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H_∞-norm and invariant manifolds of systems with state delays

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Abstract

The problem of finding bounds on the H_∞-norm of systems with a finite number of point delays and distributed delay is considered. Sufficient conditions for the system to possess an H_∞-norm which is less or equal to a prescribed bound are obtained in terms of Riccati partial differential equations (RPDE's). We show that the existence of a solution to the RPDE's is equivalent to the existence of a stable manifold of the associated Hamiltonian system. For small delays the existence of the stable manifold is equivalent to the existence of a stable manifold of the ordinary differential equations that govern the flow on the slow manifold of the Hamiltonian system. This leads to an algebraic, finite-dimensional, criterion for systems with small delays.

Keywords: Time-delay systems; Linear H_∞-control; Invariant manifolds; Small delay; Singular perturbations

1. Problem formulation

Throughout this paper we denote by $|\cdot|$ the Euclidean norm of a vector or the appropriate norm of a matrix. Let $L_2[0,\infty)$ be the space of the square integrable functions with the norm $\|\cdot\|_{L_2}$ and let $C[a,b]$ be the space of the continuous functions on $[a,b]$ with the norm $\|\cdot\|_c$.

Consider the system

$$\dot{x}(t) = L(x_t(\cdot)) + Bw(t), \tag{1.1a}$$

$$z(t) = Cx(t), \tag{1.1b}$$

where $x(t)\in\mathbf{R}^n$ is the state vector, $x_t = x(t+\theta)$, $\theta\in[-h,0]$, $w(t)\in\mathbf{R}^k$ is the disturbance, $z(t)\in\mathbf{R}^p$ is the observation, B and C are matrices of the appropriate dimensions. The R^n-valued function $L(\cdot)$ which carries R^n-valued functions on $[-h,0]$ into R^n is defined as follows:

$$L(\phi(\cdot)) = \sum_{i=0}^{k} A_i\phi(-h_i) + \int_{-h}^{0} A_{01}(s)\phi(s)\,\mathrm{d}s, \tag{1.2}$$

where $-h=-h_k<-h_{k-1}<\cdots<-h_1<-h_0=0, A_0, A_1,\ldots,A_k$ are constant matrices, A_{01} is a square integrable matrix function. Given $\gamma>0$, and assuming that $w\in L_2[0,\infty)$ we consider the following performance index

$$J = \|z\|_{L_2}^2 - \gamma^2\|w\|_{L_2}^2. \tag{1.3}$$

We assume that

A1. The system (1.1) is internally stable.

The problem is to find conditions which will ensure that $J\leqslant 0$ for all $w\in L_2[0,\infty)$ and for the zero initial conditions $x(\tau)=0,\ \tau\leqslant 0$. This means that the H_∞ norm of Eqs. (1.1a) and (1.1b) which is defined by the supremum over $w\in L_2[0,\infty)$ of the ratio

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between $\|z\|_{L_2}$ and $\|w\|_{L_2}$ is not greater than γ. Such a problem arises e.g. when a controller $u(t) = Kx(t) + \int_{-h}^{0} Q(s)x(t+s)\,ds$ (see e.g. [8]) is seen for the following system with point delays

$$\dot{x}(t) = L(x_t(\cdot)) + B_1 w(t) + B_2 u(t),$$

where it is required that the H_∞-norm level of the transference T_{zw} from w to the controlled output z, given by (1.1b), will not be greater than γ. Such conditions have been obtained in terms of Riccati operator equations in [5]. In [7] the delay-independent conditions have been derived, while in [10] both delay-dependent and delay-independent conditions have been introduced in terms of linear matrix inequalities. In [13] and [6] such conditions have been obtained for systems with delay in the input only.

In the present paper, we derive sufficient conditions in terms of RPDE's that are similar to those obtained in [14] where the LQ optimal control problem has been studied. Similar conditions in terms of inequalities have been obtained in [3] for the case of one point delay. We show that the solvability of the RPDE's is equivalent to the existence of a special stable manifold of the associated Hamiltonian system. This fact is a time-delay counterpart of the analogous result of [1]. Finally, we prove that for small delays the existence of the special stable manifold is equivalent to the existence of a stable manifold of the ordinary differential equations that govern the flow on the slow manifold of the Hamiltonian system. For systems with small delays we obtain a finite dimensional criterion in terms of matrix transcendental equations. We present a numerical example showing the efficiency of our method.

