Test Problem: Rayleigh-Bénard convection

Henk A. Dijkstra

Institute for Marine and Atmospheric Research Utrecht – Utrecht University
Princetonplein 5, 3584 CC Utrecht, The Netherlands

1 Introduction

In this note, several computations are suggested to compare the capabilities of different continuation codes. In the first section, some physical background is given of the problem, followed by the model formulation. In the last section, specific computations are described and points of comparison are highlighted. The participants are free to choose discretization, resolution, graphical output, machines etc. However, when presenting the results during the Colloquium, it should be clear what is actually done.

2 Description of the Physics

The study of the physical problems in the area of cellular convection is motivated by results from a conceptually simple experiment (Fig. 1). A container which may have rectangular or circular cylindrical shape is filled with a relatively viscous liquid, such as silicone oil. Above the upper surface of the liquid is an ambient gas, for example air and the temperature far the gas-liquid interface is nearly constant. When the initially motionless liquid is heated from below, the liquid remains motionless below a critical value of the vertical temperature gradient. The heat transfer through the layer is only by heat conduction. When the temperature gradient slightly exceeds the critical value, the liquid is set into motion and after a while the flow organizes itself into cellular patterns. The motion of the liquid can also be detected by measuring the horizontally averaged vertical heat flux. A measure for the increase of heat transport due to convection is the Nusselt number Nu which is unity in case of conduction only. In Fig. 2, Nu is plotted as a function of a measure of the vertical temperature gradient. The onset of convection in the liquid is shown by the increase of Nu above unity.

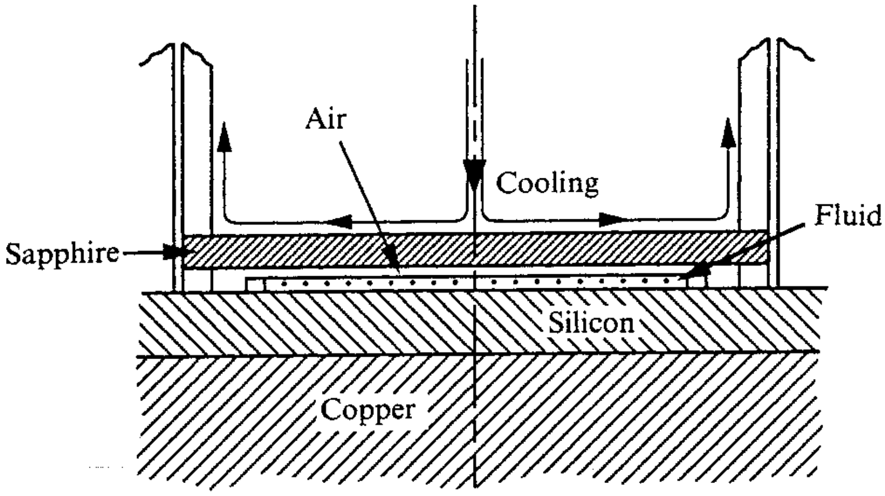

Figure 1: *Sketch of the experimental set-up (from [1]); the liquid is situated on the (heated) silicon block and separated from the (cooled) sapphire block by a small air gap.*

3 Model

In this test problem, we will consider the computation of bifurcation points marking the transition from a motionless to a convecting liquid as it is heated from below in a bounded container. The equations governing the flow are the continuity equation

$$\nabla \cdot \mathbf{v} = 0 \tag{1a}$$

the momentum balances,

$$\rho_0 \left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla \mathbf{v} \right] = -\nabla p + \mu \nabla^2 \mathbf{v} - \rho g \mathbf{e}_3 \tag{1b}$$

and the thermal energy balance

$$\rho_0 C_p \left[\frac{\partial T}{\partial t} + \mathbf{v}.\nabla T \right] = \lambda_T \nabla^2 T \tag{1c}$$

In these equations, (x, y, z) are the Cartesian coordinates of a point in the liquid layer, t denotes time, $\mathbf{v} = (u, v, w)$ is the velocity vector, p denotes pressure, $\mathbf{e}_3$ the unit vector in z-direction and T is the temperature. Finally, ρ_0, g, C_p, μ and λ_T are the reference density, the acceleration due to gravity, the specific heat, the dynamic viscosity and the thermal conductivity, respectively. The thermal diffusivity κ and kinematic viscosity ν are given by $\nu = \frac{\mu}{\rho_0}$ and $\kappa = \frac{\lambda_T}{\rho_0 C_p}$. All these quantities will be assumed constant. A linear equation of state $\rho = \rho_0(1 - \alpha_T(T - T_0))$ is assumed. In the equations above, the Boussinesq approximation is applied which is adequate here since the density variations are small with respect to ρ_0. In this approximation, density variations are only considered in the body force term of (1b) and apart from this, the liquid is considered incompressible.

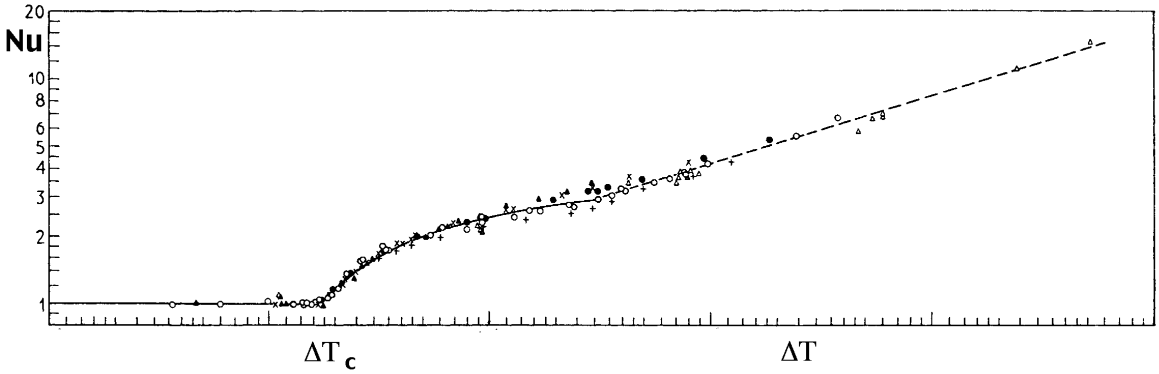

Figure 2: *Plot of the Nusselt number (see text) as a function of the vertical temperature gradient* ΔT *(from [2]);* $Nu = 1$ *if the heat transport is by conduction only and* Nu *increases if there is convection in the liquid;* ΔT_c *is the critical temperature gradient.*

