Linear Stability Analysis of Lid Driven and Convective Flows Accelerated by an Efficient Fully Coupled Time-Marching Algorithm

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Outline

- **Pressure-velocity coupled formulation of the Navier-Stokes equations**
- **Benchmark problems**
- **Analytical Solution Accelerated (ASA) smoother**
- **3D Domain Decomposition and Scalability Properties**
- **Application to the linear stability analysis**
- **Conclusions**

Incompressible N-S Equations – Numerical Challenge

Continuity - $\nabla \cdot \boldsymbol{u} = 0$

Momentum- $\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \boldsymbol{u}$

- No separate equation for pressure
- No boundary conditions for pressure

Incompressible N-S Equations – Numerical Challenge (Cont.)

Pressure Projection Approach

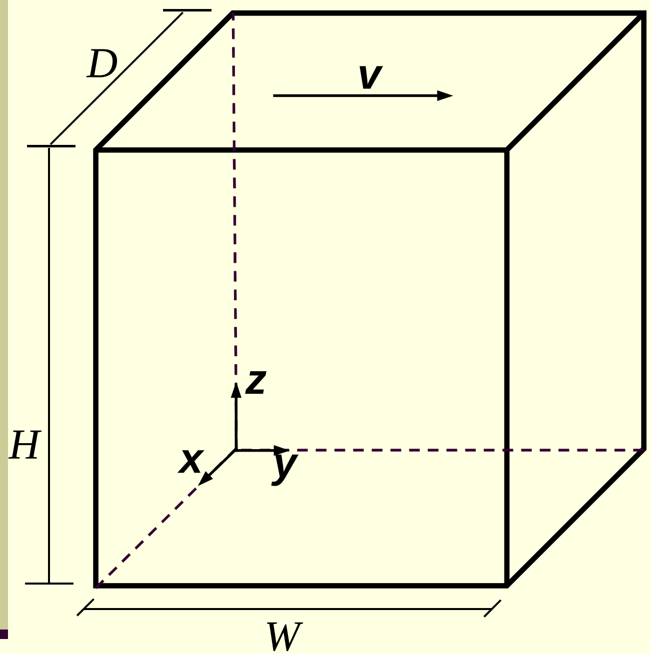
- ✓ High numerical robustness
- ✓ Low memory consumption
- ✘ Slow rate of numerical convergence
- ✘ Non-physical pressure field
- ✘ Not applicable for liquid – solid interface problems

Pressure–Velocity Coupled Approach

- ✓ High rate of numerical convergence
- ✓ The "most natural " way to solve N-S equations
- ✓ The obtained pressure is physical
- ✘ High memory consumption
- ✘ Not as numerically robust as pressure projection methods

Benchmark Problems

Lid-Driven Cubic Cavity



$$\nabla \cdot \boldsymbol{u} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{u}$$

✓**Explicit Discretization**

$$(\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n$$

▪**Semi-Implicit Discretization**

$$(\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^{n+1}$$

Realistic Boundary Conditions:

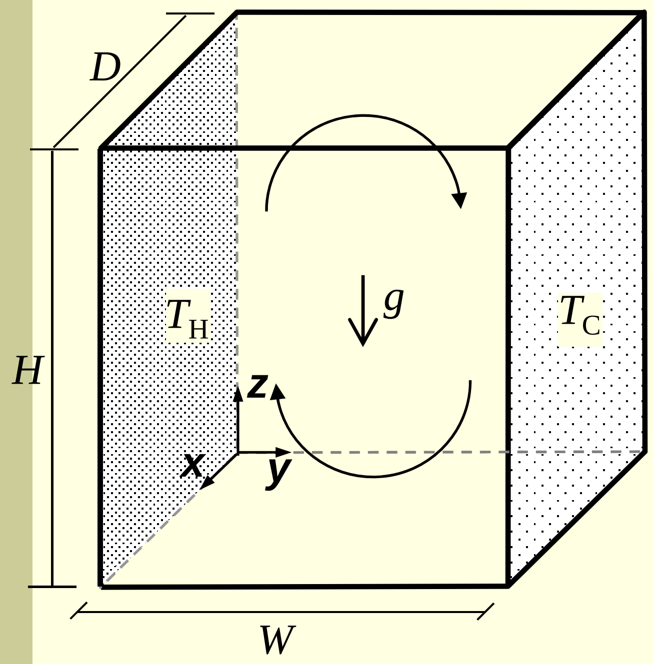
$\boldsymbol{u} = 0$ - at all static walls no slip/no penetration

$\boldsymbol{u}|_{z=H/W} = v$ -at the moving wall the flow velocity is equal to that of the moving wall itself

No boundary condition for pressure is needed

Benchmark Problems (Cont.)