2. Main results

Let $S = \gamma^{-2}BB'$, where prime denotes the transpose of the matrix. Consider the following RPDE's with respect to the $n \times n$-matrices $P, Q(\xi)$ and $R(\xi, s)$:

$$A_0'P + PA_0 + \sum_{i=1}^{k} A_i'Q'(-h_i) + \sum_{i=1}^{k} Q(-h_i)A_i + PSP + C'C + \int_{-h}^{0} Q(\theta)A_{01}(\theta)\,d\theta + \int_{-h}^{0} A_{01}'(\theta)Q'(\theta)\,d\theta = 0, \tag{2.1a}$$

$$\dot{Q}(\xi) = -(A_0' + PS)Q(\xi) - \sum_{i=1}^{k} A_i'R(-h_i, \xi) - \int_{-h}^{0} A_{01}'(s)R(s, \xi)\,ds, \tag{2.1b}$$

$$\frac{\partial}{\partial \xi}R(\xi, s) + \frac{\partial}{\partial s}R(\xi, s) = -Q'(\xi)SQ(s), \tag{2.1c}$$

$$P = Q'(0), \tag{2.1d}$$

$$Q(\xi) = R(0, \xi). \tag{2.1e}$$

A solution of Eqs. (2.1a)–(2.1e) is a triple of $n \times n$-matrices $\{P, Q(\xi), R(\xi, s)\}$ $\xi \in [-h, 0], s \in [-h, 0]$, where $Q(\xi)$ and $R(\xi, s)$ are continuous and piecewise continuously differentiable functions of their arguments, satisfying Eqs. (2.1a)–(2.1e) for almost every ξ and s.

Consider the associated Hamiltonian system:

$$\dot{x}(t) = L(x_t(\cdot)) + Sy(t), \tag{2.2a}$$

$$\dot{y}(t) = -C'Cx(t) - L'(y^t(\cdot)), \tag{2.2b}$$

where $y^t = y(t + \zeta)$, $\zeta \in [0, h]$ and for $\psi : [0, h] \to R^n$

$$L'(\psi(\cdot)) = \sum_{i=0}^{k} A_i'\psi(h_i) + \int_{-h}^{0} A_{01}'(s)\psi(-s)\,ds.$$

Notice that Eq. (2.2b) depends on the future values of the adjoint vector y (similar to the case of the state delay LQ problem [7]). A solution of Eqs. (2.2a) and (2.2b) on the segment $[0, T]$ $(T > 0)$ is a pair of continuous functions $x : [-h, T] \to R^n$ and $y : [0, T + h] \to R^n$, that is Lipschitz continuous and satisfies Eqs. (2.2a) and (2.2b) on $[0, T]$. Denote by

$$F\phi(\xi) = \sum_{i=1}^{k} A_i\phi(-h_i - \xi)\chi_i(\xi) + \int_{-h}^{\xi} A_{01}(p)\phi(p - \xi)\,dp, \tag{2.3}$$

where $\phi \in L_2[-h, 0]$ and χ_i is the indicator function for the set $[-h_i, 0]$, i.e. $\chi_i(\xi) = 1$ if $\xi \in [-h_i, 0]$ and $\chi_i(\xi) = 0$ otherwise. We look for an invariant manifold of Eqs. (2.2a) and (2.2b) of the form

$$y(t - \xi) = Q'(\xi)x(t) + \int_{-h}^{0} R(\xi, \tau)Fx_t(\tau)\,d\tau, \quad \xi \in [-h, 0]. \tag{2.4a}$$

Setting $\xi=0$ in Eq. (2.4a) we get from Eqs. (2.1d) and (2.1e)

$$y(t)=Px(t)+\int_{-h}^{0} Q(\tau)Fx_t(\tau)\,\mathrm{d}\tau. \tag{2.4b}$$

The flow on this manifold is governed by the time-delay equation that results from substituting Eq. (2.4b) in Eq. (2.2a)

$$\dot{x}(t)=L(x_t(\cdot))+SPx(t)+S\int_{-h}^{0} Q(\tau)Fx_t(\tau)\,\mathrm{d}\tau. \tag{2.5}$$

Our results are stated in the following three theorems. The proofs of these theorems are given in the appendix. The sufficient conditions for the H_∞-norm of Eqs. (1.1a) and (1.1b) to be less than or equal to γ are:

Theorem 1. *Assume that* A1 *is valid. Let Eqs.* (2.1a)–(2.1e) *have a solution such that* (2.5) *is asymptotically stable. Then,* $J\leqslant 0$ *for all* $w\in L_2[0,\infty)$ *and* $P\geqslant 0$.

We show next that the solvability of the RPDE's (2.1a–e) is equivalent to the existence of the invariant manifold of Eq. (2.4a) to the Hamiltonian system of Eqs. (2.2a) and (2.2b).

Theorem 2. *The system of Eqs.* (2.1a)–(2.1e) *has a solution iff the Hamiltonian system* (2.2) *has an invariant manifold of the form* (2.4*a*) *such that Eq.* (2.4b) *holds, where* $Q(\xi)$ *and* $R(\xi,s)$ *are piecewise-continuously differentiable functions.*

For systems with small time-delay the existence of the stable manifold (i.e. of the invariant manifold with asymptotically stable flow) of the Hamiltonian system is equivalent to the existence of a stable manifold of the ordinary differential equations that govern the flow on the slow manifold of the Hamiltonian system. Note that Eq. (2.5) is a usual time-delay system (that does not depend on future values of x) and the notion of asymptotic stability for such systems is well known (see e.g. [8]). We derive the following algebraic delay-dependent criterion:

