Linear Stability Analysis of Lid Driven Flows Accelerated by an Efficient Fully Coupled Time-Marching Algorithm

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Outline

- **Pressure-velocity coupled formulation of the Navier-Stokes equations**
- **Benchmark problem**
- **Full Pressure Coupled Direct (FPCD) time integration**
- **Application to the steady state solution**
- **Application to the linear stability analysis**
- **Conclusions**

Incompressible N-S Equations – Numerical Challenge

Continuity - $$\nabla \cdot \boldsymbol{u} = 0$$

Momentum- $$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{u}$$

- ➢No separate equation for pressure
- ➢No boundary conditions for pressure

Incompressible N-S Equations – Numerical Challenge (Cont.)

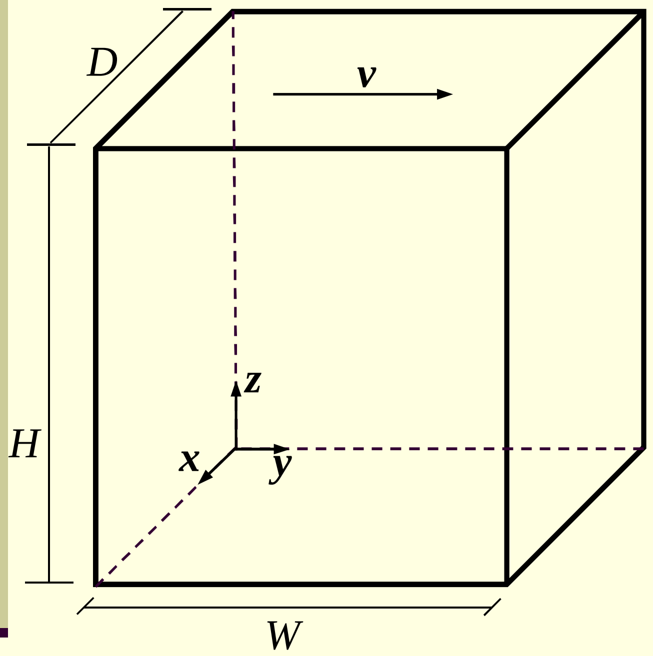
Pressure-Velocity Decoupling Approach

- ✓ High numerical robustness
- ✓ Low memory consumption
- ✘ Slow rate of numerical convergence
- ✘ Non-physical pressure field
- ✘ Not applicable for flow–structures interaction problems

Pressure–Velocity Coupled Approach

- ✓ High rate of numerical convergence
- ✓ The "most natural " way to solve N-S equations
- ✓ The obtained pressure is physical
- ✘ High memory consumption
- ✘ Not as numerically robust as pressure projection methods

Lid-Driven Rectangular and Cubic Cavity



$$\nabla \cdot \boldsymbol{u} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{u}$$

✓**Explicit Discretization**

$$(\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n$$

▪**Semi-Implicit Discretization**

$$(\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^{n+1}$$

Realistic Boundary Conditions:

$\boldsymbol{u} = 0$ - at all static walls no slip/no penetration

$\boldsymbol{u}|_{z=H/W} = v$ -at the moving wall the flow velocity is equal to that of the moving wall itself

No boundary condition for pressure is needed

Discretization in time and space

Second order backward differentiation - $\frac{\partial f^{n+1}}{\partial t} = \frac{3f^{n+1} - 4f^{n} + f^{n-1}}{2\Delta t} + O\left(\Delta t^{2}\right)$