Differently Heated Rectangular and Cubic Cavity (Boussinesq Approximation)



$$\nabla \cdot \boldsymbol{u} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \sqrt{\frac{1}{\mathrm{Gr}}}\nabla^2 \boldsymbol{u} + \theta \vec{e}_z$$

$$\frac{\partial \theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)\theta = \frac{1}{\mathrm{Pr}\sqrt{\mathrm{Gr}}}\nabla^2 \theta$$

✓**Explicit Discretization**

$$(\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^{\boldsymbol{n}} \quad (\boldsymbol{u}^n \cdot \nabla)\theta^{\boldsymbol{n+1}}$$

▪**Semi-Implicit Discretization**

$$(\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^{\boldsymbol{n+1}} \quad (\boldsymbol{u}^n \cdot \nabla)\theta^{\boldsymbol{n+1}}$$

Boundary Conditions:

$\theta|_{y=0} = 1, \; \theta|_{y=H/W} = 0$ **-isothermal vertical walls,** $\frac{\partial \theta}{\partial \boldsymbol{n}} = 0 \; or \; \theta = 1 - y$ **-horizontal and lateral walls**

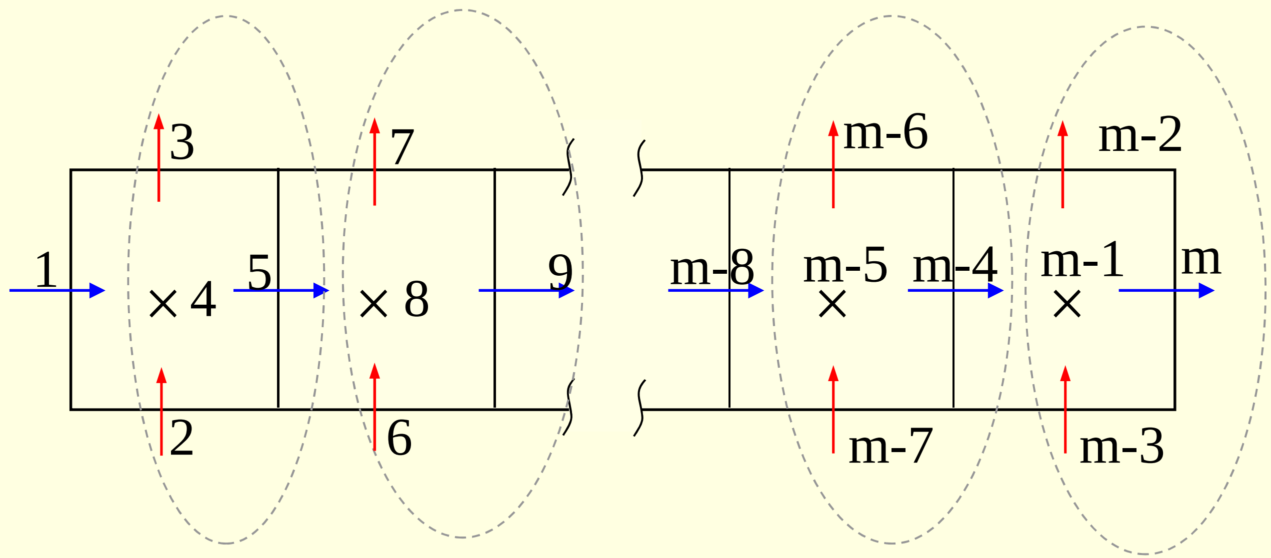
$\boldsymbol{u} = 0$ **-at all walls,**

No boundary condition for pressure is needed

Accelerated Coupled Line Gauss-Seidel Smoother (ASA-CLGS) -2D

Zeng and Wesseling (1993) – CLGS: Horizontal (vertical) sweeping with horizontally (vertically) adjacent pressure linkage

Feldman and Gelfgat (2008) – ASA-CLGS:Horizontal (vertical) sweeping without horizontally (vertically) adjacent pressure linkage



CLGS and ASA-CLGS Efficiency Estimation for 2D

Zeng and Wesseling (1993) – CLGS:

Block 3-diagonal matrix or 7-diagonal matrix

LU decomposition

$$\approx O(15M)$$

Feldman and Gelfgat (2008) – ASA-CLGS (6-Diagonal Matrix)

$$p'_{k-1} = (c_1^L v'_{k-4} + R_{k-1}^L + \sum_{i=2}^{4} c_i^L R_{k-i}^L) / c_5^L$$