Theorem 3. *Consider the following nonlinear algebraic system*:

$$Z=SY+\sum_{i=0}^{k} A_i\,\mathrm{e}^{-Zh_i}+\int_{-h}^{0} A_{01}(s)\,\mathrm{e}^{Zs}\,\mathrm{d}s, \tag{2.6a}$$

$$YZ+C'C+\sum_{i=0}^{k} A_i'Y\mathrm{e}^{Zh_i}+\int_{-h}^{0} A_{01}'(s)Y\mathrm{e}^{-Zs}\,\mathrm{d}s=0, \tag{2.6b}$$

where Z *and* Y *are* $n\times n$*-matrices. There exists* $h_1>0$ *such that for all* $h\in(0,h_1)$

(i) *Eqs.* (2.1a)–(2.1e) *has a solution and Eq.* (2.5) *is asymptotically stable iff Eqs.* (2.6a) and (2.6b) *has a solution such that the matrix* Z *is Hurwitz*;
(ii) *the system* (1.1) *has an* H_∞*-norm less than or equal to* γ *if the system of Eqs.* (2.6a) and (2.6b) *has a solution such that the matrix* Z *is Hurwitz.*

System (2.6) is linear in Y and transcendental in Z. Note that transcendental equations criteria are known for the stability analysis of time-delay systems (see e.g. [8]) and for solvability of H_∞ operator Riccati equations in the case of the systems with delay in the input only [6]. Setting $h=0$ in Eqs. (1.1a), (1.1b), (2.6a) and (2.6b) we get

$$\dot{x}(t)=\left(\sum_{i=0}^{k} A_i\right)x(t)+Bw(t), \tag{2.7a}$$

$$Z=SY+\sum_{i=0}^{k} A_i, \tag{2.7b}$$

$$Y\left(\sum_{i=0}^{k} A_i+SY\right)+C'C+\sum_{i=0}^{k} A_i'Y=0. \tag{2.7c}$$

We see that Eq. (2.7c) is the usual Riccati algebraic equation (RAE) (see e.g. [1]) corresponding to the system of Eqs. (2.7a) and (1.1b) without delay. If the RAE (2.7c) has a solution with Hurwitz Z, then by the implicit function theorem, Eqs. (2.6a) and (2.6b) has a solution with Hurwitz Z for all small enough h. If $\sum_{i=0}^{k} A_i$ is Hurwitz then A1 is valid for all small enough h (see e.g. [4]). Thus, we get

Corollary 1. *If* $\sum_{i=0}^{k} A_i$ *is Hurwitz and RAE* (2.7*c*) *has a stabilizing solution*, *then for all small enough*

Table 1

h	0.1	0.2	0.3	0.4	0.5	0.6
γ_0	0.455	0.417	0.385	0.358	0.3334	0.313
γ^*	0.4545	0.4167	0.3846	0.3571	0.3333	0.3125

$h \geqslant 0$ Eqs. (1.1a) and (1.1b) *has H_∞-norm less than or equal to γ.*

Example 1. Consider the system

$$\dot{x}(t) = -0.7x(t) - 0.3x(t-h) - \int_{-h}^{0} x(t+\theta)\,\mathrm{d}\theta + 0.5w(t), \quad z(t) = x(t).$$

For values of h given in Table 1 we verify that the system is internally stable. Applying Theorem 3 we solve Eqs. (2.6a) and (2.6b) (by using FSOLVE function of Matlab) and obtain the minimum achievable values γ_0 of γ (see Table 1). The initial approximation to γ, Y and Z in the case of $h = 0.1$ is obtained by solving Eqs. (2.6a) and (2.6b) for $h = 0$, namely by solving Eqs. (2.7c) and (2.7b). Once this initial approximation is found Eqs. (2.6a) and (2.6b) is solved by FSOLVE and a solution for $h = 0.1$ is obtained. If the resulting $Z < 0$ ($Z \geqslant 0$), then we decrease (increase) γ and solve Eqs. (2.6a) and (2.6b), using for the initial approximation the values of Z and Y that have been obtained. Increasing h we use γ, Z and Y for the previous h. Thus, for $h = 0.5$ choosing $\gamma = 0.3334$ we found $Z = -0.0507 < 0$ and $Y = 0.6507$ and hence the conditions of Theorem 3 hold. Decreasing γ and choosing $\gamma = 0.3332$ we get $Z = 3 \times 10^{-8}$ and $Y = 0.6661$. Since for $\gamma = 0.3332$ $Z > 0$, we take $\gamma_0 = 0.3334$.

To compare the obtained values of γ_0 with the actual H_∞-norm of the system, we have found the peak values γ^* (see Table 1) of the frequency response of the transfer function $T_{zw}(s) = 0.5s[s^2 + 0.7s + 0.3s\exp(-hs) + 1 - \exp(-hs)]^{-1}$. We see that γ_0 is close to γ^*.

Applying now Corollary 1 we put $h = 0$ in Eqs. (2.6a) and (2.6b) and get $\gamma_0 = 0.5$. As it is evident from Table 1, this value is not valid even for $h = 0.1$. For $h > 0$ the delay-dependent criterion of Theorem 3 should be used. Note that the delay-independent criteria of [7, 10], which are valid for all h, cannot be used for γ close to γ^*, since γ^* depends on h.

