.4) *are asymptotically stable. This solution can be approximated by Eqs.* (2.5a)–(2.5c), *uniformly on ζ and θ.*
(ii) *The controller of Eq.* (2.3) *solves the H_∞-control problem. It can be approximated by*

$$u(x_t) = u_m(x_t) + O(h^{m+1}), \tag{2.9a}$$

$$u_m(x_t) = -\sum_{i=0}^{m} h^i B' \left[P_i x(t) + \int_{-1}^{0} Q_{i-1}(\zeta) x(t + h\zeta)\,d\zeta \right], \tag{2.9b}$$

where $Q_{-1} = 0$ and $|O(h^{m+1})| \leqslant c h^{m+1} |x_t|$ $(c > 0)$. The approximate controller of Eq. (2.9b) *guarantees a performance level of $\gamma + O(h^{m+1})$.*

It follows from Theorem 1 that a high-order approximate controller improves the performance polynomially in the small time-delay h. We study the performance of the system under the zero-order controller

$$u_0(t) = -B'P_0 x(t). \tag{2.10}$$

This controller solves the H_∞-control problem for Eqs. (1.1a) and (1.1b) without delay. Denote

$$A_0 = A_1 - BB'P_0. \tag{2.11}$$

Let $X(t)$ be the fundamental matrix of the system

$$\dot{x}(t) = A_0 x(t) + A_2 x(t-h), \tag{2.12}$$

i.e. $X(t) = 0$ for $t < 0$, $X(0) = I$ and $X(t)$ satisfies Eq. (2.12) for $t \geqslant 0$. Then under A1 there exist positive numbers α, β_0, β and δ such that for small enough h the following inequalities are valid:

$$|e^{(A_0 + A_2)t}| \leqslant \beta_0 e^{-\alpha t}, \tag{2.13a}$$

$$|X(t)| \leqslant \beta e^{-\delta t}. \tag{2.13b}$$

Lemma 2. *The following values of β and δ can be chosen*

$$\beta = \beta_0[1 + (e^{\alpha h} - 1)/\alpha], \qquad \delta = \alpha - \beta_0 |A_2| \cdot (|A_0| + |A_2| e^{\alpha h}) \cdot (e^{\alpha h} - 1)/\alpha, \tag{2.14}$$

such that Eq. (2.13b) *holds for all h preserving $\delta > 0$.*

Theorem 2. *For a prechosen γ the following holds under* A1:
(i) *For small enough h the controller of Eq.* (2.10) *leads to a γ performance level.*
(ii) *For all h such that Eq.* (2.13b) *holds the controller of Eq.* (2.10) *guarantees a performance level $\bar{\gamma}$, where*

$$\bar{\gamma}^2 = 2h^2 \beta^4 |C|^2 |A_2|^2 |D|^2 / \delta^2 + \gamma^2. \tag{2.15}$$

It follows from Theorem 2 that the controller of Eq. (2.10) guarantees γ performance level for all small time delays and it guarantees a performance level $\bar{\gamma}$ for all delays that preserve the internal stability of the system. Note that $\bar{\gamma} \to \gamma$ for $h \to 0$.

Example 1. Consider the system of Eqs. (1.1a) and (1.1b), with

$$A_1 = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad C = [1 \;\; -0.5], \quad B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad D = \begin{pmatrix} -0.5 \\ 1 \end{pmatrix}.$$

Table 1

h	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
γ_1	0.4215	0.4083	0.3983	0.3953	0.3959	0.3973	0.4012	0.4171	0.4330	0.4368

Assumption A1 holds for $\gamma > 0.411$. Choosing $\gamma = 0.45$, we find from Eqs. (2.7) and (2.8), using $\Delta_0 = 0$ and $\Delta_1 = 0.5DD'$, that

$$P_0 = \begin{pmatrix} 1.2935 & -0.5172 \\ -0.5172 & 0.3964 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 2.955 & -1.454 \\ -1.454 & 0.8255 \end{pmatrix}.$$

We consider the zero-order controller of Eq. (2.10) and the first-order controller

$$u_1(t) = -B'\left[(P_0 + hP_1)x(t) + hP_0A_2 \int_{-1}^{0} x(t+h\theta)\,d\theta\right].$$