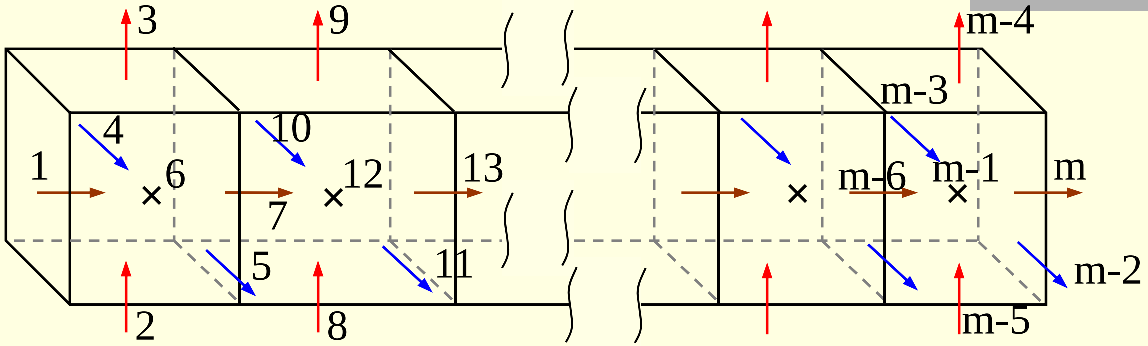
$$\begin{bmatrix} v'_5 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} c_6^1 \\ c_7^1 \\ c_8^1 \end{bmatrix} \times p'_4 + \begin{bmatrix} c_9^1 R_5^L \\ c_{10}^1 R_2^L \\ c_{11}^1 R_3^L \end{bmatrix}$$

$$\approx O(5M)$$

Thomas Algorithm (3-Diagonal Matrix)

$$\approx O(5M)$$

ASA-CLGS -Efficiency Estimation for 3D



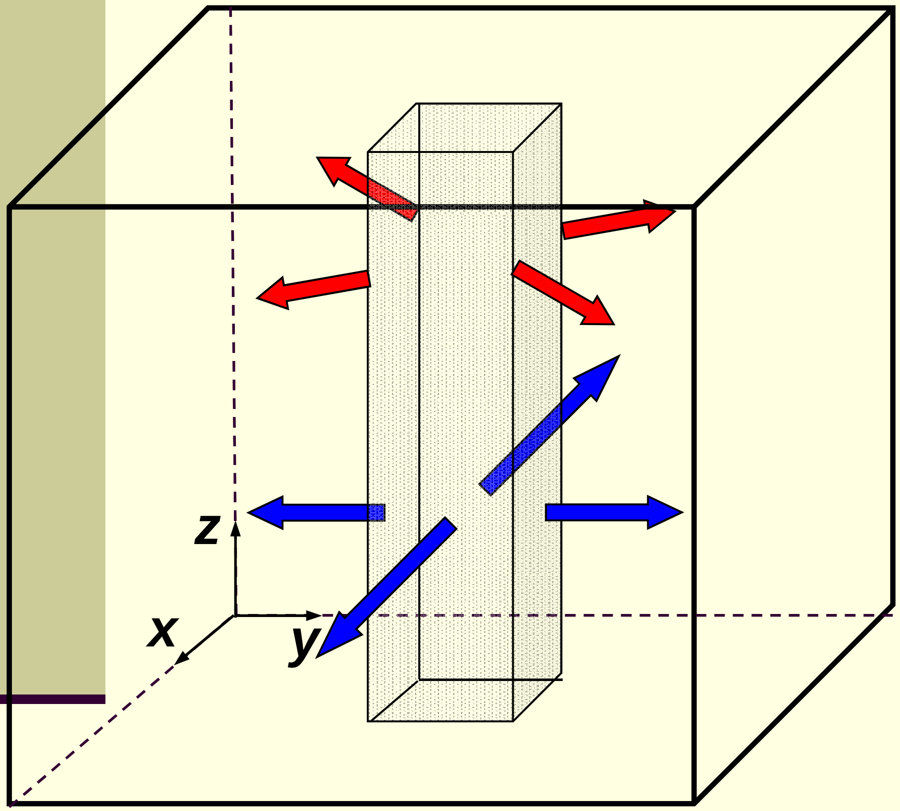
$$p'_p = \left(c_1^I w'_d + R_p^I + c_2^I R_e^I + c_3^I R_w^I + c_4^I R_n^I + c_5^I R_5^I + c_6^I R_d^I\right) / c_7^I$$

$$\begin{bmatrix} w'_d \\ v'_s \\ v'_n \\ u'_w \\ u'_e \end{bmatrix} = \begin{bmatrix} c_8^I \\ c_9^I \\ c_{10}^I \\ c_{11}^I \\ c_{12}^I \end{bmatrix} \times p'_p + \begin{bmatrix} c_{13}^I R_d^I \\ c_{14}^I R_s^I \\ c_{15}^I R_n^I \\ c_{16}^I R_w^I \\ c_{17}^I R_e^I \end{bmatrix}$$