Appendix

Theorem 1 is proved similarly to Lemma 1 of [3] by choosing

$$V(x_t) = x(t)'Px(t) + 2x'(t)\int_{-h}^{0} Q(\xi)Fx_t(\xi)\,\mathrm{d}\xi + \int_{-h}^{0}\int_{-h}^{0} F'x_t(s)R(s,\xi)Fx_t(\xi)\,\mathrm{d}s\,\mathrm{d}\xi. \quad \text{(A.1)}$$

Proof of Theorem 2 (*Sufficiency*)**.** Let the system (2.2) possess the invariant manifold of Eqs. (2.4a) and (2.4b). We shall show that $P, Q(\xi)$ and $R(\xi, s)$ satisfy Eqs. (2.1a)–(2.1e). We differentiate both sides of Eq. (2.4a) with respect to ξ and t. Since $\partial/\partial\xi[y(t-\xi)] = -\partial/\partial t[y(t-\xi)]$, we have due to Eq. (2.5)

$$\begin{aligned}
\dot{Q}'(\xi)x(t) &+ \int_{-h}^{0} \frac{\partial}{\partial \xi}R(\xi,\tau)x(t+\tau)\,\mathrm{d}\tau \\
&= -Q'(\xi)\left[L(x_t(\cdot)) + SPx(t) + S\int_{-h}^{0} Q(\theta)Fx_t(\theta)\,\mathrm{d}\theta\right] \\
&\quad - \int_{-h}^{0} R(\xi,\tau)\frac{\mathrm{d}}{\mathrm{d}t}Fx_t(\tau)\,\mathrm{d}\tau. \qquad \text{(A.2)}
\end{aligned}$$

Integrating by parts in the right-hand side (RHS) of Eq. (A.2)

$$\begin{aligned}
\int_{-h}^{0} & R(\xi,\tau)\frac{\mathrm{d}}{\mathrm{d}t}Fx_t(\tau)\,\mathrm{d}\tau \\
&= -\int_{-h}^{0} R(\xi,\tau)\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}[Fx_t(\tau)] - A_{01}(\tau)x(t)\right\}\,\mathrm{d}\tau \\
&= -R(\xi,0)[L(x_t(\cdot)) - A_0x(t)] \\
&\quad + \sum_{i=1}^{k} R(\xi,-h_i)A_ix(t) \\
&\quad + \int_{-h}^{0} \frac{\partial}{\partial\tau}R(\xi,\tau)Fx_t(\tau)\,\mathrm{d}\tau \\
&\quad + \int_{-h}^{0} R(\xi,\tau)A_{01}(\tau)\,\mathrm{d}\tau\; x(t),
\end{aligned}$$

and equating terms containing $x(t)$ and Fx_t, we get Eqs. (2.1c) and (2.1e) and the transpose of Eq. (2.1b) (since $R(\xi,\tau) = R'(\tau,\xi)$). The relation (2.1d) follows from Eq. (2.4b).

To derive Eq. (2.1a) we differentiate (2.4b) with respect to t. Using Eqs. (2.2a), (2.2b) (2.3), (2.1d), (2.1e) and (2.4a) we then find

$$\begin{aligned} &P\left\{L(x_t(\cdot))+S\left[Px(t)+\int_{-h}^{0}Q(\xi)Fx_t(\xi)\,\mathrm{d}\xi\right]\right\}\\ &\quad+\int_{-h}^{0}Q(\xi)\frac{\mathrm{d}}{\mathrm{d}t}Fx_t(\xi)\,\mathrm{d}\xi\\ &\quad=-C'Cx(t)\\ &\quad-\sum_{i=0}^{k}A_i'\left[Q'(-h_i)x(t)+\int_{-h}^{0}R(-h_i,\xi)Fx_t(\xi)\,\mathrm{d}\xi\right]\\ &\quad-\int_{-h}^{0}A_{01}'(s)\left[Q'(s)x(t)+\int_{-h}^{0}R(s,\xi)Fx_t(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s. \end{aligned} \tag{A.3}$$

Integrating by parts in Eq. (A.3) and equating terms containing $x(t)$ we obtain Eq. (2.1a).

(*Necessity*). Let $P, Q(\xi), R(\xi,s)$ satisfy Eqs. (2.1a)–(2.1e) and $x(t)$ satisfy Eq. (2.5). We have to prove that y^t of Eqs. (2.4a) and (2.4b) satisfies Eq. (2.2b). We differentiate Eq. (2.4b) with respect to t and obtain that $\dot{y}$ is equal to the LHS of Eq. (A.3). Integrating by parts and applying Eq. (2.1d) we find

$$\begin{aligned} \dot{y}(t)=&\left[PSP+PA_0+\sum_{i=1}^{2}Q(-h_i)A_i\right.\\ &\left.+\int_{-h}^{0}Q(\theta)A_{01}(\theta)\,\mathrm{d}\theta\right]x(t)\\ &+\int_{-h}^{0}[\dot{Q}(\xi)+PSQ(\xi)]Fx_t(\xi)\,\mathrm{d}\xi. \end{aligned}$$

The RHS of the latter equation resulting from Eqs. (2.1a) and (2.1b) is equal to the RHS of Eq. (A.3). The latter implies Eq. (2.2b) for y^t that is given by Eq. (2.4a). □

Proof of Theorem 3. Our proof consists of two steps. The first one is a modification of results of [2], where a more general nonlinear system of neutral type is considered. The main results here are formulated in Propositions 1 and 2.

