Statistical study of undulator radiated power by a classical detection system in the mm-wave regime

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The statistics of FEL spontaneous emission power detected with a detector integration time much larger than the slippage time has been measured in many previous works at high frequencies. In such cases the quantum (shot) noise generated in the detection process is dominant. We have measured spontaneous emission in the Israeli electrostatic accelerator FEL (EA-FEL) operating in the mm-wave lengths. In this regime the detector is based on a diode rectifier for which the detector quantum noise is negligible. The measurements were repeated numerous times in order to create a sample space with sufficient data enabling evaluation of the statistical features of the radiated power. The probability density function of the radiated power was found and its moments were calculated. The results of analytical and numerical models are compared to those obtained in experimental measurements.

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I. INTRODUCTION

Electron devices such as microwave tubes and free-electron lasers (FELs) utilize distributed interaction between an electron beam and an electromagnetic wave. Random electron distribution in the *e*-beam causes fluctuations in current density, identified as shot noise in the beam current. Electrons passing through a magnetic undulator emit radiation, which is called undulator synchrotron radiation [1]. The electromagnetic fields excited by each electron add incoherently, resulting in spontaneous emission having a certain power spectral density [2].

Saldin [3] derived an analytical proportionality relation between the radiation field and the beam current in the frequency domain, applicable for both spontaneous emission and self-amplified spontaneous emission (SASE) which is generated in the high gain regime. Saleh [4] made a distinction between the case of classical detection, as was used in this work, and the case of photon counting by a quantum detector. In the case of detection by use of a diode rectifier, the variance of the number of emitted photons k is given by

$$\sigma_k^2 = \frac{\overline{k^2}}{M} \propto \sigma_{P_T}^2 = \frac{\overline{P_T}^2}{M} \tag{1}$$

and k is described by the Gamma distribution where M is the number of modes. P_T is the detected power [Eq. (12)]. In the case of quantum detection, the variance is given by

$$\sigma_k^2 = \overline{k^2} - \bar{k}^2 = \bar{k} + \bar{k}^2, \tag{2}$$

where the number of photons k is distributed by the Bose-Einstein or geometric probability density function. The above is for a single mode, while in the multimode case the radiation is described by the negative-binomial distribution, with variance

$$\sigma_k^2 = \overline{k^2} - \bar{k}^2 = \bar{k} + \frac{\bar{k}^2}{M}, \tag{3}$$

where M is the number of modes.

Comparison of (3) and (1) shows that in the case of photon counting the term $\bar{k}$ is added, which is the so-called quantum noise. In this case the quantum noise is a significant part of the total noise, whereas for the diode rectifier it is negligible.

Tanabe *et al* [5], at the Brookhaven National Laboratories, measured the negative-binomial distribution the number of photons spontaneously emitted by their FEL.

In this work we study the statistical distribution of the spontaneous radiation power measured in the mm-wave regime for our electrostatic accelerator FEL (EA-FEL).

II. SPONTANEOUS EMISSION

The total spontaneous emission field is given by a summation of the field radiated by each electron entering the wiggler. The field contributions of different wave packets are independent random variables. Therefore, following the central limit theorem [6], for large n the probability density function of the field is Gaussian with zero mean and variance σ_E^2.

The power spectral density that results from the sum of electromagnetic fields excited by electrons passing through a magnetic undulator added incoherently is according to [2]

$$\frac{dP_{\rm sp}}{df}(L_w) = \tau_{\rm sp} P_{\rm sp}(L_w)\,{\rm sinc}^2\left(\frac{1}{2}\theta L_w\right), \tag{4}$$

where $P_{\rm sp}(L_w)$ is the expected value of the total spontaneous emission power at the end of the undulator of length L_w, $\tau_{\rm sp} = |\frac{L_w}{v_z} - \frac{L_w}{v_g}|$ is the slippage time, and $\theta = \frac{\omega}{v_{z0}} - (k_{zq} + k_w)$ is the detuning parameter. In the above equations v_{z0} is the electron axial velocity, v_g is the group velocity of the radiation wave packet, k_z is the axial wave number, k_w is the wiggler wave number, and ω is the angular frequency. The spontaneous emission null-to-null bandwidth is $\frac{2}{\tau_{\rm sp}} \simeq \frac{2f_0}{N_w}$, in which f_0 corresponds to the central frequency for which the detuning is $\theta = 0$ and N_w is the number of periods in the FEL wiggler.

