Linear Stability Analysis of Lid Driven and Convective Flows Accelerated by an Efficient Fully Coupled Time-Marching Algorithm

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Outline

- **Pressure-velocity coupled formulation of the Navier-Stokes equations**
- **The direct and the multigrid approach**
- **Numerical technique**
- **Analytical Solution Accelerated (ASA) smoother**
- **Comparison to existing benchmark solutions**
- **Application to the linear stability analysis**
- **Conclusions**

Incompressible N-S Equations – Numerical Challenge

Continuity - $\nabla \cdot \boldsymbol{u} = 0$

Momentum- $\dfrac{\partial \boldsymbol{u}}{\partial t} + \left(\boldsymbol{u} \cdot \nabla\right)\boldsymbol{u} = -\nabla p + \dfrac{1}{\text{Re}} \nabla^2 \boldsymbol{u}$

- No separate equation for pressure
- No boundary conditions for pressure

Incompressible N-S Equations – Numerical Challenge (Cont.)

Pressure Projection Approach

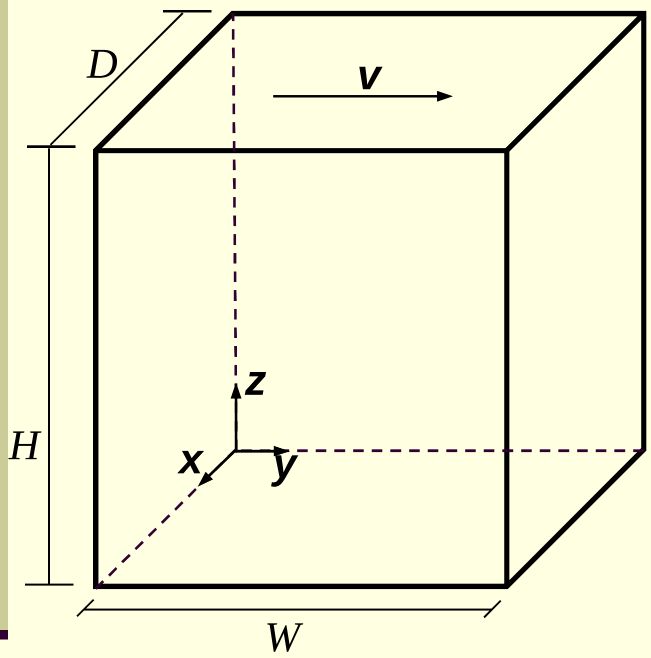
- ✓ High numerical robustness
- ✓ Low memory consumption
- ✘ Slow rate of numerical convergence
- ✘ Non-physical pressure field
- ✘ Not applicable for liquid – solid interface problems

Pressure–Velocity Coupled Approach

- ✓ High rate of numerical convergence
- ✓ The "most natural " way to solve N-S equations
- ✓ The obtained pressure is physical
- ✘ High memory consumption
- ✘ Not as numerically robust as pressure projection methods

Benchmark Problems

Lid-Driven Cubic Cavity



$$\nabla \cdot \boldsymbol{u} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{u}$$

✓**Explicit Discretization**

$$\left(\boldsymbol{u}^n \cdot \nabla\right)\boldsymbol{u}^n$$

▪**Semi-Implicit Discretization**

$$\left(\boldsymbol{u}^n \cdot \nabla\right)\boldsymbol{u}^{n+1}$$

Realistic Boundary Conditions:

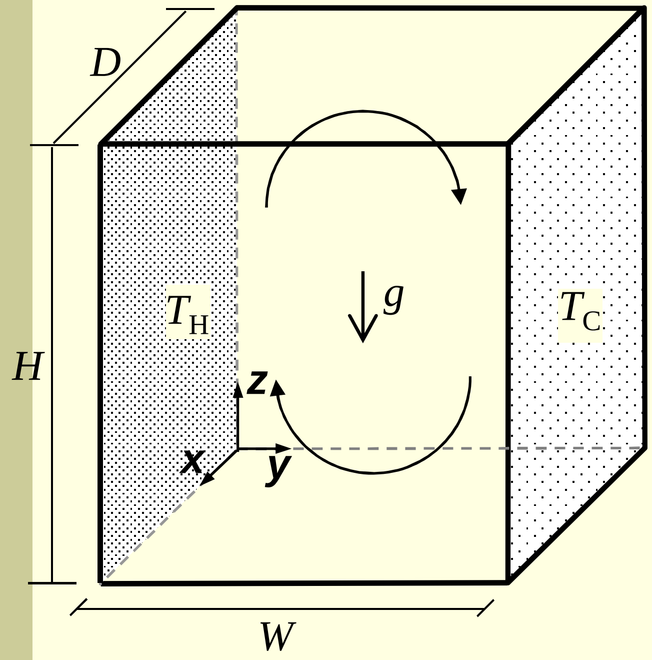
$\boldsymbol{u} = 0$ - at all static walls no slip/no penetration

$\boldsymbol{u}|_{z=H/W} = v$ -at the moving wall the flow velocity is equal to that of the moving wall itself

No boundary condition for pressure is needed

Benchmark Problems (Cont.)

Differently Heated Rectangular and Cubic Cavity (Boussinesq Approximation)



$$\nabla \cdot \boldsymbol{u} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \sqrt{\frac{1}{\mathrm{Gr}}}\nabla^2 \boldsymbol{u} + \theta \vec{e}_z$$

$$\frac{\partial \theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)\theta = \frac{1}{\mathrm{Pr}\sqrt{\mathrm{Gr}}}\nabla^2 \theta$$

✓**Explicit Discretization**

$$(\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n \quad (\boldsymbol{u}^n \cdot \nabla)\theta^{n+1}$$

▪**Semi-Implicit Discretization**

$$(\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^{n+1} \quad (\boldsymbol{u}^n \cdot \nabla)\theta^{n+1}$$

Boundary Conditions:

$\theta|_{y=0} = 1,\ \theta|_{y=H/W} = 0$ **-isothermal vertical walls,** $\frac{\partial \theta}{\partial \boldsymbol{n}} = 0\ or\ \theta = 1 - y$ **-horizontal and lateral walls**

$\boldsymbol{u} = 0$ **-at all walls,**

No boundary condition for pressure is needed

Time and spatial discretization

Second order backward differentiation - $\frac{\partial f^{n+1}}{\partial t} = \frac{3f^{n+1} - 4f^{n} + f^{n-1}}{2\Delta t} + O\left(\Delta t^2\right)$

Stokes operator linearization ➡ Temperature – velocity decoupling

Energy -

$$\left(a^{\theta}_{(\mathrm{i,j,k})} - \frac{3}{2\Delta\tau}\right)\theta^{\mathrm{n+1}}_{(\mathrm{i,j,k})} + \sum_{\mathrm{i,j,k}} a^{\theta}_{\mathrm{i,j,k}}\theta^{\mathrm{n+1}}_{\mathrm{i,j,k}} = RHP^{\mathrm{n}}_{\theta}$$