Step 1: *Slow–fast decomposition of* Eqs. (2.2a) and (2.2b). For small h Eqs. (2.2a) and (2.2b) is a singularly perturbed system (see e.g. [9]). We shall decompose Eqs. (2.2a) and (2.2b) into a purely slow ordinary differential equation and a purely fast integral equation. We apply the invariant manifolds approach of [11]. For ordinary singularly perturbed systems such a decomposition can also be obtained by Chang's transformation (see e.g. [12]). Let $T(t): C[-h,0] \to C[-h,0]$, $t \geqslant 0$ and $S(s): C[0,h] \to C[0,h]$, $s \leqslant 0$ be semigroups of shift operators, corresponding to the equation $\dot{x}(t)=0$ defined by

$$T(t)\phi(\theta)=\begin{cases}\phi(t+\theta) & \text{if } t+\theta<0,\\ \phi(0) & \text{if } t+\theta\geqslant 0,\end{cases}$$

$$S(s)\psi(\zeta)=\begin{cases}\psi(s+\zeta), & \text{if } s+\zeta>0,\\ \psi(0), & \text{if } s+\zeta\leqslant 0.\end{cases}$$

Applying the variation of constants formula [4] on Eqs. (2.2a) and (2.2b), we obtain that for all $t_2 \geqslant 0$ and $t \in [0,t_2]$ Eqs. (2.2a) and (2.2b) is equivalent to the following integro-differential system:

$$x_t=T(t)x_0+\int_0^t T(t-s)X_0[L(x_s(\cdot))+Sy(s)]\,\mathrm{d}s, \tag{A.4a}$$

$$\begin{aligned} y^t=&S(t_2-t)y^{t_2}\\ &+\int_{t_2}^{t}S(t-s)Y_0[-C'Cx(s)-L'(y^s(\cdot))]\,\mathrm{d}s, \end{aligned} \tag{A.4b}$$

where $x_t=x(t+\theta)$, $y^t=y(t+\zeta)$, $\theta\in[-h,0]$, $\zeta\in[0,h]$, $X_0(\theta)=Y_0(-\theta)=0, \theta\in[-h,0)$, and $X_0(0)=Y_0(0)=I$. For small enough h Eqs. (2.2a) and (2.2b) has a slow manifold [2]

$$\begin{aligned} x_t(\theta)&=x(t)+H_1(\theta)\begin{pmatrix}x(t)\\ y(t)\end{pmatrix},\\ y^t(\zeta)&=y(t)+H_2(\zeta)\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}, \end{aligned} \tag{A.5}$$

where $\theta\in[-h,0]$, $\zeta\in[0,h]$. The $n\times 2n$-matrices H_1 and H_2 satisfy

$$\begin{aligned} &\begin{pmatrix}(I-X_0,\ O)+H_1\\ (0,\ I-Y_0)+H_2\end{pmatrix}\left[\begin{pmatrix}LI & S\\ -C'C & -(LI)'\end{pmatrix}\right.\\ &\quad\left.+\begin{pmatrix}L(H_1(\cdot))\\ -L'(H_2(\cdot))\end{pmatrix}\right]=\begin{pmatrix}\mathscr{A}H_1\\ \mathscr{B}H_2\end{pmatrix}, \end{aligned} \tag{A.6a}$$

$$H_1(0)=H_2(0)=0. \tag{A.6b}$$

where $LI = \sum_{i=0}^{k} A_i + \int_{-h}^{0} A_{0i}(s)\,\mathrm{d}s$ and

$$\mathscr{A}\phi(\theta) = \begin{cases} \dot\phi(\theta) & \text{if } \theta < 0, \\ 0 & \text{if } \theta = 0, \end{cases}$$

$$\mathscr{B}\psi(\zeta) = \begin{cases} \dot\psi(\zeta) & \text{if } \zeta > 0, \\ 0 & \text{if } \zeta = 0. \end{cases}$$

Note that Eqs. (A.6a) and (A.6b) can be derived by substitution of Eq. (A.5) into Eqs. (A.4a) and (A.4b) and differentiation on t. The flow on this manifold is governed by

$$\begin{pmatrix} \dot u(t) \\ \dot v(t) \end{pmatrix} = \left[\begin{pmatrix} L(H_1(\cdot)) \\ -L'(H_2(\cdot)) \end{pmatrix} + \begin{pmatrix} LI & S \\ -C'C & -(LI)' \end{pmatrix} \right] \times \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}. \tag{A.7}$$