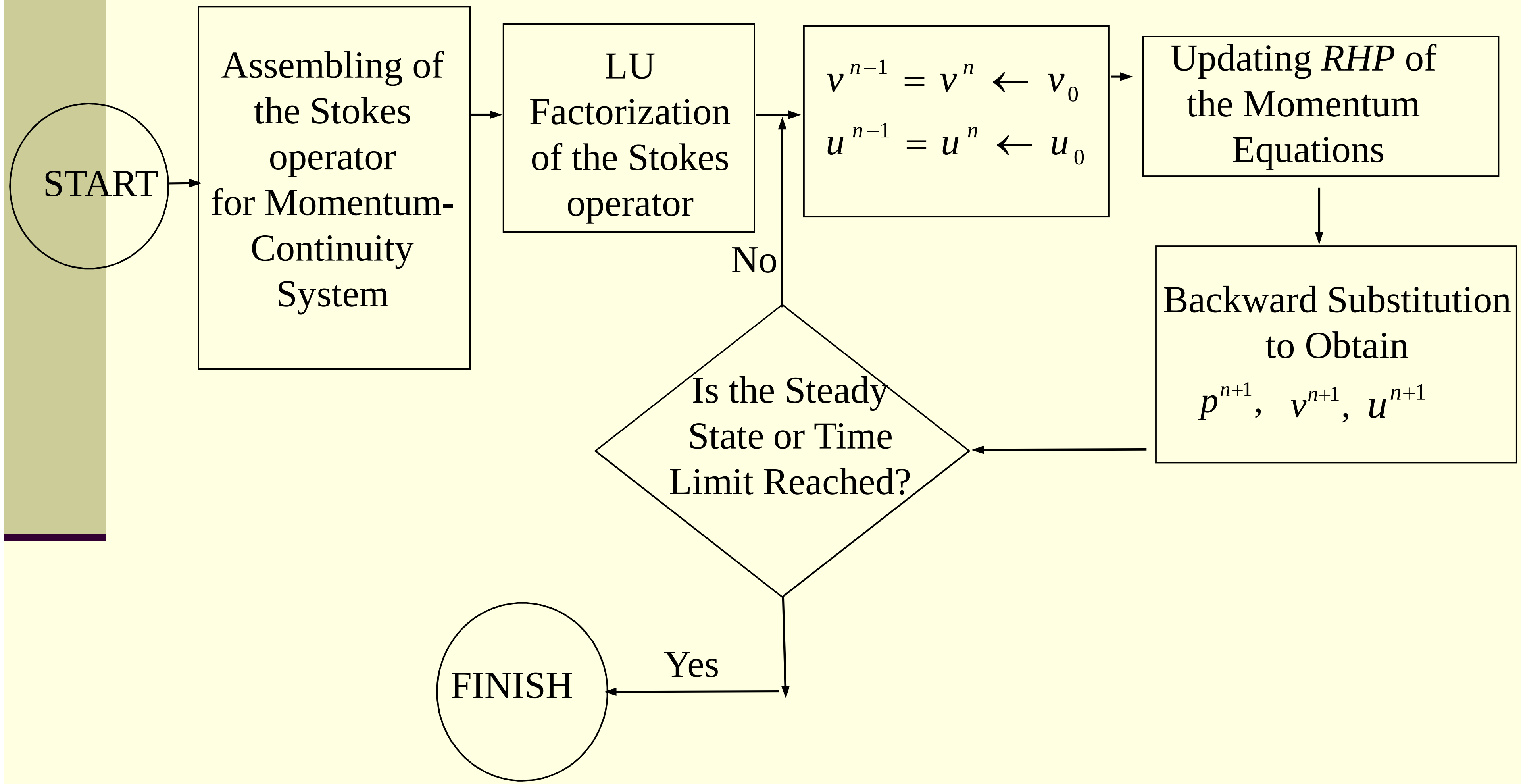
Continuity - $$\frac{\left(u^{\mathrm{n+1}}_{\mathrm{(i,j,k)}} - u^{\mathrm{n+1}}_{\mathrm{(i-1,j,k)}}\right)}{Hx(i-1)} + \frac{\left(v^{\mathrm{n+1}}_{\mathrm{(i,j,k)}} - v^{\mathrm{n+1}}_{\mathrm{(i,j-1,k)}}\right)}{Hy(j-1)} + \frac{\left(w^{\mathrm{n+1}}_{\mathrm{(i,j,k)}} - w^{\mathrm{n+1}}_{\mathrm{(i,j,k-1)}}\right)}{Hz(k-1)} = 0$$

Linearized Navier-Stokes equation; l.h.s. = Stokes operator

Momentum- $$\left(a^{\mathbf{u}}_{\mathrm{(i,j,k)}} - \frac{3}{2\Delta\tau}\right)\boldsymbol{u}^{\mathrm{n+1}}_{\mathrm{(i,j,k)}} + \sum_{\mathrm{(i,j,k)}} a^{\mathrm{u}}_{\mathrm{(i,j,k)}}\boldsymbol{u}^{\mathrm{n+1}}_{\mathrm{(i,j,k)}} - \nabla p^{(n+1)} = RHP^{\mathrm{n}}_{\mathbf{u}}$$

Conservative second order control volume method

The Full Pressure Coupled Direct (FPCD) Time Integration



Obtaining Steady State Solution

Newton iteration for steady state solution

$$(N_U + L)u = (N + L)U \qquad U \leftarrow U - u$$

For large Δt $\left(I - \Delta t L\right)^{-1} \Delta t \approx L^{-1}$

is a preconditioner for $N_U + L$

Krylov Basis Method (BiCG*stab*)