Continuity -

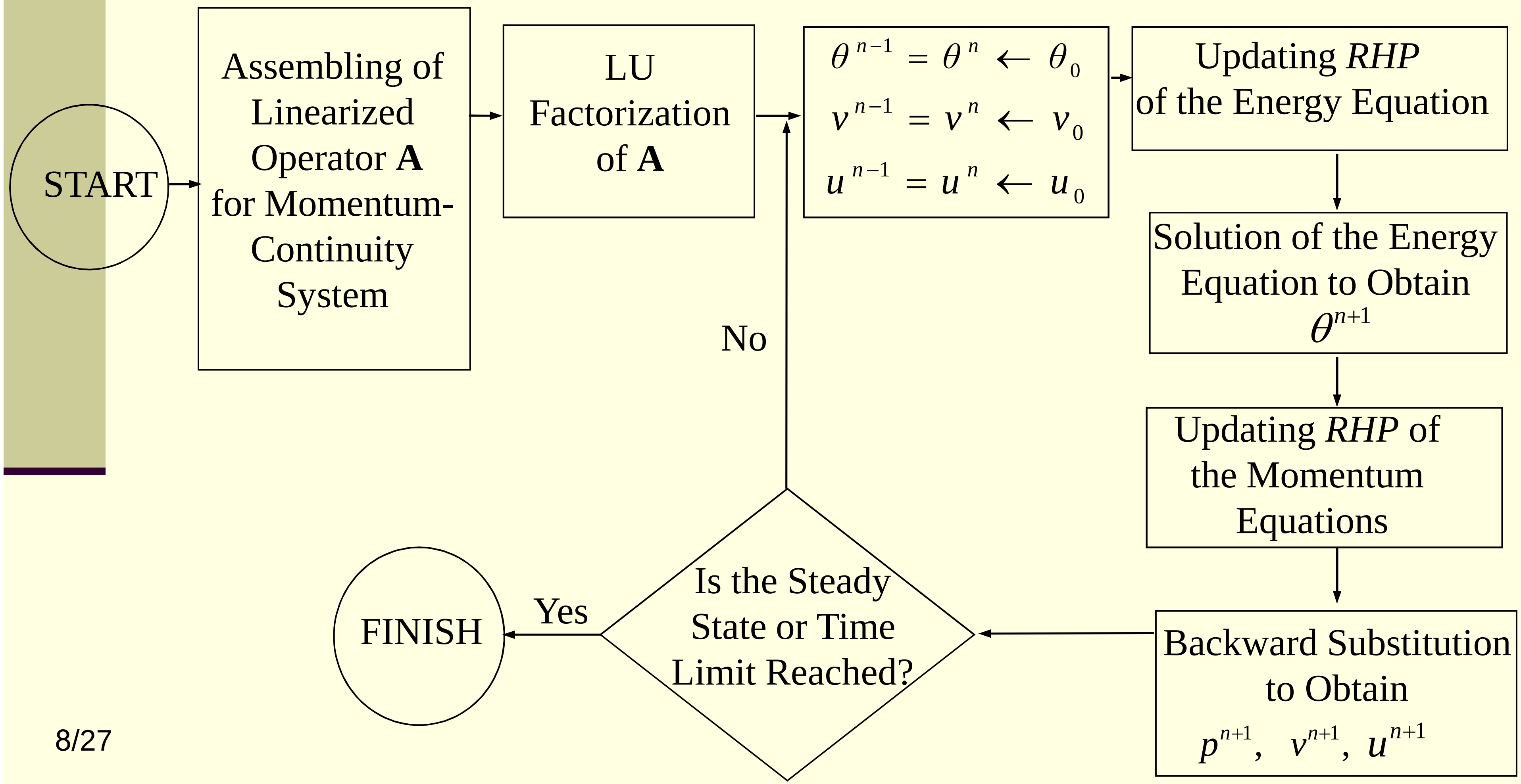
$$\frac{\left(u^{\mathrm{n+1}}_{(\mathrm{i,j,k})} - u^{\mathrm{n+1}}_{(\mathrm{i-1,j,k})}\right)}{Hx(i-1)} + \frac{\left(v^{\mathrm{n+1}}_{(\mathrm{i,j,k})} - v^{\mathrm{n+1}}_{(\mathrm{i,j-1,k})}\right)}{Hy(j-1)} + \frac{\left(w^{\mathrm{n+1}}_{(\mathrm{i,j,k})} - w^{\mathrm{n+1}}_{(\mathrm{i,j,k-1})}\right)}{Hz(k-1)} = 0$$

Momentum-

$$\left(a^{\mathbf{u}}_{(\mathrm{i,j,k})} - \frac{3}{2\Delta\tau}\right)\boldsymbol{u}^{\mathrm{n+1}}_{(\mathrm{i,j,k})} + \sum_{(\mathrm{i,j,k})} a^{\mathrm{u}}_{(\mathrm{i,j,k})}\boldsymbol{u}^{\mathrm{n+1}}_{(\mathrm{i,j,k})} - \nabla p^{(n+1)} = RHP^{\mathrm{n}}_{\mathbf{u}}$$

Conservative second order control volume method

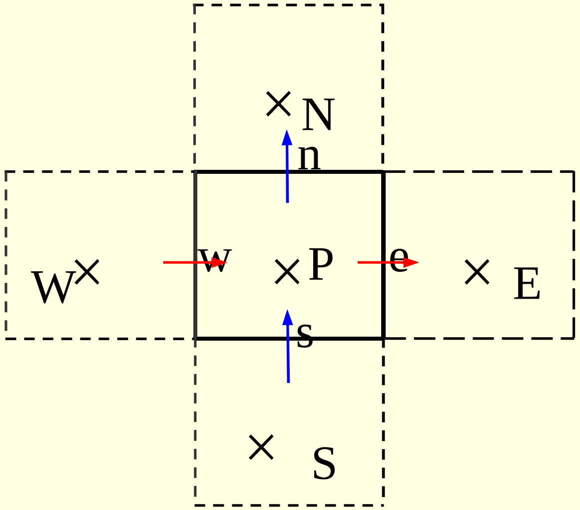
The Full Pressure Coupled Direct (FPCD) Solution



The Multigrid Algorithm

Symmetrical Coupled Gauss-Seidel Smoothing Operator (SCGS)

S.P. Vanka (1985) – analytical solution for a ***single*** finite volume



$$(u,v)^{\text{new}} = (u,v)^{\text{old}} + r_{(\text{u,v})}(u,v)'$$

$$p^{\text{new}} = p^{\text{old}} + r_{\text{p}} p'$$

$$A_1 = a_e^u - \frac{3}{2\Delta\tau} \qquad A_3 = a_w^u - \frac{3}{2\Delta\tau}$$

$$A_5 = a_n^u - \frac{3}{2\Delta\tau} \qquad A_9 = a_s^u - \frac{3}{2\Delta\tau}$$

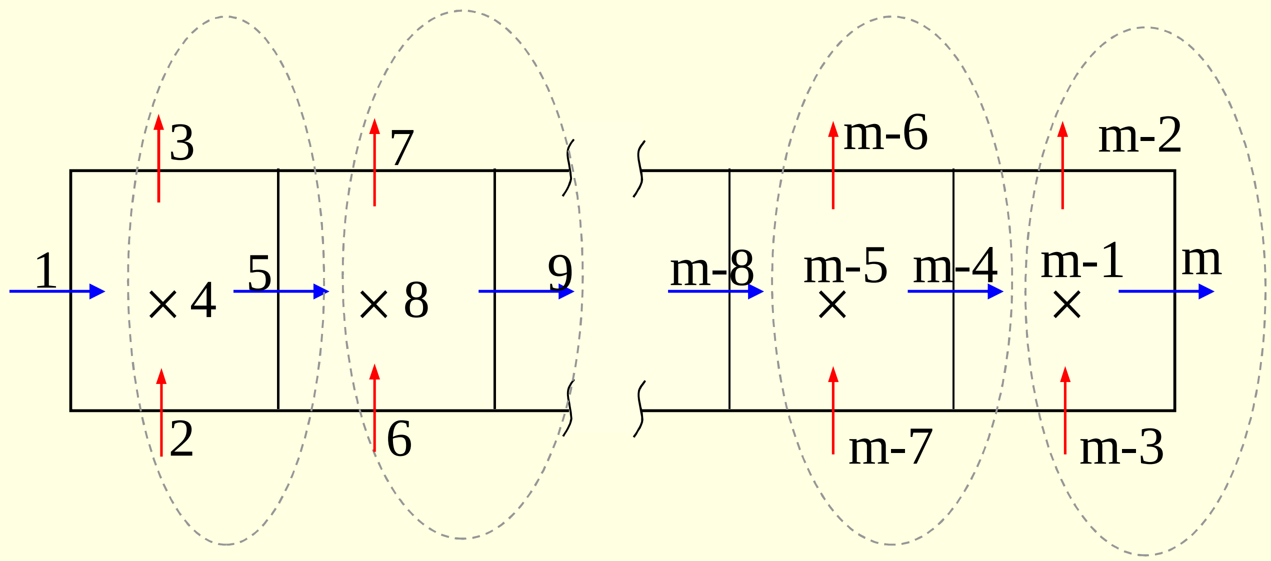
$$\begin{bmatrix} A_1 & 0 & 0 & 0 & A_2 \\ 0 & A_3 & 0 & 0 & A_4 \\ 0 & 0 & A_5 & 0 & A_6 \\ 0 & 0 & 0 & A_9 & A_{10} \\ A_7 & -A_7 & A_8 & -A_8 & 0 \end{bmatrix} \times \begin{bmatrix} u'_{\text{e}} \\ u'_{\text{w}} \\ v'_{\text{n}} \\ v'_{\text{s}} \\ p'_{\text{P}} \end{bmatrix} = \begin{bmatrix} R_{\text{ue}} \\ R_{\text{uw}} \\ R_{\text{vn}} \\ R_{\text{vs}} \\ R_{\text{cp}} \end{bmatrix}$$

for Stokes operator and constant time step formulation A_1, A_3, A_5, A_9 are constants