Changing variables in Eqs. (A.4a) and (A.4b)

$$\begin{aligned} z_t &= x_t - x(t) - H_1 \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \\ q^t &= y^t - y(t) - H_2 \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \end{aligned} \tag{A.8}$$

and using the following formulas of [2]

$$\begin{aligned} &T(t)\left[x_0 + H_1 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right] - x(t) - H_1 \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\ &\quad = \int_0^t T(t-s)\mathscr{A}H_1 \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \mathrm{d}s \\ &\qquad - \int_0^t T(t-s)\left[\dot x(s) + H_1 \begin{pmatrix} \dot x(s) \\ \dot y(s) \end{pmatrix}\right] \mathrm{d}s, \end{aligned} \tag{A.9}$$

$$\begin{aligned} &S(t_2 - t)\left[y_0 + H_2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right] - y(t) - H_2 \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\ &\quad = \int_{t_2}^t S(t-s)\mathscr{B}H_2 \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \\ &\qquad - S(t-s)\left[\dot y(s) + H_2 \begin{pmatrix} \dot x(s) \\ \dot y(s) \end{pmatrix}\right] \mathrm{d}s, \end{aligned} \tag{A.10}$$

we obtain the system

$$\begin{pmatrix} \dot x(t) \\ \dot y(t) \end{pmatrix} = \left[\begin{pmatrix} L(H_1(\cdot)) \\ -L'(H_2(\cdot)) \end{pmatrix} + \begin{pmatrix} LI & S \\ -C'C & -(LI)' \end{pmatrix} \right] \times \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} L(z_t(\cdot)) \\ -L'(q^t(\cdot)) \end{pmatrix}, \tag{A.11a}$$

$$z_t = T(t)z_0 - \int_0^t T(t-s)[H_1 + (I - X_0, 0)] \times \begin{pmatrix} L(z_s(\cdot)) \\ -L'(q^s(\cdot)) \end{pmatrix} \mathrm{d}s, \tag{A.11b}$$

$$q^t = S(t_2 - t)q^{t_2} - \int_{t_2}^t S(t-s)[H_2 + (0, I - Y_0)] \times \begin{pmatrix} L(z_s(\cdot)) \\ -L'(q^s(\cdot)) \end{pmatrix} \mathrm{d}s. \tag{A.11c}$$

Note that $z_t(0) = q^t(0) = 0$. Moreover, for all $z \in C[-h,0]$, $z(0) = 0$ and $q \in C[0,h]$, $q(0) = 0$ we have $T(t)z = 0$ for $t > h$ and $S(t)q = 0$ for $t < -h$. Therefore for all small enough h

$$\|T(t)z\|_c \leqslant K\mathrm{e}^{-t/h}\|z\|_c, \quad t > 0;$$

$$\|S(t)q\|_c \leqslant K\mathrm{e}^{t/h}\|q\|_c, \quad t < 0, \ K > 1.$$

Then, for small h, Eqs. (A.11a)–(A.11c) has the stable manifold [2]

$$x(t) = G_1 z_t, \quad y(t) = G_2 z_t, \quad q^t = G_3 z_t. \tag{A.12}$$

The flow on this manifold is governed by the equation that follows from Eqs. (A.11b) and (A.12)

$$z_t = T(t)z_0 - \int_0^t T(t-s)[H_1 + (I - X_0, 0)] \times \begin{pmatrix} L(z_s(\cdot)) \\ -L'G_3(\cdot)z_s \end{pmatrix} \mathrm{d}s. \tag{A.13}$$

The linear bounded operators $G_i : R^n \to C[-h,0]$, $i = 1,2$ and $G_3 : C[-h,0] \to C[0,h]$ satisfy the following equations for all continuously differentiable $z \in C[-h,0]$, $z(0) = 0$ [2]:

$$\begin{aligned} &\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \left\{ \mathscr{A}z - [H_1 + (I - X_0, 0)] \begin{pmatrix} L(z(\cdot)) \\ -L'(G_3(\cdot)z) \end{pmatrix} \right\} \\ &\quad = \begin{pmatrix} L(H_1(\cdot)) \\ -L'(H_2(\cdot)) \end{pmatrix} \begin{pmatrix} G_1 z \\ G_2 z \end{pmatrix} \\ &\qquad + \begin{pmatrix} LI \cdot G_1 z + Lz(\cdot) + SG_2 z \\ -(LI)'G_2 z - L'G_3(\cdot)z - C'CG_1 z \end{pmatrix}, \end{aligned} \tag{A.14a}$$

$$G_3\left\{\mathscr{A}z-[H_1+(I-X_0,0)]\begin{pmatrix}L(z(\cdot))\\ -L'(G_3(\cdot)z)\end{pmatrix}\right\}$$

$$=\mathscr{B}G_3z-(H_2+(0,I-Y_0))\begin{pmatrix}L(z(\cdot))\\ -L'(G_3(\cdot)z)\end{pmatrix}. \tag{A.14b}$$

Eqs. (A.14a) and (A.14b) result from substitution of Eq. (A.12) into Eqs. (A.11a)–(A.11c) and differentiation on t.

Proposition 1. *For all small enough h the operators G_1, G_2 and G_3 can be represented in the form of*

$$\begin{aligned}G_iz&=\int_{-h}^{0}g_i(\theta)Fz(\theta)\,\mathrm{d}\theta,\quad i=1,2;\\ G_3(\zeta)z&=\int_{-h}^{0}g_3(\zeta,\theta)Fz(\theta)\,\mathrm{d}\theta,\ \zeta\in[0,h],\end{aligned} \tag{A.15}$$

where g_1, g_2 and g_3 are continuous and piecewise continuously differentiable functions of their arguments.

