AN ACCELERATED MULTIGRID APPROACH FOR TIME-INTEGRATION OF INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Outline

- **Advantages of Multigrid methods**
- **Numerical technique**
- **Accelerated Semi Analytic (ASA) smoother**
- **Comparison with existing benchmark solutions**
- **Conclusions**

Why Multigrid ?

- ☑ Highly effective for linear and non-linear problems

  (semi-implicit discretization of the convective terms)

- ☑ Memory unrestricted for the state of the art computer recourses
- ☑ Can be easily parallelized (OpenMP or MPI approach)
- ✘ Numerical convergence rate is a problem and is grid dependent
- ✘ Numerical convergence rate strongly depends on the time step size
- ✘ Sophisticated programming is needed

Time and spatial discretization

Second order backward differentiation - $\frac{\partial f^{n+1}}{\partial t} = \frac{3f^{n+1} - 4f^{n} + f^{n-1}}{2\Delta t} + O\left(\Delta t^2\right)$

Energy - $$\left(a^{\theta}_{(\mathrm{i,j,k})} - \frac{3}{2\Delta\tau}\right)\theta^{\mathrm{n+1}}_{(\mathrm{i,j,k})} + \sum_{\mathrm{i,j,k}} a^{\theta}_{\mathrm{i,j,k}}\theta^{\mathrm{n+1}}_{\mathrm{i,j,k}} = RHP^{\mathrm{n}}_{\theta}$$

Temperature – velocity decoupling

Continuity - $$\frac{\left(u^{\mathrm{n+1}}_{(\mathrm{i,j,k})} - u^{\mathrm{n+1}}_{(\mathrm{i\text{-}1,j,k})}\right)}{Hx(i-1)} + \frac{\left(v^{\mathrm{n+1}}_{(\mathrm{i,j,k})} - v^{\mathrm{n+1}}_{(\mathrm{i,j\text{-}1,k})}\right)}{Hy(j-1)} + \frac{\left(w^{\mathrm{n+1}}_{(\mathrm{i,j,k})} - w^{\mathrm{n+1}}_{(\mathrm{i,j,k\text{-}1})}\right)}{Hz(k-1)} = 0$$

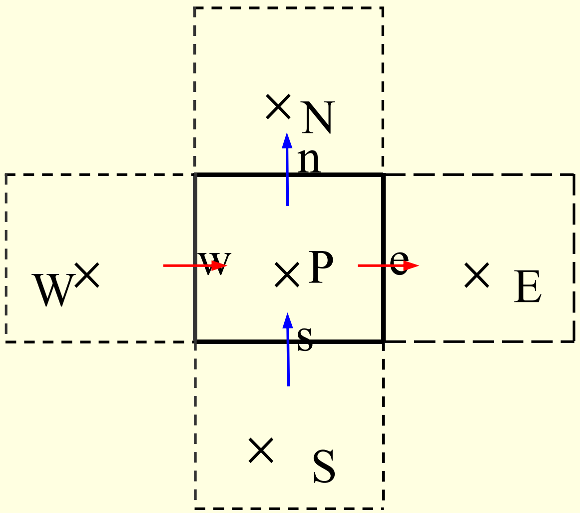
Stokes operator linearization

Momentum- $$\left(a^{\mathbf{u}}_{(\mathrm{i,j,k})} - \frac{3}{2\Delta\tau}\right)\boldsymbol{u}^{\mathrm{n+1}}_{(\mathrm{i,j,k})} + \sum_{(\mathrm{i,j,k})} a^{\mathrm{u}}_{(\mathrm{i,j,k})}\boldsymbol{u}^{\mathrm{n+1}}_{(\mathrm{i,j,k})} - \nabla p^{(n+1)} = RHP^{\mathrm{n}}_{\mathbf{u}}$$

Conservative second order control volume method

Symmetrical coupled Gauss-Seidel smoothing operator (SCGS)

S.P. Vanka (1985) – analytical solution for a ***single*** finite volume



$$(u,v)^{\text{new}} = (u,v)^{\text{old}} + r_{(\text{u,v})}(u,v)'$$

$$p^{\text{new}} = p^{\text{old}} + r_{\text{p}}p'$$

$$A_1 = a_{\text{w}}^{\text{u}} - \frac{3}{2\Delta\tau} \qquad A_2 = a_{\text{e}}^{\text{u}} - \frac{3}{2\Delta\tau}$$

$$A_3 = a_{\text{s}}^{\text{u}} - \frac{3}{2\Delta\tau} \qquad A_4 = a_{\text{n}}^{\text{u}} - \frac{3}{2\Delta\tau}$$