Accelerated Coupled Line Gauss-Seidel Smoother (ASA-CLGS) -2D

Zeng and Wesseling (1993) – CLGS: Horizontal (vertical) sweeping with horizontally (vertically) adjacent pressure linkage

Feldman and Gelfgat (2008) – ASA-CLGS:Horizontal (vertical) sweeping without horizontally (vertically) adjacent pressure linkage



Accelerated Coupled Line Gauss-Seidel Smoother (ASA-CLGS) -2D, (Cont)

Zeng and Wesseling (1993) – CLGS:

$$\ldots$$

$$A^{(x)}_{i+1/2,j}u'_{i+1/2,j}+B^{(x)}_{i+1/2,j}p'_{i,j}=R^{(x)}_{i+1/2,j}$$

$$A^{(x)}_{i-1/2,j}u'_{i-1/2,j}-B^{(x)}_{i-1/2,j}p'_{i,j}=R^{(x)}_{i-1/2,j}$$

$$A^{(y)}_{i,j+1/2}v'_{ij+1/2}-B^{(y)}_{i,j+1/2}\left(p'_{i,j+1}-p'_{i,j}\right)=R^{(y)}_{i,j+1/2}$$

$$A^{(x)}_{i,j}\left(u'_{i+1/2,j}-u'_{i-1/2,j}\right)+A^{(y)}_{i,j}\left(v'_{i,j+1/2}-v'_{i,j-1/2}\right)=0$$

$$\ldots$$

Feldman and Gelfgat (2008) – ASA-CLGS:

$$\ldots$$

$$A^{(x)}_{i+1/2,j}u'_{i+1/2,j}+B^{(x)}_{i+1/2,j}p'_{i,j}=R^{(x)}_{i+1/2,j}$$

$$A^{(x)}_{i-1/2,j}u'_{i-1/2,j}-B^{(x)}_{i-1/2,j}p'_{i,j}=R^{(x)}_{i-1/2,j}$$

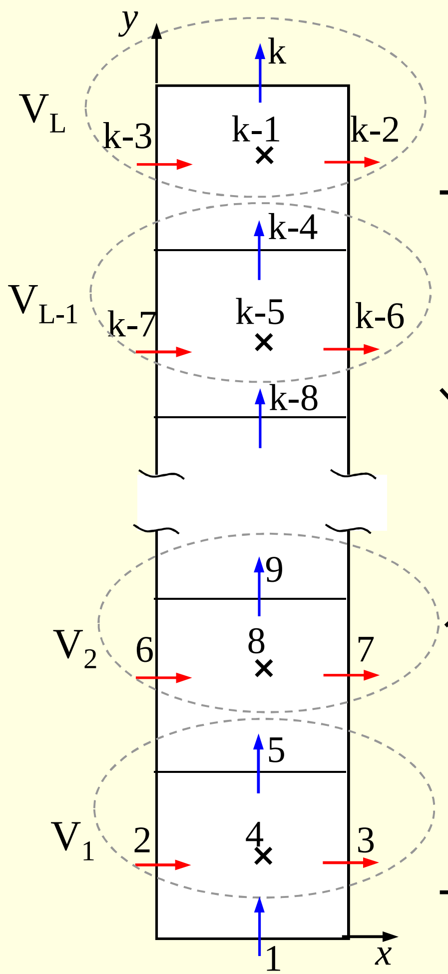
$$A^{(y)}_{i,j+1/2}v'_{ij+1/2}+B^{(y)}_{i,j+1/2}p'_{i,j}=\widetilde{R}^{(y)}_{i,j+1/2}$$

$$A^{(x)}_{i,j}\left(u'_{i+1/2,j}-u'_{i-1/2,j}\right)+A^{(y)}_{i,j}\left(v'_{i,j+1/2}-v'_{i,j-1/2}\right)=0$$

$$\ldots$$

where $\widetilde{R}^{(y)}_{i,j+1/2}=R^{(y)}_{i,j+1/2}+B^{(y)}_{i,j+1/2}p'_{i,j+1}$

A Schematic Description of ASA-CLGS Smoother



$$\begin{bmatrix} A_1^L & 0 & 0 & 0 & A_2^L \\ 0 & A_3^L & 0 & 0 & A_4^L \\ 0 & 0 & A_5^L & 0 & A_6^L \\ 0 & 0 & 0 & 1 & 0 \\ A_7^L & -A_7^L & A_8^L & -A_8^L & 0 \end{bmatrix} \times \begin{bmatrix} u'_{\text{k-2}} \\ u'_{\text{k-3}} \\ v'_{\text{k}} \\ v'_{\text{k-4}} \\ p'_{\text{k-1}} \end{bmatrix} = \begin{bmatrix} R_{\text{k-2}} \\ R_{\text{k-3}} \\ R_{\text{k}} \\ R_{\text{k-4}} \\ R_{\text{k-1}} \end{bmatrix} \Rightarrow \begin{array}{l} u'_{\text{k-2}} = \alpha_{\text{k-2}}^L p'_{\text{k-1}} + \beta_{k-2}^L \\ u'_{\text{k-3}} = \alpha_{\text{k-3}}^L p'_{\text{k-1}} + \beta_{k-3}^L \\ v'_{\text{k}} = \textit{from b.c.} \\ v'_{\text{k-4}} \ \textit{from volume } L-1 \\ p'_{\text{k-1}} = \alpha_{\text{k-1}}^L v'_{\text{k-4}} + \beta_{k-1}^L \end{array}$$

$$\begin{bmatrix} A_1^I & 0 & 0 & 0 & A_2^I \\ 0 & A_3^I & 0 & 0 & A_4^I \\ 0 & 0 & A_5^I & 0 & A_6^I \\ 0 & 0 & 0 & 1 & 0 \\ A_7^I & -A_7^I & A_8^I & -A_8^I & 0 \end{bmatrix} \times \begin{bmatrix} u'_{\text{m-2}} \\ u'_{\text{m-3}} \\ v'_{\text{m}} \\ v'_{\text{m-4}} \\ p'_{\text{m-1}} \end{bmatrix} = \begin{bmatrix} R_{\text{m-2}} \\ R_{\text{m-3}} \\ R_{\text{m}} \\ R_{\text{m-4}} \\ R_{\text{m-1}} \end{bmatrix} \Rightarrow \begin{array}{l} u'_{\text{m-2}} = \alpha_{\text{m-2}}^L p'_{\text{m-1}} + \beta_{m-2}^L \\ u'_{\text{m-3}} = \alpha_{\text{m-3}}^L p'_{\text{m-1}} + \beta_{m-3}^L \\ v'_{\text{m}} = \alpha_{\text{m}}^L p'_{\text{m-1}} + \beta_{m-4}^L \\ v'_{\text{m-4}} \ \textit{from volume } M-1 \\ p'_{\text{m-1}} = \alpha_{\text{m-1}}^L v'_{\text{m-4}} + \beta_{m-1}^L \end{array}$$

$$\begin{bmatrix} A_1^1 & 0 & 0 & 0 & A_2^1 \\ 0 & A_3^1 & 0 & 0 & A_4^1 \\ 0 & 0 & A_5^1 & 0 & A_6^1 \\ 0 & 0 & 0 & 1 & 0 \\ A_7^1 & -A_7^1 & A_8^1 & -A_8^1 & 0 \end{bmatrix} \times \begin{bmatrix} u'_3 \\ u'_2 \\ v'_5 \\ v'_1 \\ p'_4 \end{bmatrix} = \begin{bmatrix} R_3 \\ R_2 \\ R_5 \\ R_1 \\ R_4 \end{bmatrix} \Rightarrow \begin{array}{l} u'_3 = \alpha_3^L p'_4 + \beta_3^L \\ u'_2 = \alpha_2^L p'_4 + \beta_2^L \\ v'_5 = \alpha_5^L p'_4 + \beta_5^L \\ v'_1 \ \textit{from b.c.} \\ p'_4 = \alpha_4^L v'_1 + \beta_4^L \end{array}$$