In an FEL utilizing a magnetostatic planar wiggler, the total average power of the spontaneous emission is given by [2]

$$P_{\rm sp}(L_w) = \frac{1}{8} A_{JJ}^2 \frac{eI_0}{\tau_{\rm sp}} \left(\frac{a_w}{\gamma\beta_z}\right)^2 \frac{Z_{01}}{A_{\rm em}} L_w^2, \tag{5}$$

where

$$A_{JJ} = J_0\left[\frac{a_w^2}{2(a_w^2+2)}\right] - J_1\left[\frac{a_w^2}{2(a_w^2+2)}\right]. \tag{6}$$

In our case, since $a_w \ll 1$, we obtain that $A_{JJ} \cong 1$.

The spontaneous emission average power is proportional to eI_0, where I_0 is the DC beam current and e is the electron charge. The wiggler parameter is a_w, $\beta_z = \frac{v_{z0}}{c}$ in which c is the velocity of light in vacuum, and the relativistic Lorentz factor is defined as

$$\gamma = \sqrt{\frac{1}{1 - v^2/c^2}}.$$

The mode effective area is given by

TABLE I. Typical operation parameters of the Israeli EA-FEL.

Magnetostatic planar wiggler	
Magnetic induction	$B_w = 2$ kG
Period	4.444 cm
Number of periods	20
Waveguide	
Parallel curved plated	$R = 17.2$ mm, $b = 10.7$ mm
Effective mode dimensions	$a_{\rm eff} \times b_{\rm eff} = 1.01$ cm $\times$ 0.9005 cm
Radiation mode	TE_{01}
Accelerator	
Beam energy	1.4 MeV
Beam radius	$r_b = 2$ mm
Beam current	$I_0 = 1$ A
Spontaneous emission (theory)	
Slippage time	$\tau_{\rm sp} = \lvert \frac{L_w}{v_z} - \frac{L_w}{v_g} \rvert = 0.112$ ns
Theoretical spontaneous emission power generation	68 μW
Power expected at the detector	0.5 μW

$$A_{\rm em,01} = \frac{a_{\rm eff} b_{\rm eff}}{2} = 4.55 \times 10^{-5}\ {\rm m}^2 \tag{7}$$

and the mode impedance is given by

$$Z_{01} = Z_0 \frac{k}{\sqrt{k^2 - k_c^2}}, \tag{8}$$

where k is the free space wave number,

$$k = \frac{\omega}{c}, \tag{9}$$

k_c is the cutoff wave number,

$$k_c = \frac{2\pi f_c}{c}, \tag{10}$$

f_c is the cutoff frequency,

$$f_c \approx c \frac{\pi}{b_{\rm eff}}, \tag{11}$$

and $b_{\rm eff}$ is the smaller waveguide dimension (Table I).

In the low gain limit, the spontaneous emission power grows linearly with the interaction length L_W.

The spontaneous power spectral density calculated for the electrostatic accelerator FEL with the parameters given in Table I is described in Fig. 1. This EA-FEL is designed to operate in the mm-wave regime.

A. Probability density of the radiated power

The total spontaneous emission power averaged over a time interval T is

$$P_T = \frac{1}{2Z_0} \frac{1}{T} \int_0^T E^2(t) dt. \tag{12}$$

It is a random variable with mean $\bar{P}_T = P_{\rm sp}$ characterized statistically by the Gamma distri-