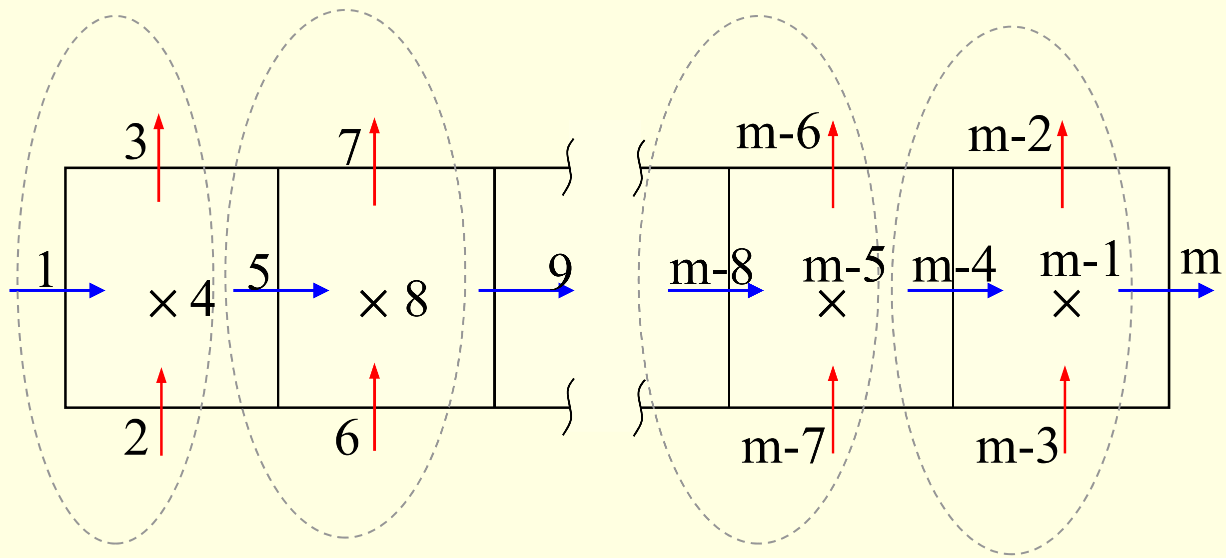
$$\begin{bmatrix} A_1 & 0 & 0 & 0 & -1/(x_{\text{P}}-x_{\text{W}}) \\ 0 & A_2 & 0 & 0 & 1/(x_{\text{E}}-x_{\text{P}}) \\ 0 & 0 & A_3 & 0 & -1/(y_{\text{P}}-y_{\text{S}}) \\ 0 & 0 & 0 & A_4 & 1/(y_{\text{N}}-y_{\text{P}}) \\ -1/(x_{\text{e}}-x_{\text{w}}) & 1/(x_{\text{e}}-x_{\text{w}}) & -1/(y_{\text{n}}-y_{\text{s}}) & 1/(y_{\text{n}}-y_{\text{s}}) & 0 \end{bmatrix} \times \begin{bmatrix} u'^{\,n+1}_{\text{w}} \\ u'^{\,n+1}_{\text{e}} \\ v'^{\,n+1}_{\text{s}} \\ v'^{\,n+1}_{\text{n}} \\ p'^{\,n+1}_{\text{P}} \end{bmatrix} = \begin{bmatrix} R_{\text{uw}} \\ R_{\text{ue}} \\ R_{\text{vs}} \\ R_{\text{vn}} \\ R_{\text{p}} \end{bmatrix}$$

for Stokes operator formulation $A_1 \div A_4$ are constants

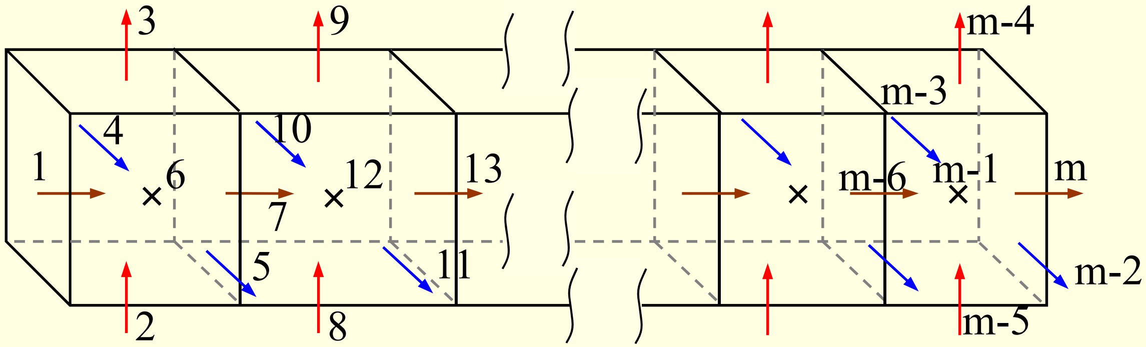
Accelerated coupled line Gauss-Seidel smoother (ASA-CLGS) -2D