CLGS and ASA-CLGS Efficiency Estimation for 2D

Zeng and Wesseling (1993) – CLGS:

↓

Block 3-diagonal matrix or 7-diagonal matrix

↓

LU decomposition

↓

$\approx O(15M)$

Feldman and Gelfgat (2008) – ASA-CLGS (6-Diagonal Matrix)

↓

$$p'_{k-1} = (c_1^L v'_{k-4} + R_{k-1}^L + \sum_{i=2}^{4} c_i^L R_{k-i}^L)/c_5^L$$

↓

$$\begin{bmatrix} v'_5 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} c_6^1 \\ c_7^1 \\ c_8^1 \end{bmatrix} \times p'_4 + \begin{bmatrix} c_9^1 R_5^L \\ c_{10}^1 R_2^L \\ c_{11}^1 R_3^L \end{bmatrix}$$

↓

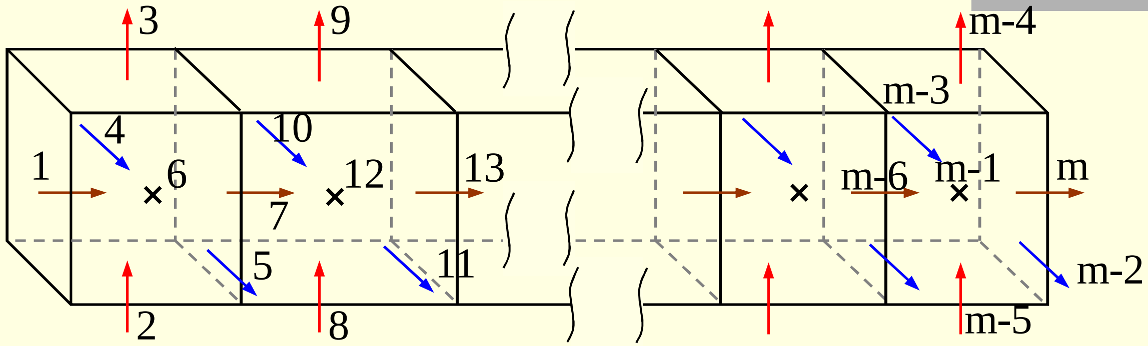
$\approx O(5M)$ ⟷ $\approx O(5M)$

Thomas Algorithm (3-Diagonal Matrix)

↓

$\approx O(5M)$

ASA-CLGS -Efficiency Estimation for 3D



$$p'_p = \left(c_1^I w'_d + R_p^I + c_2^I R_e^I + c_3^I R_w^I + c_4^I R_n^I + c_5^I R_5^I + c_6^I R_d^I\right) / c_7^I$$

$$\begin{bmatrix} w'_d \\ v'_s \\ v'_n \\ u'_w \\ u'_e \end{bmatrix} = \begin{bmatrix} c_8^I \\ c_9^I \\ c_{10}^I \\ c_{11}^I \\ c_{12}^I \end{bmatrix} \times p'_p + \begin{bmatrix} c_{13}^I R_d^I \\ c_{14}^I R_s^I \\ c_{15}^I R_n^I \\ c_{16}^I R_w^I \\ c_{17}^I R_e^I \end{bmatrix}$$

6 corrections for a single volume result in 17 multiplications and divisions and 11 summations

$$\approx O(5M)$$

Advantages of ASA-CLGS Approach

Zeng and Wesseling (CLGS, 1993)

Feldman and Gelfgat (ASA-CLGS, 2008)

☑ **Still effective for stretched grids.**

☑ **Still effective for flows with a dominating direction**

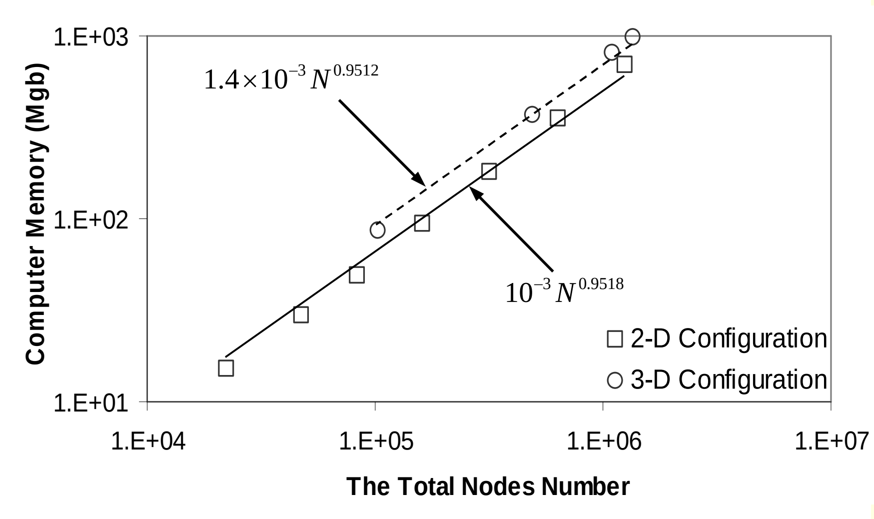
✘ **Block three-diagonal system is to be solved numerically.**

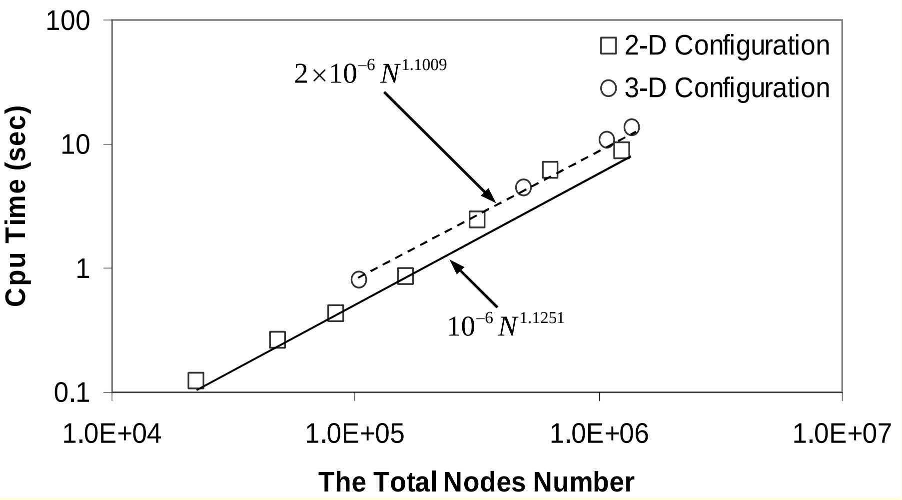
✘ **Increased number of arithmetic operations when transferring from 2D to 3D geometry**

☑ **There exists an analytical solution for the entire corrections row (column).**

☑ **Only *O*(5M) operations are needed to obtain the entire row (column) corrections per one sweep for both 2D and 3D geometries**

The Multigrid Characteristics





☑ Approximately *O*(N) of the CPU memory and time consumption for both 2D and 3D configurations

FPCD and Multigrid Approaches – Pros and Cons

FPCD

☑Independent of operating conditions : Δt magnitude, Re, Gr numbers

☑Good Scalability of LU decomposition

✘ Extremely memory demanding for 3D (3D calculations is still a challenge)

✘ Still insufficient scalability of back substitution process

Multigrid

☑ Very effective for small time steps.

☑ Very good scalability for both 2D and 3D configurations.