Applying these controllers on Eqs. (1.1a) and (1.1b), we obtain closed-loop systems with the following transfer function matrices:

$$T_i(s) = \begin{pmatrix} C \\ U_i \end{pmatrix}(sI - A_1 - A_2e^{-hs} - BU_i)^{-1}D, \quad i = 0, 1,$$

where $U_0 = -B'P_0$ and $U_1 = -B'[P_0 + hP_1 + P_0A_2(1 - e^{-hs})s^{-1}]$. We compute the H_∞-norms of the two closed-loop systems by finding the peak values γ_i of the maximum singular values of the resulting $T_i(j\phi)$, $\phi \in R$, $i = 0, 1$. For $h = 0.1, 0.2, \ldots, 1$ the resulting γ_0 is 0.4436 independently of h, while the values of γ_1 depend on h. The latter are given in the Table 1. We find from Table 1 that for all h under consideration $\gamma_1 < \gamma_0$ and u_1 thus improves the performance.

Applying Theorem 2, we obtain a bound on the performance level of the system under the controller of Eq. (2.10). We have $\alpha = 1.3682$ and $\beta_0 = 3.6691$. Choosing $h = 0.01$ we define from Eqs. (2.14) and (2.15) $\beta = 3.7062$, $\delta = 0.6863$ and $\bar{\gamma} = 0.7281$. We see that $\bar{\gamma}$ is not too close to γ but it is easy to compute.

3. Conclusions

A H_∞ controller for a state-delay linear system has been presented. This controller can be found from the RPDIs. For small enough delays an approximate solution to the RPDIs has been constructed by an expansion in powers of the delay. It has been proved, and also illustrated by a numerical example, that the higher-order accuracy controller improves the performance. The system performance under the zero-order controller has been studied. Explicit simple formula for an upper bound on this performance has been obtained for all the delays that preserve the internal stability of the system. The results can be easily generalized to the case of the finite number of discrete delays and distributed delay.

Appendix

Proof of Lemma 1. We prove the lemma by applying the arguments of [8].

Choose

$$V(x_t) = x(t)'Px(t) + 2x' \int_{-h}^{0} Q(\xi)x(t+\xi)\,d\xi + \int_{-h}^{0}\int_{-h}^{0} x'(t+s)R(s,\xi)x(t+\xi)\,ds\,d\xi, \tag{A.1}$$

where $x_t = x(t+\xi)$, $\xi \in [-h, 0]$ and where the matrices P, Q and R satisfy Eqs. (2.1a)–(2.1e).

Let $x(t)$ be a solution of Eq. (1.1a). Then, differentiating $V(x_t)$ with respect to t, and integrating by parts, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_t) = 2[A_1x(t) + A_2x(t-h) + Bu + Dw]'\left[Px(t) + \int_{-h}^{0} Q(\xi)x(t+\xi)\,\mathrm{d}\xi\right]$$
$$+ 2x'(t)Q(\xi)x(t+\xi)|_{-h}^{0} - 2x'\int_{-h}^{0}\dot{Q}(\xi)x(t+\xi)\,\mathrm{d}\xi$$
$$+ 2\int_{-h}^{0} x'(t+s)[R(s,\xi)x(t+\xi)|_{-h}^{0}]\,\mathrm{d}s$$
$$- \int_{-h}^{0}\int_{-h}^{0} x'(t+s)\left[\frac{\partial}{\partial\xi}R(\xi,s) + \frac{\partial}{\partial s}R(\xi,s)\right]x(t+\xi)\,\mathrm{d}\xi\,\mathrm{d}s. \tag{A.2}$$

Denoting

$$w^* = \gamma^{-2}B'\left[Px(t) + \int_{-h}^{0} Q(\xi)x(t+\xi)\,\mathrm{d}\xi\right].$$