Zeng and Wesseling (1993) – CLGS: Horizontal (vertical) sweeping with horizontally (vertically) adjacent pressure linkage

Feldman and Gelfgat (2008) – ASA-CLGS:Horizontal (vertical) sweeping without horizontally (vertically) adjacent pressure linkage



Accelerated coupled line Gauss-Seidel smoother (ASA-CLGS) -3D



Zeng and Wesseling (CLGS, 1993)   Feldman and Gelfgat (ASA-CLGS, 2008)

☑ **Still effective for stretched grids.**

☑ **Still effective for flows with a dominating direction**

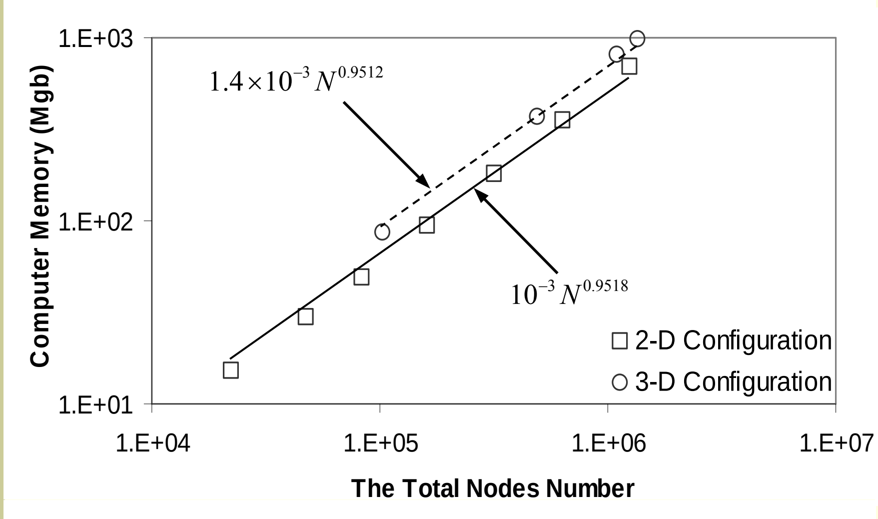
✘ **Block three-diagonal system is to be solved numerically.**

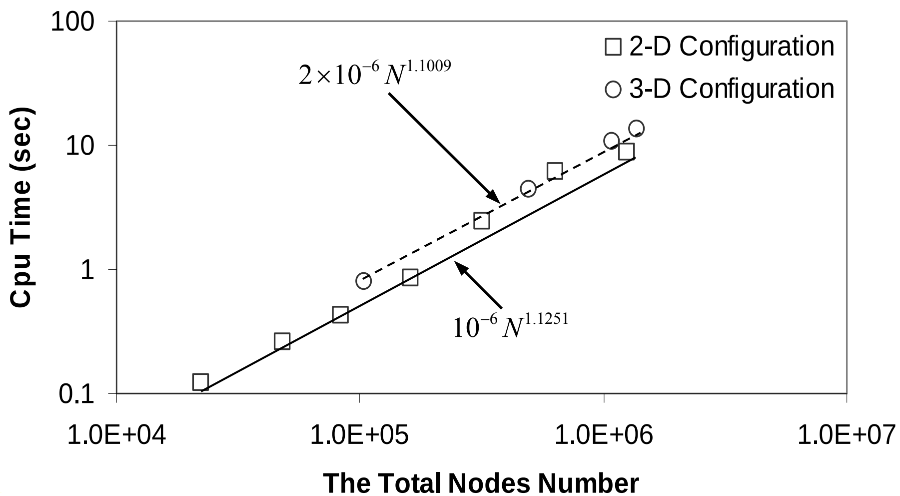
✘ **Increasing amount of arithmetic operations when passing from 2D to 3D geometry or when solving non-linear problems**

☑ **There exists an analytical solution for the entire corrections row (column).**

☑ **Only *O*(5m) operations are needed to obtain the entire row (column) corrections per one sweep for both 2D and 3D geometries**