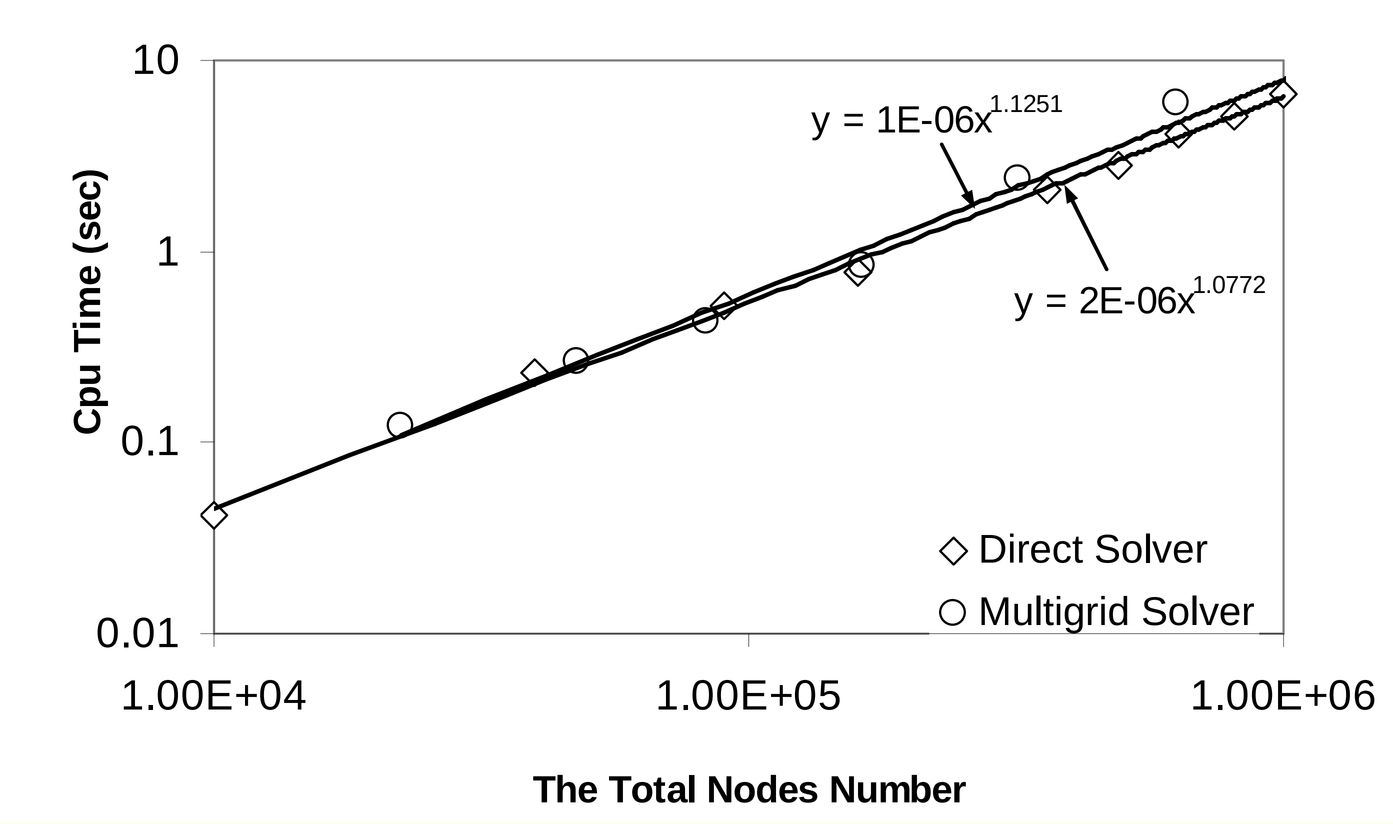
☑ Small memory consumption.

✘ Performance of the method depends on operating conditions

✘ Performnce of the method depends on initial guess

FPCD and Multigrid Approaches – Pros and Cons (Cont)

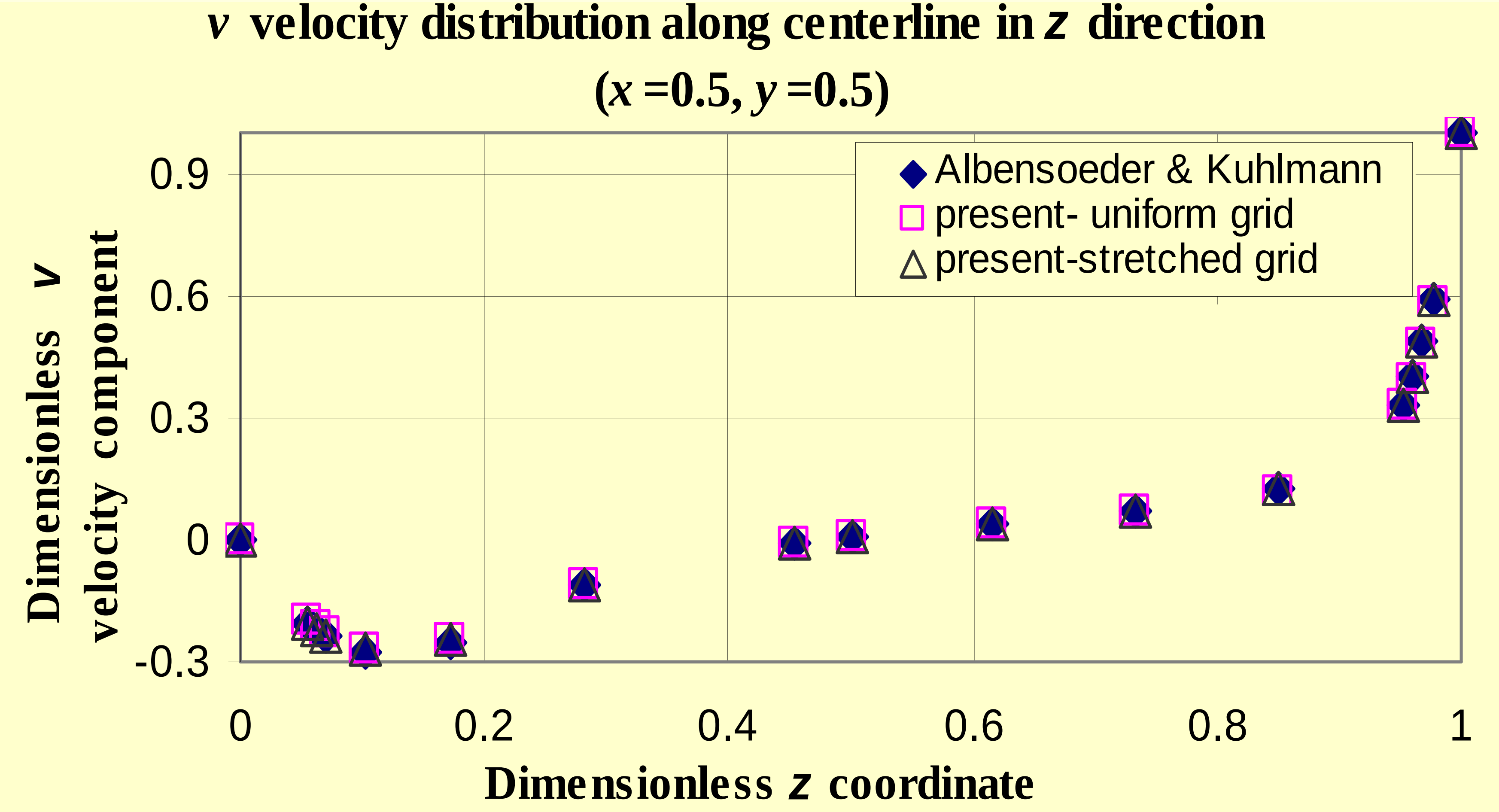
CPU time consumption for one time step -2D configuration



Cubic lid- driven cavity, grid resolution 103^3

Comparison with Albensoeder & Kuhlmann, 2005.