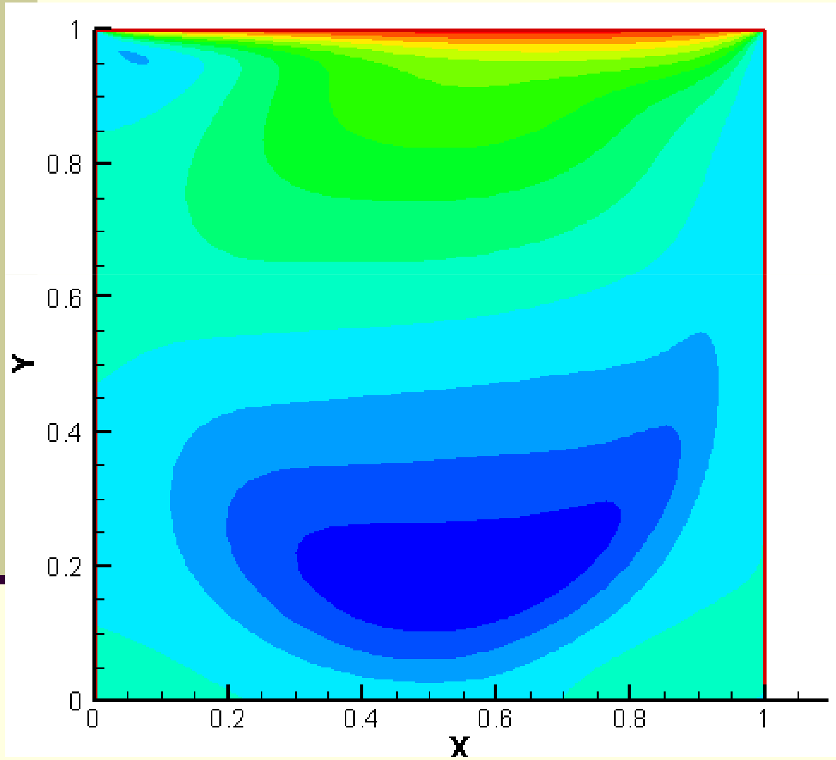
$$\underbrace{\left[\left(I - \Delta t L\right)^{-1}\left(I + \Delta t N_U - I\right)\right]}_{\text{Difference between two consecutive linearized time steps}} u = \underbrace{\left[\left(I - \Delta t L\right)^{-1}\left(I + \Delta t N\left(U\right) - I\right)\right]}_{\text{Difference between two consecutive time steps}} U$$

Difference between two consecutive linearized time steps

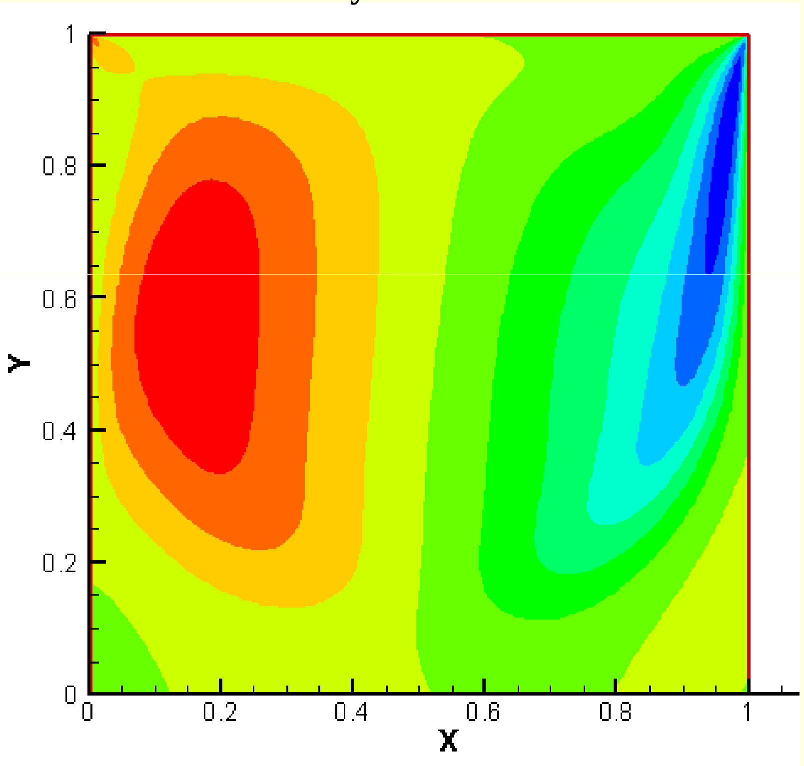
Difference between two consecutive time steps