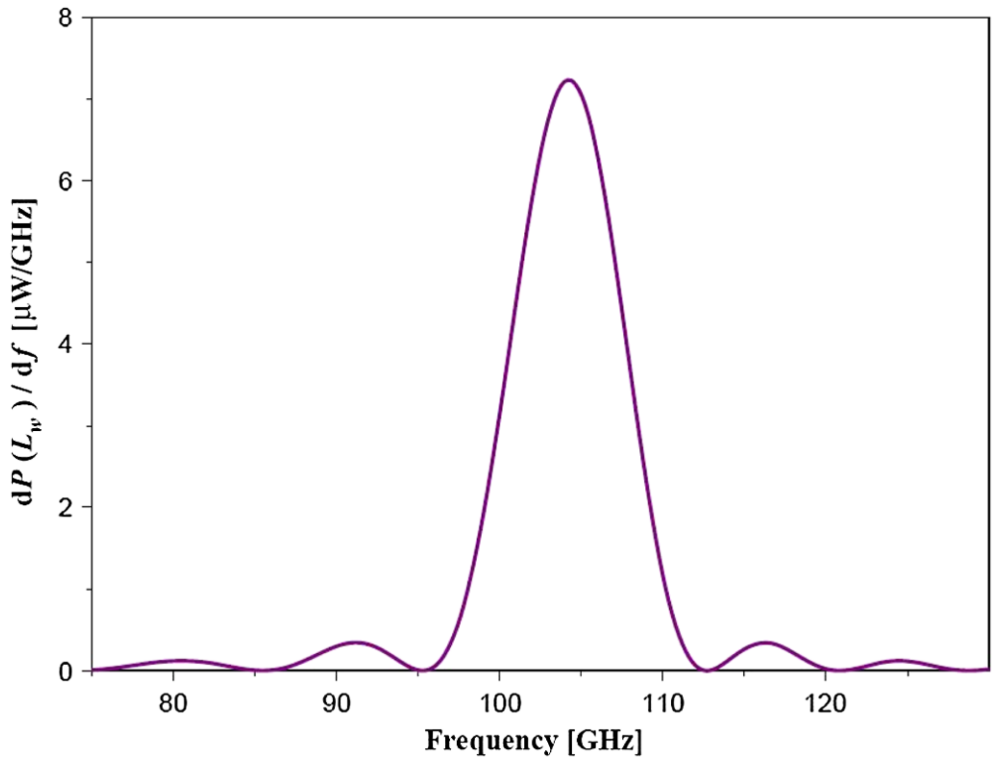

FIG. 1. Power spectral density of spontaneous emission for typical parameters.

bution [3,4,6]

$$f_{P_T}(P_T) = \begin{cases} (\frac{M}{P_{\rm sp}})^M \frac{P_T^{M-1} e^{-M(P_T/P_{\rm sp})}}{\Gamma(M)} & P_T \geq 0 \\ 0 & P_T < 0. \end{cases} \quad (13)$$

Γ stands for the Gamma function. The parameter M is the number of modes within the detection integration time. In this context we define modes as random wave packets of coherence time $\tau_{\rm sp}$ (not resonator modes!) and is given by (15) below.

It is the number of coherence lengths [3,7] within the integration interval T.

As $M \rightarrow 1$, the negative-exponential distribution is obtained. For very high values of M, this is the sum of a very large number of random variables, and, according to the central limit theorem, the distribution becomes Gaussian.

The variance of the power is

$$\sigma_{P_T}^2 = \frac{P_{\rm sp}^2}{M}. \quad (14)$$

III. EXPERIMENTAL RESULTS

In order to compare theoretical and numerical results with experiment, additional components of the measurement setup must be taken into account. Figure 2 illustrates the measurement setup. The detector is a GaAs diode detector. Its -3 dB video bandwidth is $B = 10$ MHz. The integration interval's length is $T = 1/2\pi B = 16$ ns. Since the round-trip time of the resonator is approximately 10 ns, and since the radiation pulse of one electron is the sequence of fully correlated pulses of length $\tau_{\rm sp}$, delayed by 10 ns from each other, two side peaks of correlation function give contribution to the integral. Thus [6]

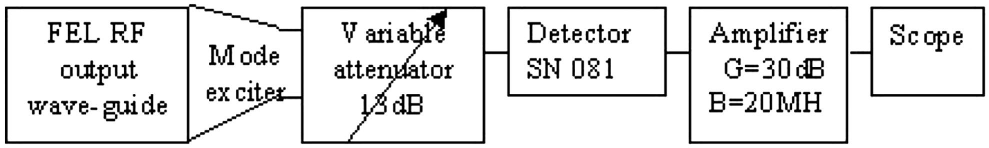

FIG. 2. Measurement setup.

$$M = \frac{T}{3\tau_{\mathrm{sp}}}. \qquad (15)$$

Taking τ_{sp} from Table I and using (15), we find $M = 47$.