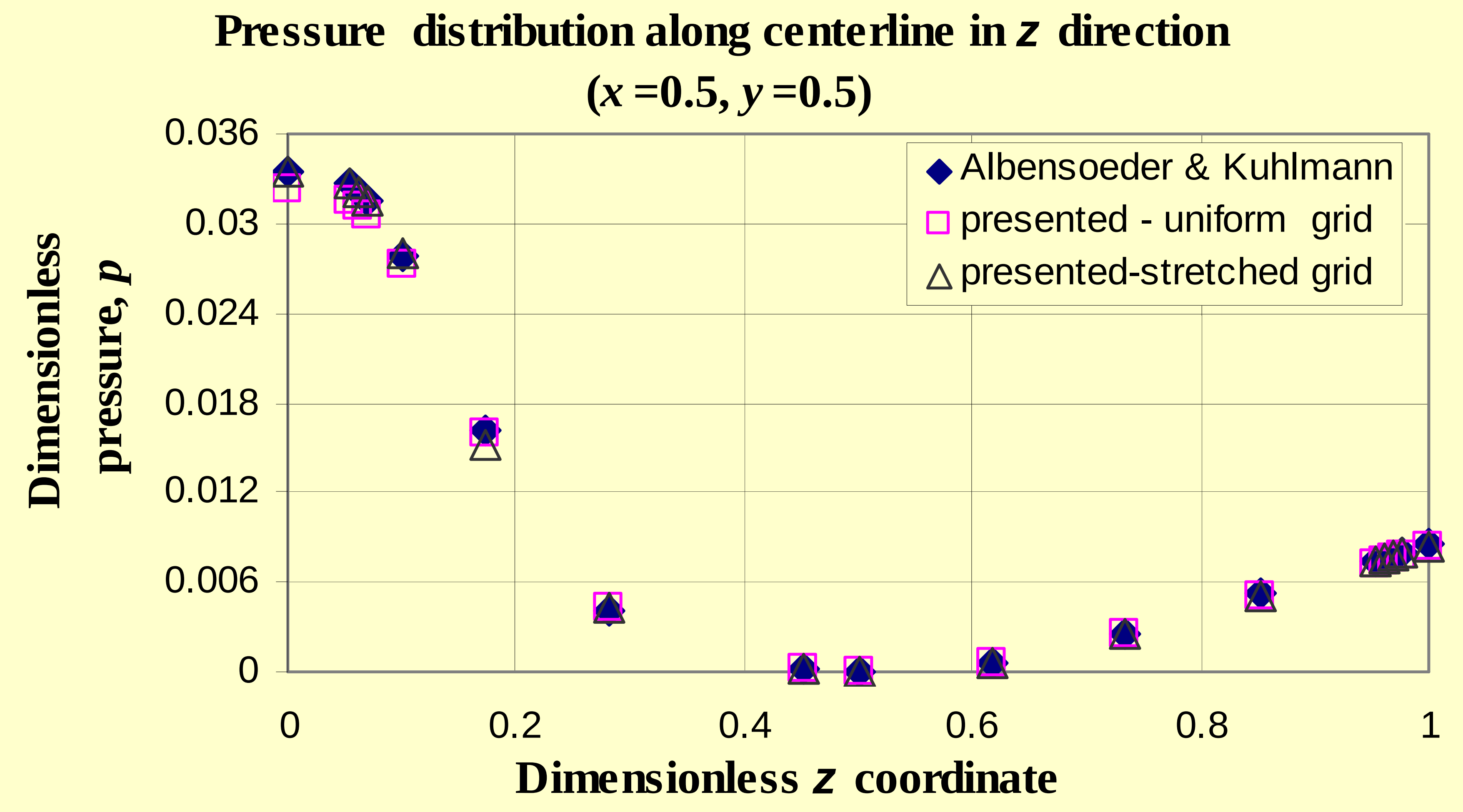
Flow at Re= 1000



Cubic lid-driven cavity, grid resolution 103^3 (cont)

Comparison with Albensoeder & Kuhlmann, 2005.

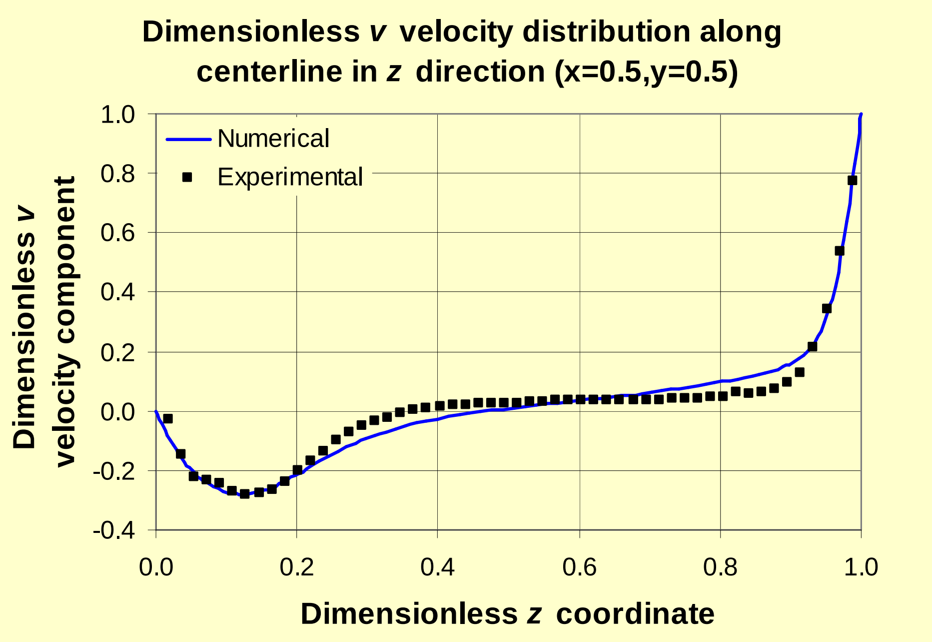
Flow at Re= 1000

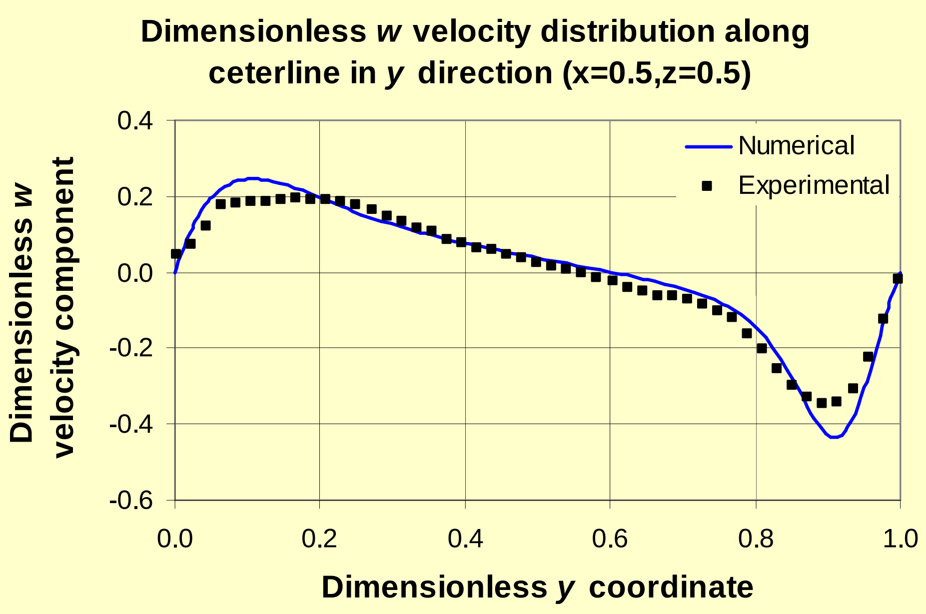


Cubic lid-driven cavity, grid resolution 103^3 (Cont.2)

Comparison with Experiments of A. Liberzon, 2008.