Lid Driven Cavity- Steady State

V_x , *Re*=1000

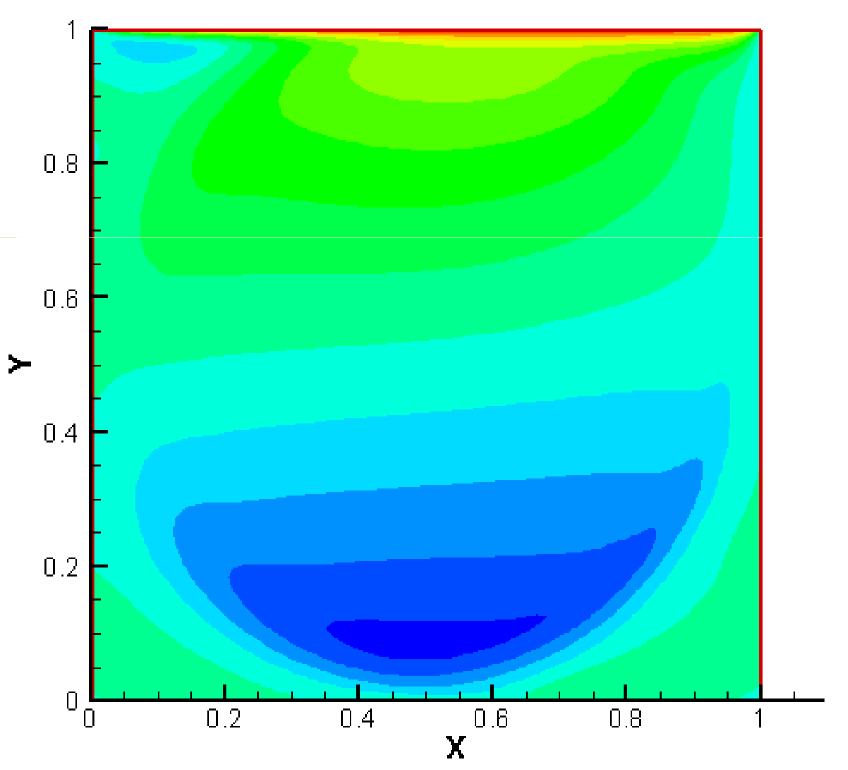


V_y , *Re*=1000

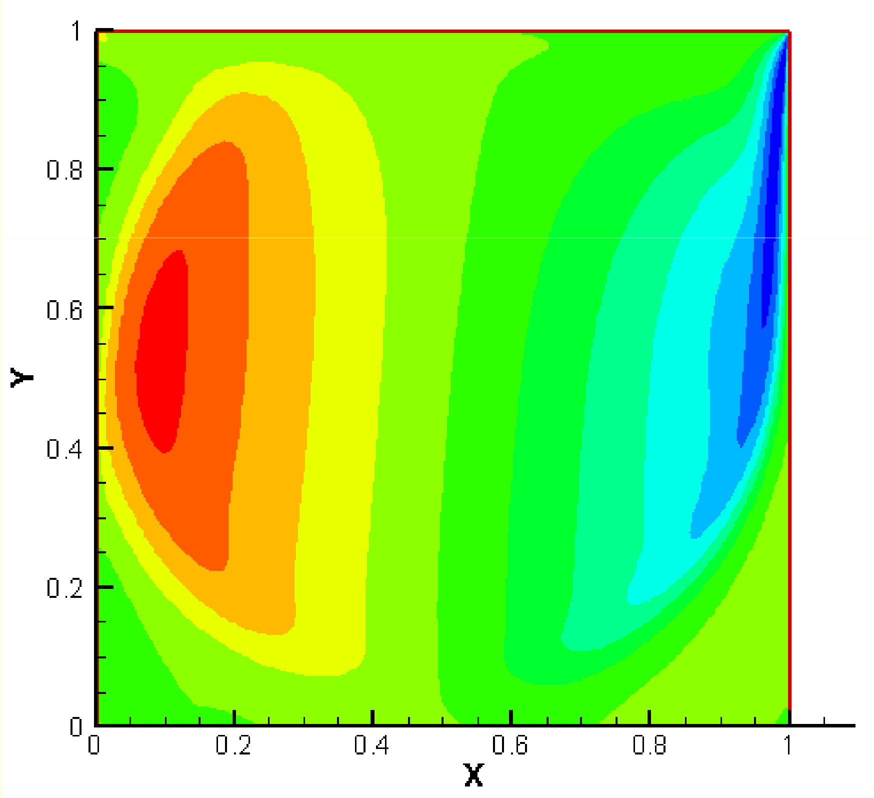


Lid Driven Cavity- Steady State (Cont1)