Figure 3 shows the model for the diode detector with its equivalent noise sources [8]. No bias voltage is applied. The spreading resistance, R_s, lies typically between 1 Ω and 50 Ω. This is the Ohmic resistance associated with the semiconductor outside the junction and the contacts. The dynamic resistance of the junction, R_j, at zero bias is 2.2 kΩ. C_j is the junction capacitance and its value is 30 fF. R_L, the load resistance, is the amplifier's input impedance. Its value is 20 kΩ. Total detection system responsivity is $R = v_{\mathrm{out}}/P_{\mathrm{rf}} = 1$ μV/nW.

We measured the power of each of one-hundred subsequent pulses of spontaneous emission, trying to maintain FEL conditions as stable as possible from one pulse to the next. The power at 3 μs after the beginning of the pulse, when the power has already reached its peak, was taken as the random variable. Figure 4 shows a few examples of the pulses captured. The average power obtained was $P_{\mathrm{sp}} = 484$ nW, and the standard deviation was $\sigma_{P_T} = 77$ nW. The power measured at the detector is substantially lower than the theoretical value (see Table I), which was calculated from Eq. (5) for the power generation inside the cavity due to the cavity out-coupling coefficient $1 - R = 0.07$ and substantial transmission loss of the waveguide transmission line.

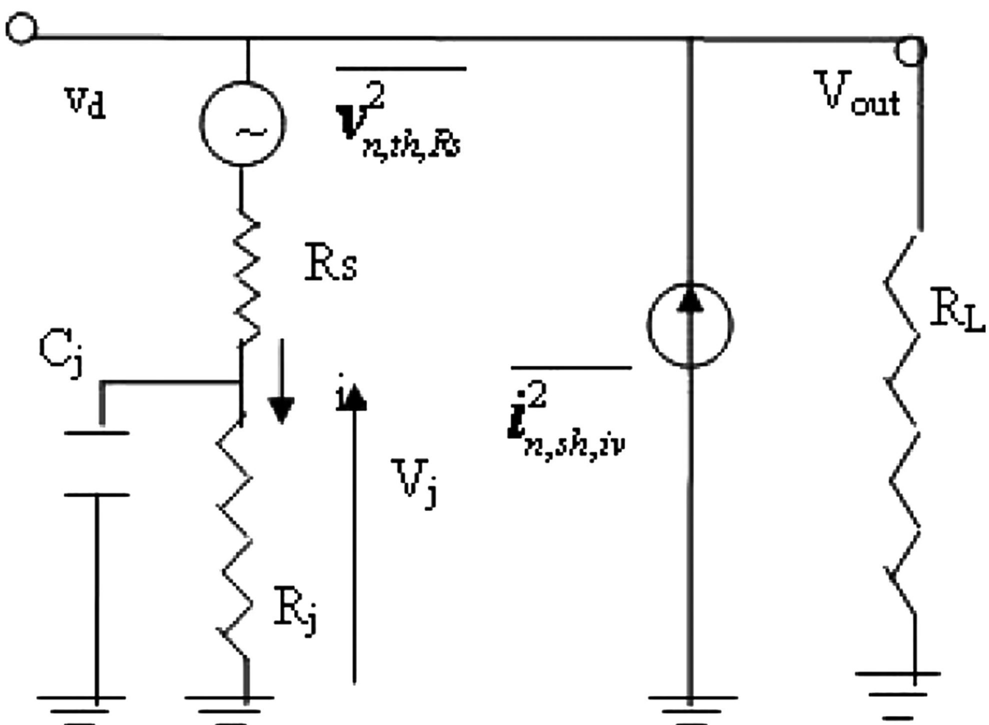

FIG. 3. Diode detector with equivalent noise sources.