Flow at Re= 1000

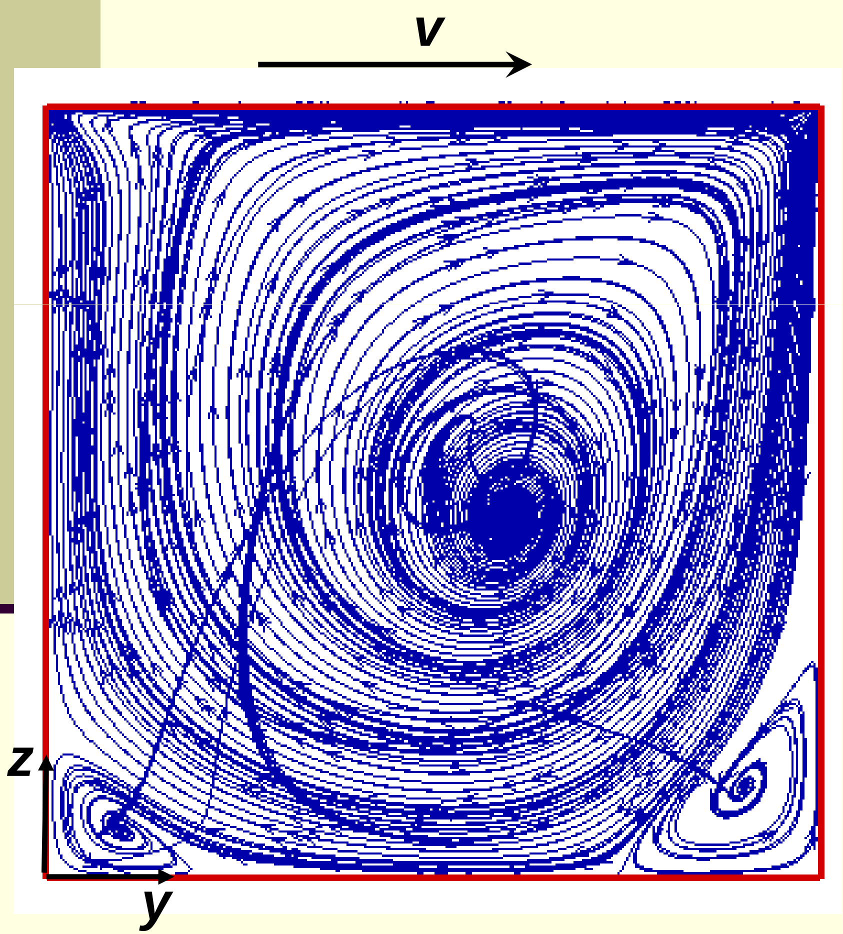


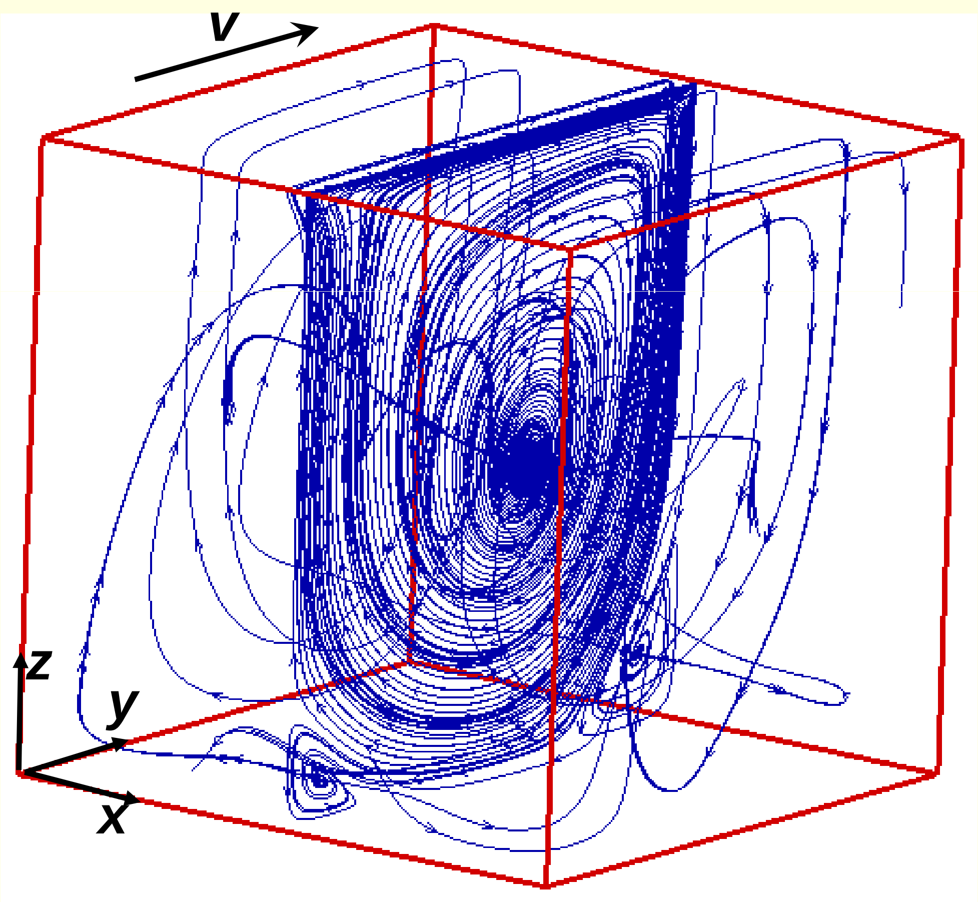


Subproject : which resolution is necessary to fit experimental data with larger Reynolds number ?