V_x, ***Re*=4000**

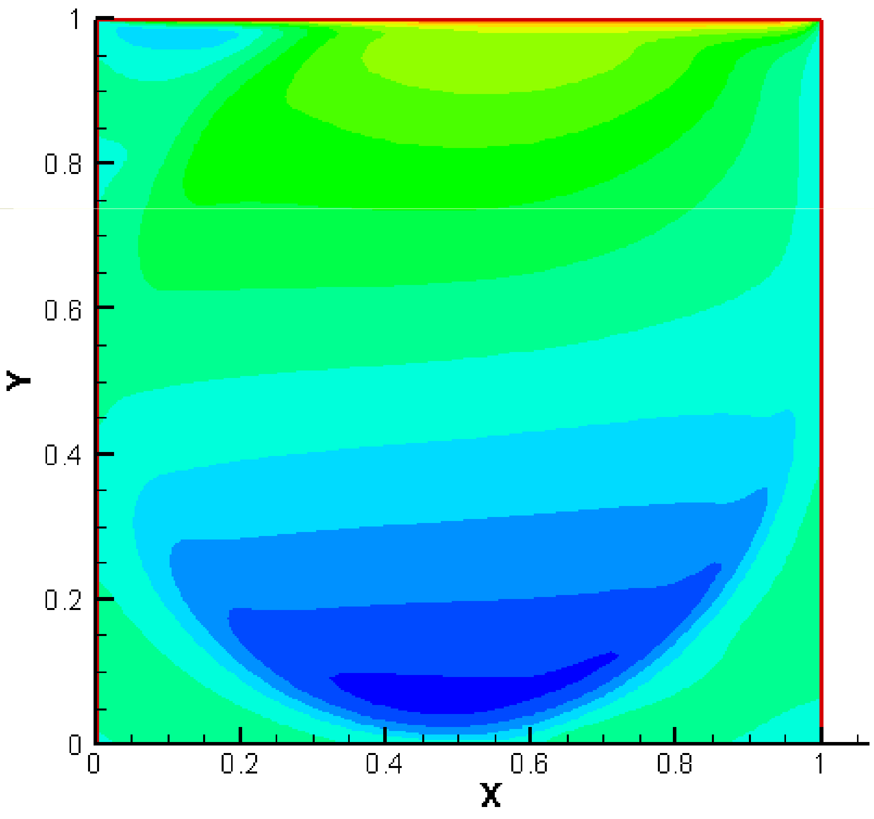


V_y, ***Re*=4000**

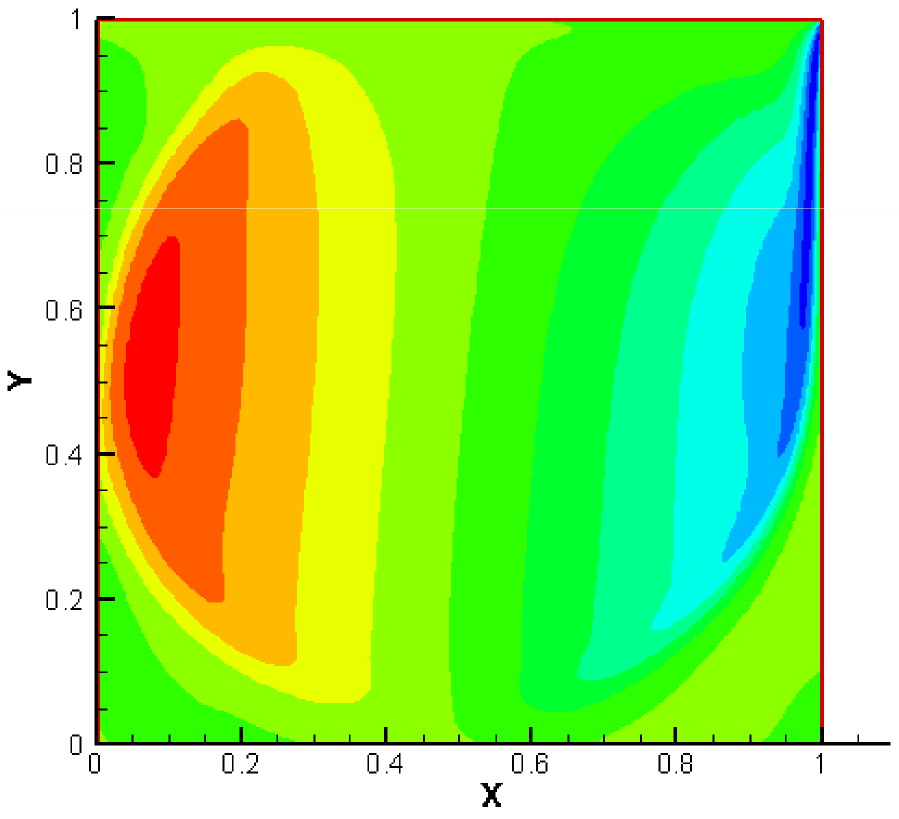


Lid Driven Cavity- Steady State (Cont2

V_x , *Re*=8000



V_y , *Re*=8000



Application to the Linear Stability Analysis

Inverse formulation with Arnoldi iteration

$$u_{n+1} = \left(N_U + L\right)^{-1} u_n$$

Krylov Basis Method (BICG)