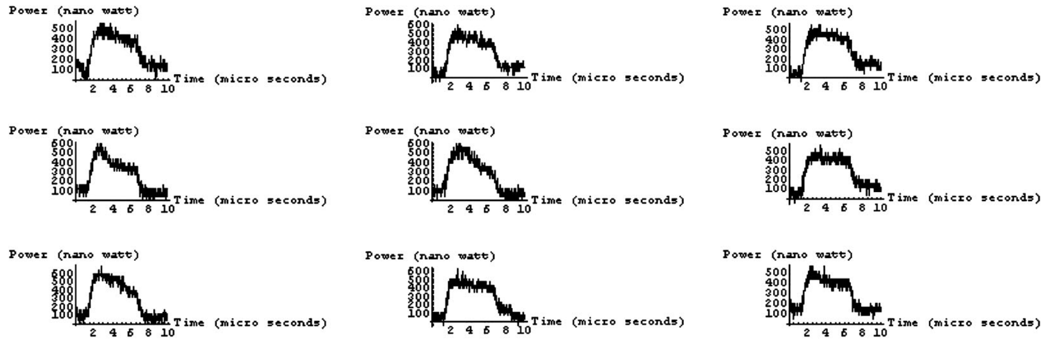

FIG. 4. A few of the pulse signals.

Theoretically, a Gamma distribution is expected for the detected power [Eq. (13)], with the parameter M being the number of modes [Eq. (15)].

We need now to eliminate the effect of the detector shot noise from the variance of the measured distribution. The detector output voltage is the sum of two additive components:

$$V_{\text{field}} = A_v \mathcal{R} P_T \tag{16}$$

$$V_{n,\text{out}} = \sqrt{\overline{\Delta v_{j,R_s}}^2} F A_v, \tag{17}$$

where v_{field} is the output voltage of the amplifier at the detector output, corresponding to the radiated power, $\mathcal{R}$ is the responsivity of the detector, A_v is the voltage gain of the amplifier, and $V_{n,\text{out}}$ is the detector noise voltage at the amplifier output. F is the noise figure.

The standard deviation of the measured detector noise power was $\sigma_n = \sqrt{\frac{\overline{v_n^2}}{\mathcal{R}}} = 45$ nW. The standard deviation of the measured data distribution (blue curve in Fig. 5) is $\sigma_{P_T} = 77$ nW. Since the detector noise and the radiation noise are statistically independent, we estimate the variance of the net radiation power distribution as the difference of the variances of the two distributions:

$$\sigma_{\text{sp}} = \sqrt{\sigma_{PT}^2 - \overline{\sigma_n^2}} = 63 \text{ nW}.$$

This estimate should be compared to the calculated theoretical standard deviation that is calculated from (14) with the measured $P_{\text{sp}} = 484$ nW and the calculated$M = 47$ [Eq. (15)].

The theoretical standard deviation that should be expected for this value of M by (14) and (15) is

$$\sigma_{P_T} = \frac{P_{\text{sp}}}{\sqrt{M}} = 70.6 \text{ nW}.$$

Figure 5 displays the following distributions: (i) The blue curve displays the distribution of the measured data. (ii) The dashed curve displays a normal distribution best fit to the experimental curve with $P_{\text{sp}} = 484$ nW and $\sigma_{P_T} = 77$ nW. (iii) The green curve displays the normal distribution of the net radiation power (detector noise subtracted) with $P_{\text{sp}} = 484$ nW and $\sigma_{\text{sp}} = 63$ nW. (iv) The red curve displays the theoretical Gamma distribution for parameter $P_{\text{sp}} = 484$ nW and $M = 47$, which correspond in reality (since $M \gg 1$) to a Gaussian distribution with $\sigma_{\text{sp}} = 70.6$ nW.

Clearly, the theoretical distribution of the radiation power (red curve) is only slightly narrower than the power without subtraction of the detector noise. This widening must be attributed to technical noise related to the instability of the accelerator parameters including