The Multigrid Characteristics



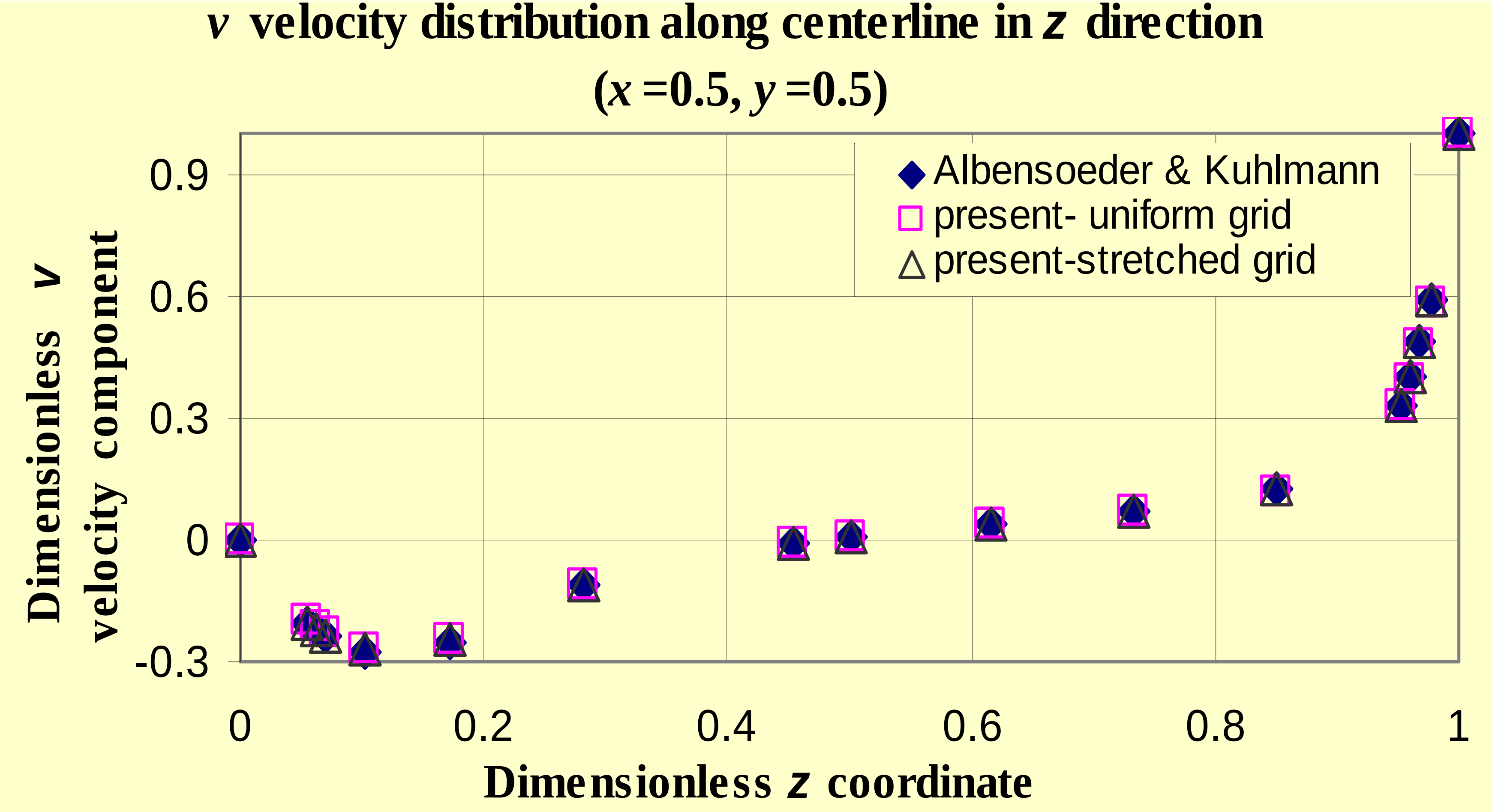


- ☑ Approximately *O*(N) of the CPU memory and time consumption for both 2D and 3D configurations

Cubic lid- driven cavity, grid resolution 103^3

Comparison with Albensoeder & Kuhlmann, 2005.