$$\underbrace{\left[\left(I - \Delta t L\right)^{-1}\left(I + \Delta t \overbrace{N_U - I\right)\right] u_{n+1} = \left(I}^{\text{Krylov Basis Method (BICG)}} - \Delta t L\right)}_{\text{Difference between two consecutive linearized time steps}} u_{n+1} = \underbrace{\left(I - \Delta t L\right)^{-1}}_{\text{Difference between two consecutive time steps of the Stokes operator}} \Delta t u_n$$

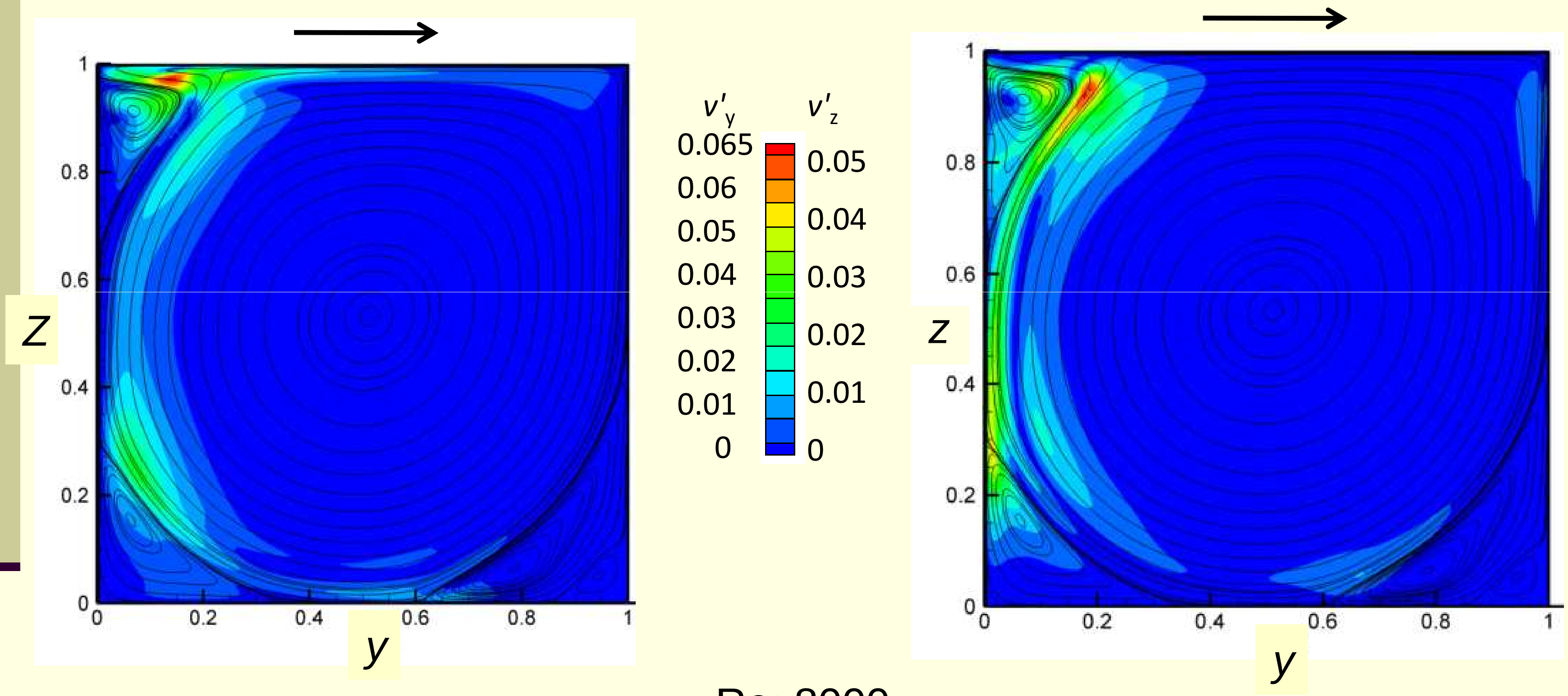
Difference between two consecutive linearized time steps