Let the gas-liquid interface be located at $z = d$ and nondeformable, then the boundary conditions become:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0\ ;\ \ -\lambda_T \frac{\partial T}{\partial z} = h(T - T_A) \tag{2a}$$

where h is an interfacial heat transfer coefficient and T_A is the temperature of the gas far from the interface. The lower boundary is a very good conducting boundary and therefore the temperature is constant. Moreover, no-slip conditions apply and hence,

$$z = 0 : T = T_B\ ;\ \mathbf{v} = 0 \tag{2b}$$

On the lateral walls (at $x = 0, L_x$ and $y = 0, L_y$) no-flux and no-slip conditions are prescribed, i.e.

$$x = 0, L_x : u = v = w = \frac{\partial T}{\partial x} = 0 \tag{3a}$$

$$y = 0, L_y : u = v = w = \frac{\partial T}{\partial y} = 0 \tag{3a}$$

4 Motionless solution

For $\bar{\mathbf{v}} = 0$, there is a steady state given by

$$\bar{T}(z) = T_B - \beta z \ ; \ \beta = \frac{h(T_B - T_A)}{\lambda_T + hd} \tag{4a}$$

where the quantity β is the vertical temperature gradient over the layer. The pressure is readily determined from (1b) and if one chooses $T_0 = T_A$ one obtains

$$\bar{p}(z) = p_0 + \rho_0 g([\alpha_T(T_B - T_A) - 1]z - \alpha_T\beta\frac{z^2}{2}) \tag{4b}$$

This motionless solution is characterized by only conductive heat transfer and is easily realized in laboratory experiments.

5 Non-dimensional equations

The equations are non-dimensionalized using scales $\frac{\kappa}{d}$ for velocity, $\frac{d^2}{\kappa}$ for time and d for length. Moreover a dimensionless temperature $\hat{T}$ is introduced through $T = (T_B - T_A)\hat{T} + T_A$. This leads to the non-dimensional problem

$$Pr^{-1}\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v}\right] = -\nabla p + \nabla^2\mathbf{v} + Ra\hat{T}\mathbf{e}_3 \tag{5a}$$

$$\frac{\partial \hat{T}}{\partial t} + \mathbf{v}.\nabla\hat{T} = \nabla^2\hat{T} \tag{5b}$$

The dimensionless boundary conditions become

$$z = 1 : \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0; \frac{\partial \hat{T}}{\partial z} = -Bi\hat{T} \tag{6a}$$

$$z = 0 : \hat{T} = 1 \ ; \ \mathbf{v} = 0 \tag{6b}$$

$$x = 0, A_x : u = v = w = \frac{\partial \hat{T}}{\partial x} = 0 \tag{6c}$$

$$y = 0, A_y : u = v = w = \frac{\partial \hat{T}}{\partial y} = 0 \tag{6d}$$

In the equations (5-6), the dimensionless parameters Pr (Prandtl), Ra (Rayleigh), A_x, A_y and Bi (Biot) appear which are defined as

$$Ra = \frac{\alpha_T g(T_B - T_A)d^3}{\nu\kappa}; \ Pr = \frac{\nu}{\kappa}; \ Bi = \frac{hd}{\lambda_T}; \ A_x = L_x/d; \ A_y = L_y/d \tag{7}$$

and hence there are five parameters in this system of equations. This number reduces to four in the two-dimensional case.

6 Computations

The dimensionless motionless solution is given by

$$\bar{T}(z) = 1 - \frac{Bi}{Bi+1} z \tag{8}$$

6.1 Two-dimensional case

The solution (8) is a solution for all values of Ra, but it becomes unstable if Ra increases above a certain critical value, say Ra_c.

Problem 1: Compute Ra_c as a function of $A_x \; \varepsilon \; [1, 10]$ for fixed $Bi = 1$.

Problem 2: Compute Ra_c as a function of $Bi \; \varepsilon \; [0, 10]$ for fixed $A_x = 10$.

Problem 3: Compute the pattern of the critical mode for the case $A_x = 10, Bi = 1$.

6.2 Three-dimensional case

The solution (8) is also a solution for all values of Ra, and it becomes again unstable if Ra increases above a certain critical value, say Ra_c.

Problem 4: Compute Ra_c as a function of $A_x = A_y \; \varepsilon \; [1, 8]$ for fixed $Bi = 1$.

Problem 5: Compute the patterns of the critical modes for the case $A_y = 4, Bi = 1$ for several values of $A_x \; \varepsilon \; [1, 8]$.

6.3 Presentation

In presenting the results, the following guidelines may be helpful.

1. Sketch the discretization and resolution used.

2. Describe the type of continuation method and the path followed through parameter space.

3. Describe the basic (linear systems and eigenvalue) solvers.

4. Provide the CPU time per steady solution/eigenvalue computation and mention the machine on which the calculations have been done.

5. Describe typical problems encountered.

I have not yet done all the computations myself !

Happy Computations

References

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