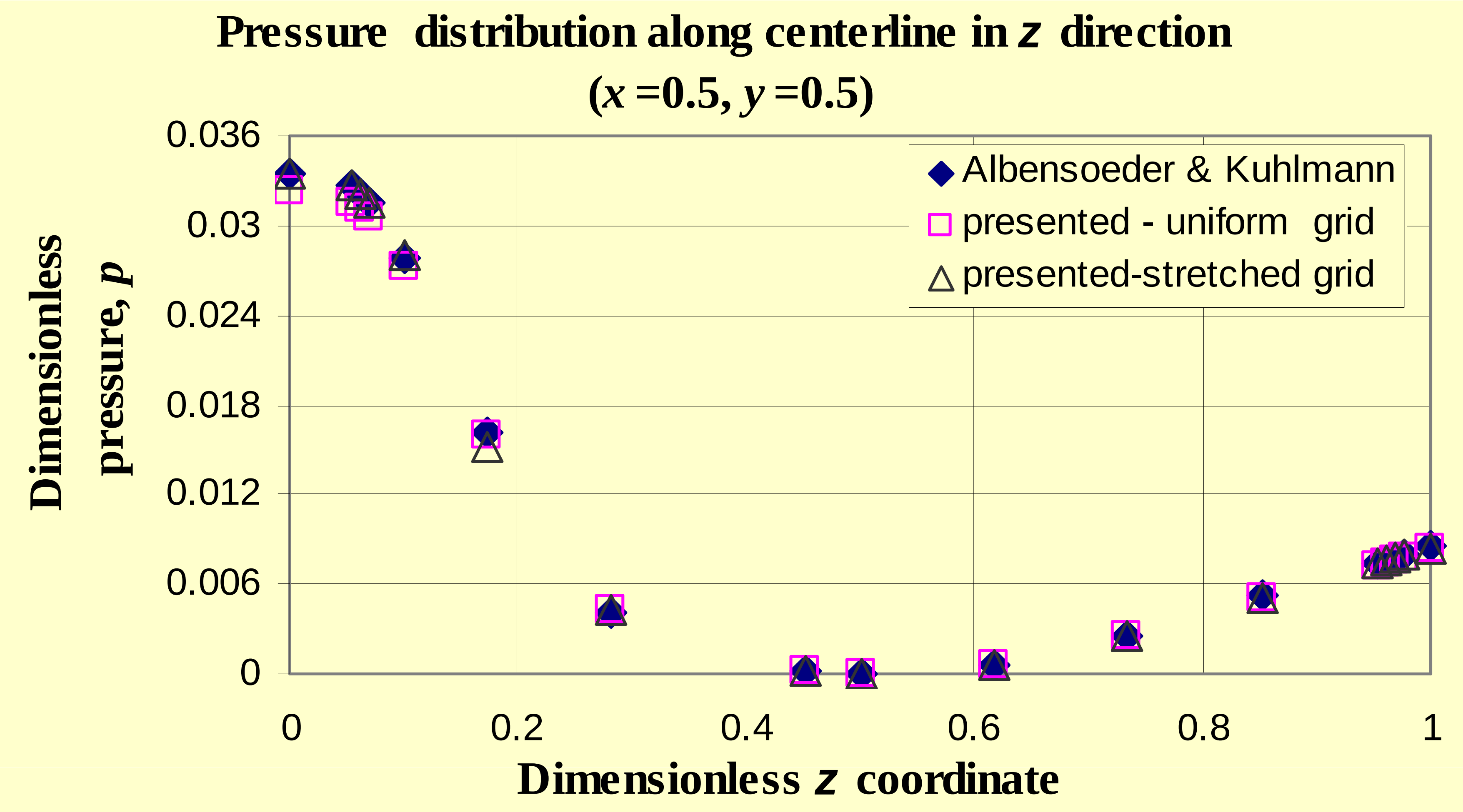
flow at Re= 1000



Cubic lid-driven cavity, grid resolution 103^3 (cont)

Comparison with Albensoeder & Kuhlmann, 2005.