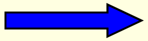
6 corrections for a single volume result in 17 multiplications and divisions and 11 summations

$$\approx O(5M)$$

Domain Decomposition for 3D Configuration

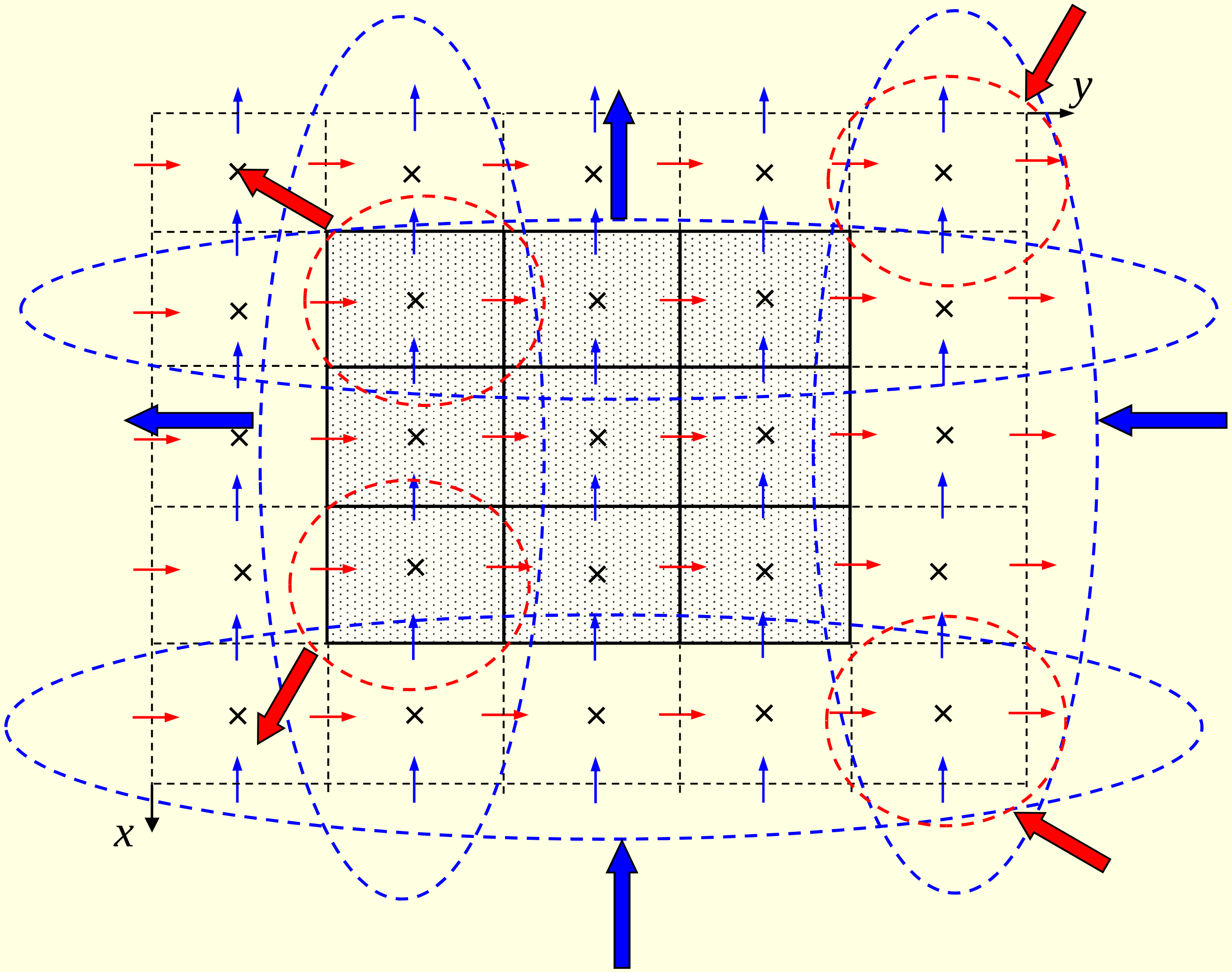


✓ Existance of Analytical Solution for the Whole Column Allows for 2D Virtual Topology of 3D Configuration.