Difference between two consecutive time steps of the Stokes operator

Good performance for 2D configuration

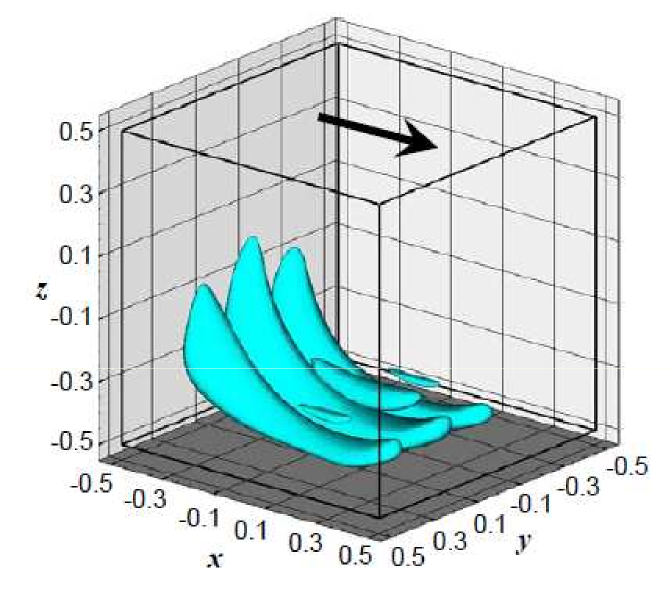
Still a challenge for 3D configuration

Application to the Linear Stability Analysis (Cont)