We obtain from Eqs. (A.2) and (2.1a)–(2.1e)

$$\frac{\mathrm{d}V}{\mathrm{d}t} = x'(t)(A_1'P + PA_1 + Q(0) + Q'(0))x(t) + 2\gamma^2w'w^* + 2u'u^*$$
$$+ 2x'(t)\int_{-h}^{0}[A_1'Q(\xi) - \dot{Q} + R(0,\xi)]x(t+\xi)\,\mathrm{d}\xi$$
$$- \int_{-h}^{0} x'(t+s)\left[\frac{\partial}{\partial\xi}R(\xi,s) + \frac{\partial}{\partial s}R(\xi,s)\right]x(t+\xi)\,\mathrm{d}\xi\,\mathrm{d}s$$
$$\leqslant x'(t)(-C'C - PSP)x(t) + 2\gamma^2w'w^* - 2u'u^* - 2x'(t)\int_{-h}^{0}PSQ(\xi)x(t+\xi)\,\mathrm{d}\xi$$
$$- \int_{-h}^{0}\int_{-h}^{0} x'(t+s)Q'(\xi)SQ(s)x(t+\xi)\,\mathrm{d}\xi\,\mathrm{d}s$$
$$= -x'(t)C'Cx(t) - \gamma^2(w-w^*)'(w-w^*) + \gamma^2w'w + (u-u^*)'(u-u^*) - u'u. \tag{A.3}$$

We first show that the asymptotic stability of Eq. (2.4) implies that $V(x_0)\geqslant 0$ for all $x_0 \in C[-h,0]$. Let $x_2(t)$ be a solution of Eq. (2.4) with an initial function $x_0 \in C[-h,0]$. Then, setting $w=0$ and $u=u^*$ in Eq. (A.3), we obtain

$$\frac{\mathrm{d}V(x_{2t})}{\mathrm{d}t} \leqslant 0.$$

Integrating the latter inequality on t from 0 to ∞ we obtain that $V(x_0)\geqslant 0$ for all $x_0 \in C[-h,0]$ since $x(\infty)=0$. Hence, $P\geqslant 0$.

Let $x(t)$ be a solution of Eq. (1.1a) with $x_0 = 0$. Then, from Eq. (A.3), it follows that for all $w \in L_2[0,\infty)$

$$\int_0^{\infty}\left(\frac{\mathrm{d}V}{\mathrm{d}t} - \gamma^2w'w + z'z\right)\mathrm{d}t \leqslant 0.$$

The latter inequality implies that $J\leqslant 0$, since $V\geqslant 0$ and $V(0)=0$. □

Remarks. Note that equality in Eqs. (2.1a)–(2.1e) can be obtained from the integral kernel representation of the solution to the Riccati operator equation of [1, 6].

Proof of Theorem 1. (i) To prove the validity of Eqs. (2.5a)–(2.5c) we consider the equations for the remainders

$$h^{m+1}P_{m+1}=P-\sum_{i=0}^{m}h^iP_i,\qquad h^{m+1}Q_{m+1}(\zeta)=Q(h\zeta)-\sum_{i=0}^{m}h^iQ_i(\zeta),$$

$$h^{m+1}R_{m+1}(\zeta,\theta)=R(h\zeta,h\theta)-\sum_{i=0}^{m}h^iQ_i(h\zeta,h\theta)$$

in these expansions:

$$P_{m+1}M+M'P_{m+1}+Q'_{m+1}(0)-A'_2P_{m+1}+Q_{m+1}(0)-P_{m+1}A_2+E_m(h,hP_{m+1})=0, \tag{A.4a}$$

$$\dot{Q}_{m+1}(\zeta)=G_m(h,\zeta,hP_{m+1},hQ_{m-1}(\zeta))+hR_{m+1}(0,\zeta), \tag{A.4b}$$

$$\frac{\partial}{\partial\zeta}R_{m+1}(\zeta,\theta)+\frac{\partial}{\partial\zeta}R_{m+1}(\zeta,\theta)+K_m(h,\zeta,\theta,hQ_{m+1}(\zeta),hQ_{m+1}(\theta))=0, \tag{A.4c}$$

$$A'_2P_{m+1}=Q'_{m+1}(-1), \tag{A.4d}$$

$$A'_2Q_{m+1}(\zeta)=R_{m+1}(-1,\zeta). \tag{A.4e}$$

Note that P_{m+1}, Q_{m+1} and R_{m+1} depend on h. The known matrix functions E_m, G_m and K_m are continuous on h,ζ,θ and contain linear and quadratic terms in hP_{m+1} and hQ_{m+1} terms.