Proof. We start with G_3. Substituting Eq. (A.15) into Eq. (A.14b), further integrating by parts and equating terms containing $Fz(\theta)$ and $L(z(\cdot))$ we get

$$\frac{\partial}{\partial\zeta}g_3(\zeta,\theta)-\frac{\partial}{\partial\theta}g_3(\zeta,\theta)$$

$$=\left[\int_{-h}^{0}g_3(\zeta,s)FH_{12}(s)\,\mathrm{d}s-H_{22}(\zeta)-I\right]$$

$$\times L'(g_3(\cdot,\theta)), \tag{A.16a}$$

$$g_3(\zeta,0)=-\int_{-h}^{0}g_3(\zeta,s)F[H_{11}(s)+I]\,\mathrm{d}s+H_{21}(\zeta), \tag{A.16b}$$

$$g_3(0,\theta)=0. \tag{A.16c}$$

where $H_i=\mathrm{col}\{H_{i1},H_{i2}\},\quad i=1,2$. Note that Eq. (A.16c) follows from $q^t(0)=0$. Applying the characteristics method on Eqs. (A.16a)–(A.16c) we get the following equivalent to Eqs. (A.16a)–(A.16c) integral equations:

$$g_3(\zeta,\theta)$$

$$=\begin{cases}\int_0^{\zeta}f(s,-s+\theta+\zeta)\,\mathrm{d}s, & \text{if } \zeta\leqslant-\theta,\\ g_3(\zeta+\theta,0)\\ \quad+\int_0^{-\theta}f(s+\zeta+\theta,-s)\,\mathrm{d}s, & \text{if } \zeta>-\theta,\end{cases} \tag{A.17a}$$

$$f(\zeta,\theta)=\left[\int_{-h}^{0}g_3(\zeta,s)FH_{12}(s)\,\mathrm{d}s-H_{22}(\zeta)-I\right]$$

$$\times L'(g_3(\cdot,\theta)). \tag{A.17b}$$

Substituting Eqs. (A.16b) and (A.17b) into Eq. (A.17a), and applying the contraction principle argument to the resulting equation, one can show that for small enough h this equation has a unique continuous solution $g_3(\zeta,\theta)$, $\zeta\in[0,h]$, $\theta\in[-h,0]$. This solution is continuously differentiable function for $\zeta\neq-\theta$. Really, differentiating Eqs. (A.17a) and (A.17b) on θ and ζ, we get a system of integral equations, which, for small h, possesses a unique solution by contraction principle argument. Thus, the second of Eq. (A.15) is proved.

The existence of g_1 and g_2 can be proved similarly to the existence of g_3. □

Applying another change of variables

$$\begin{aligned}u(t)&=x(t)-G_1z_t,\\ v(t)&=y(t)-G_2z_t,\quad r^t=q^t-G_3z_t\end{aligned} \tag{A.18}$$

we obtain

$$\begin{pmatrix}\dot u(t)\\ \dot v(t)\end{pmatrix}=\left[\begin{pmatrix}L(H_1(\cdot))\\ -L'(H_2(\cdot))\end{pmatrix}+\begin{pmatrix}LI & S\\ -C'C & -(LI)'\end{pmatrix}\right]$$

$$\times\begin{pmatrix}u(t)\\ v(t)\end{pmatrix}-\left[I+\begin{pmatrix}G_1\\ G_2\end{pmatrix}H_{12}\right]L'(r^t(\cdot)), \tag{A.19a}$$

$$z_t=T(t)z_0-\int_0^t T(t-s)[H_1+(I-X_0,0)]$$

$$\times\begin{pmatrix}L(z_s(\cdot))\\ -L'(G_3(\cdot)z_s+r^s(\cdot))\end{pmatrix}\mathrm{d}s, \tag{A.19b}$$

$$r^t = S(t_2 - t)r^{t_2} + \int_{t_2}^{t} S(t-s)[H_{22} + I - Y_0 - G_3 H_{12}]L'(r^s(\cdot))\,\mathrm{d}s. \tag{A.19c}$$

Solutions of Eq. (A.19c) exponentially decay as $t \to -\infty$ [2]. Hence all the stable solutions of Eqs. (A.19a)–(A.19c) correspond to $r^t \equiv 0$, and Eqs. (A.18) and (A.8) yield

$$\begin{pmatrix} x_t(\theta) \\ y^t(\zeta) \end{pmatrix} = \left[I_{2n} + \begin{pmatrix} H_1(\theta) \\ H_2(\zeta) \end{pmatrix} \right] \begin{pmatrix} u(t) + G_1 z_t \\ v(t) + G_2 z_t \end{pmatrix} + \begin{pmatrix} z_t(\theta) \\ G_3(\zeta) z_t \end{pmatrix}. \tag{A.20}$$

We thus proved the following

Proposition 2. *For all small enough values of h the stable solutions of Eqs.* (2.2a) and (2.2b) *can be represented in the form* (A.20), *where u, v and z satisfy Eqs.* (A.7) and (A.13).