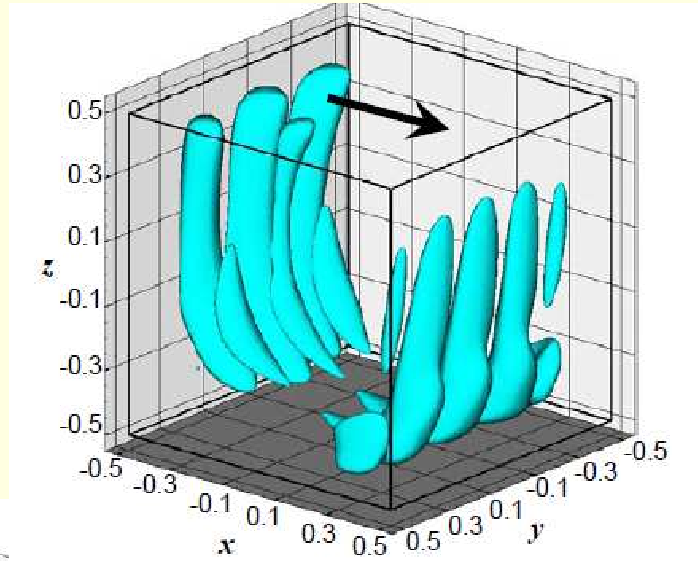


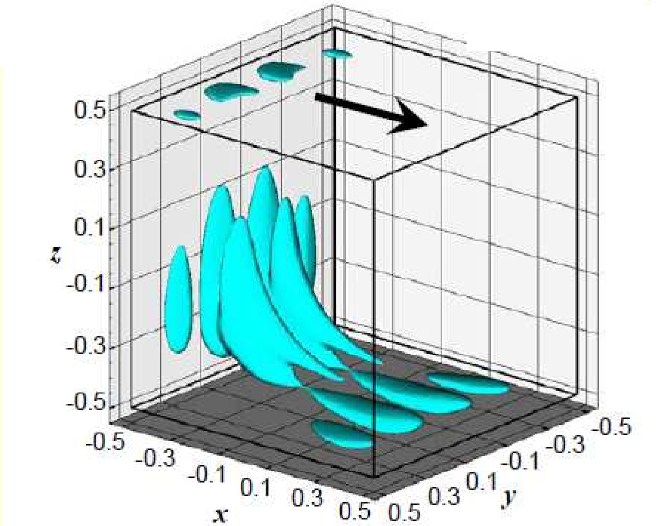
Re≈8000

3D instability: the most unstable eigenvector

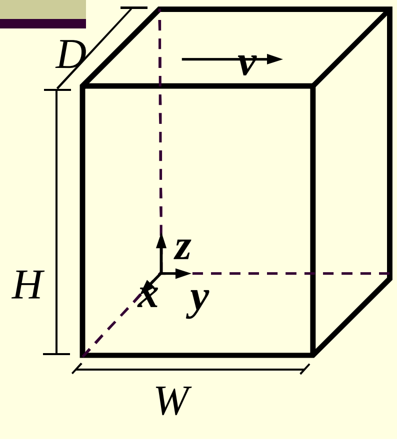


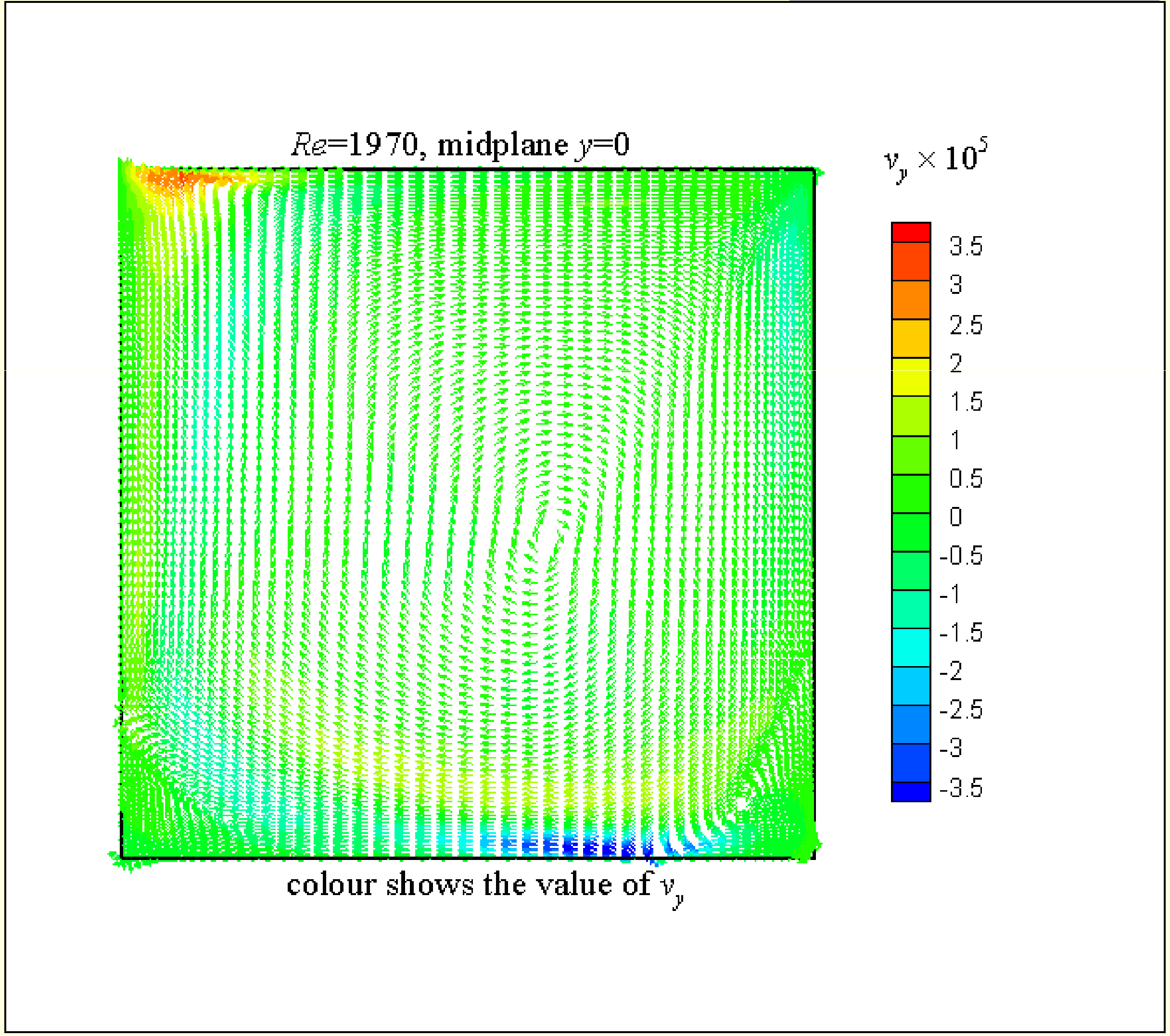
Re≈1920





3D time-dependent computation Pressure-velocity coupled + multigrid





Conclusions

- ✓ **The FPCD approach, utilizing the *LU* decomposition of the Stokes operator, shows competitive computational times for two dimensional problems, but remains restricted by the available computer memory when is applied to three-dimensional models.**

- ✓ **A great advantage of the FPCD approach is a constant and a priori known CPU time consumed at each time step. Apparently it is not a case for any iterative solver.**

- ✓ **The approach may be easily parallelized taking advantage of using massively parallel platforms and allowing its extension to 3-D configurations.**

- ✓ **The approach easily extended to Newton iteration based steady state solves and stability solvers based on inverse Arnoldi iteration**

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Thank You