The system of Eqs. (A.4a)–(A.4e) imply the following integral system for P_{m+1} and Q_{m+1} determination:

$$P_{m+1}=\int_0^\infty e^{M's}[Q'_{m+1}(0)-A'_2P_{m+1}+Q_{m+1}(0)-P_{m+1}A_2+E_m(h,hP_{m+1})]e^{Ms}\,ds, \tag{A.5a}$$

$$Q_{m+1}(\zeta)=P_{m+1}A_2+\int_0^\zeta[G_m(h,s,hP_{m+1},hQ_{m+1}(s))+hR_{m+1}(0,s)]\,ds, \tag{A.5b}$$

$$R_{m+1}(0,s)=Q'_{m+1}(-s-1)A_2+\int_0^{s+1}K_m(h,t-s-1,t-1,hQ_{m+1}(t-s-1),hQ_{m+1}(t-1))\,dt. \tag{A.5c}$$

Substituting Eq. (A.5c) in Eq. (A.5b) and applying the contraction principle argument to the resulting system of Eqs. (A.5a) and (A.5b), one can show that for all small enough h the latter system has a unique solution P_{m+1},Q_{m+1} continuously depending on h and ζ. Notice that Q_{m+1} is continuously differentiable since the right-hand side of Eq. (A.5b) has a continuous derivative with respect to ζ. Then, Eqs. (A.4c) and (A.4e) have a unique continuous in h, ζ and θ solution given by

$$R_{m+1}(\zeta,\theta)=\begin{cases}A'_2Q_{m+1}(\theta-\zeta-1)+\int_{-1}^{\zeta}f(\theta,t+\theta-\zeta)\,dt & \text{if } \zeta\leqslant\theta,\\ A'_2Q_{m+1}(\zeta)+\int_{-1}^{\theta}f(t-\theta+\zeta,t)\,dt & \text{if } \zeta\geqslant\theta,\end{cases}$$

where $f(\zeta,\theta)=K_m(h,\zeta,\theta,hQ_{m+1}(\zeta),hQ_{m+1}(\theta))$. Hence, for all small enough h Eqs. (2.1a)–(2.1e) has a solution and the approximation (2.5a)–(2.5c) is uniform on ζ,θ.

Under A1, Eqs. (2.2) and (2.4), where $h=0$, are asymptotically stable. Then, for small h, these equations are asymptotically stable. This completes the proof of (i).

The approximation of Eqs. (2.9a) and (2.9b) follows from Eqs. (2.3) and (2.5a)–(2.5e). The rest of the proof is similar to [3]. We apply Eq. (2.3) on Eqs. (1.1a) and (1.1b):

$$\dot{x}(t)=(A_1-BB'P_0)x(t)+A_2x(t-h)+B[u(x_t)-u_0(x_t)]+Dw(t). \tag{A.6}$$

Similarly, substituting u_m for u in Eqs. (1.1a) and (1.1b), we obtain

$$\dot{x}_m(t)=(A_1-BB'P_0)x_m(t)+A_2x_m(t-h)+B[u_m(x_t)-u_0(x_t)]+Dw(t), \tag{A.7a}$$

$$z_m(t)=\text{column}\{Cx_m(t),u_m(x_{mt})\}. \tag{A.7b}$$

Thus, $y_t=x_t-x_{mt}$ satisfies

$$\dot{y}(t)=(A_1-BB'P_0)y(t)+A_2y(t-h)+B[u(x_t)-u_m(x_t)]. \tag{A.8}$$

We find that

$$\|z_m-z\|_{L_2}^2\leqslant\int_0^\infty c_1[\|y_t\|^2+h^{m+1}\|x_t\|^2]\,\mathrm{d}t\leqslant c_1[\|y\|_{L_2}\|x_t\|_{L_2}+h^{m+1}\|x_t\|_{L_2}^2],\quad c_1>0, \tag{A.9}$$

where $\|x_t\|_{L_2}=\max_{\theta\in[-h,0]}\|x_t(\theta)\|_{L_2}$. Evidently, $\|x(t)\|_{L_2}\leqslant\|x_t\|_{L_2}$. Under A1 the matrix $A_1+A_2-BB'P_0$ is Hurwitz. Thus, for all small enough h (2.13b) holds. By the variation of constants formula [4] Eq. (A.6) is equivalent to the following integral equation:

$$x_t=\int_0^t X_{t-s}\{B[u(x_s)-u_0(x_s)]+Dw(s)\}\,\mathrm{d}s. \tag{A.10}$$

From the latter equation and Eq. (A.10) we obtain

$$\|x_t\|_{L_2}^2\leqslant\int_0^\infty\int_0^t\int_0^t c'\mathrm{e}^{-\delta(2t-r-p)}[|w(p)|+h\|x_p\|][|w(r)|+h\|x_r\|]\,\mathrm{d}r\,\mathrm{d}p\,\mathrm{d}t, \tag{A.11}$$

where $c'>0$. Further, estimating from above the product of the square brackets by $|w(p)|^2+|w(r)|^2+h^2[\|x_p\|^2+\|x_r\|^2]$ and reversing the order of integration we conclude that

$$\begin{aligned}\|x_t\|_{L_2}^2&\leqslant 2c'\int_0^\infty\int_p^\infty\int_0^t \mathrm{e}^{-\delta(2t-r-p)}\,\mathrm{d}r\,\mathrm{d}t\cdot[\|w(p)\|_{L_2}^2+h^2\|x_p\|_{L_2}^2]\,\mathrm{d}p\\&\leqslant 2c'/\delta^2[\|w\|_{L_2}^2+h^2\|x_t\|_{L_2}^2].\end{aligned} \tag{A.12}$$

Therefore, for small h we have $\|x_t\|_{L_2}^2\leqslant c\|w\|_{L_2}^2$, $c>0$. Similarly, one can derive $\|x_{mt}\|_{L_2}^2\leqslant c\|w\|_{L_2}^2$, and $\|y_t\|_{L_2}^2\leqslant ch^{2m+2}\|x_t\|_{L_2}^2\leqslant c^2h^{2m+2}\|w\|_{L_2}^2$. The latter inequalities, together with Eq. (A.9), imply $\|z_m\|_{L_2}=\|z\|_{L_2}+\mathrm{O}(h^{m+1})\|w\|_{L_2}^2$. Since $\|z\|_{L_2}^2\leqslant\gamma^2\|w\|_{L_2}^2$, we derive $\|z_m\|_{L_2}^2\leqslant[\gamma^2+\mathrm{O}(h^{m+1})]\|w\|_{L_2}^2=[\gamma+\mathrm{O}(h^{m+1})]^2\|w\|_{L_2}^2$. □

Proof of Lemma 2. Since Eq. (2.12) can be written in the form

$$\dot{x}(t)=(A_0+A_2)x(t)+A_2[x(t-h)-x(t)], \tag{A.13}$$

the matrix $X(t)$ satisfies the integral equation

$$X(t)=\mathrm{e}^{(A_0+A_2)t}+\int_0^t \mathrm{e}^{(A_0+A_2)(t-s)}A_2[X(s-h)-X(s)]\,\mathrm{d}s,\quad t\geqslant 0. \tag{A.14}$$

From Eqs. (A.14) and (2.13a) and the relation

$$X(s-h)-X(s)=\begin{cases}\int_s^{s-h}[A_0X(\tau)+A_2X(\tau-h)]\,\mathrm{d}\tau & \text{if } s\geqslant h,\\ \int_s^{s-h}[A_0X(\tau)+A_2X(\tau-h)]\,\mathrm{d}\tau+I & \text{if } 0\leqslant s<h,\end{cases}$$