Step 2: *Existence of a stable manifold of* Eqs. (2.2a) and (2.2b). We are looking for a stable manifold $v = Yu$ of Eq. (A.7). Denoting

$$\begin{pmatrix} U(\theta) \\ V(\zeta) \end{pmatrix} = \left[I_{2n} + \begin{pmatrix} H_1(\theta) \\ H_2(\zeta) \end{pmatrix} \right] \begin{pmatrix} I \\ Y \end{pmatrix}. \tag{A.21}$$

We differentiate the relation $v(t) = Yu(t)$ with respect to t and apply Eq. (A.7), where $v = Yu$. We obtain

$$Y[L(U(\cdot)) + SY] = -L'(V(\cdot)) - C'C. \tag{A.22}$$

Differentiating Eq. (A.21) on θ and ζ, and applying Eq. (A.6a) multiplied by $\mathrm{col}\{I, Y\}$ and Eq. (A.22), we have

$$\dot{U}(\theta) = U(\theta)[L(U(\cdot)) + SY], \quad \theta < 0, \tag{A.23a}$$

$$\dot{V}(\zeta) = V(\zeta)[L(U(\cdot)) + SY], \quad \zeta > 0. \tag{A.23b}$$

Solving Eqs. (A.23a) and (A.23b) with the initial conditions $U(0) = I$, $V(0) = Y$, which follow from Eqs. (A.21) and (A.6b), we find

$$U(\theta) = \mathrm{e}^{Z\theta}, \tag{A.24a}$$

$$V(\zeta) = Y\mathrm{e}^{Z\zeta}, \tag{A.24b}$$

$$Z = L(U(\cdot)) + SY. \tag{A.24c}$$

Then Eq. (2.6a) follows from Eqs. (A.24c) and (A.24a), while Eq. (2.6b) follows from Eqs. (A.22) and (A.24b).

Conversely, if Y and Z satisfy Eqs. (2.6a) and (2.6b) then Eqs. (A.24c) and (A.22) hold, where U and V are given by Eqs. (A.24a) and (A.24b) and thus satisfy Eqs. (A.23a) and (A.23b). Then Eq. (A.21) follows from Eqs. (A.6a), (A.22), (A.23a) and (A.23b). Hence, Eq. (A.22) means that $v = Yu$ is an invariant manifold of Eq. (A.7).

Proposition 3. *For all small enough values of h the following statements are equivalent*:

(A) *The algebraic system* (2.6) *has a solution such that the matrix Z is stable*;

(B) *Eq.* (A.7) *has a stable manifold* $v = Yu$ *with a flow governed by* $\dot{u} = Zu$;

(C) *The system* (2.2) *has a stable manifold of Eq.* (2.4a).

Proof. (A) $\Leftrightarrow$ (B) was proved just before Proposition 3. We shall prove that (B) $\Rightarrow$ (C). Substitute Eq. (A.20), where $v = Yu$, and Eq. (A.21) into Eq. (2.4a)

$$\begin{aligned} &\int_{-h}^{0} R(\xi,\tau)F \left\{ U(\tau)u + [(I,O) + H_1(\tau)] \begin{pmatrix} G_1 z \\ G_2 z \end{pmatrix} + z(\tau) \right\} \mathrm{d}\tau + Q'(\xi)[u + G_1 z] \\ &\quad = V(-\xi)u + [(O,I) + H_2(\xi)] \begin{pmatrix} G_1 z \\ G_2 z \end{pmatrix} + G_3(-\xi)z. \end{aligned} \tag{A.25}$$

We will prove that there exist $Q(\xi)$ and $R(\xi,s)$ such that they satisfy Eq. (2.4a). We equate in Eq. (A.25) terms containing Fz and u and obtain the following two coupled equations:

$$\begin{aligned} &\left\{ \int_{-h}^{0} R(\xi,\tau)F[(I,0) + H_1(\tau)]\,\mathrm{d}\tau - (0,I) - H_2(\xi) \right\} \times \begin{pmatrix} g_1(s) \\ g_2(s) \end{pmatrix} + Q'(\xi)g_1(s) + R(\xi,s) \\ &\quad = g_3(-\xi,s), \end{aligned} \tag{A.26a}$$

$$\int_{-h}^{0} R(\xi,\tau)FU(\tau)\,\mathrm{d}\tau + Q'(\xi) = V(-\xi). \tag{A.26b}$$

For small h Eqs. (A.26a) and (A.26b) has a unique solution $Q(\xi), R(\xi,s)$ continuous on its arguments. Hence, $Q(\xi)$ and $R(\xi,s)$ satisfy Eq. (2.4a).

To prove (C) $\Rightarrow$ (B) we suppose that there exist $Q(\xi), R(\xi, s)$ such that Eq. (2.4a) holds. Substituting the RHS of Eq. (A.20) into Eq. (2.4a), we obtain an equation that can be solved with respect to v for small h. This solution $v = Yu + Xz$ defines the stable manifold of Eq. (A.7). Moreover, $X = 0$ since Eqs. (A.7) and (A.13) are decoupled. $\square$

Theorem 3 follows from Theorem 2, Proposition 3 and Theorem 1. $\square$

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