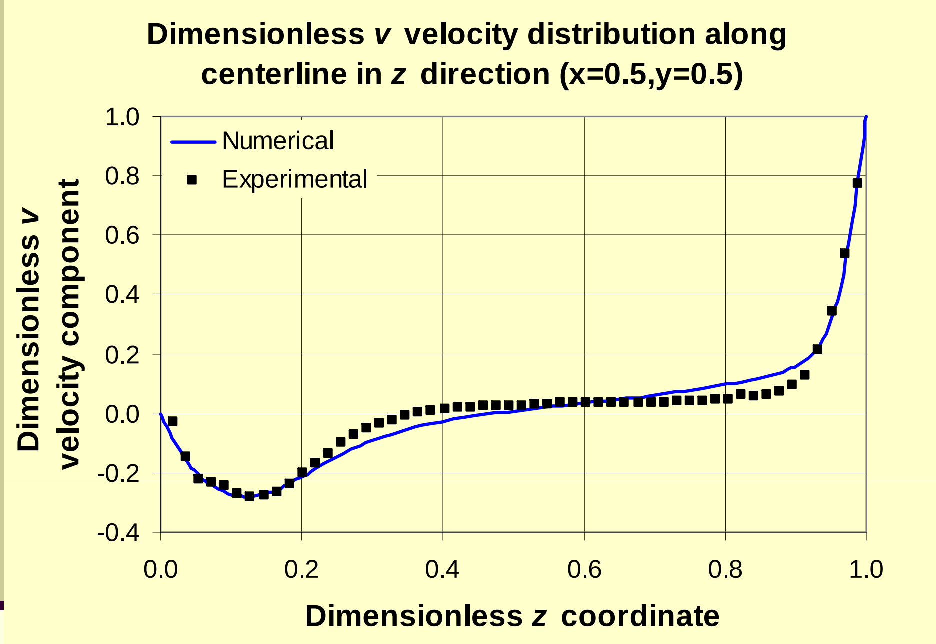
flow at Re= 1000

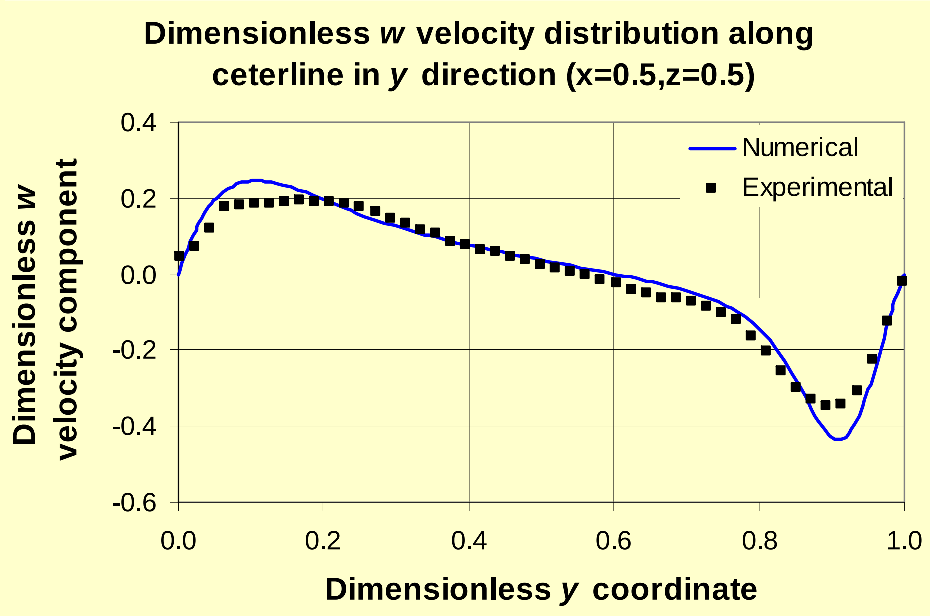


Cubic lid-driven cavity, grid resolution 103^3 (Cont.2)

Comparison with experiments of A. Liberzon, 2008.