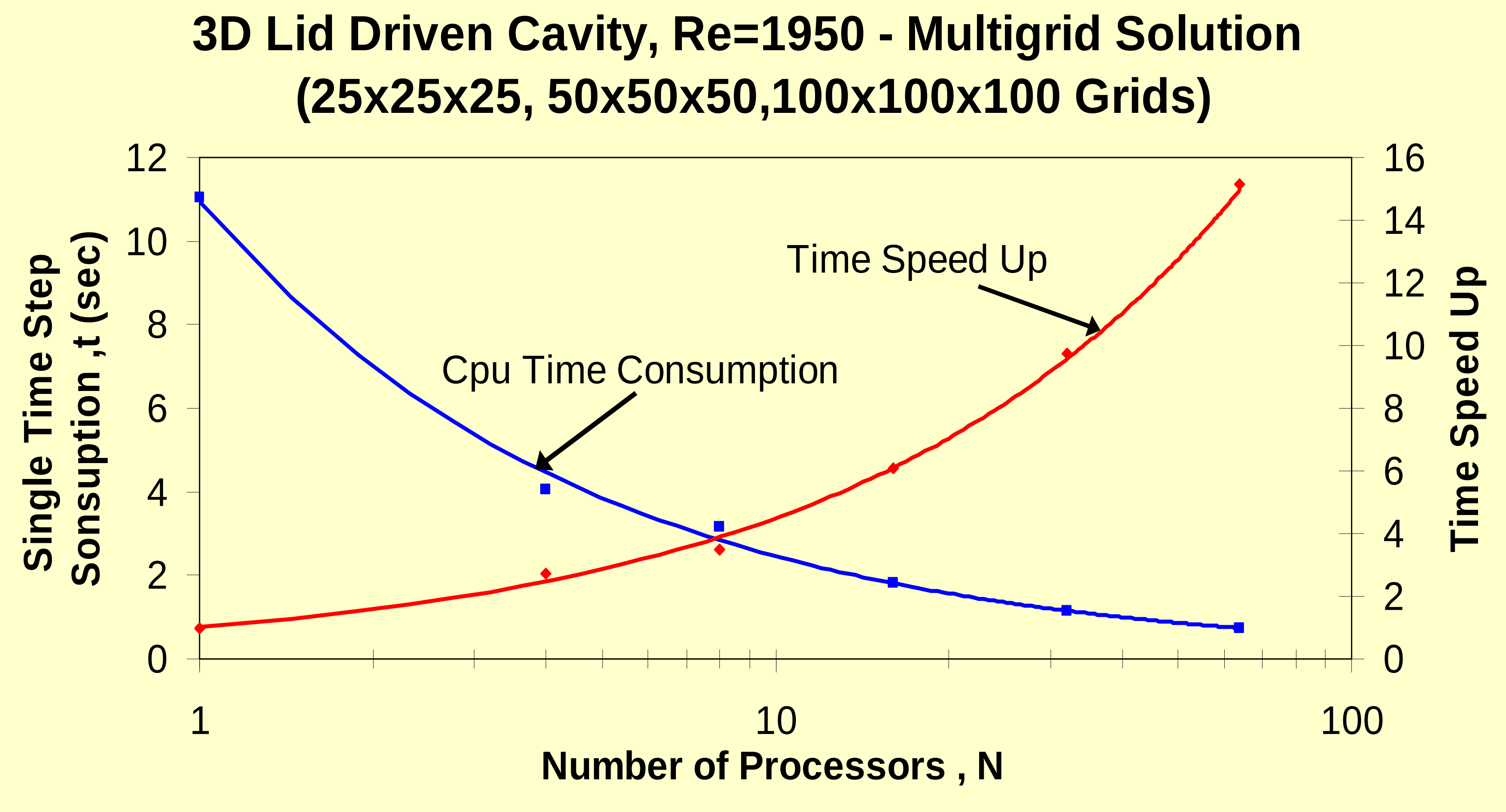
 All volumes located at the sub -volume faces exchange data with neighbors

 All volumes located at the sub -volume vertical edges exchange data with diagonal neighbors