we obtain

$$\begin{aligned}|X(t)| &\leqslant \beta_0 e^{-\alpha t} + \beta_0 \int_0^h e^{-\alpha(t-s)}\,ds + \int_0^t \beta_0 e^{-\alpha(t-s)}|A_2| \int_s^{s-h} [|A_0|\,|X(\tau)| + |A_2|\,|X(\tau-h)|]\,d\tau\,ds \\ &\leqslant \beta_0 e^{-\alpha t}[1 + (e^{\alpha h}-1)/\alpha] + \int_0^t \beta_0 |A_2| [|A_0|\,|X(\tau)| + |A_2|\,|X(\tau-h)|] \int_\tau^{\tau+h} e^{-\alpha(t-s)}\,ds\,d\tau \\ &= \beta_0 e^{-\alpha t}[1 + (e^{\alpha h}-1)/\alpha] + \beta_0 |A_2| (e^{\alpha h}-1)/\alpha \cdot \int_0^t e^{-\alpha(t-\tau)}[|A_0|\,|X(\tau)| + |A_2|\,|X(\tau-h)|]\,d\tau \\ &\leqslant \beta_0 e^{-\alpha t}[1 + (e^{\alpha h}-1)/\alpha] + \beta_0 |A_2| (e^{\alpha h}-1)/\alpha \cdot [|A_0| + |A_2| e^{\alpha h}] \int_0^t e^{-\alpha(t-\tau)}|X(\tau)|\,d\tau. \qquad \text{(A.15)}\end{aligned}$$

Eq. (A.15) and Gronwall inequality yield Eqs. (2.13b) and (2.14). □

Proof of Theorem 2. (i) Applying Eq. (2.10) to (2.1e), we obtain a system that has an L_2-gain less or equal to γ if the corresponding RPDI of Eqs. (2.1a)–(2.1e), where $A_1 = A_0$, $C = \text{column}\{C, -B'P_0\}$, $S = DD'/\gamma^2$, has a solution such that the corresponding Eqs. (2.2) and (2.4) are asymptotically stable. The existence of such a solution can be proved similar to (i) of Theorem 1.

(ii) Let $x(t)$ be a solution of Eqs. (1.1a), and (2.10) with $h>0$ such that Eqs. (2.13a) and (2.13b) holds and let $y(t)$ be a solution of Eqs. (1.1a) and (2.10) with $h = 0$. Then, $v(t) = x(t) - y(t)$ satisfies the following equation:

$$\dot v(t) = A_0 v(t) + A_2 v(t-h) + A_2[y(t-h) - y(t)], \quad v_0 = 0, \qquad \text{(A.16)}$$

where

$$y(t) - y(t-h) = \int_{t-h}^t e^{(A_0+A_2)(t-s)} D w(s)\,ds. \qquad \text{(A.17)}$$

From Eqs. (A.17) and (2.13b) it follows that

$$|y(t) - y(t-h)| \leqslant \beta |D| \int_{t-h}^t |w(s)|\,ds. \qquad \text{(A.18)}$$

By the variation of constants formula, Eq. (A.16) is equivalent to the integral equation:

$$v(t) = \int_0^t X(t-s) A_2 [y(s-h) - y(s)]\,ds. \qquad \text{(A.19)}$$

From Eqs. (A.18), (A.19) and (2.13b) we obtain

$$\begin{aligned}\|v\|_{L_2}^2 &\leqslant \beta^4 |A_2|^2 |D|^2 \int_0^\infty \int_0^t e^{-\delta(t-s)} \int_{s-h}^s |w(\tau)|\,d\tau\,ds \cdot \int_0^t e^{-\delta(t-p)} \int_{p-h}^p |w(r)|\,dr\,dp\,dt \\ &\leqslant h^2 \beta^4 |A_2|^2 |D|^2 \int_0^\infty \int_0^t e^{-\delta(t-\tau)}|w(\tau)|\,d\tau \cdot \int_0^t e^{-\delta(t-p)}|w(p)|\,dp\,dt \\ &= h^2 \beta^4 |A_2|^2 |D|^2 \int_0^\infty dt \int_0^t dp \int_0^t e^{-\delta(2t-\tau-p)}|w(\tau)|\,|w(p)|\,d\tau \\ &\leqslant 2h^2 \beta^4 |A_2|^2 |D|^2 \int_0^\infty \int_p^\infty \int_0^t e^{-\delta(2t-\tau-p)}\,d\tau\,dt |w(p)|^2\,dp \\ &\leqslant 2h^2 \beta^4 |A_2|^2 |D|^2/\delta^2 \|w\|_{L_2}^2.\end{aligned}$$

Hence,

$$\|z\|_{L_2}^2 \leqslant \|Cv\|_{L_2}^2 + \|Cy\|_{L_2}^2 \leqslant \gamma'^2 \|w\|_{L_2}^2. \qquad \square$$

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