Flow Visualization of Cubic Lid-Driven Cavity. Steady State Flow, 103^3 Nodes

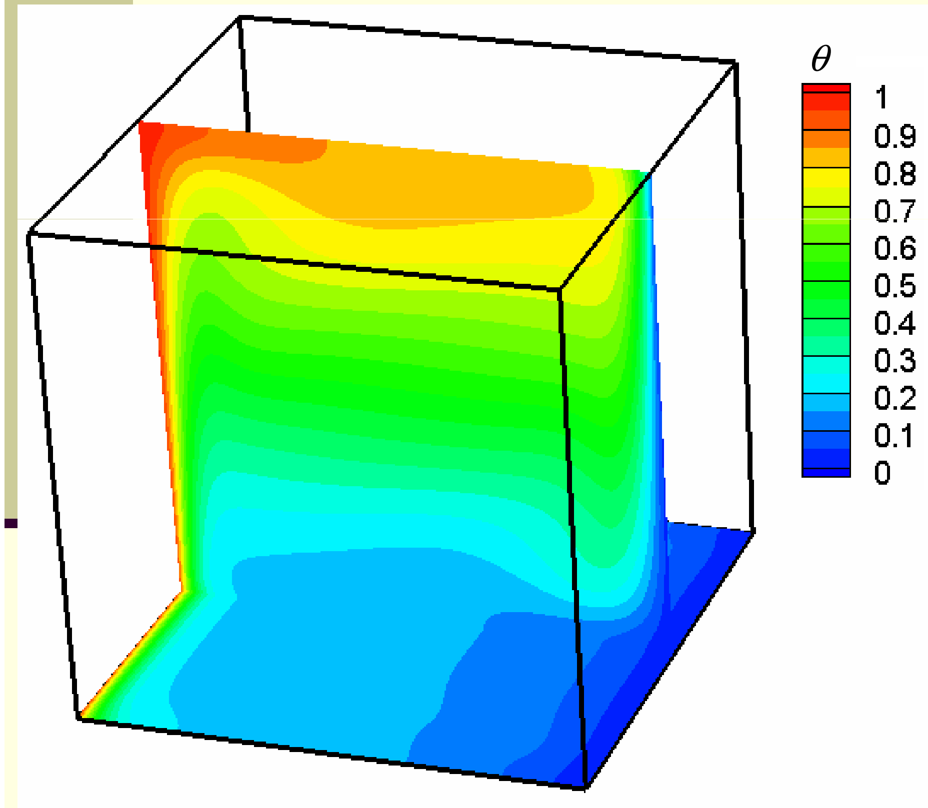
flow at Re= 10^3

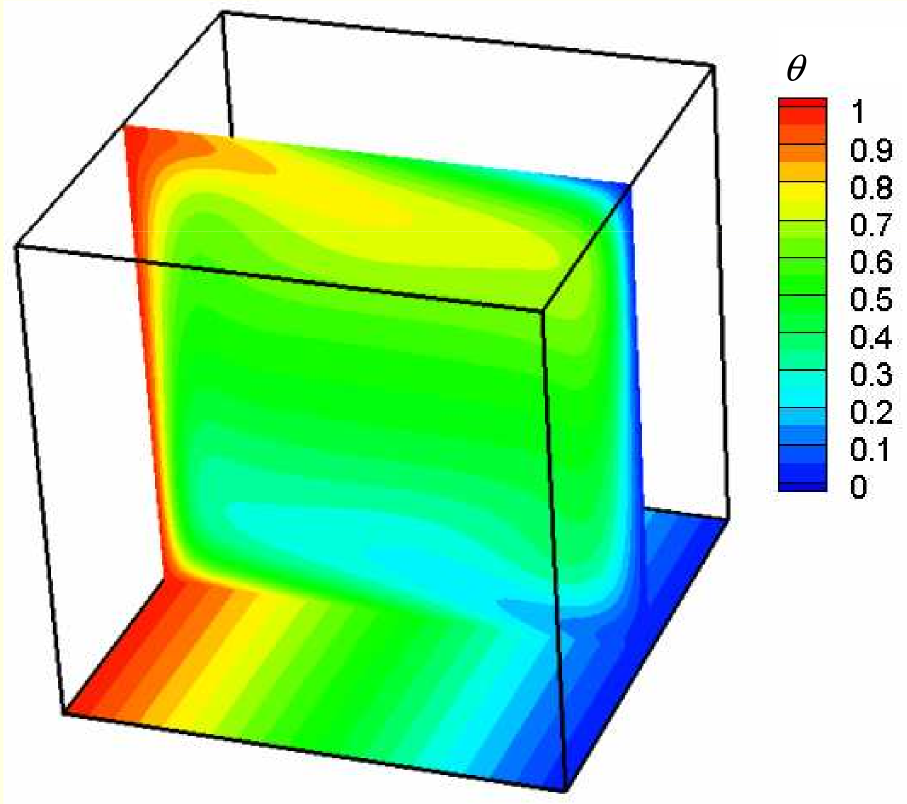




Temperature Distribution in a Laterally Heated Cubic Cavity, 103^3 Nodes

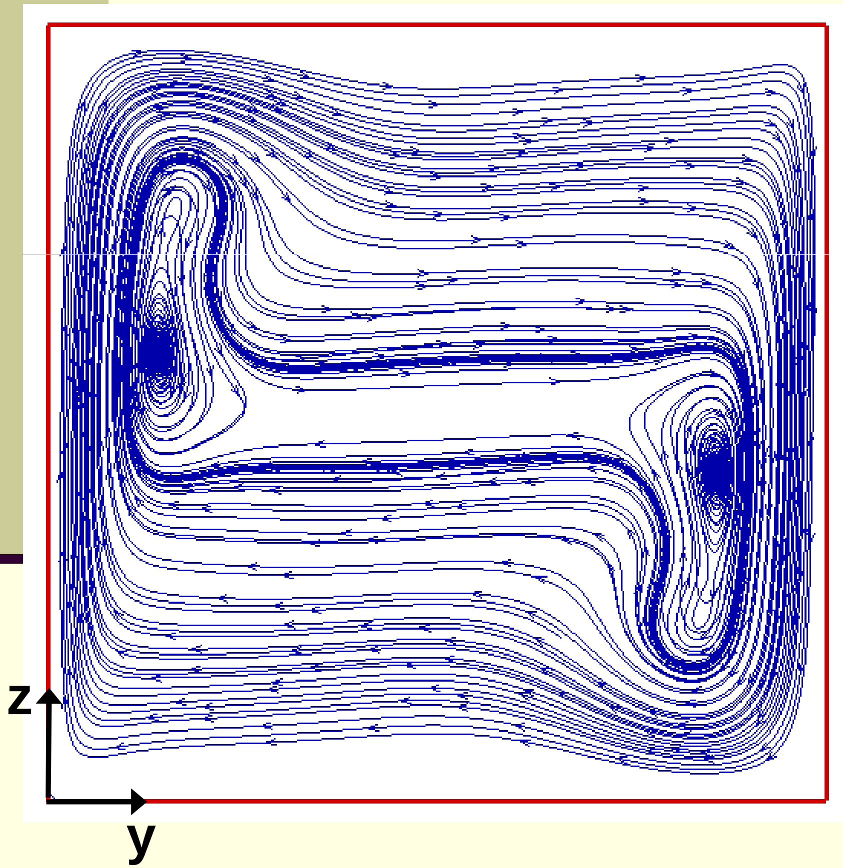
flow at Ra= 10^6

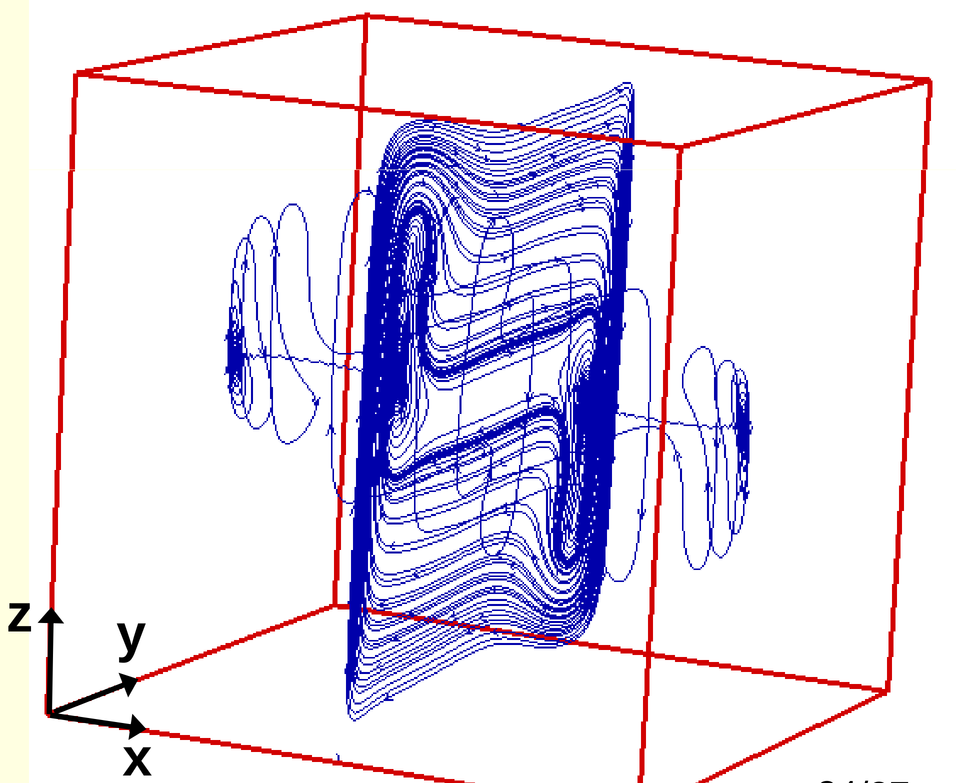




Flow Visualization of Laterally Heated Cavity. Steady State Flow, 103^3 Nodes

flow at Ra= 10^6





Application to the Stability Analysis

Newton iteration for steady state solution

L.S. Tuckerman 1999

$$(N_U + L)u = (N + L)U \qquad U \leftarrow U - u$$

$$\underbrace{\left[\left(I - \Delta t L\right)^{-1}\left(I + \Delta t N_U - I\right)\right]}_{\text{Difference between two consecutive linearized time steps}} u = \underbrace{\left[\left(I - \Delta t L\right)^{-1}\left(I + \Delta t N\left(U\right) - I\right)\right]}_{\text{Difference between two consecutive time steps}} U$$

Difference between two consecutive linearized time steps

Difference between two consecutive time steps

For large Δt $\left(I - \Delta t L\right)^{-1} \approx L^{-1}$

is a preconditioner for $N_U + L$

Application to the Stability Analysis (Cont)

Inverse power method for the leading eigen value

L.S. Tuckerman 1999

$$u_{n+1} = \left(N_U + L\right)^{-1} u_n$$

$$\underbrace{\left[\left(I - \Delta t L\right)^{-1}\left(I + \Delta t N_U - I\right)\right]}_{\text{Difference between two consecutive linearized time steps}} u_{n+1} = \underbrace{\left(I - \Delta t L\right)^{-1}}_{\text{Difference between two consecutive time steps of the Stokes operator}} \Delta t u_n$$

Difference between two consecutive linearized time steps

Difference between two consecutive time steps of the Stokes operator

Good performance for 2D configuration

Still a challenge for 3D configuration

Conclusions

- ✓ **An Accelerated Semi-Analytical Coupled Line Implicit Gauss-Seidel Smoother (ASA-CLGS) and Full Pressure Coupled Direct Solution (FPDS) were developed and implemented for the solution of incompressibel N-S equations.**
- ✓ **The Navier-Stokes and Boussinesq equations are solved without pressure-velocity decoupling.**
- ✓ **The code was verified on existing benchmark solutions for the lid-driven and thermally driven cavities.**
- ✓ **The potential implementation of the developed time marching solvers to the linear stability analysis was studied.**
- ✓ **The characteristic CPU times consumed for a single time step per one node and per one CPU are of order 5 $\times 10^{-3}$ msec and 10^{-2} msec for 2D and 3D calculations, respectively.**