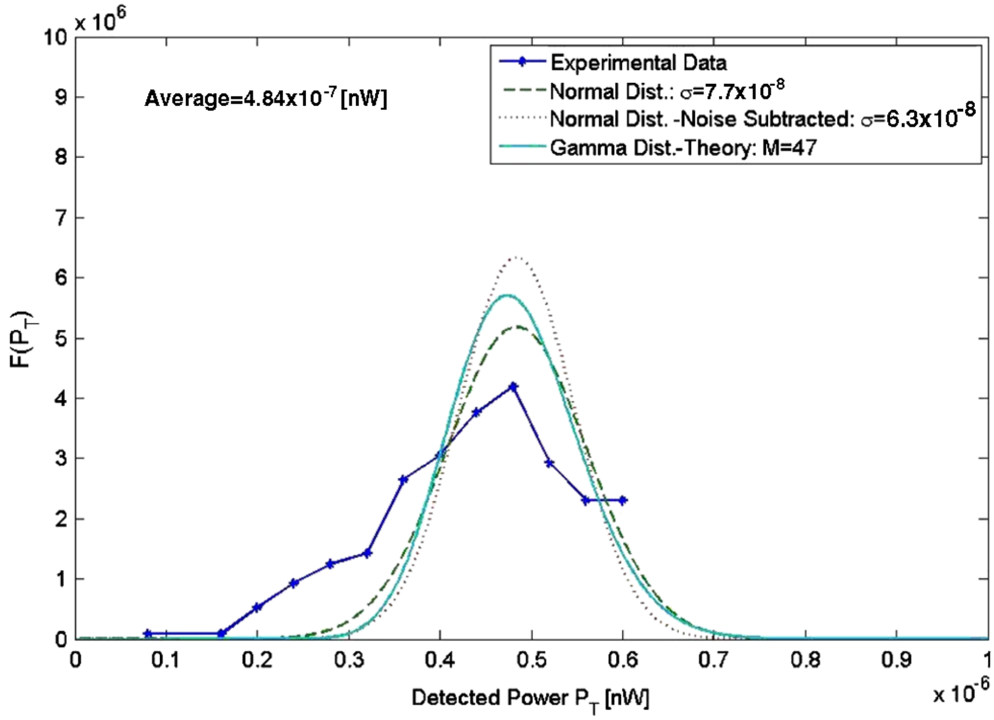

FIG. 5. (Color) Comparison of the experimental data to a Gamma distribution with the theoretical value $M = 47$, the Gaussian approximation for the latter, with standard deviation 77 nW and the normal distribution with standard deviation 63 nW

instability of the accelerator terminal voltage, the *e*-beam current, and the *e*-beam deviated trajectories due to the electron-optics dispersion.

The results of the present experiment indicate that the measured statistical power distribution reflects the inherent features of the spontaneous emission originating from the *e*-beam shot noise. With proper stabilization of the electrostatic accelerator even better results may be achievable in the future.

The classical statistical optics that we implement in this article for the first time to FEL oscillators, predicts that the inherent statistical distribution is a Gamma distribution function. At the experimental parameters of the present setup, the number of modes correspond to $M > 1$ and the distribution appears Gaussian (with or without the technical noise). Non-Gaussian distribution is definitely possible at the low frequency regime (mm and THz waves). However, their observation requires faster detectors ($T \ll 1$ ns) or a longer wiggler.

Using the experimental values for the power mean and variance, $P_{\mathrm{sp}} = 484$ nW and $\sigma_{P_T} = 77$ nW, we obtain by (14)

$$M = \left(\frac{P_{\mathrm{sp}}}{\sigma_{P_T}}\right)^2 = 40$$

which is close to the theoretical value 47 calculated above.

It should be noted that the noise voltage is also assumed Gaussian and, thus, the detected noise power also has the Gamma distribution. Finally, the sum of the two power components is Gamma distributed.

The Gamma distribution functions corresponding to $M = 40$ and $M = 47$ are shown on top of the experimental histogram in Fig. 5, as is the Gaussian approximation for the Gamma distribution with $M = 40$. Since $M \gg 1$, the Gamma distribution tends to be Gaussian.

There is a reasonable fit between the width of the theoretically calculated distribution functions and the experimental histograms. The figure clearly shows that for the value $M = 40$ the Gamma distribution function is closely approximated by the Gaussian function.

The theoretical calculated total generated power is according to Eq. (5) $P_{sp}(L_w) = 68$ uW. The overall losses, including resonator round-trip Ohmic losses and out-coupling losses (together 80%) and transport losses (96%), are 21 dB. Accordingly, the measured power at the detector was 0.5 uW.

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