3D Configuration- Data Exchange Principle

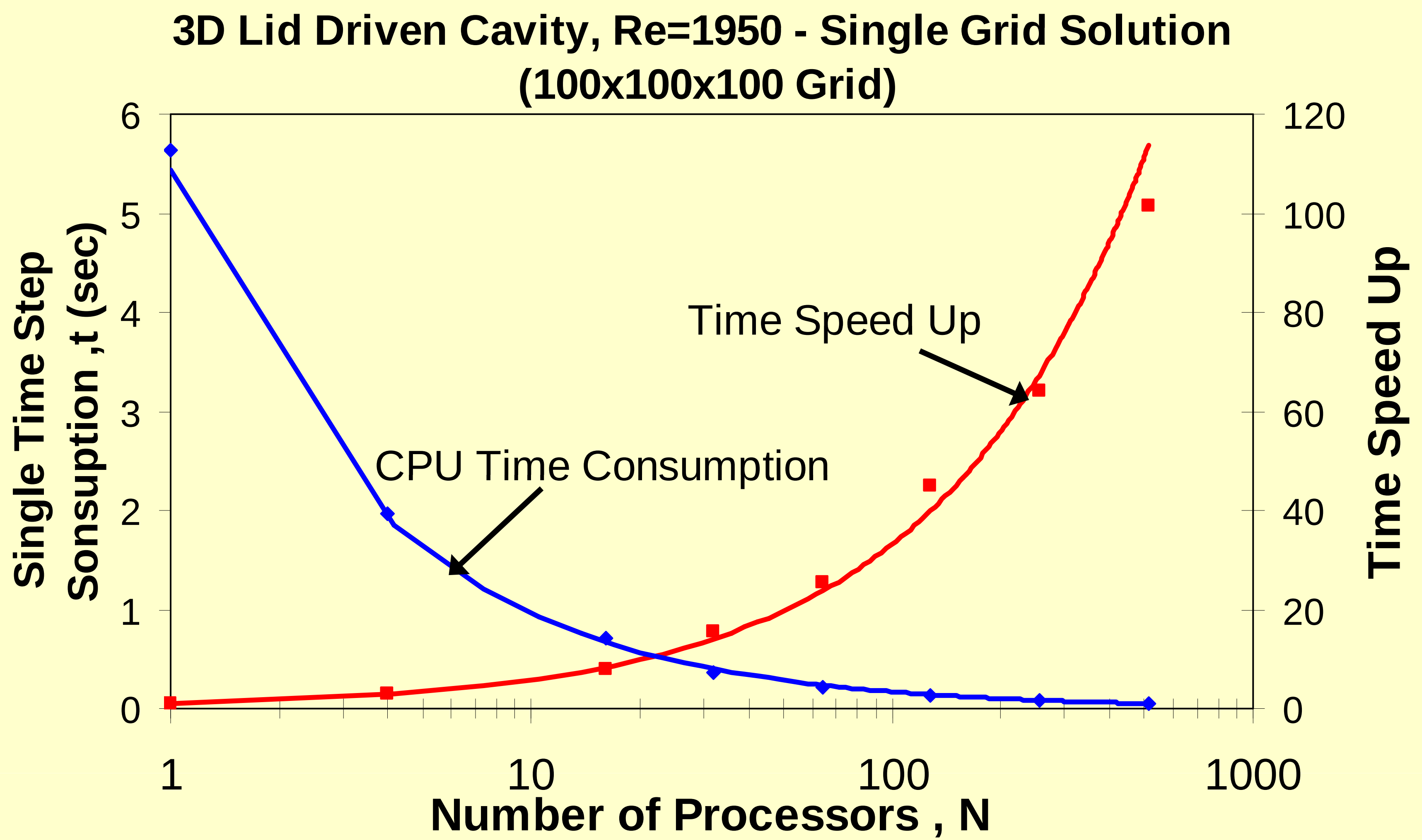


The Method Scalability Characteristics



Number of CPU is restricted by the coarsest level (8x8=64 CPU)