flow at Re= 1000

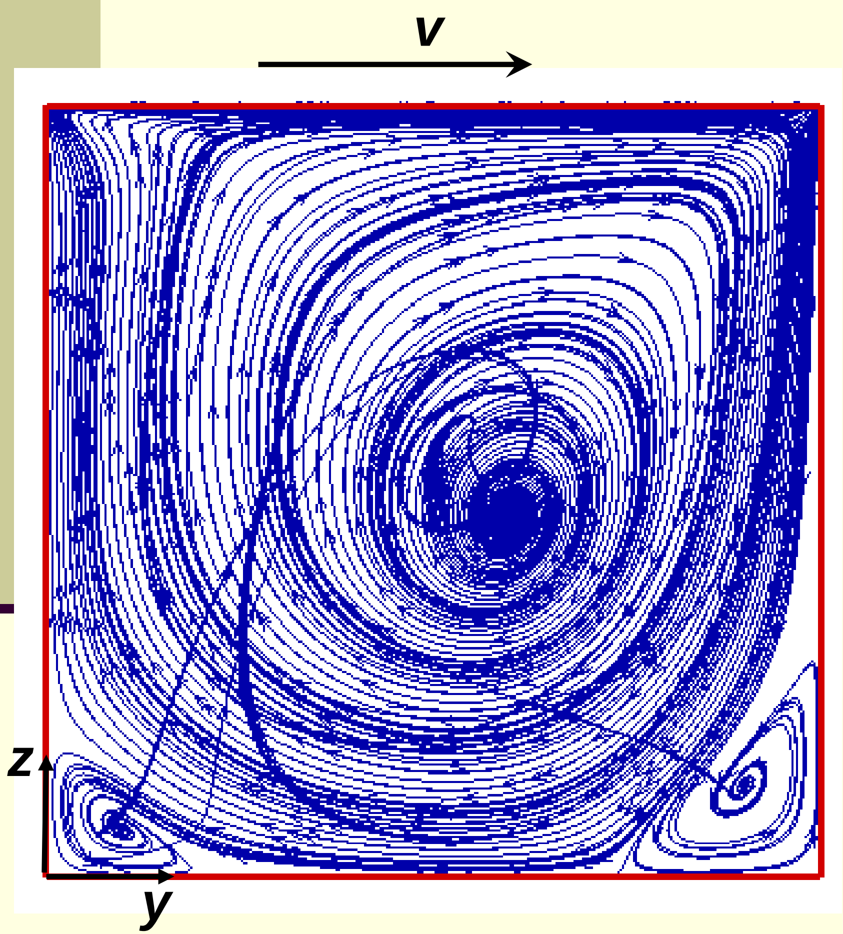


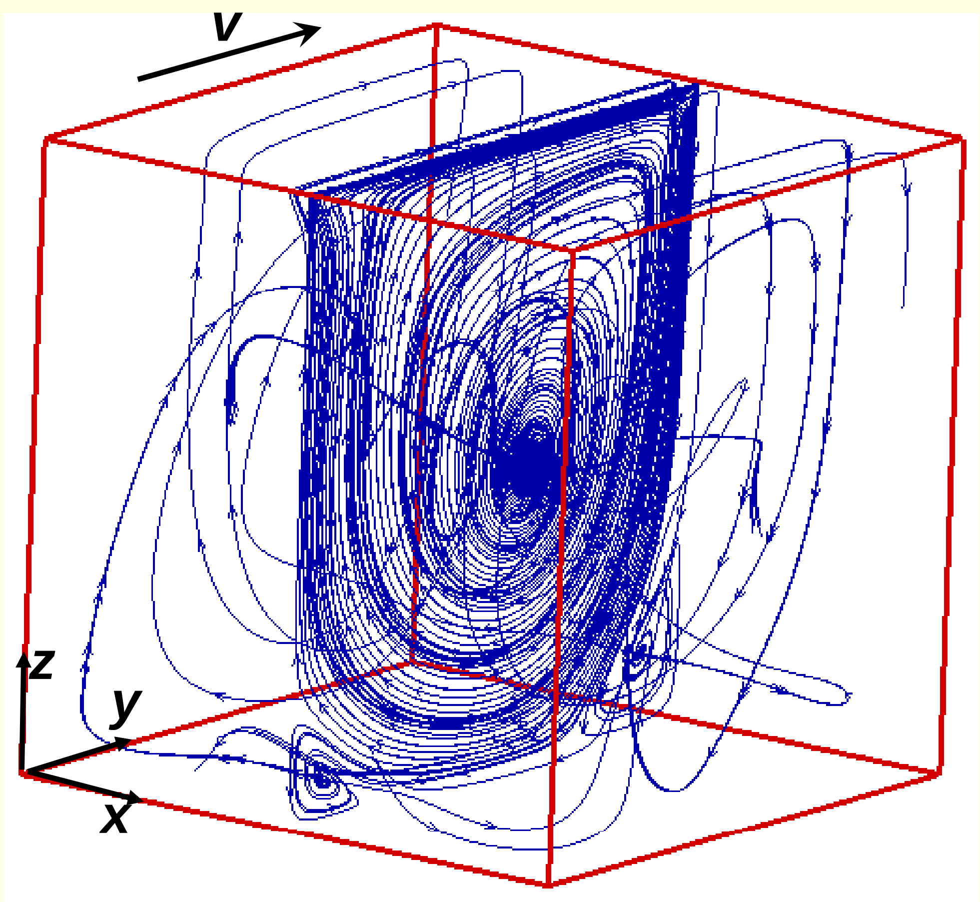


Subproject : which resolution is necessary to fit experimental data with larger Reynolds number ?