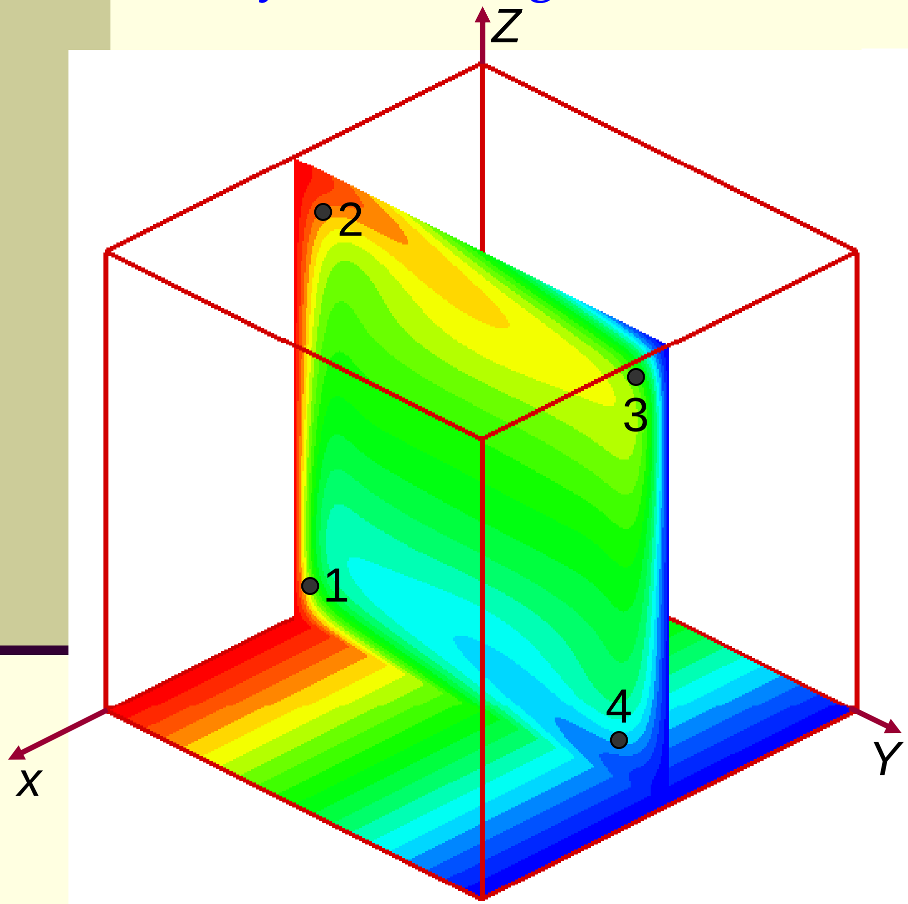
The Method Scalability Characteristics (Cont.)

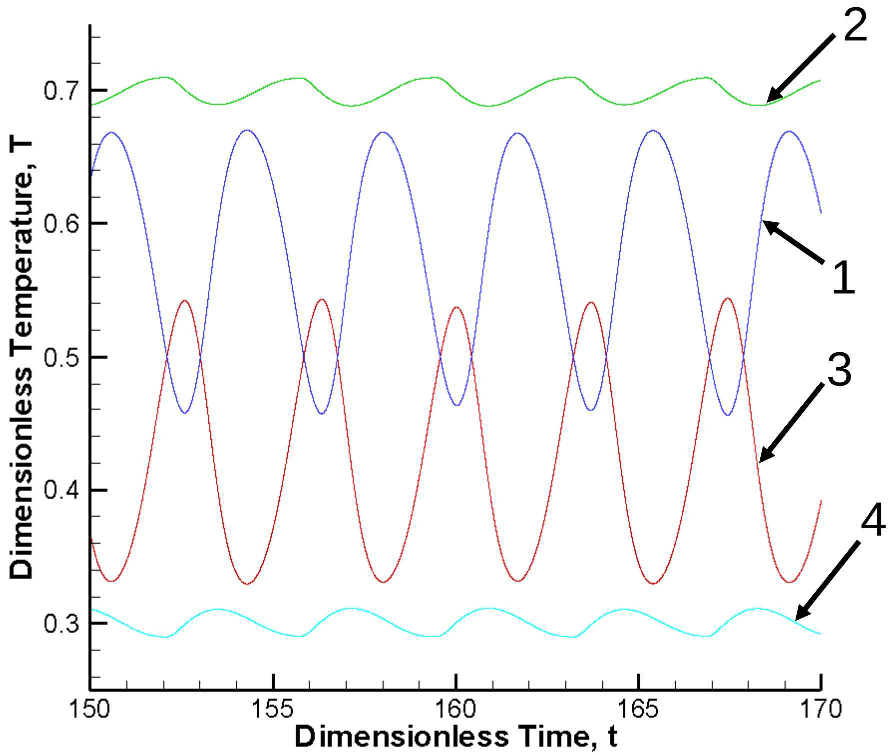


512 CPU ➡ 55.6 msec per time step ➡ only 16 hours for 10^6 time steps

Differentially Heated Cavity, Gr=3.5x10^6

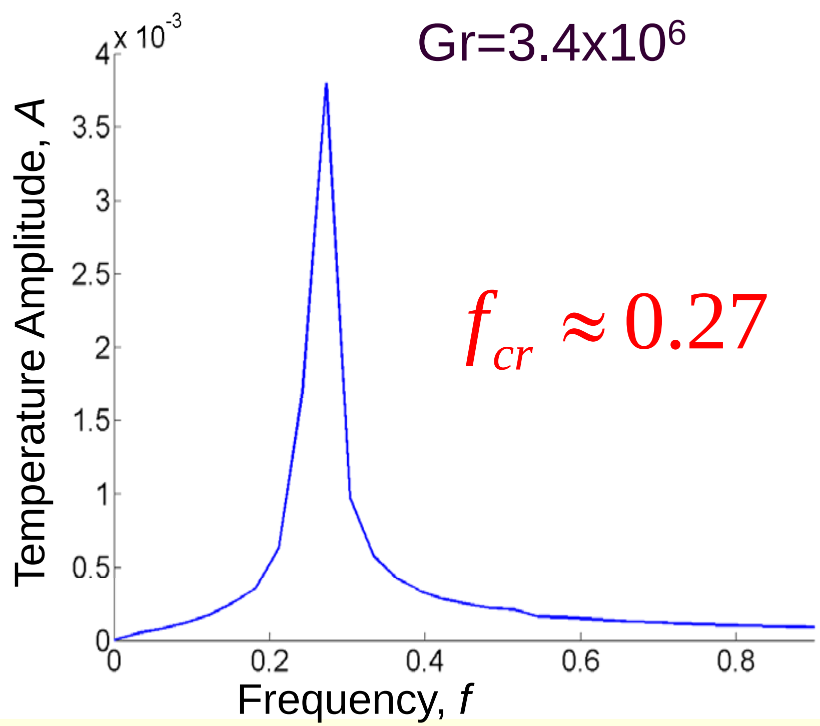
Perfectly Conducting Lateral Walls **DNS Results for Middle Plane Points**

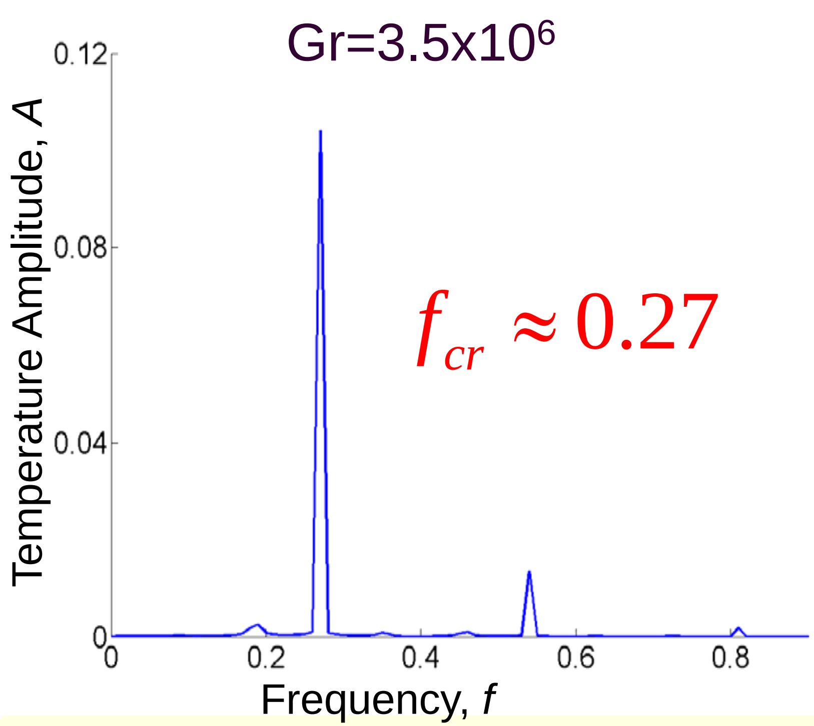




Centro symmetry is preserved (opposite phases for opposite corners)

Leading Mod Frequency and Estimation of Critical Gr





According to the Hopf Theorem $Gr - Gr_{cr} \approx \mu A^2 \Rightarrow Gr_{cr} \approx 3.35 \times 10^6$

Experimental Results of Jones and Briggs, 1989 : $f_{cr} \approx 0.25$

$Gr_{cr} \approx 4.44 \times 10^6$

Conclusions

- ✓ **An Accelerated Semi-Analytical Coupled Line Implicit Gauss-Seidel Smoother (ASA-CLGS) and Full Pressure Coupled Direct Solution (FPDS) were developed and implemented for the solution of incompressibel N-S equations.**
- ✓ **The Navier-Stokes and Boussinesq equations are solved without pressure-velocity decoupling.**
- ✓ **The code was verified on existing benchmark solutions for the lid-driven and thermally driven cavities.**
- ✓ **The potential implementation of the developed time marching solvers to the linear stability analysis was studied.**
- ✓ **The characteristic CPU times consumed for a single time step per one node and per one CPU are of order 5 $\times 10^{-3}$ msec and 10^{-2} msec for 2D and 3D calculations, respectively.**