Flow visualization of cubic lid-driven cavity. steady state flow, 103^3 nodes

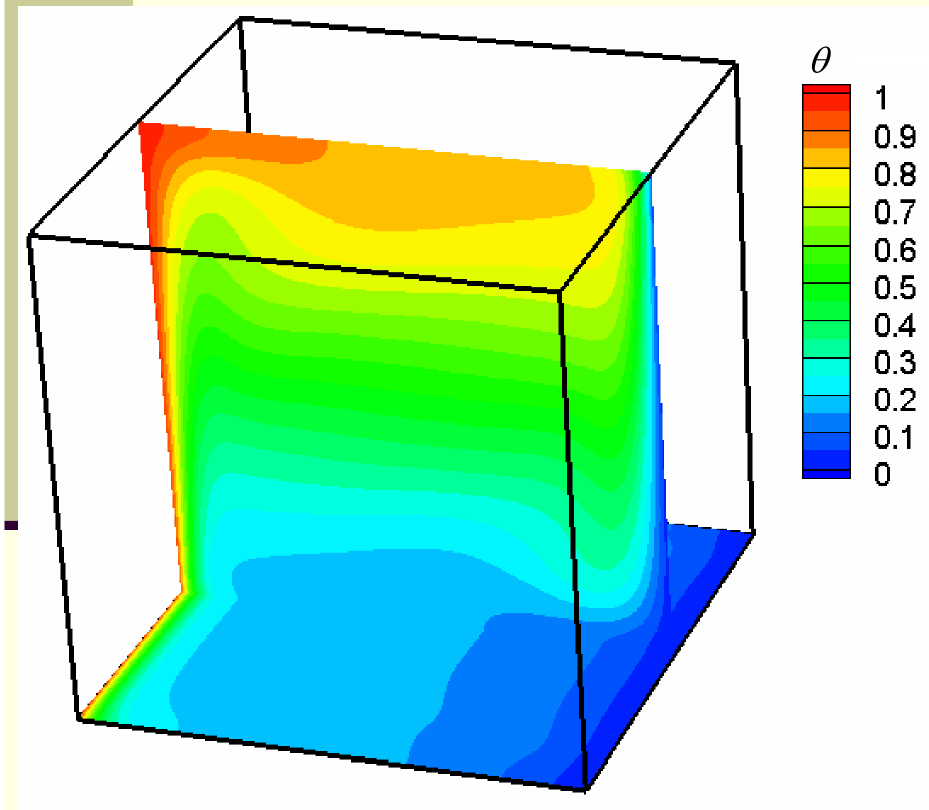
flow at Re= 10^3

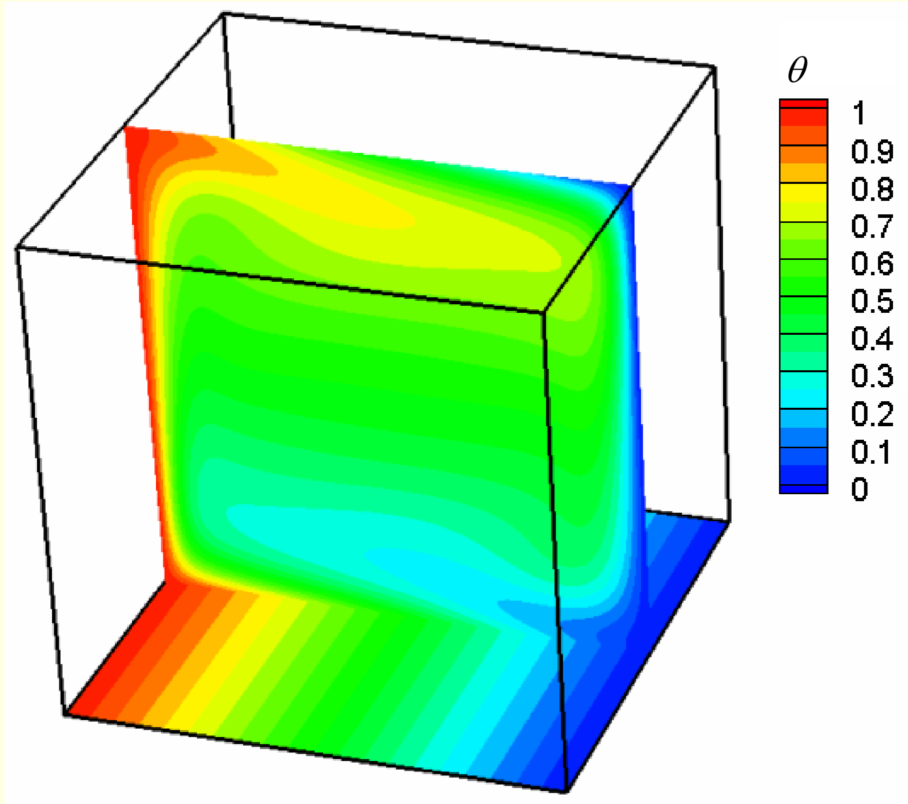




Temperature distribution in a laterally heated cubic cavity, 103^3 nodes

flow at Ra= 10^6





Conclusions

- ✓ **An Accelerated Semi-Analytical Coupled Line Implicit Gauss-Seidel Smoother (ASA-CLGS) was developed and implemented in the inner iteration of the multigrid approach.**
- ✓ **The Navier-Stokes and Boussinesq equations are solved without pressure-velocity decoupling.**
- ✓ **The code was validated on existing benchmark solutions for the lid-driven and thermally driven cavities.**
- ✓ **The approach does not require too large computational recourses allowing to perform 3D calculations on a regular PC.**
- ✓ **The characteristic CPU times consumed for a single time step per one node and per one CPU are of order 5 $\times 10^{-3}$ msec and 10^{-2} msec for 2D and 3D calculations, respectively.**